Math 5220

Representation Theory

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Notes by Ben Salisbury

Contents

Introduction

The goal of this course will be to give the participant a working knowledge of semisimple Lie algebras over the complex numbers. Lie algebras arise as the tangent space to Lie (or algebraic) groups at the identity, along with some extra structure coming from the group law. The study of their structure and representations leads to connections with algebraic geometry and combinatorics, most notably root systems. It also gives a good introduction to general techniques of representation theory which appear in the study of other algebraic objects, (e.g., finite groups, Lie groups, algebraic groups, quivers, finite-dimensional associative algebras, Hecke algebras, etc. . .)

We will follow closely the text by Humphreys [**8**] on Lie algebras to get a classification of semisimple complex Lie algebras in terms of Dynkin diagrams, then move on to a study of weights and representations. I would like to place a much stronger emphasis on examples than the text does, so we will also use the book of Fulton and Harris [**4**] on representation theory. The only prerequisite is a graduate-level course in abstract algebra. Students who are currently taking abstract algebra are welcome to speak with me to check if their background will be sufficient.

R. Kinser

Homework Assignments

The first homework assignment is to review the following facts and concepts, which should be familiar to you from linear algebra. We will use these results without proof throughout the course. The base field is **C** below.

- Eigenvectors and eigenvalues of an endomorphism.
- Jordan canonical form of an endomorphism.
- Minimal and characteristic polynomial of an endomorphism.
- An endomorphism is diagonalizable if and only if its minimal polynomial has distinct roots.
- Two diagonalizable endomorphisms which commute can be simultaneously diagonalized. So if V is a finite-dimensional vector space, and $H \subset \mathrm{End}(V)$ is a subspace consisting of commuting diagonalizable operators, there exists a basis of V with respect to which every element of H is diagonal.
- Bilinear forms: symmetric, positive-definite, nondegenerate. Symplectic forms will be important examples, but not directly used in the theoretical development.
- Tensor product and direct sum of vector spaces.

Exercises marked with an asterisk are more important (not more difficult). Also, look for the ones that are interesting to you. If you like characteristic $p > 0$, there are many exercises illustrating the difference in the theory. Pick some and do them. (All exercises are from [**8**].)

Assignment	Submitted	Suggested
1	1.3*, 1.11, 2.6, 3.2*, 3.4	1.1, 1.5, 1.7, 1.9, 1.12, 2.1, 3.1, 3.3
2	4.1*, 4.5, 5.5*, 6.1*, 6.5	4.7, 5.1, 5.3, 5.8, 6.2*, 6.3*, 6.7

CHAPTER 1

Lie Algebras

1. Definitions

Our base field will be $F = \mathbf{C}$ unless otherwise noted.

1.1. Definition. A ***Lie algebra*** is a $\mathbf{C}$-vector space $\mathfrak{g}$ along with a bracket operation $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$ such that

(1) $[\cdot,\cdot]$ is bilinear;
(2) $[x,x]=0$ for all $x\in\mathfrak{g}$;
(3) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ for all $x,y,z\in\mathfrak{g}$. This last identity is known as the ***Jacobi identity***.

The second condition implies that $[\cdot,\cdot]$ is skew-symmetric; in other words, $[x,y]=-[y,x]$ for all $x,y\in\mathfrak{g}$. We will use the convention that $\mathfrak{g}$ is finite-dimensional. A useful property is that if the first property holds, then it suffices to check the second and third properties for a basis of $\mathfrak{g}$.

1.2. Example. Let $\mathfrak{g}=\mathbf{R}^3$ and define the bracket by $[v,w]=v\times w$ (the usual cross product) for all $v,w\in\mathbf{R}^3$.

1.3. Example. Let $\mathfrak{g}$ be any vector space and let $[\cdot,\cdot] \equiv 0$ on $\mathfrak{g}$. Then the bracket satisfies the properties above. This Lie algebra is called the abelian Lie algebra.

1.4. Example. Let V be an n-dimensional $\mathbf{C}$-vector space and let $\mathfrak{g} = \mathrm{End}(V) \cong \mathrm{Mat}_n(\mathbf{C})$. We define the bracket operation on $\mathfrak{g}$ to be $[A,B] = AB - BA$ for any $A, B \in \mathfrak{g}$. When we want to emphasize the Lie algebra structure of this particular set, we write $\mathfrak{g} = \mathfrak{gl}_n$.

1.5. Definition. A ***subalgebra*** $\mathfrak{h} \subset \mathfrak{g}$ is a vector subspace which is closed under the bracket operation.

1.6. Example. We define the upper triangular matrices to be

$$\mathfrak{t}_n = \left\{ \begin{pmatrix} * & * & \cdots & * \\ & * & \cdots & * \\ & & \ddots & * \\ & & & * \end{pmatrix} \right\}$$

The nilpotent matrices and the diagonal matrices are defined by

$$\mathfrak{n}_n = \left\{ \begin{pmatrix} 0 & * & \cdots & * \\ & 0 & \cdots & * \\ & & \ddots & * \\ & & & 0 \end{pmatrix} \right\}, \qquad \mathfrak{d}_n = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\},$$

respectively. Then $\mathfrak{n}_n \subset \mathfrak{gl}_n$ and $\mathfrak{d}_n \subset \mathfrak{t}_n \subset \mathfrak{gl}_n$.

2. Classical Lie algebras

The special linear Lie algebra is defined to be

$$\mathfrak{sl}_n = \{x \in \mathfrak{gl}_n : \mathrm{Tr}(x) = 0\}.$$

This Lie algebra has a nice basis, given in section 1.2 of [**8**], with dimension $n^2 - 1$. As we will see later, $\mathfrak{sl}_n$ is a "type A_{n-1}" Lie algebra. For practice, it is in the reader's best interest to prove *every* theorem for $\mathfrak{sl}_2$.

Define $J = \left(\begin{smallmatrix} O & I_n \\ -I_n & O \end{smallmatrix}\right)$. The symplectic Lie algebra is defined to be

$$\mathfrak{sp}_{2n} = \{x \in \mathfrak{gl}_{2n} : x^\top J + Jx = 0\}.$$

This is called "type C_n."

The orthogonal algebra is defined to be

$$\mathfrak{so}_n = \{x \in \mathfrak{gl}_n : xB + Bx = 0\},$$

where

$$B = \begin{pmatrix} O & I_k & O \\ I_k & O & O \\ O & O & 1 \end{pmatrix}$$

if $n = 2k+1$, and $B = \left(\begin{smallmatrix} O & I_k \\ I_k & O \end{smallmatrix}\right)$ if $n = 2k$. The case when $n = 2k+1$, we say $\mathfrak{so}_n$ is "type B_k," and when $n = 2k$, we say $\mathfrak{so}_n$ is "type D_k."

3. Derivations

If A is any algebra, a ***derivation*** of A is a linear map $\delta\colon A \longrightarrow A$ that satisfies the Leibniz rule,

$$\delta(ab) = a\delta(b) + \delta(a)b.$$

The collection of all derivations of some fixed algebra A, written $\mathrm{Der}(A)$, forms a Lie algebra. It is clear that $\mathrm{Der}(A)$ is a $\mathbf{C}$-vector space since $\mathrm{Der}(A) \subset \mathrm{End}_{\mathbf{C}}(A)$. Just as we did in the vector space case, the commutator on $\mathrm{Der}(A)$ is defined by $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$. It is left as an exercise to show that $[\delta_1, \delta_2] \in \mathrm{Der}(A)$.

1.7. Example. Let $A = \mathfrak{g}$ be a Lie algebra, and let $x \in \mathfrak{g}$. Define a function $\mathrm{ad}(x)\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ by $y \mapsto [x, y]$. One can show (and should be checked as an exercise) that $\mathrm{ad}(x)$ satisfies the Leibniz rule, so $\mathrm{ad}(x)$ is a derivation of $\mathfrak{g}$. Thus we have a linear map $\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathrm{Der}(\mathfrak{g})$, called the adjoint representation.

4. Structure constants

A (complex) Lie algebra can be described in a way different from the axiomatic definition above. Choose a basis $\{x_1, \ldots, x_n\}$ for $\mathfrak{g}$. Then the bracket is determined by

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^k x_k, \qquad c_{ij}^k \in \mathbf{C}.$$

Thus, any Lie algebra of dimension n is completely determined by these n^3 numbers.

5. Algebraic structure

1.8. DEFINITION. An ***ideal*** $\mathfrak{a} \subset \mathfrak{g}$ is a subspace such that, for all $x \in \mathfrak{a}$ and $y \in \mathfrak{g}$, we have $[x, y] \in \mathfrak{a}$. By anticommutativity, this is equivalent to the requirement $[y, x] \in \mathfrak{a}$.

This definition is analogous to two-sided ideals in a ring, or normal subgroups of a group. Insofar, if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals, then $\mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{a} + \mathfrak{b}$ are ideals. Additionally, if $\mathfrak{a}$ an ideal and A is a subalgebra, then $\mathfrak{a} + A$ is a subalgebra.

1.9. EXAMPLE. The ***center*** of a Lie algebra $\mathfrak{g}$ is defined as

$$Z(\mathfrak{g}) := \{z \in \mathfrak{g} : [x, z] = 0 \text{ for all } x \in \mathfrak{g}\}.$$

This is the kernel of the linear map $\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathrm{Der}(\mathfrak{g})$.

1.10. EXAMPLE. The ***derived algebra***, denoted $[\mathfrak{g}, \mathfrak{g}]$ or $\mathfrak{g}'$, is the subspace generated by $\{[x, y] : x, y \in \mathfrak{g}\}$. That is, it is the space of finite linear combinations of commutators. If $\mathfrak{g}$ is abelian, then $[\mathfrak{g}, \mathfrak{g}] = 0$. Also, for any classical algebra, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. It is a good exercise for the reader to check this equality for $\mathfrak{g} = \mathfrak{sl}_n$.

1.11. DEFINITION. A Lie algebra is ***simple*** if it is nonabelian and has no ideals besides the zero ideal and the whole Lie algebra.

1.12. EXAMPLE. Each classical algebra is simple; however, this is highly nontrivial. We will show, here, that $\mathfrak{sl}_2$ is simple. Let $\mathfrak{a}$ be a nonzero ideal in $\mathfrak{sl}_2$. To verify that $\mathfrak{sl}_2$ is simple, we need to show that $\mathfrak{a} = \mathfrak{sl}_2$. It is enough to show that any of x, y, or h is in $\mathfrak{a}$, where $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Indeed, if $h \in \mathfrak{a}$, then $[h, x] = 2x$ and $[h, y] = -2y$, which implies $2x \in \mathfrak{a}$ and $-2y \in \mathfrak{a}$. If $x \in \mathfrak{a}$ or $y \in \mathfrak{a}$, then $[x, y] = h$ implies $h \in \mathfrak{a}$. Thus $\mathfrak{a} = \mathfrak{sl}_2$ in these cases. (This is the same method to approach Exercise 2.6 of [**8**].)

In general, take a typical nonzero $v = ax + by + ch$ from $\mathfrak{a}$. Then $(\mathrm{ad}\, x)^2 v = [x, [x, v]] \in \mathfrak{a}$. But $(\mathrm{ad}\, x)^2 v = -2bx$, so $-2bx \in \mathfrak{a}$. Similarly,

$(\operatorname{ad} y)^2 v = -2ay \in \mathfrak{a}$. If either a or b is nonzero, then either $x \in \mathfrak{a}$ or $y \in \mathfrak{a}$. [Here, we are assuming that the characteristic of the field is not 2. For ease, suppose $F = \mathbf{C}$.] If both $a = b = 0$, then $v = ch$ is a nonzero element of $\mathfrak{a}$, so $h \in \mathfrak{a}$ and $\mathfrak{a} = \mathfrak{sl}_2$.

1.13. NONEXAMPLE. When $\mathfrak{g} = \mathfrak{gl}_n$, then $\mathfrak{g}$ is not simple because its center $Z(\mathfrak{g}) = \mathfrak{s}_n = \{aI_n : a \in F\}$ is a nontrivial ideal.

1.14. DEFINITION. If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, then the ***quotient*** $\mathfrak{g}/\mathfrak{a}$ has a natural Lie algebra structure given by

$$[x \bmod \mathfrak{a}, y \bmod \mathfrak{a}] = [x, y] \bmod \mathfrak{a}.$$

1.15. DEFINITION. A ***homomorphism*** between Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is a linear map $\varphi\colon \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ such that

$$\varphi([x, y]) = [\varphi(x), \varphi(y)], \qquad x, y \in \mathfrak{g}_1.$$

Note that $\ker \varphi$ is an ideal in $\mathfrak{g}_1$ and $\operatorname{im} \varphi$ is a subalgebra of $\mathfrak{g}_2$. We'll constantly use terminology and facts about homomorphisms that were stated in the vector space case. For example, the homomorphism theorems hold in the Lie algebra case. We will not prove these results here.

1.16. DEFINITION. A ***representation*** of $\mathfrak{g}$ is a homomorphism $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, where V is a $\mathbf{C}$-vector space.

1.17. EXAMPLE. The adjoint representation is a representation. In particular, $\operatorname{ad}\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$. To prove this, we need to check that $\operatorname{ad}([x, y]) = [\operatorname{ad}(x), \operatorname{ad}(y)]$ in $\mathfrak{gl}(\mathfrak{g})$. Let $z \in \mathfrak{g}$. We have $\operatorname{ad}([x, y])(z) = [[x, y], z]$ and

$$\begin{aligned}
[\operatorname{ad}(x), \operatorname{ad}(y)](z) &= \operatorname{ad}(x) \operatorname{ad}(y)(z) - \operatorname{ad}(y) \operatorname{ad}(x)(z) \\
&= [x, [y, z]] - [y, [x, z]] \\
&= [x, [y, z]] + [y, [z, x]] \\
&= -[z, [x, y]] = [[x, y], z].
\end{aligned}$$

Thus ad is a representation.

1.18. Example. The classical algebras all have a ***standard representation***, given by the inclusion map $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, for some vector space V.

6. Solvable and nilpotent Lie algebras

1.19. Example. Throughout this section, consider this motivating example. Consider the Lie algebra of upper triangular matrices in $\mathfrak{gl}_3$. Let

$$x = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}, \qquad y = \begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix}$$

be elements in $\mathfrak{t}_3$. Then,

$$[x, y] = \begin{pmatrix} 0 & b(d' - a') + b'(a - d) & c(f' - a') + c'(a - f) + be' - b'e \\ 0 & 0 & e(f' - d') + e'(d - f) \\ 0 & 0 & 0 \end{pmatrix},$$

so the derived algebra, denoted here by $\mathfrak{t}_3^{(1)}$, is the set of all elements of the form

$$\mathfrak{t}_3^{(1)} = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Further, computing the derived algebra of this subalgebra gives the set of all elements of the form

$$\mathfrak{t}_3^{(2)} = \left\{ \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

It is clear that $\mathfrak{t}_3^{(3)} = 0$. We wish to study any Lie algebra which satisfies the property that $\mathfrak{g}^{(n)} = 0$ for some n. This property is called solvability, which we define more precisely below.

Let $\mathfrak{g}$ be any Lie algebra. Recall that the derived algebra is an ideal in $\mathfrak{g}$. Denote this ideal by $\mathfrak{g}^{(1)}$. Recursively, define $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$, for $i > 0$. By convention, we denote $\mathfrak{g}^{(0)} = \mathfrak{g}$.

1.20. Definition. The chain of ideals $\mathfrak{g} = \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots$ is called the ***derived series*** of $\mathfrak{g}$. We say that $\mathfrak{g}$ is ***solvable*** if $\mathfrak{g}^{(n)} = 0$ for some n.

1.21. Example (cf. Exercise 3.1 in [**8**]). Each $\mathfrak{g}^{(i)}$ is an ideal of $\mathfrak{g}$. Indeed, we use induction on i, and the base case is clear. Assume that $\mathfrak{g}^{(i)}$ is an ideal. To show $\mathfrak{g}^{(i+1)}$ is an ideal, note that it is spanned by elements of the form $[x, y]$ with $x, y \in \mathfrak{g}^{(i)}$. So it is enough to show that $[z, [x, y]] \in \mathfrak{g}^{(i+1)}$ for all such $x, y \in \mathfrak{g}^{(i)}$ and all $z \in \mathfrak{g}$. But

$$[z, [x, y]] = -[x, [y, z]] - [y, [z, x]]$$

by the Jacobi identity. Then $[y, z]$ and $[z, x]$ are in $\mathfrak{g}^{(i)}$ by the induction hypothesis. Since $x, y \in \mathfrak{g}^{(i)}$, we have both $[x, [y, z]]$ and $[y, [z, x]]$ are in $\mathfrak{g}^{(i+1)}$, and thus the linear combination above is in $\mathfrak{g}^{(i+1)}$.

1.22. Example. Every abelian Lie algebra is solvable.

1.23. Example. If $\mathfrak{g}$ is a simple Lie algebra, then $\mathfrak{g}$ is not solvable.

1.24. Example. As we saw in Example 1.19, the Lie algebra $\mathfrak{t}_3$ is solvable. More generally, $\mathfrak{t}_n$ is solvable for any $n \geqslant 2$. Define e_{ij} to be the $n \times n$ matrix with a 1 in the (i, j)-position and zeros elsewhere. Then $\{e_{ij} : 1 \leqslant i \leqslant j \leqslant n\}$ is a basis of $\mathfrak{t}_n$. By directly computing the derived algebra (which is omitted here), we see that $\mathfrak{t}_n = \mathfrak{n}_n$ because $\mathfrak{n}_n$ has basis $\{e_{ij} : i < j\}$ and each $e_{ij} = [e_{ii}, e_{ij}]$. Define the level of e_{ij} to be $j - i$. The basic idea is that we get a basis of $\mathfrak{t}_n^{(i)}$ consisting of e_{ij} of level at least 2^i. Hence $\mathfrak{t}_n^{(k)} = 0$ for $k \gg 0$.

1.25. Proposition. *Let $\mathfrak{g}$ be a Lie algebra.*

(1) *If $\mathfrak{g}$ is solvable, then any subalgebra and any quotient of $\mathfrak{g}$ is solvable.*
(2) *If $\mathfrak{a} \subset \mathfrak{g}$ is a solvable ideal such that $\mathfrak{g}/\mathfrak{a}$ is solvable, then $\mathfrak{g}$ is solvable.*
(3) *If $\mathfrak{a}$ and $\mathfrak{b}$ are both solvable ideals in $\mathfrak{g}$, then $\mathfrak{a} + \mathfrak{b}$ is a solvable ideal.*

1.26. Remark. The first two assertions can be written as for any short exact sequence $0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{a} \longrightarrow 0$, the outer two terms are solvable if and only if the middle term is solvable.

Proof. To prove (1), we prove something more general. That is, we prove $\mathfrak{g}^{(i)}$ is functorial in $\mathfrak{g}$. This means that if $\varphi\colon \mathfrak{g} \longrightarrow \mathfrak{g}_1$ is a Lie algebra homomorphism, then $\varphi(\mathfrak{g}^{(i)}) \subset \mathfrak{g}_1^{(i)}$ for all i. This can be proved using induction on i. Then apply to inclusions and quotients. In particular, when φ is onto or injective, then $\varphi^{(i)}\colon \mathfrak{g}^{(i)} \longrightarrow \mathfrak{g}_1^{(i)}$ is onto or injective, respectively.

To prove (2), let $\pi\colon \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{a}$ be the reduction map. Then $\mathfrak{g}/\mathfrak{a}$ solvable implies $(\mathfrak{g}/\mathfrak{a})^{(n)} = 0$ for some n. But $\pi(\mathfrak{g}^{(n)}) \subset (\mathfrak{g}/\mathfrak{a})^{(n)} = 0$, which implies $\mathfrak{g}^{(n)} \subset \ker \pi = \mathfrak{a}$. Since $\mathfrak{a}$ is solvable, we have $\mathfrak{a}^{(m)} = 0$ for some m, so $(\mathfrak{g}^{(n)})^{(m)} \subset \mathfrak{a}^{(m)} = 0$. Since $(\mathfrak{g}^{(n)})^{(m)} = \mathfrak{g}^{(n+m)}$, it follows that $\mathfrak{g}$ is solvable.

Lastly, we prove (3) using the isomorphism theorems. We have the isomorphism $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \cong \mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})$. The quotient on the right is solvable by part (1) and $\mathfrak{a}$ is solvable by assumption, so $\mathfrak{a}+\mathfrak{b}$ is solvable by part (2). ■

1.27. Remark. By part (3), any Lie algebra $\mathfrak{g}$ has a *unique* maximal solvable ideal, called the ***radical*** of $\mathfrak{g}$, denoted $\operatorname{rad}\mathfrak{g}$. Equivalently, $\operatorname{rad}\mathfrak{g}$ is the sum of all solvable ideals in $\mathfrak{g}$.

1.28. Definition. If $\operatorname{rad}\mathfrak{g} = 0$, then we say that $\mathfrak{g}$ is ***semisimple***.

1.29. Example. Every simple Lie algebra is semisimple.

1.30. Example. By part (2) of Proposition 1.25, for any $\mathfrak{g}$, one can show that $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is semisimple.

1.31. Definition. The ***descending central series*** of a Lie algebra $\mathfrak{g}$ is recursively defined by $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}]$, $\mathfrak{g}^2 = [\mathfrak{g},\mathfrak{g}^1]$, and $\mathfrak{g}^i = [\mathfrak{g},\mathfrak{g}^{i-1}]$ for $i > 1$. We say $\mathfrak{g}$ is ***nilpotent*** if $\mathfrak{g}^n = 0$ for some n.

1.32. Example. Since $\mathfrak{g}^{(i)} \subset \mathfrak{g}^i$ for all i, any nilpotent algebra is solvable.

1.33. Example. The nilpotent matrices in $\mathfrak{gl}_n$ form a nilpotent Lie algebra.

1.34. Example. The Lie algebra $\mathfrak{t}_n$ is solvable but not nilpotent.

1.35. Proposition. *Let $\mathfrak{g}$ be a Lie algebra.*

(1) *If $\mathfrak{g}$ is nilpotent, then any subalgebra or quotient algebra is quotient.*
(2) *If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then $\mathfrak{g}$ is nilpotent.*
(3) *Any nilpotent algebra has a nontrivial center.*

Proof. (1) The central series construction is functorial in $\mathfrak{g}$, so $\varphi\colon \mathfrak{g}_1 \longrightarrow \mathfrak{g}$ induces a map $\varphi^i\colon \mathfrak{g}_1^i \longrightarrow \mathfrak{g}^i$ which is injective (resp. surjective) when φ is injective (resp. surjective). The result follows.

(2) Similarly, $(\mathfrak{g}/Z(\mathfrak{g}))^n = 0$ for some n, so $\mathfrak{g}^n \subset Z(\mathfrak{g})$. But then

$$\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n] \subset [\mathfrak{g}, Z(\mathfrak{g})] = 0.$$

(3) The next to last nonzero term in the central series must be contained in the center. ■

7. Nilpotence and ad

Our short term goal is to take these ideas to prove the following classical theorems. We combine the two statements into one statement:

If $\mathfrak{g}$ is a solvable (resp. nilpotent) subalgebra of $\mathfrak{gl}(V)$, then $\mathfrak{g}$ is contained in $\mathfrak{t}_n$ (resp. $\mathfrak{n}_n$) for some choice of basis for V.

This will apply to images of representations $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

If $\mathfrak{g}$ is nilpotent with $\mathfrak{g}^n = 0$, then for any $y \in \mathfrak{g}$ and any sequence $x_1, \ldots, x_n \in \mathfrak{g}$, we have

$$[x_1, [\cdots, [x_{n-1}, [x_n, y]] \cdots]] = 0$$

Written more succinctly, we have

$$\mathrm{ad}(x_1) \cdots \mathrm{ad}(x_n)(y) = 0.$$

In particular, if $x_i = x$ for all i and some $x \in \mathfrak{g}$, then $(\mathrm{ad}\, x)^n(y) = 0$. So $\mathrm{ad}(x)$ is nilpotent in $\mathrm{End}_{\mathbf{C}}(\mathfrak{g})$, for any $x \in \mathfrak{g}$.

1.36. Definition. If $\mathrm{ad}(x)$ is nilpotent as an element of $\mathrm{End}_{\mathbf{C}}(\mathfrak{g})$, then we say x is ad-***nilpotent***.

1.37. Lemma. *If $x \in \mathfrak{gl}(V)$ is nilpotent, then $\mathrm{ad}(x)$ is nilpotent.*

Proof. Any $x \in \mathfrak{gl}(V)$ gives a map $\lambda_x \colon \mathrm{End}_{\mathbf{C}}(V) \longrightarrow \mathrm{End}_{\mathbf{C}}(V)$ by $y \mapsto xy$. Similarly, we have a map $\rho_x \colon \mathrm{End}_{\mathbf{C}}(V) \longrightarrow \mathrm{End}_{\mathbf{C}}(V)$ by $\rho_x(y) = yx$. Each of these maps is nilpotent and they commute; that is, $\lambda_x\rho_x = \rho_x\lambda_x$. Then

$$(\lambda_x - \rho_x)^n = \sum_{i=1}^{n} \binom{n}{i} \lambda_x^i \rho_x^{n-i}$$

is zero for large enough n. So $\lambda_x - \rho_x$ is nilpotent. But

$$\lambda_x - \rho_x = \mathrm{ad}(x). \qquad \blacksquare$$

1.38. Theorem. *Let V be a nonzero vector space and let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a subalgebra consisting of nilpotent endomorphisms. Then $\mathfrak{g}$ has a common eigenvector; i.e., there exists some $v \in V$ such that $\mathfrak{g}(v) = 0$.*

1.39. Nonexample. Let $V = \mathbf{C}^2$ and $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then x and y have no common eigenvectors. This does not contradict the theorem because there is no subalgebra of $\mathfrak{gl}_2$ consisting of of nilpotent endomorphisms containing both x and y.

Proof. We will use induction on $\dim \mathfrak{g}$.

Step 1. We will show that $\mathfrak{g}$ has a codimension one ideal. Let $\mathfrak{h}$ be a *maximal* proper subalgebra of $\mathfrak{g}$. There is a representation of $\mathfrak{h}$, namely $\rho \colon \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ by $h \mapsto (x \bmod \mathfrak{h} \mapsto [h, x] \bmod \mathfrak{h})$. It is left to the reader to check that this map is well-defined, using that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Since $\mathfrak{h}$ consists of nilpotent endomorphisms, by Lemma 1.37 we get $\operatorname{im}\rho \subset \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ consists of nilpotents (and $\dim \operatorname{im}\rho < \dim \mathfrak{g}$). By induction, we get a nonzero vector $x_0 \bmod \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$, which implies that $x_0 \notin \mathfrak{h}$, such that $[h, x_0] \bmod \mathfrak{h} = \mathfrak{h}$ for all $h \in \mathfrak{h}$. Thus $[h, x_0] \in \mathfrak{h}$. In other words,

$$x_0 \in \{y \in \mathfrak{g} : [h, y] \in \mathfrak{h} \text{ for all } h \in \mathfrak{h}\},$$

called the ***normalizer*** of $\mathfrak{h}$ in $\mathfrak{g}$. (It is the largest subalgebra of $\mathfrak{g}$ in which $\mathfrak{h}$ is an ideal.) Pictorially, we have

$$\begin{tikzcd} \mathfrak{g} \arrow[d, dash] \\ N_{\mathfrak{g}}(\mathfrak{h}) \arrow[d, dash] \\ \mathfrak{h} \end{tikzcd}$$

The condition $\mathfrak{h} \neq N_{\mathfrak{g}}(\mathfrak{h})$ forces equality:

$$\begin{tikzcd} \mathfrak{g} \arrow[d, equal] \\ N_{\mathfrak{g}}(\mathfrak{h}) \arrow[d, dash] \\ \mathfrak{h} \end{tikzcd}$$

Hence $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{h}$ is an ideal, if $\dim \mathfrak{g}/\mathfrak{h} > 1$, then any nonzero x mod $\mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$ would give a subalgebra $\mathfrak{h} \subsetneq \mathfrak{h} + \mathbf{C}x \subsetneq \mathfrak{g}$, which contradicts the maximality of $\mathfrak{h}$.

Step 2. Let $W = \{v \in V : \mathfrak{h}v = 0\}$, which is nonempty by the induction hypothesis. For all $x \in \mathfrak{g}$ and $w \in W$, we find that $x(w) \in W$, since

$$hx(w) = xh(w) - [x,h](w)$$

for all $h \in \mathfrak{h}$. The first term is zero by definition of W and the second is zero because $\mathfrak{h}$ is an ideal. Now choose $z \in \mathfrak{g} \setminus \mathfrak{h}$ so $\mathfrak{g} = \mathfrak{h} + \mathbf{C}z$. Then z has an eigenvector $w_0 \in W$ since z acts on W, and so $\mathfrak{g}w_0 = 0$. Thus w_0 is the common eigenvector. ∎

As a benefit to the reader, and to better understand this proof, try to prove the theorem from scratch using a more naive method. For instance, try using a basic induction the $\dim \mathfrak{g}$ or $\dim V$. This will/should illustrate the need for a more complicated proof.

1.40. Corollary. *Under the same conditions as the theorem, there exists a flag $V_0 \subset \cdots \subset V_n$ such that $\mathfrak{g}(V_i) \subset V_{i-1}$. This gives a basis of V with respect to $\mathfrak{g} \subset \mathfrak{n}_n$.*

Proof. By the theorem, take $v \in V$ such that $\mathfrak{g}(v) = 0$. Set $V_1 = \mathbf{C}v$. Then $\mathfrak{g}$ acts on $W := V/V_1$ by nilpotent endomorphisms. By induction on $\dim V$ we get a flag $0 = W_0 \subset W_1 \subset \cdots \subset W_{n-1}$ such that $\mathfrak{g}(W_i) \subset W_{i-1}$. Take $V_i = \pi^{-1}(W_{i-1})$ where $\pi\colon V \twoheadrightarrow W$ is the canonical reduction map given by

$$\begin{pmatrix} 0 & * & \cdots & * \\ & 0 & \cdots & * \\ & & \ddots & * \\ & & & 0 \end{pmatrix}. \qquad \blacksquare$$

1.41. Corollary (Engel's Theorem). *For any Lie algebra $\mathfrak{g}$, if every element of $\mathfrak{g}$ is* ad*-nilpotent, then $\mathfrak{g}$ is nilpotent.*

Proof. We have $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$. By hypothesis, $\mathrm{ad}(\mathfrak{g})$ is a subalgebra of nilpotent endomorphisms of $\mathfrak{g}$. Indeed, there exists some nonzero $x \in \mathfrak{g}$ such that $[\mathfrak{g}, x] = 0$; equivalently, such that $\mathrm{ad}(\mathfrak{g})(x) = 0$. But the kernel of ad is $Z(\mathfrak{g})$, so $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}/Z(\mathfrak{g})$. By the theorem, $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{n}_n$ in some basis, hence nilpotent. Since $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, we have $\mathfrak{g}$ is nilpotent. $\blacksquare$

1.42. Theorem. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be solvable, where V is a nonzero vector space. Then V contains a common eigenvector for $\mathfrak{g}$.*

1.43. Example. Let $\mathfrak{g} = \mathfrak{t}_3 \cap \mathfrak{sl}_3$. The basis for which is

$$\{h_1, h_2, e_{12}, e_{13}, e_{23}\}.$$

We can compute $[h_1, e_{12}] = 2e_{12}$, $[h_2, e_{12}] = -e_{12}$, and

$$[e_{13}, e_{12}] = [e_{13}, e_{23}] = [e_{12}, e_{23}] = 0.$$

So e_{12} is a common eigenvector for $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$.

Proof. *Step* 1. We find a codimension one ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is solvable, we have $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$. Then $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is a nonzero abelian algebra.

Hence any subspace is an abelian ideal. Any codimension one subspace of $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ then lifts to a codimension one ideal in $[\mathfrak{g},\mathfrak{g}] \subsetneq \mathfrak{h} \subsetneq \mathfrak{g}$.

Step 2. By induction on $\dim \mathfrak{g}$, $\mathfrak{h}$ has a common eigenvector, say $v_0 \in V$. There exists a linear functional $\lambda\colon \mathfrak{h} \longrightarrow \mathbf{C}$ which satisfies $h(v_0) = \lambda(h)v_0$ for all $h \in \mathfrak{h}$.[1] For this fixed λ, consider

$$W := \{w \in V : h(w) = \lambda(h)w \text{ for all } h \in \mathfrak{h}\}.$$

Notice $v_0 \in W$, so W is nonzero.

Step 3. Postponed for now. In the meantime, assume $\mathfrak{g}(W) \subset W$.

Step 4. As before, $\mathfrak{g} = \mathfrak{h} + \mathbf{C}z$ with $z(W) \subset W$ and z has an eigenvector (since $\mathbf{C}$ is algebraically closed). Say $z(w_0) = cw_0$ for some $c \in \mathbf{C}$. Then for any $x = h + \alpha z \in \mathfrak{g}$, where $h \in \mathfrak{h}$, we get

$$z(w_0) = h(w_0) + \alpha z(w_0) = \lambda(h)w_0 + \alpha c w_0 = (\lambda(h) + \alpha c)w_0. \qquad \blacksquare$$

1.44. Corollary (Lie's Theorem). *If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a solvable Lie algebra, then there is a basis of V with respect to which $\mathfrak{g} \subset \mathfrak{t}_n$.*

To finish the proof of Lie's theorem, we use a lemma from [**4**].

1.45. Lemma. *Let $\mathfrak{h} \subset \mathfrak{g}$ be an ideal, let V be a representation of $\mathfrak{g}$, and let $\lambda\colon \mathfrak{h} \longrightarrow \mathbf{C}$ be linear. Define*

$$W = \{w \in V : h(w) = \lambda(h)w \textit{ for all } h \in \mathfrak{h}\}.$$

Then $xW \subset W$ for all $x \in \mathfrak{g}$.

Proof. For any nonzero $w \in W$ and $x \in \mathfrak{g}$, test by applying $h \in \mathfrak{h}$:

$$h(x(w)) = x(h(w)) + [h,x](w) = \lambda(h)x(w) + \lambda([h,x])w.$$

By the definition of W, we have $x(w) \in W$ if and only if $\lambda([h,x]) = 0$ for all $h \in \mathfrak{h}$. Consider the subspace U spanned by $\{w, x(w), x^2(w), \dots\}$. This subspace is constructed so that $xU \subset U$. Also, we can inductively see $hU \subset U$ for all $h \in \mathfrak{h}$. We have

$$h(x^i(w)) = h(x(x^{i-1}(w))) = x(h(x^{i-1}(w))) + [h,x](x^{i-1}(w)).$$

[1] This is a crucial step! In our example, we have $v_0 = e_{12}$ and $\lambda = 2h_1^\vee - h_2^\vee$, where $h_1^\vee$ and $h_2^\vee$ are elements of the dual basis.

The first term on the right-hand side is an element of U, and the second term is an element of U by induction. The matrix for h in this basis is given by

$$h = \begin{array}{c} \\ w \\ xw \\ x^2w \\ \vdots \\ x^{n-1}w \end{array} \begin{array}{c} \begin{array}{ccccc} w & xw & x^2w & \cdots & x^{n-1}w \end{array} \\ \begin{pmatrix} \lambda(h) & * & * & \cdots & * \\ 0 & \lambda(h) & * & \cdots & * \\ 0 & 0 & \lambda(h) & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda(h) \end{pmatrix} \end{array}.$$

So $h \in \mathfrak{h}$ is an upper triangular matrix with respect to this basis. Hence $\mathrm{Tr}_U(h) = n\lambda(h)$, which implies $\mathrm{Tr}_U([h,x]) = n\lambda([h,x])$. On the other hand,

$$\mathrm{Tr}_U([h,x]) = \mathrm{Tr}_U(hx - xh) = \mathrm{Tr}_U(hx) - \mathrm{Tr}_U(xh) = 0.$$

Hence $n\lambda([h,x]) = 0$, which implies $\lambda([h,x]) = 0$ for all $h \in \mathfrak{h}$ and $x \in \mathfrak{g}$. ■

1.46. Corollary. *If $\mathfrak{g}$ is a solvable Lie algebra, then there exists a chain of ideals $0 = \mathfrak{g}_0 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ such that $\dim \mathfrak{g}_i = i$.*

Proof. We have $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{t}_n \subset \mathfrak{gl}(\mathfrak{g})$, and a subspace $U \subset \mathfrak{g}$ is stable under ad if and only if it is an ideal. ■

1.47. Corollary. *If $\mathfrak{g}$ is solvable, then $\mathrm{ad}(\mathfrak{g})(x)$ is nilpotent for all $x \in [\mathfrak{g}, \mathfrak{g}]$.*

Proof. We have $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is solvable, so we again get a basis of $\mathfrak{g}$ for which $\mathrm{ad}(y)$ is upper triangular for all y. Then $\mathrm{ad}([y,z]) = [\mathrm{ad}(y), \mathrm{ad}(z)]$ is strictly upper triangular, and hence nilpotent. But $[\mathfrak{g}, \mathfrak{g}]$ is spanned by such $[y,z]$. ■

In particular, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent whenever $\mathfrak{g}$ is solvable. Indeed, for $x \in \mathfrak{g}$, the restriction $\mathrm{ad}([\mathfrak{g}, \mathfrak{g}])(x)$ is nilpotent. So by Engel's theorem, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

8. Jordan-Chevalley decomposition

In this section, we work over any algebraically closed field F. Recall that any matrix over F is conjugate to one with diagonal blocks of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Note that $x \in \mathrm{End}_F(V)$ is ***semisimple*** if and only if it is diagonalizable.

1.48. Proposition. *Let V be a finite-dimensional vector space over F and let $x \in \mathrm{End}_V(F)$.*

(1) *There exists a unique commuting endomorphisms x_s and x_n in $\mathrm{End}_F(V)$ such that $x = x_s + s_n$, where x_s is semisimple and x_n is nilpotent.*
(2) *There exist $p, q \in F[T]$ with zero constant terms such that $p(x) = x_s$ and $q(x) = x_n$.*
(3) *If $A \subset B \subset V$ are subspaces such that $xB \subset A$, then $x_sB \subset A$ and $x_nB \subset A$.*

The decomposition is called the ***Jordan decomposition***.

1.49. Lemma. *Let x be an endomorphism of V with Jordan decomposition $x = x_s + x_n$. Then $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$ is the Jordan decomposition of $\mathrm{ad}(x)$ in $\mathrm{End}(\mathrm{End}(V))$.*

Proof. We need to verify that $\mathrm{ad}(x_s)$ is semisimple, $\mathrm{ad}(x_n)$ is nilpotent, and $\mathrm{ad}(x_s)$ and $\mathrm{ad}(x_n)$ commute. We have already shown that if x_n, then $\mathrm{ad}(x_n)$ is nilpotent. Let $\{v_1, \ldots, v_n\}$ be a basis of V such that $x_s(v_i) = a_i v_i$. Identify $\mathrm{End}(V)$ with Mat_n using the basis $\{v_1, \ldots, v_n\}$, and let $\{e_{ij}\}$ be the standard basis of Mat_n under the identification $\mathrm{End}(V) \cong \mathrm{Mat}_n$. In particular, $e_{ij} = \delta_{jk} v_i$. Then

$$\mathrm{ad}(x_s)(e_{ij}) = (a_i - a_j) e_{ij}.$$

Indeed, apply this to some v_k:

$$\begin{aligned}(\mathrm{ad}(x_s)(e_{ij}))(v_k) &= [x_s, e_{ij}](v_k) \\ &= x_s e_{ij}(v_k) - e_{ij} x_s(v_j) \\ &= (a_i - a_j) e_{ij}(v_k).\end{aligned}$$

So this shows $\{e_{ij}\}$ is a basis of eigenvectors for $\mathrm{ad}(x_s)$. Hence $\mathrm{ad}(x_s)$ is semisimple. To show that $\mathrm{ad}(x_s)$ and $\mathrm{ad}(x_n)$ commute, observe

$$[\mathrm{ad}(x_s), \mathrm{ad}(x_n)] = \mathrm{ad}([x_s, x_n]) = 0. \qquad \blacksquare$$

1.50. Lemma. *For any finite-dimensional F-algebra A, we have the set of derivations* $\mathrm{Der}(A) \subset \mathrm{End}_F(A)$ *and* $\mathrm{Der}(A)$ *contains the semisimple and nilpotent parts of its elements.*

Proof. Let $\delta \in \mathrm{Der}(A)$ and let $\delta = \sigma + \nu$ be its Jordan decomposition in $\mathrm{End}(A)$. Since $\nu = \delta - \sigma$, it suffices to show that $\sigma \in \mathrm{Der}(A)$. For any $\lambda \in F$, set

$$A_\lambda = \{a \in A : (\delta - \lambda I)^k(x) = 0 \text{ for some } k\}.$$

For example, consider

$$\delta = \left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right).$$

Then

$$\delta - 2I = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

So using a basis $\{e_1, e_2, e_3, e_4\}$, then $A_2 = \langle e_1, e_2, e_3 \rangle$. Similarly,

$$\delta - 3I = \left(\begin{array}{cc|cc} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

so $A_3 = \langle e_4 \rangle$. These are called generalized eigenspaces of δ, and $A_\lambda \neq 0$ if and only if λ is an eigenvalue. Moreover, $A = \bigoplus_\lambda A_\lambda$.

Now $\sigma(x) = \lambda x$ for any $x \in A_\lambda$. Furthermore, one can show $A_\lambda A_\mu = A_{\lambda+\mu}$ using

$$\big(\delta - (\lambda+\mu)I\big)^n(xy) = \sum_{i=0}^{n} \binom{n}{i} (\delta - \lambda I)^{n-i}(x) \cdot (\delta - \mu I)^i(y).$$

We want to show σ is a derivation. Let $x \in A_\lambda$ and $y \in A_\mu$. Then $\sigma(xy) = (\lambda+\mu)xy$ and $\sigma(x)y + x\sigma(y) = \lambda xy + \mu xy = (\lambda+\mu)xy$. But the property of being a derivation can be checked on a basis, and since $A = \bigoplus_\lambda A_\lambda$, we are done. ■

9. Cartan's criterion and the Killing form

1.51. Lemma. *Let $A \subset B \subset \mathfrak{gl}(V)$ be subspaces and set*

$$M = \{x \in \mathfrak{gl}(V) : [x, B] \subset A\}.$$

Then any $x \in M$ which satisfies $\mathrm{Tr}(xy) = 0$, for all $y \in M$, is nilpotent.

1.52. Theorem (Cartan's criterion). *For $\mathfrak{g} \subset \mathfrak{gl}(V)$, if $\mathrm{Tr}(xy) = 0$ for all $x \in [\mathfrak{g},\mathfrak{g}]$ and $y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable.*

Proof. It is enough to show $x \in [\mathfrak{g},\mathfrak{g}]$ is nilpotent. Then $\mathrm{ad}(x)$ is nilpotent for all $x \in [\mathfrak{g},\mathfrak{g}]$, so $[\mathfrak{g},\mathfrak{g}]$ is nilpotent by Engel's theorem. Then use Proposition 1.25 to get

$$0 \longrightarrow [\mathfrak{g},\mathfrak{g}] \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \longrightarrow 0.$$

To show $x \in [\mathfrak{g},\mathfrak{g}]$ is nilpotent, we use the lemma with $V = \mathfrak{g} = B$ and $A = [\mathfrak{g},\mathfrak{g}]$. Then $M = \{x \in \mathfrak{gl}(V) : [x,\mathfrak{g}] \subset [\mathfrak{g},\mathfrak{g}]\}$ and $M \supset \mathfrak{g}$. We assumed $\mathrm{Tr}(xy) = 0$ for $x \in [\mathfrak{g},\mathfrak{g}]$ and $y \in \mathfrak{g}$, but we need $\mathrm{Tr}(xy) = 0$ for all $y \in M$. Let $y \in M$ be arbitrary and $x = [u,v]$ be a typical generator of $[\mathfrak{g},\mathfrak{g}]$. Then

$$\mathrm{Tr}(xy) = \mathrm{Tr}([u,v]y) = \mathrm{Tr}(u[v,y]) = \mathrm{Tr}([v,y]u),$$

but $[v,y] \in [\mathfrak{g},\mathfrak{g}]$ and $u \in \mathfrak{g}$, so $\mathrm{Tr}(xy) = 0$ by our assumption. Now apply the lemma to $x \in [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g} \subset M$ to get x is nilpotent. ■

We can apply Cartan's criterion to any $\mathfrak{g}$, not just $\mathfrak{g} \subset \mathfrak{gl}(V)$, because $\mathfrak{g}$ is solvable if and only if $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(V)$ is solvable. I.e., if $\mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$ for $x \in [\mathfrak{g},\mathfrak{g}]$ and $y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable. Also note that the converse of Cartan's criterion is true, which says if $\mathfrak{g} \subset \mathfrak{gl}(V)$ is solvable, then $\mathrm{Tr}(xy) = 0$ for $x \in [\mathfrak{g},\mathfrak{g}]$ and $y \in \mathfrak{g}$. This converse follows using Lie's theorem.

We now define the Killing form, named after the German mathematician Wilhelm Killing. He developed the theory of Lie algebras independently at the same time as Sophus Lie, however did not write proofs which were completely rigorous nor as timely as Lie. It was Elie Cartan's task, Killing's graduate student, to help develop the rigorous theory. For more information, see the Wikipedia article at `http://en.wikipedia.org/wiki/Wilhelm_Killing`.

Our goal is to use the Cartan criterion to get a criterion for semisimplicity.

1.53. Definition. Let $\mathfrak{g}$ be any Lie algebra. The ***Killing form*** on $\mathfrak{g}$ is a symmetric bilinear form $\kappa\colon \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathbf{C}$ by $(x,y) \mapsto \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$.

The Killing form is an example of an associative bilinear form. That is, for $x,y,z \in \mathfrak{g}$, we have $\kappa([x,z],y) = \kappa(x,[z,y])$.

1.54. Lemma. *Let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal. Then $\kappa_{\mathfrak{a}} = \kappa|_{\mathfrak{a}\times\mathfrak{a}}$, where $\kappa_{\mathfrak{a}}$ is the Killing form on $\mathfrak{a}$.*

Proof. Recall that if $\varphi \in \mathrm{End}(V)$ such that $\mathrm{im}\,\varphi \subset W \subsetneq V$, then $\mathrm{Tr}(\varphi) = \mathrm{Tr}(\varphi|_W)$. So for $x,y \in \mathfrak{a}$, we get $\mathrm{ad}(x), \mathrm{ad}(y) \in \mathrm{End}(\mathfrak{g})$ such that $\mathrm{im}(\mathrm{ad}(x))$ and $\mathrm{im}(\mathrm{ad}(y))$ are contained in $\mathfrak{a}$. Then

$$\kappa(x,y) = \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)|_{\mathfrak{a}})$$

and $(\mathrm{ad}(y)\,\mathrm{ad}(y))|_{\mathfrak{a}} = (\mathrm{ad}(x)|_{\mathfrak{a}})(\mathrm{ad}(y)|_{\mathfrak{a}})$, whose trace is $\kappa_{\mathfrak{a}}$. ∎

We recall some facts (and vocabulary) from the theory of bilinear forms. If $\beta\colon V \times V \longrightarrow \mathbf{C}$ is any symmetric bilinear form, we say β is ***nondegenerate*** when

$$\mathrm{rad}\,\beta = \{v \in V : \beta(v,\cdot) \equiv 0\}$$

is zero. If $\{v_1, \dots, v_n\}$ is a basis of V, we get a matrix $B = \big(\beta(v_i, v_j)\big)_{i,j}$. We can determine nondegeneracy by taking the determinant of B. Indeed, β is nondegenerate if and only if $\det B \neq 0$. In this case, we get an isomorphism $V \longrightarrow V^\vee$ sending $x \mapsto [y \mapsto \beta(x,y)]$. Lastly, $\operatorname{rad}\kappa$ is an ideal in $\mathfrak{g}$.

1.55. Theorem. *A Lie algebra is semisimple if and only if its Killing form is nongenerate.*

Proof. Let $S = \operatorname{rad}\kappa$ and $\operatorname{rad}\mathfrak{g} = 0$. Then $\operatorname{Tr}(\operatorname{ad}(x)\operatorname{ad}(y)) = 0$ for every $x \in S \supset [S,S]$ and all $y \in \mathfrak{g} \supset S$. By Cartan's criterion, S is solvable. Since S is an ideal, we get $S \subset \operatorname{rad}\mathfrak{g} = 0$, which proves κ is nondegenerate.

Conversely, suppose $S = \operatorname{rad}\kappa = 0$ and let $\mathfrak{a} \subset \mathfrak{g}$ be an *abelian* ideal. For $x \in \mathfrak{a}$ and any $y \in \mathfrak{g}$, we get a sequence

$$\mathfrak{g} \xrightarrow{\operatorname{ad}(y)} \mathfrak{g} \xrightarrow{\operatorname{ad}(x)} \mathfrak{a} \xrightarrow{\operatorname{ad}(y)} \mathfrak{a} \xrightarrow{\operatorname{ad}(x)} 0$$

Thus $(\operatorname{ad}(x)\operatorname{ad}(y))^2 = 0$ is nilpotent and has trace zero for $x \in \mathfrak{a}$ and $y \in \mathfrak{g}$. Thus $x \in \operatorname{rad}\kappa$, so $x = 0$, which implies $\mathfrak{a} = 0$. ■

Note that it is not necessarily true that $\operatorname{rad}\kappa = \operatorname{rad}\mathfrak{g}$. But, it is the case that $\operatorname{rad}\kappa \subset \operatorname{rad}\mathfrak{g}$.

10. Simple ideals

We say $\mathfrak{g}$ is a direct sum of ideals and write $\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$ if it is a direct sum as vector spaces. This means $[\mathfrak{a}_i, \mathfrak{a}_j] = \mathfrak{a}_i \cap \mathfrak{a}_j = 0$. So the structure of $\mathfrak{g}$ is completely determined by the structure of the $\mathfrak{a}_i$'s.

1.56. Theorem. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then it has a unique set of ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ such that $\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$.*

Proof. *Step* 1. We show every proper ideal $\mathfrak{a} \subset \mathfrak{g}$ has a complementary ideal. Let $\mathfrak{a}^\perp = \{x \in \mathfrak{g} : \kappa(x,y) = 0 \text{ for all } y \in \mathfrak{a}\}$. Notice that $\mathfrak{a}^\perp$ is an ideal by associativity (the proof is similar to the proof that $\operatorname{rad}\mathfrak{g}$ is an ideal). Moreover, $\mathfrak{a} \cap \mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$, so $\kappa_{\mathfrak{a}\cap\mathfrak{a}^\perp} = \kappa|_{\mathfrak{a}\cap\mathfrak{a}^\perp \times \mathfrak{a}\cap\mathfrak{a}^\perp}$. But $\kappa(x,y) = 0$ for $x, y \in \mathfrak{a} \cap \mathfrak{a}^\perp$, so $\mathfrak{a} \cap \mathfrak{a}^\perp$ is a solvable ideal. Thus $\mathfrak{a} \cap \mathfrak{a}^\perp = 0$ by semisimplicity of $\mathfrak{g}$. Since κ is nondegenerate, we have $\dim \mathfrak{g} = \dim \mathfrak{a} + \dim \mathfrak{a}^\perp$. Hence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$.

Step 2. Let $\mathfrak{a}_1 \subset \mathfrak{g}$ be a minimal (proper) ideal. Then $\mathfrak{a}_1^\perp$ is an ideal with $\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_1^\perp$. Every ideal of $\mathfrak{a}_1$ or $\mathfrak{a}_1^\perp$ is an ideal of $\mathfrak{g}$. In particular, the radicals of $\mathfrak{a}_1$ and $\mathfrak{a}_1^\perp$ are solvable ideals of $\mathfrak{g}$, so $\operatorname{rad} \mathfrak{a}_1 = \operatorname{rad} \mathfrak{a}_1^\perp = 0$. By the induction hypothesis, $\mathfrak{a}_1^\perp$ decomposes as $\mathfrak{a}_1^\perp = \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_n$, a direct sum of simple ideals. Thus $\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$.

Uniqueness. If $\mathfrak{a}$ is any simple ideal of $\mathfrak{g}$, then $[\mathfrak{a}, \mathfrak{g}]$ is a nonzero ideal contained in $\mathfrak{a}$. Hence $[\mathfrak{a}, \mathfrak{g}] = \mathfrak{a}$. On the other hand, we have $[\mathfrak{a}, \mathfrak{g}] = [\mathfrak{a}, \mathfrak{a}_1] \oplus \cdots \oplus [\mathfrak{a}, \mathfrak{a}_n]$, so $[\mathfrak{a}, \mathfrak{g}] = \mathfrak{a} = [\mathfrak{a}, \mathfrak{a}_i]$ for some i, and $[\mathfrak{a}, \mathfrak{a}_j] = 0$ for $j \neq i$. But $[\mathfrak{a}, \mathfrak{a}_i] = \mathfrak{a}$, so $\mathfrak{a} = \mathfrak{a}_i$ for some i. ■

1.57. Nonexample. Let $\mathfrak{g} = \mathfrak{t}_3$. Let $\mathfrak{a}$ be the strictly upper triangular matrices. Then $\mathfrak{a}$ has no complementary ideal. More generally, $[\mathfrak{g}, \mathfrak{g}]$ has no complement in $\mathfrak{g}$ when $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$.

1.58. Corollary. *If $\mathfrak{g}$ is semisimple, then every ideal and quotient is semisimple and every ideal is the sum of some collection of the $\mathfrak{a}_j$.*

1.59. Theorem. *If $\mathfrak{g}$ is semisimple, then* $\operatorname{ad}(\mathfrak{g}) = \operatorname{Der}(\mathfrak{g})$.

Proof. Notice that for any Lie algebra $\mathfrak{g}$, one can show for any $\delta \in \operatorname{Der}(\mathfrak{g})$ and $x \in \mathfrak{g}$, we have $[\delta, \operatorname{ad}(x)] = \operatorname{ad}(\delta(x))$ in $\mathfrak{gl}(\mathfrak{g})$. This implies $\operatorname{ad}(\mathfrak{g})$ is an ideal in $\operatorname{Der}(\mathfrak{g})$.

Since $\mathfrak{g}$ is semisimple, we have $Z(\mathfrak{g}) = 0$, so the map $\mathfrak{g} \longrightarrow \operatorname{ad}(\mathfrak{g})$ is an isomorphism. Since $\operatorname{ad}(\mathfrak{g})$ is an ideal, we have

$$\kappa_{\operatorname{ad}(\mathfrak{g})} = \kappa_{\operatorname{Der}(\mathfrak{g})}\big|_{\operatorname{ad}(\mathfrak{g}) \times \operatorname{ad}(\mathfrak{g})},$$

so $\operatorname{ad}(\mathfrak{g})^\perp$ satisfies $\operatorname{ad}(\mathfrak{g}) \cap \operatorname{ad}(\mathfrak{g})^\perp = \varnothing$. Using the same idea as the last proof, we have $\operatorname{Der}(\mathfrak{g}) = \operatorname{ad}(\mathfrak{g}) \oplus \operatorname{ad}(\mathfrak{g})^\perp$. Moreover, for any $\delta \in \operatorname{ad}(\mathfrak{g})^\perp$ (which we want to be zero) we get $0 = [\delta, \operatorname{ad}(x)] = \operatorname{ad}(\delta(x))$ for all $x \in \mathfrak{g}$. Since ad is injective, we have $\delta(x) = 0$ for all x, which implies $\delta = 0$. Hence $\operatorname{ad}(\mathfrak{g})^\perp = 0$, and $\operatorname{ad}(\mathfrak{g}) = \operatorname{Der}(\mathfrak{g})$. ■

We turn our attention to the so-called ***abstract Jordan decomposition***. For a semisimple Lie algebra $\mathfrak{g}$, we get an isomorphism $\operatorname{ad}\colon \mathfrak{g} \longrightarrow \operatorname{Der}(\mathfrak{g})$, so we get a decomposition $\operatorname{ad}(x) = (\operatorname{ad} x)_s + (\operatorname{ad} x)_n$

in $\mathfrak{gl}(\mathfrak{g})$. In fact, both $(\operatorname{ad} x)_s$ and $(\operatorname{ad} x)_n$ both lie in $\operatorname{Der}(\mathfrak{g})$ by Lemma 1.50. So we can define a unique x_s and x_n from $\mathfrak{g}$ such that $x = x_s + x_n$ and $[x_s, x_n] = 0$. In particular, we have $\operatorname{ad}(x_s) = (\operatorname{ad} x)_s$ and $\operatorname{ad}(x_n) = (\operatorname{ad} x)_n$.

Note that if $\mathfrak{g} \subset \mathfrak{gl}(V)$ is semisimple, then we will show when we have developed more theory that the two notions of Jordan decompositions agree.

CHAPTER 2

Representation Theory

1. Modules

2.1. Definition. Let $\mathfrak{g}$ be any Lie algebra. A $\mathfrak{g}$-***module*** V is a bilinear map $\mathfrak{g} \times V \longrightarrow V$ by $(x, v) \mapsto x \cdot v$ which satisfies

$$[x, y]v = xyv - yxv$$

for all $x, y \in \mathfrak{g}$ and $v \in V$.

The map notation is usually omitted, as the meaning is clear from the context. This module idea is equivalent to a representation. Specifically, if $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, then the $\mathfrak{g}$-module structure is given by $x \cdot v = \rho(x)v$.

For $\mathfrak{g}$-modules V and W, a homomorphism of $\mathfrak{g}$-modules, denoted $f \in \mathrm{Hom}_{\mathfrak{g}}(V, W)$, is a linear map such that $f(x \cdot v) = xf(v)$ for all $x \in \mathfrak{g}$ and $v \in V$.

2.2. Definition. A module is ***irreducible*** if it has no nonzero submodules.

2.3. Example. Any Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-module via the adjoint representation. That is, $x \cdot y = \mathrm{ad}(x)(y) = [x, y]$. The $\mathfrak{g}$-submodules of $\mathfrak{g}$ are the ideals of $\mathfrak{g}$. The irreducibles are the simple ideals.

2.4. Definition. We say a module V is ***completely reducible*** (or semisimple) if there exist irreducible submodules $V_1, \dots, V_n$ such that $V = V_1 \oplus \cdots \oplus V_n$.

Equivalently, this happens when every submodule $W \subset V$ has a complement W'.

2.5. Example. If $\mathfrak{g}$ is semisimple, it is completely reducible as a module over itself.

2.6. Theorem (Schur's Lemma). *Let $\mathfrak{g}$ be a Lie algebra. If V and W are irreducible $\mathfrak{g}$-modules, then $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ contains nonzero elements if and only if $V \cong W$. Moreover, if $V = W$, then every $\mathfrak{g}$-module map $f \colon V \longrightarrow W$ is a scalar multiple of* id.

Proof. For $f \in \mathrm{Hom}_{\mathfrak{g}}(V, W)$, both $\ker f \subset V$ and $\operatorname{im} f \subset W$ are $\mathfrak{g}$-submodules. Since V is irreducible, then $f \equiv 0$ or f is injective. Similarly, the irreducibility of W implies $f \equiv 0$ or f is surjective. So any nonzero f is an isomorphism.

Now let $f \in \mathrm{Hom}_{\mathfrak{g}}(V, V) = \mathrm{End}_{\mathfrak{g}}(V)$. Then f has an eigenvalue $\lambda \in \mathbf{C}$. Thus $f - \lambda\,\mathrm{id}$ has a nonzero kernel, which implies $f - \lambda\,\mathrm{id} = 0$ by the first part. Hence $f = \lambda\,\mathrm{id}$. ∎

The dual module $V^\vee$ of a $\mathfrak{g}$-module V is acted on via

$$(xf)(v) = -f(xv).$$

It is left to the reader to verify that this operation satisfies the $\mathfrak{g}$-module structure.

For $\mathfrak{g}$-modules V and W, we get a $\mathfrak{g}$-module structure on the tensor product $V \otimes_{\mathbf{C}} W$ via

$$x(v \otimes w) = xv \otimes w + v \otimes xw.$$

Again, verify that this is a $\mathfrak{g}$-module. This action comes from the tensor product of group representations being $g(v \otimes w) = gv \otimes gw$. One can check that taking a suitable derivative of the group case will yield the Lie algebra action.

For any vector spaces V and W, we have a (very useful!) identification $\mathrm{Hom}_{\mathbf{C}}(V, W) \cong V^\vee \otimes_{\mathbf{C}} W$. To see this, let $\{v_1, \dots, v_n\}$ be a basis for V and $\{w_1, \dots, w_m\}$ be a basis of W. Then $\{v_1^\vee, \dots, v_n^\vee\}$ is a the dual basis of V. Define a linear map $\varphi\colon V^\vee \otimes W \longrightarrow \mathrm{Hom}_{\mathbf{C}}(V, W)$ by

$$\sum_{i,j} c_{ij} v_i^\vee \otimes w_j \mapsto \Big[v \mapsto \sum_{i,j} c_{ij} v_i^\vee(v) w_j \Big].$$

It is a good exercise to verify that this map is an isomorphism. It is even more beneficial to construct the inverse map. Try it!

If V and W are $\mathfrak{g}$-modules, we can an induced $\mathfrak{g}$-module structure on $\mathrm{Hom}_{\mathbf{C}}(V, W)$ via this canonical isomorphism. We find that for any $f \in \mathrm{Hom}_{\mathbf{C}}(V, W)$, the action is explicitly given by

$$(x \cdot f)(v) = xf(v) - f(xv).$$

We can see that $x \cdot f = 0$, for all $x \in \mathfrak{g}$, if and only if $xf(v) = f(xv)$. This implies $f \in \mathrm{Hom}_{\mathfrak{g}}(V, W) \subset \mathrm{Hom}_{\mathbf{C}}(V, W)$.

2.7. Lemma. *If $\mathfrak{g}$ is solvable, then every irreducible representation of $\mathfrak{g}$ is 1-dimensional.*

Proof. Let V be an irreducible representation of $\mathfrak{g}$. Then Lie's theorem implies we can make the action upper triangular. Then the space spanned by the first basis vector, call this vector v_1, is a nonzero submodule of V. The irreducibility of V implies that $\mathrm{Span}(v_1) = V$. ■

2.8. Lemma. *For any Lie algebra $\mathfrak{g}$ and 1-dimensional $\mathfrak{g}$-module V, the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$ acts trivially.*

Proof. Each element of $\mathfrak{g}$ acts by scalar multiplication since V is 1-dimensional. So $[x, y]v = xyv = yxv = 0$ by the commutativity of scalar multiplication. ■

2. Weyl's theorem

Unless otherwise noted, for this section, suppose $\mathfrak{g}$ is a semisimple Lie algebra.

First, we introduce the notion of a Casimir operator. Recall that we used the Killing form to study ideals in a semisimple Lie algebra. We will do something similar here.

We start with a so-called ***faithful representation***, which is a representation $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ such that $\ker \rho = 0$. From here on, we will assume all our representations are faithful, unless otherwise noted. Define a bilinear form $\beta_\rho\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbf{C}$ by $(x, y) \mapsto \operatorname{Tr}\big(\rho(x)\rho(y)\big)$. This associative bilinear form is called the ***trace form*** of the representation. Notice that $\beta_\rho = \kappa$ when $\rho = \operatorname{ad}$. By associativity, we have $\operatorname{rad}\beta_\rho$ is an ideal. Perhaps most importantly (for us), by the Cartan criterion, we have $\rho(\operatorname{rad}\beta_\rho)$ is solvable, so $\operatorname{rad}\beta_\rho \cong \rho(\operatorname{rad}\beta_\rho) = 0$. Hence β is nondegenerate.

Now let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$. We can get a dual basis (in $\mathfrak{g}$!) of $\mathfrak{g}$ with respect to this form. Call this dual basis $\{y_1, \ldots, y_n\}$. The ***Casimir operator*** is defined to be

$$c_\rho = \sum_{i=1}^{n} \rho(x_i)\rho(y_i) \in \operatorname{End}_{\mathbf{C}}(V).$$

Beware! It is not always the case that c_ρ is the image of some element of $\mathfrak{g}$. However, it remains to verify $c_\rho \in \operatorname{End}_{\mathbf{C}}(V)$; that is, we need to show $c_\rho(\rho(x)(v)) = \rho(x)(c_\rho(v))$. But this is true precisely when c_ρ commutes with $\rho(x)$ for all x, which is true if and only if $[c_\rho, \rho(x)] = 0$ in $\operatorname{End}_{\mathbf{C}}(V)$ for all $x \in \mathfrak{g}$.

Let's compare $[x, \cdot]$ on different elements. We have

$$[x, x_i] = \sum_{j=1}^{n} a_{ij}x_j, \qquad [x, y_i] = \sum_{j=1}^{n} b_{ij}y_j, \qquad a_{ij}, b_{ij} \in \mathbf{C}.$$

To pick out a_{ik}, apply $\beta(\cdot, y_k)$ to get $\beta\big([x, x_i], y_k\big) = a_{ik}$. Then, by associativity and bilinearity, we have

$$\beta(-[x_i, x], y_k) = -\beta\big(x_i, [x, y_k]\big) = -b_{ki}.$$

Hence $a_{ik} = -b_{ki}$.

Now for any $x \in \mathfrak{g}$, look at $[\rho(x), c_\rho]$. Then each term in the sum is

$$\begin{aligned}[\rho(x), \rho(x_i)\rho(y_i)] &= [\rho(x), \rho(x_i)]\rho(y_i) + \rho(x_i)[\rho(x), \rho(y_i)] \\ &= \rho\big([x, x_i]\big)\rho(y_i) + \rho(x_i)\rho\big([x, y_i]\big) \\ &= \sum_{j=1}^{n} a_{ij}\rho(x_j)\rho(y_i) - \sum_{j=1}^{n} a_{ji}\rho(x_i)\rho(y_j).\end{aligned}$$

That was one term. Now summing over all terms we get

$$[\rho(x), c_\rho] = \sum_{i,j} a_{ij}\rho(x_j)\rho(y_i) - \sum_{i,j} a_{ji}\rho(x_i)\rho(y_j) = 0.$$

Let's investigate the properties of c_ρ. If $W \subset V$ is a submodule, then $c_\rho(W) \subset W$ by construction. If V is irreducible, then $c_\rho = \lambda\,\mathrm{id}_V$ for some $\lambda \in \mathbf{C}$ by Schur's lemma. Hence $\mathrm{Tr}(c_\rho) = \lambda \dim V$. But

$$\mathrm{Tr}(c_\rho) = \sum_{i=1}^{n} \mathrm{Tr}\left(\rho(x_i)\rho(y_i)\right) = n = \dim \mathfrak{g}.$$

Thus

$$c_\rho = \left(\frac{\dim \mathfrak{g}}{\dim V}\right) \mathrm{id}_V\,.$$

In our construction above, the definition of c_ρ depended on a choice of basis for V. We now present a basis-free description of the Casimir element. Recall the natural isomorphism $\mathrm{End}_{\mathbf{C}}(\mathfrak{g}) \longrightarrow \mathfrak{g}^\vee \otimes_{\mathbf{C}} \mathfrak{g}$. We also have a natural map $\mathfrak{g}^\vee \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ via the bilinear form. That is, β gives rise to an isomorphism $\mathfrak{g} \longrightarrow \mathfrak{g}^\vee$ via $x \mapsto \beta(x, \cdot)$. Hence

$$\mathrm{End}_{\mathbf{C}}(\mathfrak{g}) \longrightarrow \mathfrak{g}^\vee \otimes_{\mathbf{C}} \mathfrak{g} \xrightarrow{\beta \otimes \mathrm{id}_{\mathfrak{g}}} \mathfrak{g}^{\otimes 2} \xrightarrow{\rho \otimes \rho} \mathfrak{gl}(V)^{\otimes 2} \longrightarrow \mathfrak{gl}(V)$$

via

$$\beta(x, \cdot) \otimes y \mapsto x \otimes y \mapsto \rho(x) \otimes \rho(y) \mapsto \rho(x)\rho(y).$$

Let Ψ denote this entire composition. Then define $c_\rho = \Psi(\mathrm{id}_{\mathfrak{g}})$. It is left to the reader to verify this assertion.

2.9. Theorem (Weyl). *Every representation of a semisimple Lie algebra is completely reducible; i.e., the representation is semisimple.*

We can state this theorem in a slightly different way. If $\mathfrak{g}$ is a semisimple Lie algebra and V is a finite-dimensional representation containing a submodule W, then W has a complementary submodule W'; i.e, a submodule W' such that $V = W \oplus W'$.

Proof. Let $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra $\mathfrak{g}$.

Case 1. Suppose V has an irreducible submodule $W \subset V$ of codimension 1. We know $c_\rho(W) \subset W$, so $c_\rho|_W = \lambda \operatorname{id}_W$ for some $\lambda \in \mathbf{C}$. But c_ρ acts trivially on $V/W \cong \mathbf{C}$ because it is 1-dimensional, and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ since $\mathfrak{g}$ is semisimple. We can view the action of c_ρ in the following matrix:

$$\begin{array}{cc} & \begin{array}{cc} W & \quad V/W \end{array} \\ \begin{array}{c} W \\ \\ V/W \end{array} & \begin{pmatrix} \lambda & & & \\ & \ddots & & \\ & & \lambda & \\ & & & 0 \end{pmatrix} \end{array}.$$

Thus $c_\rho(V/W) = 0$, which implies $c_\rho(V) \subset W$. Hence $\ker c_\rho \neq 0$ is a nonzero submodule of V. If $\ker c_\rho \cap W \neq 0$, then $\lambda = 0$ and $\operatorname{Tr}(c_\rho) = 0$, which contradicts $\operatorname{Tr}(c_\rho) = \dim \mathfrak{g} \neq 0$. Whence $\lambda \neq 0$ and $\ker c_\rho \cap W = 0$, so $V = \ker c_\rho \oplus W$.

Case 2. Let W be a submodule of codimension 1 which is not irreducible. We proceed by induction on $\dim V$. If $0 \subsetneq Z \subsetneq W$ is a submodule, then W/Z has codimension 1 in V/Z. By induction on $\dim V$, we get a complement Y/Z. Clearly $\dim Y < \dim V$, and the codimension of Z in Y is 1, so by induction (or by Case 1) we get $U \subset Y \subset V$ such that $Y = U \oplus Z$. We present our situation thus far diagrammatically in Figure 2.1. Then $U \cap W = 0$ and

$$\dim U + \dim W = \dim V,$$

so $V = U \oplus W$ as $\mathfrak{g}$-modules.

Case 3. Now let $W \subset V$ be irreducible of any codimension. Recall W is a direct summand of V if and only if there exists a $\mathfrak{g}$-module

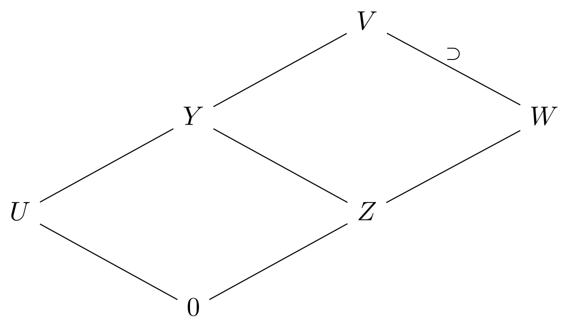


FIGURE 2.1. The collection of submodules constructed in the proof of Weyl's theorem.

homomorphism $\pi\colon V \longrightarrow W$ such that $\pi|_W = \mathrm{id}$. In such a case, we have $V = W \oplus \ker\pi$. To this end, consider

$$\varphi\colon \operatorname{Hom}_{\mathbf{C}}(V, W) \longrightarrow \operatorname{End}_{\mathbf{C}}(W)$$

via $f \mapsto f|_W$. Recall that $(x\cdot f)(v) = xf(v) - f(x\cdot v)$ makes both spaces into $\mathfrak{g}$-modules, and φ is evidently a $\mathfrak{g}$-module homomorphism (check!). Since W is irreducible, Schur's lemma implies $\operatorname{Hom}_{\mathfrak{g}}(W, W) = \mathbf{C}\,\mathrm{id}_W$ is a 1-dimensional $\mathfrak{g}$-submodule of $\operatorname{End}_{\mathbf{C}}(W)$. Set $Z = \varphi^{-1}(\operatorname{Hom}_{\mathfrak{g}}(W, W))$, which is a submodule of $\operatorname{Hom}_{\mathbf{C}}(V, W)$. Notice also that $\ker\varphi$ has codimension 1. By Case 2, $\ker\varphi$ must have a complement Y, so $Y \oplus \ker\varphi = Z$. Thus $\varphi(Y) = \operatorname{Hom}_{\mathfrak{g}}(W, W) = \mathbf{C}\,\mathrm{id}_W$, so there exists $\pi \in Y$ such that $\varphi(\pi) = \pi|_W = \mathrm{id}_W$. So $\ker\pi$ gives the complement to W.

Case 4. If W is reducible, then rerun the argument of Case 2. ■

2.10. THEOREM. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be semisimple. Then $\mathfrak{g}$ contains the semisimple and nilpotent parts of each of its elements.*

2.11. EXAMPLE. Let $\mathfrak{g} = \mathfrak{sl}(V)$. Then $\operatorname{Tr}(x) = 0$ for any $x \in \mathfrak{g}$. Since x_n is nilpotent, each of its eigenvalues is zero, so $\operatorname{Tr}(x_n) = 0$. But $\operatorname{Tr}(x_s) = \operatorname{Tr}(x - x_n) = \operatorname{Tr}(x) - \operatorname{Tr}(x_n) = 0$. Hence $x_n, x_s \in \mathfrak{g}$.

Proof. Let $x = x_s + x_n$ be the Jordan decomposition of $x \in \mathfrak{g}$. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ by semisimplicity, we have $\mathfrak{g} \subset \mathfrak{sl}(V)$. More generally, for any $\mathfrak{g}$-submodule $W \subset V$, let

$$\mathfrak{s}(W) = \{y \in \mathfrak{gl}(V) : y(W) \subset W \text{ and } \operatorname{Tr}(y|_W) = 0\}.$$

Then $\mathfrak{s}(W)$ is a subalgebra of $\mathfrak{gl}(V)$ containing $\mathfrak{g}$ and $x_s, x_n \in \mathfrak{s}(W)$. Now recall that x_s and x_n can be written as polynomials in x. Whence $[x, \mathfrak{g}] \subset \mathfrak{g}$, which implies $[x_s, \mathfrak{g}]$ and $[x_n, \mathfrak{g}]$ are both contained in $\mathfrak{g}$. Now consider the normalizer

$$N(\mathfrak{g}) = \{y \in \mathfrak{gl}(V) : [y, \mathfrak{g}] \subset \mathfrak{g}\},$$

which is a subalgebra of $\mathfrak{gl}(V)$ containing $\mathfrak{g}$ as an ideal. Moreover, both x_n and x_s are in $N(\mathfrak{g})$. We claim

$$\mathfrak{g} = \mathfrak{sl}(V) \cap N(\mathfrak{g}) \cap \Big(\bigcap_{\substack{W \subset V \\ \text{submodule}}} \mathfrak{s}(W) \Big).$$

Call the right-hand side $\mathfrak{G}$. Notice $\mathfrak{g}$ is an ideal and $\mathfrak{g}$-submodule of $\mathfrak{G} \subset N(\mathfrak{g})$. So $\mathfrak{g}$ has a complement in $\mathfrak{G}$, namely $\mathfrak{G} = U \oplus \mathfrak{g}$, as $\mathfrak{g}$-modules. Then $[\mathfrak{g}, \mathfrak{G}] \subset \mathfrak{g}$, which implies $[\mathfrak{g}, U] = 0$. We want to show $U = 0$ in $\mathfrak{gl}(V)$, so it is enough to show every $y \in U$ acts as the zero map on V. By Weyl's theorem again, write

$$V = W_1 \oplus \cdots \oplus W_n,$$

where each W_i is irreducible. Hence it is enough t show $y|_{W_i} = 0$. Since W_i is a $\mathfrak{g}$-submodule and $y \in \mathfrak{G} \subset \mathfrak{s}(W_i)$, we know $y(W_i) \subset W_i$ But $[y, \mathfrak{g}]0$, so the action of y commutes with $\mathfrak{g}$, which implies $y \in \operatorname{End}_{\mathfrak{g}}(W_i)$. Then Schur's lemma implies y acts by a scalar on W_i. But $\operatorname{Tr}(y) = 0$ on W, so $y = 0$. Hence $U = 0$ and $\mathfrak{g} = \mathfrak{G}$. ∎

3. Motivation for studying semisimple Lie algebras

Let $\mathfrak{g} = \mathbf{C}$, then $\mathfrak{g}$ is abelian (also called the trivial Lie algebra). Consider the following three representations of $\mathfrak{g}$ on $\mathbf{C}^2 = \langle e_1, e_2 \rangle$. That is, we will have three maps $\rho_i \colon \mathfrak{g} \longrightarrow \mathfrak{gl}_2(\mathbf{C})$, for $i = 1, 2, 3$, where

$$\rho_1(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \quad \rho_2(t) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad \rho_3(t) = \begin{pmatrix} t & t \\ 0 & 0 \end{pmatrix}.$$

It is left to reader to verify that these are in fact representations of $\mathfrak{g}$. In the first case, each $t \in \mathfrak{g}$ acts by a semisimple (i.e., a diagonalizable) operator on $\mathbf{C}^2$. In the second case, no element acts by a semisimple operator. In fact, they are all nilpotent matrices, and the submodule $\mathbf{C}e_1$ has no complementary submodule since

$$\rho_2(t)(\alpha e_1 + \beta e_2) = t\beta e_1.$$

The last case is the worst of all. Indeed, no nonzero $t \in \mathfrak{g}$ acts by a semisimple or nilpotent operator, and for every $t \in \mathfrak{g}$, $\rho_3(\mathfrak{g})$ does not contain the semisimple nor nilpotent parts of $\rho_3(t)$. In summary, even though $\mathfrak{g}$ is the "easiest" conceivable Lie algebra, we cannot predict anything about its representations. That is, even though $\mathfrak{g}$ has no structure, the representations of $\mathfrak{g}$ are, in some sense, arbitrarily behaved.

When $\mathfrak{g}$ is semisimple, we get a much better theory. For instance, last time we showed whenever $\mathfrak{g} \subset \mathfrak{gl}(V)$ is semisimple, $\mathfrak{g}$ contains the semisimple and nilpotent parts of its elements. We also know that every $x \in \mathfrak{g}$ has an ***absolute Jordan decomposition*** (i.e., abstract Jordan decomposition) $x = x_n + x_s$ via the adjoint action, even when it is *not* necessarily the case $\mathfrak{g} \subset \mathfrak{gl}(V)$.

Here are some exercises about semisimple Lie algebras for the reader to consider.

- [**8**, Ex. 5.8]. The Jordan decomposition is compatible with decomposition of $\mathfrak{g}$ into simple ideals. For example, if $\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ and we write $x = (x_1, x_2)$, then $x_s = ((x_1)_s, (x_2)_s))$ and $x_n = ((x_1)_n, (x_2)_n)$. To prove, use the defining property and the uniqueness of Jordan decomposition.
- The Jordan decomposition is preserved between Lie algebra homomorphisms between semisimple Lie algebras. That is,

if $\varphi\colon \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$, where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are semisimple Lie algebras, then $\varphi(x_s) = \varphi(x)_s$ and $\varphi(x_n) = \varphi(x)_n$. To prove, use the fact that $\ker\varphi$ is an ideal in $\mathfrak{g}_1$ and that $\mathfrak{g}_1$ decomposes uniquely into ideals. Then apply the previous exercise.

We will take these results for granted and prove a corollary to the theorem above.

2.12. Corollary. *If $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is any representation of a semisimple Lie algebra, then $\rho(x_s) = \rho(x)_s$ and $\rho(x_n) = \rho(x)_n$ for all $x \in \mathfrak{g}$.*

Proof. We saw before, using $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, that $\operatorname{im}\rho \subset \mathfrak{sl}(V)$. So ρ is a homomorphism between semisimple Lie algebras. Using the exercise above, we get $\rho(x_s)$ and $\rho(x_n)$ are semisimple and nilpotent, respectively, in $\mathfrak{sl}(V)$. Now apply the theorem, which says the Jordan decomposition of $\operatorname{im}\rho$ agrees with that in $\mathfrak{sl}(V)$. ∎

Let's take a look at a summary of what we've discussed thus far.

- Semisimple Lie algebras have no solvable nor abelian ideals.
- If V is a representation of a semisimple Lie algebra $\mathfrak{g}$, then any subspace $W \subset V$ which is stable under the action of $\mathfrak{g}$ has a complementary subspace which is also stable under the action of $\mathfrak{g}$. Equivalently, we can write $V = V_1 \oplus \cdots \oplus V_n$ with each V_i stable under the action of $\mathfrak{g}$ and each V_i has no proper nonzero subspace stable under $\mathfrak{g}$.
- Conversely, *only* semisimple Lie algebras have this property for *every* representation. If every representation of $\mathfrak{g}$ has this property, then so does the adjoint representation. So for any ideal $\mathfrak{a}$, we can find a complementary ideal $\mathfrak{b}$. If $\mathfrak{g}$ is not semisimple, then it has a (nonzero) abelian ideal, say $\mathfrak{a}$. So we have $\pi\colon \mathfrak{g} \longrightarrow \mathfrak{a}$, projection on the first coordinate, is a map of Lie algebras. Then any representation $\rho\colon \mathfrak{a} \longrightarrow \mathfrak{gl}(V)$ induces a representation of $\mathfrak{g}$ by

$$\rho \circ \pi\colon \mathfrak{g} \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{gl}(V),$$

usually denoted $\pi^*(\rho)$, called the pullback. But $\mathfrak{a}$ has many representations which are not completely reducible (e.g., the representation $\rho_2 \colon \mathbf{C} \longrightarrow \mathfrak{gl}_2(\mathbf{C})$ we saw earlier). These will not be completely reducible for $\mathfrak{g}$ either, so we have a contradiction.

4. The Lie algebra $\mathfrak{sl}_2$

Let's first consider the fundamental thing to study representation of $\mathfrak{sl}_2$. Since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{sl}_2$ is semisimple, its action on any $\mathfrak{sl}_2$-module is diagonalizable. This means we can write the action on V as some matrix

$$h = \left(\begin{array}{ccc|c|ccc} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ \hline & & & \ddots & & & \\ \hline & & & & \lambda_n & & \\ & & & & & \ddots & \\ & & & & & & \lambda_n \end{array}\right).$$

Hence we get a decomposition $V = \bigoplus_{\lambda \in \mathbf{C}} V_\lambda$, where

$$V_\lambda = \{v \in V : h \cdot v = \lambda v\}.$$

Recall the usual basis of $\mathfrak{sl}_2$: $x = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ with $[h, x] = 2x$, $[h, y] = -2y$, and $[x, y] = h$. If V is any representation of $\mathfrak{sl}_2$, then the action of h on V can be diagonalized. That is, there is a basis of V such that

$$h = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \qquad \lambda_i \in \mathbf{C}.$$

This results in a decomposition of V into eigenspaces V_λ as defined above. We proceed with the ***fundamental calculation*** (as defined in [**4**]). If $v \in V_\lambda$, then $xv \in V_{\lambda+2}$ and $yv \in V_{\lambda-2}$. We test by applying h:

$$hxv = xhv + [h, x]v = \lambda xv + 2xv = (2 + \lambda)xv.$$

Similarly,

$$hyv = yhv + [h, y]v = \lambda yv - 2xv = (\lambda - 2)yv.$$

Since the dimension of V is finite, there is some point $V_\lambda \neq 0$ such that $V_{\lambda+2} = 0$, so $xv = 0$ for any $v \in V_\lambda$.

From now on, let's take V to be irreducible, let $n = \lambda \in \mathbf{C}$ some maximal eigenvalue (so $V_n \neq 0$ but $V_{n+2} = 0$), and let $v_0 \in V_n$. The action of x and y can be visualized in Figure 2.2. Notice that $\bigoplus_{k\in\mathbf{Z}} V_{n+2k}$ is an $\mathfrak{sl}_2$-stable subspace of V (i.e., a submodule), so

$$V = \bigoplus_{k\in\mathbf{Z}} V_{n+2k}$$

by the irreducibility of V. We say that the eigenvalues form a ***string*** $n, n-2, n-4, \ldots$.

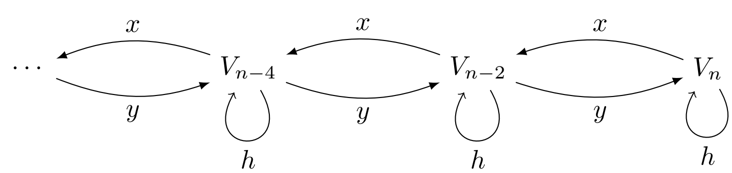


FIGURE 2.2. The action of x and y on the eigenspaces of an irreducible $\mathfrak{sl}_2$-module.

Now consider the subspace spanned by $\{v_0, yv_0, y^2v_0, \ldots\}$. We claim that this subspace is stable under the action of $\mathfrak{sl}_2$, which implies it is equal to V since V is irreducible. To show it is $\mathfrak{sl}_2$-stable, we apply the basis vectors of $\mathfrak{sl}_2$. We have

$$hy^iv_0 = (n - 2i)y^iv_0,$$

since $y^iv_0 \in V_{n-2i}$. Clearly, this subspace is y-stable, so it remains to check when x is applied. Since v_0 was chosen to be maximal, $xv_0 = 0$. Then

$$xyv_0 = yxv_0 + [x, y]v_0 = hv_0 = nv_0,$$

and

$$\begin{aligned} xy^2v_0 &= xy(yv_0) \\ &= yx(yv_0) + [x,y]yv_0 \\ &= y(nv_0) + hyv_0 \\ &= nyv_0 + (n-2)yv_0 \\ &= (n+n-2)yv_0. \end{aligned}$$

By induction, one can show the formula in general is given by

$$xy^m v_0 = m(n-m+1)y^{m-1}v_0.$$

Thus the space is stable, so it is all of V.

We conclude each $V_{n-2k} = \mathbf{C}y^k v_0$. Hence V is completely reducible and determined by the eigenvalues n, $n-2$, $n-4$, $\ldots$. If W is another $\mathfrak{sl}_2$-module with the same list of eigenvalues, say $w_0 \in W_n$, then we get an isomorphism $\varphi \colon V \longrightarrow W$ by $y^i v_0 \mapsto y^i w_0$.

Since $\dim(V) < \infty$, there is some "minimal" eigenvalue in this list, say $y^m v_0 = 0$ but $y^{m-1}v_0 \neq 0$. From the calculations above,

$$0 = xy^m v_0 = m(n-m+1)y^{m-1}v_0$$

implies that $n-m+1=0$. Therefore, $n = m-1 \in \mathbf{Z}$ and the eigenvalues of h on V are the integers

$$n, n-2, n-4, \ldots, -n+2, -n.$$

So, in fact, the entire representation V is determined completely by $n \in \mathbf{Z}$ (up to isomorphism).

In summary, given any $n \in \mathbf{Z}_+$, we can construct an irreducible $\mathfrak{sl}_2$-module $V(n)$ as above. Observe $V(n)$ has basis $\{v_0, \ldots, v_n\}$ (which is identified with the basis above) with

$$\begin{aligned} hv_i &= (n-2i)v_i \\ xv_i &= i(n-i+1)v_{i-1} \\ yv_i &= v_{i+1}. \end{aligned}$$

It is left to the reader that this defines an $\mathfrak{sl}_2$-module action.

2.13. Example. Consider the usual representation $V = \mathbf{C}^2$. Then h has eigenvalues 1 and -1, so this representation is precisely $V(1)$.

2.14. Example. Let $V = \mathfrak{sl}_2$ under the adjoint representation. Then

$$\mathrm{ad}(h) = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}.$$

The eigenvalues are -2, 0 and 2, so this representation is $V(2)$.

2.15. Example. Now suppose we have an unknown representation V and we calculate the eigenvalues of h on V (with multiplicity) to be

$$-3, -2, -1, -1, 0, 1, 1, 2, 3.$$

To get 3 and -3, we must have a direct summand isomorphic to $V(3)$. The remaining eigenvalues are

$$-2, -1, 0, 1, 2.$$

To get an eigenvalue of 2 we must have a direct summand isomorphic to $V(2)$. This leaves the eigenvalues 1 and -1, which we showed above is isomorphic to $V(1)$. Hence $V = V(1) \oplus V(2) \oplus V(3)$.

One can construct "natural" irreducible representations of $\mathfrak{sl}_2$. Let $\{a, b\}$ be a basis for $\mathbf{C}^2$ with the usual action of $\mathfrak{sl}_2$. Then we get an induced action of $\mathfrak{sl}_2$ on $\mathbf{C}[a, b]$ by identifying $\mathbf{C}[a, b]$ with $\mathrm{Sym}(\mathbf{C}^2)$. Explicitly, $z \in \mathbf{C}^2$ acts on a and b as usual (i.e., as vectors), and then extended to any polynomial by linearity via

$$z(fg) = (zf)g + f(zg)$$

for all $f, g \in \mathbf{C}[a, b]$. This action stabilizes the homogeneous polynomials of degree m, so we get finite-dimensional representations of $\mathfrak{sl}_2$ this way. For example

$$x(2a^2 + ab) = 2ab + b^2.$$

The space of homogeneous polynomials of degree m is $(m+1)$-dimensional and you can check that $h(a^{m-i}b^i) = (m - 2i)a^{m-i}b^i$. Hence this space is isomorphic to $V(m)$ as an $\mathfrak{sl}_2$-module. This construction can be generalized using Weyl's construction and/or Schur functors.

5. Irreducible representations of $\mathfrak{sl}_3$

Let $\mathfrak{g} = \mathfrak{sl}_3$ and let $\{e_{ij} : i \neq j\} \cup \{h_1, h_2\}$ be the standard basis of $\mathfrak{g}$. Since h_1 and h_2 are diagonal, they are semisimple, so their action on any representation V can be diagonalized. Moreover, $[h_1, h_2] = 0$ implies their action can be simultaneously diagonalized. So if α and β are such that $h_1 v = \alpha v$ and $h_2 v = \beta v$, then

$$(c_1 h_1 + c_2 h_2) = (c_1 \alpha + c_2 \beta) v. \tag{2.1}$$

Define $\mathfrak{h} = \langle h_1, h_2 \rangle$. This is the subalgebra of diagonal matrices in $\mathfrak{sl}_3$. We can decompose $V = \bigoplus_{\alpha \in \mathfrak{h}^\vee} V_\alpha$ by (2.1), where

$$V_\alpha = \{v \in V : hv = \alpha(h) v \text{ for all } h \in \mathfrak{h}\}.$$

In this situation, we say v is an eigenvector for $\mathfrak{h}$, we say $\alpha \in \mathfrak{h}^\vee$ is an eigenvalue (for the action of $\mathfrak{h}$ on V) if $V_\alpha \neq 0$, and V_α are eigenspaces (when $V_\alpha \neq 0$). Since we only consider finite-dimensional vector spaces, we conclude that $V_\alpha \neq 0$ for only finitely many α.

For $\mathfrak{sl}_2$, we used that $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$ take one eigenspace to another. The reason why this works is because x and y are eigenvectors for the adjoint action of h on $\mathfrak{sl}_2$. To this end, what are the eigenvectors for $\mathrm{ad}(h)$ on $\mathfrak{sl}_3$? By the first homework assignment, they are $\{e_{ij} : i \neq j\} \cup \{h_1, h_2\}$. Hence we get a decomposition

$$\mathfrak{g} = \mathfrak{sl}_3 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^\vee} \mathfrak{g}_\alpha,$$

where $x \in \mathfrak{g}_\alpha$ implies $[h, x] = \alpha(h) x$. To see which $\alpha \in \mathfrak{h}^\vee$ give nonzero $\mathfrak{g}_\alpha$, it is easier to work with another basis of $\mathfrak{h}^\vee$. The usual basis for $\mathfrak{h}$ gives

$$\mathfrak{h}^\vee = \langle \lambda_1, \lambda_2, \lambda_3 : \lambda_1 + \lambda_2 + \lambda_3 = 0 \rangle,$$

where $\lambda_i(\mathrm{diag}(a_1, a_2, a_3)) = a_i$. Then the action of $\mathfrak{h}$ looks like

$$[\mathrm{diag}(a_1, a_2, a_3), e_{ij}] = (a_i - a_j) e_{ij}, \quad i \neq j.$$

Thus the eigenvalues of $\mathfrak{h}$ on $\mathfrak{g}$ are precisely $\lambda_i - \lambda_j \in \mathfrak{h}^\vee$ for $i \neq j$. We can visualize $\mathfrak{h}^\vee$ in Figure 2.3.

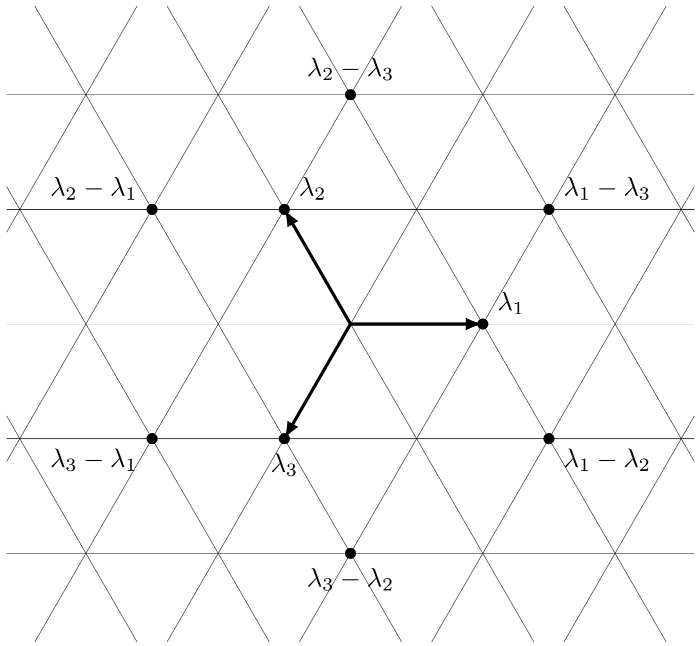


FIGURE 2.3. Root lattice of $\mathfrak{sl}_3$. Dots indicate eigenvalues of adjoint representation (roots).

We now carry out the fundamental calculation to see how e_{ij} acts. Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. Apply $h \in \mathfrak{h}$ to get

$$[h, [x, y]] = (\alpha(h) + \beta(h))[x, y],$$

so $[x, y] \in \mathfrak{g}_{\alpha+\beta}$. Said differently, $\mathrm{ad}(\mathfrak{g}_\alpha)\colon \mathfrak{g}_\beta \longrightarrow \mathfrak{g}_{\alpha+\beta}$.

Now let V be any $\mathfrak{g}$-module. We have $V = \bigoplus_{\alpha \in \mathfrak{h}^\vee} V_\alpha$. The same calculation as above shows $xv \in V_{\alpha\beta}$ for $x \in \mathfrak{g}_\alpha$ and $v \in V_\beta$. If V is irreducible and $V_\beta \neq 0$ for some β, then each α such that $V_\alpha \neq 0$ differs from β by a $\mathbf{Z}$-linear combination of the $\lambda_i - \lambda_j$. In this context, we call an eigenvalue α with $V_\alpha \neq 0$ a ***weight*** of V, we call $v \in V_\alpha$ is called a ***weight vector***, and the $V_\alpha \neq 0$ are called ***weight***

spaces. More specifically, when the representation in question is the adjoint representation, we replace the word "weight" above with the word "root."

Let V be a representation of $\mathfrak{g}$, and consider the subalgebra $\mathfrak{t}_3$ inside $\mathfrak{sl}_3$ acting on V. By Lie's theorem, there exists $v_0 \in V$ which is an eigenvector for $\mathfrak{h}$ and killed by $\mathfrak{n}_3 = \langle e_{12}, e_{13}, e_{23}\rangle$. Say $v_0 \in V_\beta \neq 0$, where $\beta \in \mathfrak{h}^\vee$. Such a β is called a ***highest weight***, and v_0 is called a ***highest weight vector***. We claim

$$V = \langle e_{21}, e_{31}, e_{32}\rangle v_0.$$

Said differently, V is generated by successively applying e_{21}, e_{31}, and e_{32} to v_0 (in any order, with repetitions allowed). Note $e_{31} = [e_{32}, e_{21}]$, so it suffices to repeatedly apply e_{32} and e_{21}. This claim can be verified in the same we verified the analogous claim for $\mathfrak{sl}_2$. Let W be this space. Then W is stable under $\mathfrak{h}$, and stable under e_{21}, e_{31} and e_{32}. We need to see that $e_{12}W \subset W$, $e_{13}W \subset W$, and $e_{23}W \subset W$. Again, it suffices to show $e_{12}W \subset W$ and $e_{23}W \subset W$ since $e_{13} = [e_{12}, e_{23}]$. Then if W is stable under the action of these elements and V is irreducible, then $W = V$.

To prove, we need the notion of a word. A ***word*** in $\{e_{21}, e_{32}\}$ is an ordered sequence of these symbols. In this language, W is obtained by applying words to v_0. We use induction on the length of the word. Let $W_n = \langle wv_0 : \ell(w) \leqslant n\rangle$. We will show $e_{12}W_n \subset W_{n-1}$ and $e_{23}W_n$. Since words end in either e_{21} or e_{32}, the two cases can be handled similar to the induction for the $\mathfrak{sl}_2$ computation. We conclude that a highest weight vector $v_0 \in V_\beta$ generates an irreducible submodule of any $\mathfrak{g}$-module.

Next, consider eigenvectors

$$v_0,\ e_{21}v_0,\ e_{21}e_{21}v_0, \ldots.$$

Let m be the smallest integer such that $e_{21}^m v_0 = 0$.[1] When is the eigenspace $V_{\beta+k(\lambda_2-\lambda_1)} = 0$? Note that the e_{21}-action stabilizes

$$W = \bigoplus_k V_{\beta+k(\lambda_2-\lambda_1)}.$$

A similar result holds for e_{12}. Let $h_{12} = [e_{12}, e_{21}]$. Then

$$\mathfrak{s}_{\lambda_2-\lambda_1} = \langle e_{12}, e_{21}, h_{12}\rangle \cong \mathfrak{sl}_2$$

acts on W. We know the eigenvalues of h_{12} on W are integers and symmetric about the origin, so the string of vectors

$$\{\beta + k(\lambda_2 - \lambda_1) : k = 0, 1, \ldots, m-1\}$$

are symmetric about the line in $\mathfrak{h}^\vee$ given by

$$\{f \in \mathfrak{h}^\vee : f(h_{12}) = 0\} = \mathbf{C}(\lambda_1 + \lambda_2).$$

We can do the same thing with e_{32} (or any e_{ij}) to get

$$\mathfrak{s}_{\lambda_i-\lambda_j} = \langle e_{ij}, e_{ji}, h_{ij} := [e_{ij}, e_{ji}]\rangle \cong \mathfrak{sl}_2\,.$$

By reflecting across lines perpendicular to $\lambda_i - \lambda_j$, we get an outer bound for the eigenvalues of V, which is a hexagon. See Figure 2.4. Moreover, if we write

$$\beta = b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 = (b_1 - b_3)\lambda_1 + (b_2 - b_3)\lambda_2,$$

for some $b_i \in \mathbf{C}$, then $\beta(h_{ij})$ is the eigenvalue of h_{ij} acting on v_0, so $\beta(h_{ij}) \in \mathbf{Z}$. Indeed, by the relation to $\mathfrak{sl}_2$-representations, we know $\beta(h_{12}) = b_1 - b_2 \in \mathbf{Z}$. But also $\beta = (b_1 - b_2)\lambda_1 - b_2\lambda_3$, so

$$\beta(h_{13}) = b_1 - b_2 - (-1)b_2 = b_1 \in \mathbf{Z}.$$

Hence $b_2 \in \mathbf{Z}$, too, and $\beta \in \mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2 \subset \mathfrak{h}^\vee$. Define $P = \mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2$; this is called the ***weight lattice***. Now start at any eigenvalue on the border in Figure 2.4 and use various $\mathfrak{s}_{\lambda_i-\lambda_j} \cong \mathfrak{sl}_2$ to get eigenvalues in the interior of the hexagon.

[1]The notation e_{21}^m does not mean take the mth power of e_{21} first and then apply to v_0. Since a representation is required to be a Lie algebra homomorphism and not a multiplicative homomorphism, there is no guarantee that e_{21}^m is the mth power. For example, consider the representation $\mathbf{C} \longrightarrow \mathfrak{gl}_2(\mathbf{C})$ by $t \mapsto \left(\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right)$.

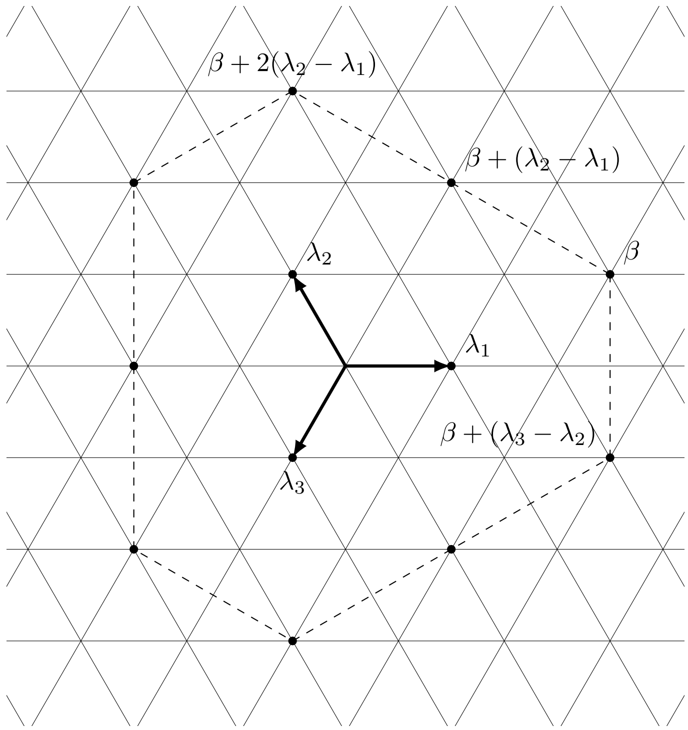


FIGURE 2.4. Eigenvalues of irreducible representation with heighest weight β must lie on intersection points within (or on boundary of) dashed line.

In summary, given any irreducible $\mathfrak{sl}_3$-module V, it has a highest weight $\beta \in P$. All other weights of V are obtained by reflecting across the lines $\langle \lambda, h_{ij} \rangle = 0$, where $h_{ij} = [e_{ij}, e_{ji}]$ and $\langle \lambda, h \rangle = \lambda(h) \in \mathbf{C}$ for $\lambda \in \mathfrak{h}^\vee$ and $h \in \mathfrak{h}$. This gives the outer corners of the hexagon, so the weights of V are the elements of the weight lattice P which are congruent to β modulo the root lattice $Q = \mathbf{Z}(\lambda_1 - \lambda_2) + \mathbf{Z}(\lambda_2 - \lambda_3)$.

New questions have arisen. Which β can occur as a highest weight? How many different irreducible representations can have the same highest weight?

To answer the latter, suppose V and W are irreducible representations of $\mathfrak{sl}_3$ with highest weight $\beta \in P$. Consider $V \oplus W$, and note that $z(v,w) = (zv, zw)$ for any $v \in V$ and $w \in W$. So if v_0 is a highest weight vector for V and w_0 is a highest weight vector for W, then (v_0, w_0) is a highest weight vector in $V \oplus W$. So (v_0, w_0) generates an irreducible representation $U = \mathfrak{sl}_3(v_0, w_0) \subset V \oplus W$. Now let $\pi_1 \colon V \oplus W \longrightarrow V$ and $\pi_2 \colon V \oplus W \longrightarrow W$ be the projection maps onto the first and second coordinate, respectively. By Schur's lemma, $\pi_1(U) \subset V$ implies $\pi_1(U) = V$ or $\pi_1(U) = 0$. But $\pi_1(v_0, w_0) = v_0 \neq 0$, so $\pi_1(U) = V$ and $U \cong V$. Similarly, $U \cong W$, so $V \cong W$.[2]

To answer the former, we need some general constructions. Let $\mathfrak{g}$ be a Lie algebra. Let $h \in \mathfrak{g}$ act semisimply on representations V and W, with eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots \mu_m$, respectively. Then we can get eigenvalues of other constructions, as shown in Table 2.1.

construction	eigenvalues
$V \oplus W$	$\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m$
$V^\vee$	$-\lambda_1, \ldots, -\lambda_n$
$V \otimes W$	$\{\lambda_i + \mu_j : 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$
$\mathrm{Sym}^k(V)$	$\left\{ \sum_{j=1}^{k} \lambda_{i_j} : \substack{i_j \in \{1,\ldots,n\} \\ \text{repetitions allowed}} \right\}$
$\bigwedge^k(V)$	$\left\{ \sum_{j=1}^{k} \lambda_{i_j} : \substack{i_j \in \{1,\ldots,n\} \\ \text{no repetitions}} \right\}$

Table 2.1. Constructions of representations for general Lie algebras.

If V is a $\mathfrak{g}$-module with basis $\{v_1, \ldots, v_n\}$, then

$$\mathrm{Sym}(V^\vee) = \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k(V^\vee) \cong \mathbf{C}[v_1, \ldots, v_n].$$

[2]This proof is not special to $\mathfrak{sl}_3$; we can apply this argument to show uniqueness for other Lie algebras, provided we first generalize the machinery developed for $\mathfrak{sl}_3$ to an arbitrary Lie algebra.

We will use this to answer the first question above (i.e., to show which $\beta \in P$ can occur as a highest weight vector). Indeed, any $\beta \in P$, where $P = \mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2 + \mathbf{Z}\lambda_3$, can be written as $\beta = a\lambda_1 - b\lambda_3$. Then β is a possible highest weight if and only if $a, b \geqslant 0$, as shown in Figure 2.5.

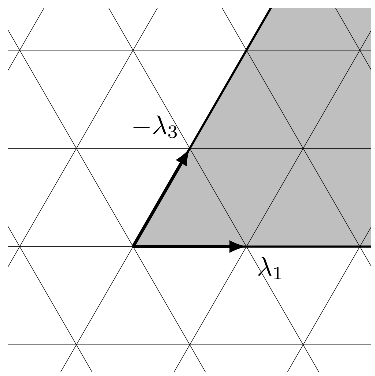


FIGURE 2.5. The region for possible highest weights.

The highest weights of $V = \mathbf{C}^3$ (i.e., the usual representation) and $V^\vee$ are exactly λ_1 and $-\lambda_3$. From Table 2.1, the representation $\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V^\vee)$ will have $a\lambda_1 - b\lambda_3$ as a highest weight.[3] So for any pair $a, b \geqslant 0$, we get a unique irreducible representation of $\mathfrak{sl}_3$. From Figure 2.4, we see $\beta = 2\lambda_1 - \lambda_3$, so this corresponds to $(a, b) = (2, 1)$.

6. Root space decompositions

Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\mathfrak{g}$ has some semisimple elements. Recall that any $x \in \mathfrak{g}$ has its semisimple part in $\mathfrak{g}$. If this semisimple part was always zero, then x would be nilpotent for all $x \in \mathfrak{g}$, to which Engel's theorem implies $\mathfrak{g}$ is nilpotent. This is a contradiction with the fact that $\mathfrak{g}$ is semisimple.

[3] This representation may not be irreducible, but then we can get an irreducible subrepresentation with highest weight $a\lambda_1 - b\lambda_3$ inside it.

A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is ***toral*** if every element $h \in \mathfrak{h}$ is semisimple. (For example, the diagonal elements in a semisimple algebra is form a toral subalgebra.)

2.16. LEMMA. *All toral algebras are abelian.*

PROOF. Let $\mathfrak{h}$ be toral and let $x \in \mathfrak{h}$. Then $\mathrm{ad}(x)|_{\mathfrak{h}}$ has an eigenvector $y \in \mathfrak{h}$; i.e., $[x, y] = \lambda y$ for some $\lambda \in \mathbf{C}$. We intend to show $\lambda = 0$. On the other hand, $\mathrm{ad}(y)|_{\mathfrak{h}}$ is also diagonalizable, say

$$\mathrm{ad}(y)|_{\mathfrak{h}} = \begin{array}{c} \begin{matrix} z_1 & \cdots & z_n & \cdots & \cdots & z_m \end{matrix} \\ \begin{pmatrix} \lambda_1 &&&&& \\ & \ddots &&&& \\ && \lambda_n &&& \\ &&& 0 && \\ &&&& \ddots & \\ &&&&& 0 \end{pmatrix} \end{array}$$

where $\lambda_i \in \mathbf{C}^\times$, in some basis $\{z_1, \ldots, z_m\}$. Write $x = c_1 z_1 + \cdots + c_m z_m$ with $c_i \in \mathbf{C}$. Then

$$[y, x] = \sum_{i=1}^{n} \lambda_i c_i z_i$$

is in the span of nonzero eigenspaces of $\mathrm{ad}(y)$. But $[x, y] = -[y, x]$ is in the span of y by the above, and $\mathrm{ad}(y)(y) = [y, y] = 0$. This is in $\ker \mathrm{ad}(y)$, so all $\lambda_i = 0$ and $\lambda = 0$. ∎

In the general setup, $\mathfrak{h} \subset \mathfrak{g}$ is a *maximal* toral subalgebra, which will play the same role the diagonal matrices of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ played in their respective representation theory. As an exercise, one should show that the diagonal matrices in any classical algebra is a maximal toral subalgebra.

The maximal toral subalgebra does not contain *all* semisimple elements of $\mathfrak{g}$. For example, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ are semisimple, but they do not commute.

Since $\mathfrak{h}$ is abelian, the action of $\mathfrak{h}$ on any representation of $\mathfrak{g}$ is diagonalizable. In particular, from the adjoint representation we get

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^\vee} \mathfrak{g}_\alpha = \mathfrak{g}_0 \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \Big),$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

and

$$\Phi = \{\alpha \in \mathfrak{h}^\vee : \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\}.$$

For example, in $\mathfrak{sl}_3$,

$$\Phi\{\alpha_i - \alpha_j : 1 \leqslant i \neq j \leqslant 3\}.$$

Elements of Φ are called ***roots***.

We know $\mathfrak{h} \subset \mathfrak{g}_0$, which we will prove soon. We'll also show (much later) that Φ contains "all the information" about $\mathfrak{g}$.

2.17. Proposition. *Let $\mathfrak{g}$ be a semisimple Lie algebra.*

(1) *For $\alpha, \beta \in \mathfrak{h}^\vee$, we get*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}.$$

(2) *If $x \in \mathfrak{g}_\alpha$ with $\alpha \neq 0$, then $\mathrm{ad}(x)$ is nilpotent.*
(3) *If $\beta \neq -\alpha$ in $\mathfrak{h}^\vee$, then $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ under the Killing form.*
(4) *The Killing form is nondegenerate on $\mathfrak{g}_0$.*

Proof. The proof of (1) is precisely the fundamental calculation we carried out for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, so we omit it here. Since $\mathfrak{g}$ is finite-dimensional, $\mathfrak{g}_{\beta+k\alpha} = 0$ for $k \gg 0$. This is the same idea as in Figure 2.4 in the $\mathfrak{sl}_3$ case, so (2) is also obvious from earlier work.

The third assertion is more interesting. Suppose $\beta \neq -\alpha$ in $\mathfrak{h}^\vee$. Then there exists $z \in \mathfrak{h}$ such that $\beta(z) \neq -\alpha(z)$. Now let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. If x and y are orthogonal, then $\kappa(x, y) = 0$. So

$$\kappa([z, x], y) = \alpha(z)\kappa(x, y).$$

By associativity of the Killing form, we have

$$\kappa(-[x,z],y) = -\kappa(x,[z,y]) = -\beta(z)\kappa(x,y).$$

Thus

$$\alpha(z)\kappa(x,y) = -\beta(z)\kappa(x,y),$$

which implies $\kappa(x,y) = 0$.

We now prove (4). Note $\mathfrak{g}_0$ is not an ideal, so we cannot apply earlier results to prove nondegeneracy. Let $z \in \mathfrak{g}_0$. Then $\kappa(z,x) = 0$ for $x \in \mathfrak{g}_\alpha$, where $\alpha \neq 0$, by part (3). So if $\kappa(z,x) = 0$ for all $x \in \mathfrak{g}_0$, then $\kappa(z,x) = 0$ for all $x \in \mathfrak{g}$, which implies $z \in \operatorname{rad}\kappa$. But $\operatorname{rad}\kappa = 0$ because $\mathfrak{g}$ is semisimple, so there must be some $x \in \mathfrak{g}_0$ such that $\kappa(z,x) \neq 0$ if $z \neq 0$. ■

2.18. Proposition. *In this setup,* $\mathfrak{g}_0 = \mathfrak{h}$.

Proof. We prove in several steps.

Step 1. We show $\mathfrak{g}_0$ contains the semisimple and nilpotent parts of its elements. We have $x \in \mathfrak{g}_0$ if and only if $[z,x] = 0$ for all $z \in \mathfrak{h}$, if and only if $\operatorname{ad}(x)(z) = 0$ for all $z \in \mathfrak{h}$. But $\operatorname{ad}(x)_s(z) = 0$ since $\operatorname{ad}(x)_s$ is a polynomial in $\operatorname{ad}(x)$ with no constant term. Then $\operatorname{ad}(x)_s = \operatorname{ad}(x_s)$, so $\operatorname{ad}(x_s)(z) = 0$, and so $x_s \in \mathfrak{g}_0$. The same idea holds for x_n.

Step 2. We claim $\mathfrak{h} \subset \mathfrak{g}_0$ contains all the semisimple elements of $\mathfrak{g}_0$. If $s \in \mathfrak{g}_0$ is semisimple but not in $\mathfrak{h}$, then $\mathfrak{h} + \mathbf{C}s$ is a larger toral subalgebra, contradicting the maximality of $\mathfrak{h}$.

Step 3. The Killing form is nondegenerate when restricted to $\mathfrak{h}$. Indeed, suppose $h \in \mathfrak{h}$ is such that $\kappa(h,y) = 0$ for all $y \in \mathfrak{h}$. If $x \in \mathfrak{g}_0$ is nilpotent, then $\operatorname{ad}(x)$ is nilpotent and $[x,y] = 0$ for all $y \in \mathfrak{h}$. But then $[\operatorname{ad}(x), \operatorname{ad}(y)] = 0$, and so $\operatorname{ad}(x)$ and $\operatorname{ad}(y)$ commute. Thus $\operatorname{ad}(x)\operatorname{ad}(y)$ is nilpotent. So $\kappa(x,y) = \operatorname{Tr}(\operatorname{ad}(x)\operatorname{ad}(y)) = 0$ for all x nilpotent and $y \in \mathfrak{h}$. Since $\mathfrak{g}_0$ is generated by $\mathfrak{h}$ and nilpotent elements, this gives that $\kappa(h,y) = 0$ for all $y \in \mathfrak{g}_0$. However, this contradicts that $\kappa|_{\mathfrak{g}_0}$ is nondegenerate. Hence $\kappa|_{\mathfrak{h}}$ is nondegenerate.

Step 4. Now we show $\mathfrak{g}_0$ is a nilpotent Lie algebra. Let $x = x_s + x_n$ be a Jordan decomposition of $x \in \mathfrak{g}_0$. We know $x_s \in \mathfrak{h}$ by Step 2, so $\operatorname{ad}(x_s)(\mathfrak{g}_0) = 0$ is nilpotent (the zero map is nilpotent). Since x_n is nilpotent, $\operatorname{ad}(x_n)$ is nilpotent. Hence $\operatorname{ad}(x_s)$ and $\operatorname{ad}(x_n)$ commute, so $\operatorname{ad}(x) = \operatorname{ad}(x_n) + \operatorname{ad}(x_s)$ is nilpotent for $x \in \mathfrak{g}_0$. Then Engel's theorem implies $\mathfrak{g}_0$ is nilpotent.

Step 5. We show $\mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0] = 0$. Indeed,

$$\kappa(\mathfrak{h}, [\mathfrak{g}_0, \mathfrak{g}_0]) = \kappa([\mathfrak{h}, \mathfrak{g}_0], \mathfrak{g}_0) = 0$$

since $[\mathfrak{h}, \mathfrak{g}_0] = 0$ already. By Step 3, $\kappa|_{\mathfrak{h}}$ is nondegenerate, so $[\mathfrak{g}_0, \mathfrak{g}_0]$ cannot contain any elements of $\mathfrak{h}$, so $\mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0] = 0$.

Step 6. $\mathfrak{g}_0$ is abelian. If $[\mathfrak{g}_0, \mathfrak{g}_0] \neq 0$, then by Proposition 1.35(3), we have $Z(\mathfrak{g}_0) \cap [\mathfrak{g}_0, \mathfrak{g}_0] \neq 0$, say it has a nonzero element z. Then z is not semisimple by combining Steps 2 and 5, so the nilpotent part of z is nonzero and $z_n \in [\mathfrak{g}_0, \mathfrak{g}_0] \cap Z(\mathfrak{g}_0)$. Since z_n is nilpotent and $z_n \in Z(\mathfrak{g})$, we have $\operatorname{ad}(z_n)\operatorname{ad}(x)$ is nilpotent for all $x \in \mathfrak{g}_0$ and $\kappa(z_n, \mathfrak{g}_0) = 0$. This is a contradiction with $\kappa|_{\mathfrak{g}_0}$ be nondegenerate.

Step 7. $\mathfrak{g}_0 = \mathfrak{h}$. If not, then $\mathfrak{g}_0$ has some nilpotent x since $\operatorname{ad}(x)\operatorname{ad}(y) = \operatorname{ad}(y)\operatorname{ad}(x)$ for all $y \in \mathfrak{g}_0$. Thus $\kappa(x, y) = 0$ for all $y \in \mathfrak{g}_0$, contradicting the nondegeneracy of κ on $\mathfrak{g}_0$. ∎

Recall that we use the Killing form to identify $\mathfrak{h}$ with $\mathfrak{h}^\vee$. Indeed, define a map $x \mapsto \kappa(x, \cdot)$ for all $x \in \mathfrak{h}$. The nondegeneracy of the Killing form implies that this map is an isomorphism of vector spaces. Conversely, for $\varphi \in \mathfrak{h}^\vee$, let t_φ be the unique element of $\mathfrak{h}$ such that $\kappa(t_\varphi, y) = \varphi(y)$ for all $y \in \mathfrak{h}$. We will use this notation in the following section.

7. Construction of $\mathfrak{sl}_2$-triples

Our goal is the following: For each $\alpha \in \Phi$, we will construct a subalgebra $\mathfrak{s}_\alpha \subset \mathfrak{g}$ with $\mathfrak{sl}_2 \cong \mathfrak{s}_\alpha$. To do so, first note the following facts.

(a) Φ spans $\mathfrak{h}^\vee$. If Φ spanned a proper subspace, then there exists an $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$. That is, $[h, x] = \alpha(h)x = 0$ for all $\alpha \in \Phi$ and $x \in \mathfrak{g}_\alpha$. But we know that $[h, x] = 0$ for all $x \in \mathfrak{h}$, so $h \in Z(\mathfrak{g})$. This contradicts the fact that $\mathfrak{g}$ is semisimple (i.e., $Z(\mathfrak{g}) = 0$).

(b) If $\alpha \in \Phi$, then $-\alpha \in \Phi$. We know that $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\mu) = 0$ for all $\mu \neq -\alpha$ by orthogonality, so if $\mathfrak{g}_{-\alpha} = 0$, then $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}) = 0$. This contradicts the nondegeneracy of the Killing form on $\mathfrak{g}$.

(c) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbf{C}t_\alpha$; specifically, $[x, y] = \kappa(x, y)t_\alpha$ for all $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$. To verify, we apply $\kappa(z, \cdot)$ to both sides with $z \in \mathfrak{h}$ arbitrary. Then

$$\kappa(z, [x, y]) = \kappa([z, x], y) = \alpha(z)\kappa(x, y)$$

and

$$\kappa(z, \kappa(x, y)t_\alpha) = \kappa(x, y)\kappa(z, t_\alpha) = \alpha(z)\kappa(x, y).$$

So $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbf{C}t_\alpha$. But if $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = 0$, then $\kappa(x, y) = 0$ for all $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, which contradicts the nondegeneracy of the Killing form.

(d) $\kappa(t_\alpha, t_\alpha) \neq 0$. If $\kappa(t_\alpha, t_\alpha) = 0$, then by definition we have $\alpha(t_\alpha) = 0$. So we can choose $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ such that $[x, y] = ct_\alpha \neq 0$ for some $c \in \mathbf{C}$. Then $[t_\alpha, x] = \alpha(t_\alpha)x = 0$, and similarly $[t_\alpha, y] = 0$. This implies the subalgebra $\mathfrak{s} = \langle x, y, t_\alpha \rangle$ is solvable. Since $\langle t_\alpha \rangle$ is an ideal, we have $\mathfrak{s}/\langle t_\alpha \rangle = \langle x, y \rangle$ is abelian. Then the adjoint action of $\mathfrak{s}$ on $\mathfrak{g}$ can be made upper triangular, so

$$\mathrm{ad}(t_\alpha) = c^{-1}\,\mathrm{ad}([x, y]) = c^{-1}[\mathrm{ad}(x), \mathrm{ad}(y)]$$

is nilpotent. This contradicts $t_\alpha \in \mathfrak{h}$ being semisimple.

Now let $x_\alpha \in \mathfrak{g}_\alpha$ be arbitrary, for some fixed root $\alpha \in \Phi$, and choose $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that

$$[x_\alpha, y_\alpha] = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}.$$

Define the element on the right-hand side to be h_α. Then

$$\alpha(h_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)}\alpha(t_\alpha) = 2.$$

From $[x_\alpha, y_\alpha] = h_\alpha$, we have

$$[h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha = 2x_\alpha$$

and

$$[h_\alpha, y_\alpha] = -\alpha(h_\alpha)y_\alpha = -2y_\alpha.$$

Thus the subalgebra $\mathfrak{s} = \langle x_\alpha, y_\alpha, h_\alpha \rangle$ is isomorphic to $\mathfrak{sl}_2$.

Now let $\alpha \in \Phi \subset \mathfrak{h}^\vee$ be fixed, and consider

$$M = \mathfrak{h} \oplus \bigoplus_{c \in \mathbf{C}^\times} \mathfrak{g}_{c\alpha}.$$

This is an $\mathfrak{s}_\alpha$-module. If x is a nonzero element in $\mathfrak{g}_{c\alpha}$, then

$$[h_\alpha, x] = c\alpha(h_\alpha)x = 2cx,$$

so $2c \in \mathbf{Z}$ by the isomorphism with $\mathfrak{sl}_2$. Recall that every irreducible $\mathfrak{sl}_2$-module has either 0 or 1 as a weight of h_α, so h_α acts by zero precisely on $\mathfrak{h} = \ker \alpha + \mathbf{C}h_\alpha$. But $\ker \alpha$ is an $\mathfrak{s}_\alpha$-module itself, with trivial action. (It suffices to check that x_α, y_α, and h_α all kill $\ker \alpha$. Moreover, $h_\alpha \in \mathfrak{s}_\alpha$, which is an $\mathfrak{s}_\alpha$-submodule of M. So M has no other submodules with an even string of eigenvalues of $\mathfrak{h}_\alpha$. In particular, 4 is not an eigenvalue of h_α, so 2α is not a root. By applying the same reasoning to $\frac{1}{2}\alpha$, we see $\frac{1}{2}\alpha$ cannot be a root either. Thus $c\alpha \in \Phi$ if and only if $c = \pm 1$. So

$$M = \ker \alpha \oplus \mathfrak{s}_\alpha$$

as an $\mathfrak{s}_\alpha$-module. But wait, there's more! As vector spaces,

$$M = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha},$$

with $\mathfrak{g}_\alpha = \mathbf{C}x_\alpha$ and $\mathfrak{g}_{-\alpha} = \mathbf{C}y_\alpha$.

To study other root spaces, say $\beta \in \Phi$, add the minimal amount of "stuff" to $\mathfrak{g}_\beta$ to get an $\mathfrak{s}_\alpha$-module. Let

$$N = \sum_{k \in \mathbf{Z}} \mathfrak{g}_{\beta + k\alpha}.$$

This is an $\mathfrak{s}_\alpha$-module. To analyze this space, we look at the eigenvalues of h_α again. We have

$$(\beta + k\alpha)(h_\alpha) = \beta(h_\alpha) + 2k \in \mathbf{Z}.$$

From previous work, each $\mathfrak{g}_{\beta+k\alpha}$ is 1-dimensional (provided it's nonzero). Both 1 and 0 cannot appear as eigenvalues simultaneously; i.e., the eigenvalues of h_α on N are

$$\beta(h_\alpha) - 2r, \ldots, \beta(h_\alpha) - 2, \beta(h_\alpha), \beta(h_\alpha) + 2, \ldots, \beta(h_\alpha) + 2q,$$

which forms an unbroken string of integers symmetric about zero. This yields $\beta(h_\alpha) = r - q \in \mathbf{Z}$ and the string of roots

$$\{\beta - r\alpha, \ldots, \beta + q\alpha\}$$

is unbroken. Finally, if γ and δ are roots such that $\gamma + \delta \in \Phi$, then $[\mathfrak{g}_\gamma, \mathfrak{g}_\delta] = \mathfrak{g}_{\gamma+\delta}$.

8. Rationality of the Killing form

Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. Recall that the identification $\mathfrak{h} \cong \mathfrak{h}^\vee$ via the Killing form κ gives a form on $\mathfrak{h}^\vee$. Define $(\cdot,\cdot)\colon \mathfrak{h}^\vee \times \mathfrak{h}^\vee \longrightarrow \mathbf{C}$ via $(\lambda, \mu) = \kappa(t_\lambda, t_\mu)$. Our goal is to show $(\cdot,\cdot)$ is a positive definite form on the real vector space spanned by the roots Φ. That is, we want to show $(\alpha, \alpha) > 0$ for all $\alpha \in \Phi$.[4]

First consider $\kappa|_{\mathfrak{h}}$. If $x \in \mathfrak{h}$, then the eigenvalues of $\operatorname{ad}(x)$ are $\{\alpha(x) : \alpha \in \Phi\} \cup \{0\}$, so for $x, y \in \mathfrak{h}$ we have

$$\kappa(x, y) = \operatorname{Tr}(\operatorname{ad}(x)\operatorname{ad}(y)) = \sum_{\alpha\in\Phi} \alpha(x)\alpha(y).$$

In particular, for $\beta \in \Phi$,

$$\kappa(h_\beta, h_\beta) = \sum_{\alpha\in\Phi} \alpha(h_\beta)^2$$

since we defined h_β so that $\alpha(h_\beta) \in \mathbf{Z}$.[5] Under the correspondence $\mathfrak{h} \cong \mathfrak{h}^\vee$, we have

$$h_\beta \longleftrightarrow \frac{2}{(\beta,\beta)}\beta.$$

Then

$$\kappa(h_\beta, h_\beta) = \left(\frac{2}{(\beta,\beta)}\beta, \frac{2}{(\beta,\beta)}\beta\right) = \frac{4}{(\beta,\beta)^2}(\beta,\beta) = \frac{4}{(\beta,\beta)} > 0,$$

[4] Notice that positive definiteness does not make sense on a $\mathbf{C}$-vector space. If $(\lambda,\lambda) > 0$, then $(i\lambda, i\lambda) = i^2(\lambda,\lambda) = -(\lambda,\lambda) < 0$.

[5] These $\alpha(h_\beta)$ are called Cartan integers.

and so $(\beta,\beta) \in \mathbf{Q}_{>0}$. We will define for future use the ***coroot*** by

$$\beta^\vee = \frac{2}{(\beta,\beta)}\beta.$$

We also see that $\mu(h_\beta) = (\mu, \beta^\vee) \in \mathbf{Z}$ for $\beta, \mu \in \Phi$. But

$$(\mu, \beta^\vee) = \frac{2(\mu,\beta)}{(\beta,\beta)}$$

implies $(\mu,\beta) \in \mathbf{Q}$. So one can say, "all the information about Φ and $(\cdot,\cdot)$ is contained in some $\mathbf{Q}$-vector space."

How large is the $\mathbf{Q}$-vector space spanned by Φ? Say $\dim_{\mathbf{C}}(\mathfrak{h}^\vee) = n$. Let $\alpha_1, \ldots, \alpha_n \in \Phi$ be a $\mathbf{C}$-basis of $\mathfrak{h}^\vee$. If β is any other root, then

$$\beta = \sum_{i=1}^{n} c_i \alpha_i, \qquad c_i \in \mathbf{C}^\times.$$

Apply $(\cdot, \alpha_j)$ to each side to get

$$(\beta, \alpha_j) = \sum_{i=1}^{n} c_i (\alpha_i, \alpha_j).$$

Then the c_i are the solutions to a system of n linear equations in n unknowns. The (α_i, α_j) are the entries of the matrix for the $\kappa|_{\mathfrak{h}^\vee}$ in the basis $\alpha_1, \ldots, \alpha_n$, so the nondegeneracy of $\kappa|_{\mathfrak{h}^\vee}$ implies this matrix is nonsingular. Hence the c_i's are unique solutions to this system. We just showed that each (β, α_j) and (α_i, α_j) are elements of $\mathbf{Q}$. Hence $c_i \in \mathbf{Q}$ for each i. This shows that the $\mathbf{Q}$-dimension of the $\mathbf{Q}$-vector space spanned by Φ is still n.

9. The Weyl group

For each $\alpha \in \Phi$, define a $\mathbf{C}$-linear map $s_\alpha \colon \mathfrak{h}^\vee \longrightarrow \mathfrak{h}^\vee$ by

$$\beta \mapsto \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - (\beta, \alpha^\vee)\alpha.$$

We start by making some observations about these functions. Notice $s_\alpha(\alpha) = -\alpha$. Also, $s_\alpha(\beta) = \beta$ if and only if $(\beta,\alpha) = 0$; i.e., s_α is reflection in the codimension one subspace $\Omega_\alpha = \{\beta \in \mathfrak{h}^\vee : (\alpha,\beta) = 0\}$. Thus s_α is an involution.

Let $W = W(\Phi)$ be the group generated by $\{s_\alpha : \alpha \in \Phi\} \subset \mathrm{GL}(\mathfrak{h}^\vee)$. This is called the ***Weyl group*** of Φ. When $\beta \in \Phi$, we get $s_\alpha(\beta) = \beta = \beta(h_\alpha)\alpha$ and $\beta(h_\alpha) = r - q$, where

$$\beta - r\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \ldots, \beta + q\alpha$$

is the α-string through β, where $r - q$ is the distance of β from the middle of the string. In particular, $\beta - \beta(h_\alpha)\alpha$ is always a root! Thus W permutes Φ! Hence W is a finite group.

CHAPTER 3

Root Systems

1. Abstract root systems

Recall that if $\mathfrak{g}$ is semisimple, then a choice of maximal toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$ leads to a set of roots $\Phi \subset \mathfrak{h}^\vee$. This does not depend on the choice of $\mathfrak{h}$. (Of course, we have Φ as a subset of $\mathfrak{h}^\vee$, so this well-definedness of Φ will require some more explanation.) Define

$$Q = \bigoplus_{\alpha \in \Phi} \mathbf{Z}\alpha$$

to be the root lattice. Let $E = \mathbf{R} \otimes_{\mathbf{Z}} Q$ be the $\mathbf{R}$-span of the roots. We know $\dim_{\mathbf{R}}(E) = \dim_{\mathbf{C}}(\mathfrak{h})$, and this common value is called the ***rank*** of $\mathfrak{g}$. This is also sometimes referred to as the rank of Φ.

We are interested in the following properties of E, equipped with the bilinear form $(\cdot, \cdot)$, and Φ.

(1) Φ is a finite set spanning E.
(2) For $\alpha \in \Phi$ and $c \in \mathbf{R}$, we have $c\alpha \in \Phi$ if and only if $c = \pm 1$.
(3) For $\alpha \in \Phi$, the reflection over the hyperplane orthogonal to α preserves Φ.

(4) For $\alpha, \beta \in \Phi$, we have

$$(\beta, \alpha^\vee) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbf{Z}.$$

3.1. Definition. Any finite subset of a Euclidean space[1] satisfying all four conditions above is called an ***abstract root system***.

Our goal, using this definition, is to classify all semisimple (complex) Lie algebras as follows:

(A) A semisimple $\mathfrak{g}$ gives an abstract root system.
(B) Find all abstract root systems.
(C) If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are semisimple Lie algebras yielding the "same" root system, then $\mathfrak{g}_1 \cong \mathfrak{g}_2$ as Lie algebras.
(D) Every abstract root system is the root system of some finite-dimensional semisimple Lie algebra.

We have already shown (A) in our previous work. The proof of (B) is essentially combinatorial, and (C) and (D) are intertwined.

Since only ratios of inner products of vectors appear in the axioms for a root system, we see that for any root system $\Phi \subset E$, we get many more root systems by scaling Φ.

3.2. Definition. Two root systems $\Phi \subset E$ and $\Phi' \subset E'$ are said to be isomorphic if there is an isomorphism $f \in \mathrm{Hom}_{\mathbf{R}}(E, E')$ such that $f(\Phi) = \Phi'$ and $(\alpha, \beta^\vee)_E = (f(\alpha), f(\beta)^\vee)_{E'}$ for all $\alpha, \beta \in \Phi$.[2]

Let's look at (B) now. What are the possible root systems? Property (4) from the axioms of a root system is very restrictive. Recall that for $v, w \in E$, we have $v \cdot w = \|v\|\|w\| \cos\theta$, where θ is the angle between v and w. So for $\alpha, \beta \in \Phi$, we get

$$(\beta, \alpha^\vee) = \frac{2(\beta,\alpha)}{\|\alpha\|^2} = \frac{2\|\beta\|}{\|\alpha\|}\cos\theta \in \mathbf{Z}.$$

Hence

$$(\beta, \alpha^\vee)(\alpha, \beta^\vee) = 4\cos^2\theta \in \mathbf{Z}.$$

---

[1]A Euclidean space is an $\mathbf{R}$-vector space with a positive-definite, symmetric bilinear form.

[2]Think of the last condition as "f preserves the angles between vectors and ratios between root lengths."

Since $0 \leqslant \cos^2\theta \leqslant 1$, we have $4\cos^2\theta = 0, 1, 2, 3, 4$. If $4\cos^2\theta = 0$, then $\cos^2\theta = 0$ which implies α and β are orthogonal, so $(\alpha, \beta^\vee) = 0$. On the other extreme, if $4\cos^2\theta = 4$, then $\cos^2\theta = 1$ and $\cos\theta = \pm 1$, which implies α and β parallel, or that $\beta = \pm\alpha$. So suppose $4\cos^2\theta = 1, 2, 3$. Since $4\cos^2\theta$ is the product of integers $(\beta, \alpha^\vee)$ and $(\alpha, \beta^\vee)$, without loss of generality we may assume $(\alpha, \beta^\vee) = \pm 1$. Moreover, $(\beta, \alpha^\vee)$ and $(\alpha, \beta^\vee)$ have the same sign. This yields six possibilities for pairs $(\alpha, \beta^\vee)$ and $(\alpha, \beta^\vee)$. For each possibility, compute θ to get $\frac{\|\beta\|}{\|\alpha\|}$. A table for such values is available in [**4**, §2.1] and [**8**, §9.4].

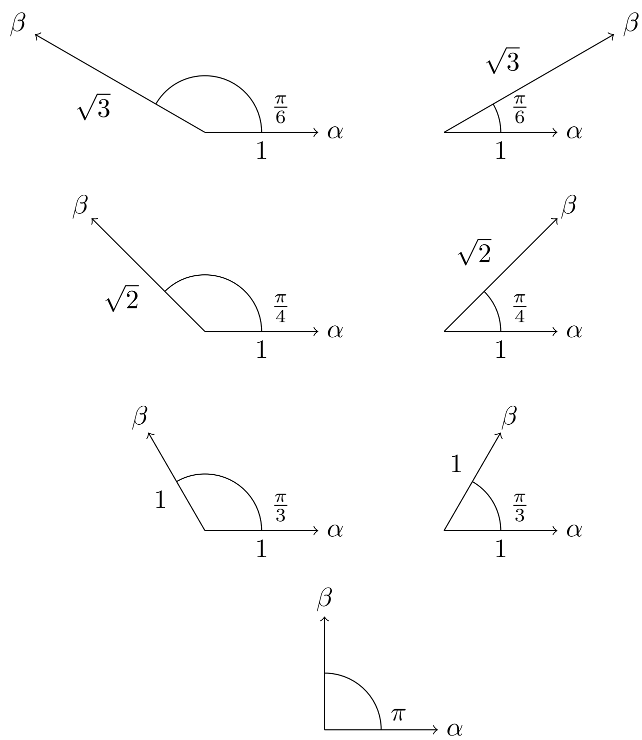
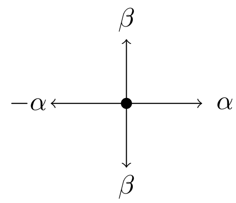


FIGURE 3.1. The possible angles between roots α and β in a root system.

3.3. EXAMPLE. We start with rank 1. This yields the root space

$$A_1 : \qquad -\alpha \longleftarrow\!\!\bullet\!\!\longrightarrow \alpha$$

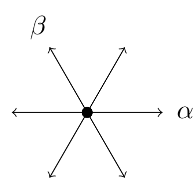
3.4. Example. Consider $A_1 \times A_1$, which is a rank two root system.



This root system is "reducible," as it decomposes into two root systems $E = E_1 \times E_2$ with every vector in Φ_1 is orthogonal to all vectors in Φ_2. We will be interested in irreducible root systems.

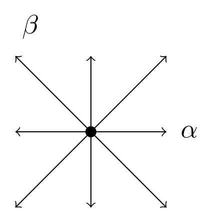
The other three rank two root systems are

A_2 :



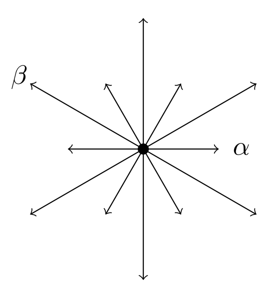
which corresponds to $\mathfrak{sl}_3$,

$B_2 = C_2$:



which corresponds to $\mathfrak{sp}_4$ and $\mathfrak{so}_5$, and

G_2 :



The Lie algebra corresponding to G_2 is 14-dimensional, and can be represented by 7×7 matrices.

2. Simple roots

Our goal is to simplify the amount of data needed to describe a root system $\Phi \subset E$ by specifying a "good" basis of E. There will be many possible choices for such a basis.

First, we choose a "direction" in E for the basis. By direction, we mean a linear functional $\ell\colon E \longrightarrow \mathbf{R}$ such that $\ell(\alpha) \neq 0$ for all $\alpha \in \Phi$. This function ℓ divides E into two halves; namely $E^+ = \{p \in E : \ell(p) > 0\}$ and $E^- = \{p \in E : \ell(p) < 0\}$. Then $\alpha \in E^+$ if and only if $-\alpha \in E^-$, so exactly half of the roots lie in each half space. Define $\Phi^+ = \Phi \cap E^+$ and $\Phi^- = \Phi \cap E^-$ to be the ***positive roots*** and ***negative roots***, respectively.[3]

3.5. Example. Here is an example of a chosen direction in G_2. Note that many different ℓ give the same sets of positive and negative

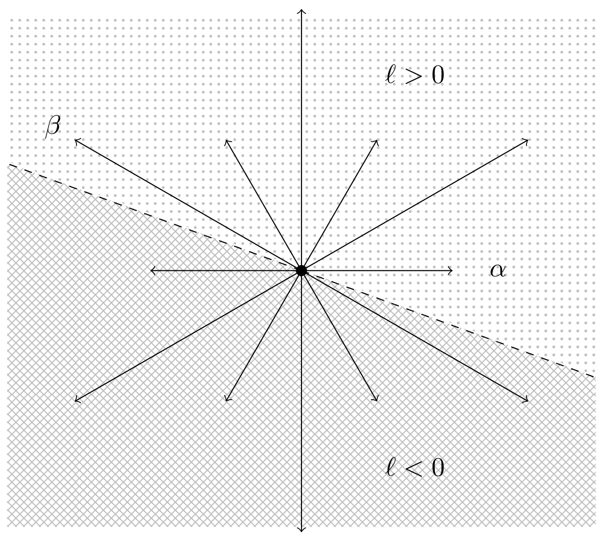


Figure 3.2. An example of picking a 'direction' ℓ in the root system G_2. The dashed line represents $\ell = 0$.

roots by "wiggling" ℓ a bit.

[3]These definitions rely on the choice of direction. We will suppress this comment from the discussion from now on.

3.6. Definition. Call $\alpha \in \Phi^+$ ***simple*** if it cannot be written as the sum of two positive roots.

In the example above, both α_1 and α_2 are simple roots. They are the roots that are "close" to ℓ.

3.7. Theorem. *In a root system $\Phi \subset E$, the set of simple roots is a basis of E. Write $\Delta = \{\alpha_1, \dots, \alpha_n\}$ for the set of simple roots. For any $\beta \in \Phi^+$, we can write $\beta = c_1\alpha_1 + \cdots + c_n\alpha_n$, where each $c_i \in \mathbf{Z}_{\geqslant 0}$.*

Proof. We prove somewhat informally.

Fact 1. Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$ and consider the α-string through β:

$$\beta - p\alpha, \beta - (p-1)\alpha, \dots, \beta - \alpha, \beta, \beta + \alpha, \dots, \beta + (q-1)\alpha, \beta + q\alpha.$$

Applying s_α to the α-string through β reverses the string, so $s_\alpha(\beta - p\alpha) = \beta + q\alpha$. By the definition of s_α, we have

$$s_\alpha(\beta - p\alpha) = \beta - (\beta, \alpha^\vee)\alpha + p\alpha.$$

Combining these two equations implies $(\beta, \alpha^\vee) = p - q$. Choosing β at the far left of a string amounts to putting $p = 0$ and we get $-(\beta, \alpha^\vee) = q$, which is the length of the string. We saw earlier that $q \leqslant 3$.

Fact 2. Let $\alpha, \beta \in \Phi$ such that $\alpha \neq \pm\beta$. Then

$$(\beta, \alpha)\begin{cases} > 0 & \Longrightarrow \alpha - \beta \in \Phi \\ < 0 & \Longrightarrow \alpha + \beta \in \Phi \\ = 0 & \Longrightarrow \alpha \pm \beta \in \Phi \text{ or } \alpha \pm \beta \notin \Phi. \end{cases}$$

Notice that (β, α) and $(\beta, \alpha^\vee)$ have the same parity. By Fact 1, we have $(\beta, \alpha^\vee) = p - q < 0$ if β left of center in the α-string through β, and $(\beta, \alpha^\vee) > 0$ if β is the right of center. This amounts to being able to add (or, respectively, subtract) α from β and still be on the α-string; i.e., adding (resp. subtracting) α still yields a root. If $(\beta, \alpha^\vee) = 0$, then α and β are orthogonal, so $p = q$ and β is in the center of the α-string. Since α-strings are symmetric, we can move both left and right (i.e., add or subtract α) or can do neither.

Fact 3. If α_i and α_j are simple roots, then $\pm(\alpha_i - \alpha_j)$ is not a root. Suppose $\alpha_i - \alpha_j \in \Phi^+$. Then $\alpha_i = (\alpha_i - \alpha_j) + \alpha_j$, which contradicts $\alpha_i \in \Delta$. If $\alpha_i - \alpha_j \in \Phi^-$, then $-(\alpha_i - \alpha_j) \in \Phi^+$, and apply the same argument.

Fact 4. The angle between two simple roots is not acute, so $(\alpha_i, \alpha_j) \leqslant 0$. Given Fact 2, if $(\alpha_i, \alpha_j) > 0$, then $\alpha_i - \alpha_j \in \Phi$, which contradicts Fact 3.

Fact 5. The simple roots are linearly independent. A relation $r_1\alpha_1 + \cdots + r_n\alpha_n = 0$, with $r_i \in \mathbf{R}$, can be rewritten as

$$a_1\alpha_1 + \cdots + a_n\alpha_n = b_1\alpha_1 + \cdots + b_n\alpha_n$$

with all $a_i, b_j \geqslant 0$ and either $a_i = 0$ or $b_i = 0$ for all $1 \leqslant i \leqslant n$. Then, by Fact 4,

$$\Big(\sum_{i=1}^{n} a_i\alpha_i, \sum_{j=1}^{n} b_j\alpha_j\Big) = \sum_{i,j} a_i b_j (\alpha_i, \beta_j) \leqslant 0.$$

Hence

$$\sum_{i=1}^{n} a_i\alpha_i = \sum_{j=1}^{n} b_j\alpha_j = 0$$

by nondegeneracy of $(\cdot,\cdot)$. Applying ℓ to these gives

$$\sum_{i=1}^{n} a_i \ell(\alpha_i) = 0.$$

Since each $\ell(\alpha_i) > 0$, all $a_i = 0$. Repeating this with the other expression shows each $b_j = 0$. Hence Δ is linearly independent.

Fact 6. Every positive root can be written as a $\mathbf{Z}_{\geqslant 0}$-linear combination of the simple roots. Suppose to the contrary that there exists $\alpha \in \Phi^+$ that cannot be written in such a way. Choose such α with $\ell(\alpha)$ minimal. Since α is not simple, we can write $\alpha = \beta + \gamma$ for some $\beta, \gamma \in \Phi^+$. Then $\ell(\beta) < \ell(\alpha)$ and $\ell(\gamma) < \ell(\alpha)$ by linearity, which contradicts the minimality of $\ell(\alpha)$. Hence α has the desired representation. Moreover, every root is in the $\mathbf{R}$-span of the simple roots, which implies Δ spans E. Hence Δ is a basis of E. ∎

From Fact 6, we can strengthen Fact 3. If $\sum a_i\alpha_i$ is a root, then all $a_i \in \mathbf{Z}_{\geqslant 0}$ or all $a_i \in \mathbf{Z}_{\leqslant 0}$. This observation allows us to define a notion of height with respect to a root. For $\beta = a_1\alpha_1 + \cdots + a_n\alpha_n$, define the ***height*** of β, denoted $\mathrm{ht}(\beta)$, is given by $a_1 + \cdots + a_n \in \mathbf{Z}$.

Now we present some facts.

- Every root system has a unique highest root.
- Although Δ depended on a choice of $\ell\colon E \longrightarrow \mathbf{R}$, if we obtain Δ' from $\ell'\colon E \longrightarrow \mathbf{R}$, then there exists $w \in W(\Phi) \subset \mathrm{GL}(E)$ such that $w(\Delta) = \Delta'$.

3. Dynkin diagrams

3.8. Definition. A (symmetrizable) generalized Cartan matrix is a matrix (a_{ij}) with integer entries satisfying the following:

(C1) $a_{ii} = 2 - 2n_i$ for some $n_i \in \mathbf{Z}_{\geqslant 0}$,
(C2) $a_{ij} \leqslant 0$ for $i \neq j$,
(C3) $a_{ij} = 0$ if and only if $a_{ji} = 0$,
(C4) there exists a diagonal matrix D and symmetric matrix S such that $A = SD$.

We say that A is positive-definite if the associated S is positive-definite. To see this material in more generality, see [**10**].

3.9. Example. Let Φ be a root system and Δ be a set of simple roots. The Cartan matrix of Φ is the matrix $A = (a_{ij})$, where $a_{ij} = (\alpha_i, \alpha_j^\vee)$. Each condition for this A is obvious, except for the fourth condition. For this axiom, use $S = \big((\alpha_i, \alpha_j)\big)$ and

$$D = \mathrm{diag}\left(\frac{2}{(\alpha_j, \alpha_j)}\right).$$

In this case, A is positive-definite because the Killing form is positive-definite.

3.10. Definition. The ***generalized Dynkin diagram*** associated to an $n \times n$ Cartan matrix $A = (a_{ij})$ is constructed as follows.

(1) Draw a node for every $i \in \{1, \ldots, n\}$. These correspond to the simple roots in the root system.

(2) Draw n_i loops at each vertex i.

(3) For $i \neq j$, draw a colored edge

$$i \xrightarrow{(|a_{ij}|, |a_{ji}|)} j$$

when $a_{ij} \neq 0$. When A comes from a root system, the only possible edges are

$$i \overset{(1,1)}{\text{———}} j \qquad i \overset{(2,1)}{\text{———}} j \qquad i \overset{(3,1)}{\text{———}} j$$

In the older notation, these three cases are written as

$$i \text{ —— } j \qquad i \Rightarrow j \qquad i \Rrightarrow j$$

The Cartan matrix can be completely recovered from the Dynkin diagram. It is a good exercise to verify a root system $\Phi \subset E$ is irreducible if and only if its Dynkin diagram is connected.[4]

3.11. THEOREM (Classification of root systems). *A Cartan matrix is positive-definite if and only if its diagram is a member of one of the four following families in Table 3.1. Moreover, each diagram in this list comes from a finite root system.*

PROOF. Reduce to these possibilities by showing the associated Cartan matrix is not positive-definite. Then give an explicit root system construction for each diagram on the list. ∎

4. Recovering a root system from the Dynkin diagram

Last time we showed how to construct the Dynkin diagrams from a root system. We are going to show that the Dynkin diagrams encode all the important information about a root system so that we can actually recover the root system from the diagram. This will closely follow [**4**, §21.3].

[4]We will/may see later that this will happen if and only if the original semisimple Lie algebra was simple. This could also be done as an exercise at this stage.

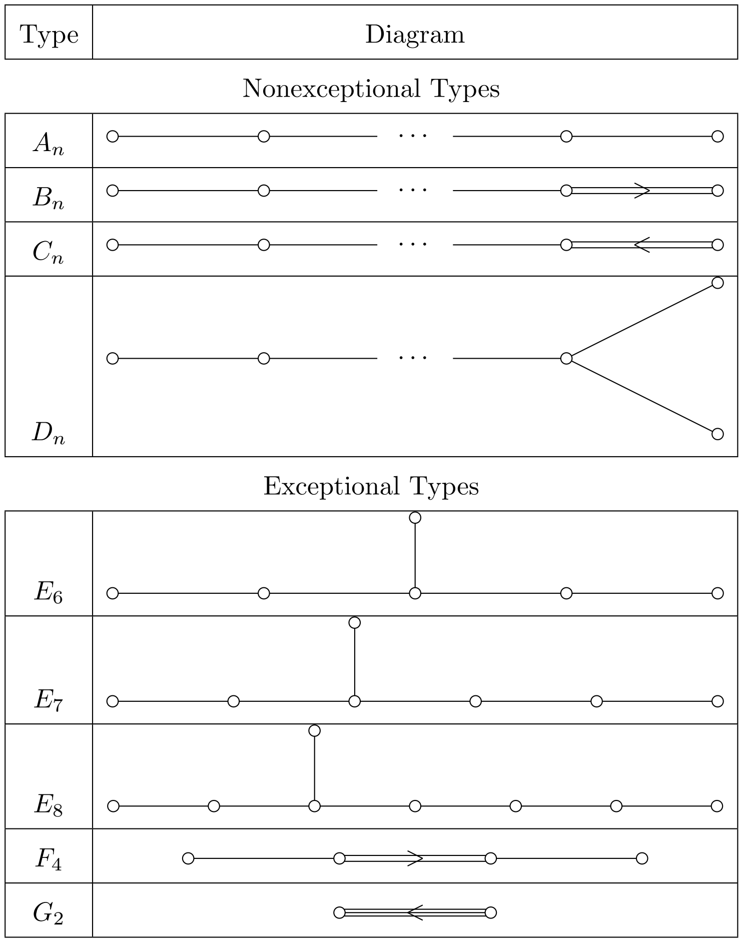


TABLE 3.1. The classification of positive-definite Cartan matrices through Dynkin diagrams.

3.12. LEMMA. *Every $\beta \in \Phi^+$ of height larger or equal to 2 is of the form $\beta = \beta' + \alpha$ for some $\beta' \in \Phi^+$.*

PROOF. We want to show that $\beta - \alpha_i \in \Phi^+$ for some i. What happens if $(\beta, \alpha_i) > 0$? If $(\beta, \alpha_i) \leqslant 0$ for all i, then the proof of Fact 5 says $\{\beta\} \cup \Delta$ is linearly independent, which yields a contradiction.

Hence $(\beta, \alpha_i) > 0$ for some i, and we must have $\beta - \alpha_i \in \Phi^+$. Hence the desired β' is $\beta - \alpha_i \in \Phi^+$. ■

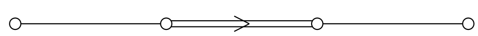
To build the roots we proceed level by level. The height one roots are exactly the simple roots. These come from the nodes of the Dynkin diagram. The height 2 roots must be of the form $\alpha_i + \alpha_j$, which are roots if and only if $(\alpha_i, \alpha_j) < 0$ by Fact 2 (i.e., coefficients of summands of a root must have the same sign). By construction of the Dynkin diagrams, $(\alpha_i, \alpha_j) < 0$ if and only if there is an edge i —— j. We now proceed inductively. If we know roots up to some height m, take each one $\beta = \sum n_i \alpha_i$ and decide if $\beta + \alpha_j \in \Phi$ for each j. By the lemma, we get *all* roots of height $m + 1$ by this method.

Writing $b - p\alpha_j, \ldots, \beta, \ldots, \beta + q\alpha_j$, an α_j-string through β. We get $\beta + \alpha_j \in \Phi$ if and only if $q \geqslant 1$. By Fact 1, $p - q = (\beta, \alpha_j^\vee)$. We know:

- the value p, since $\beta - p\alpha_j$ has smaller height,
- $(\beta, \alpha_j^\vee) = \sum_i m_i (\alpha_i, \alpha_j^\vee)$ since, from the diagram, we know $(\alpha_i, \alpha_j^\vee)$ from the edge labels.

Thus we get $q > 0$ exactly when $p > (\beta, \alpha_j^\vee)$.

3.13. Example. We do a few layers for F_4.



In practice, we usually want to write down the Cartan matrix $A = (a_{ij})$ with $a_{ij} = (\alpha_i, \alpha_j^\vee)$. For F_4, the Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

We give the positive roots in Table 3.2.

Note that this process only tells us what the roots will be if the diagram comes from a root system. The root system for F_4 come from the following data. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of $\mathbf{R}^4$.

height	positive roots
1	α_1, α_2, α_3, α_4
2	$\alpha_1+\alpha_2$, $\alpha_2+\alpha_3$, $\alpha_3+\alpha_4$
3	$\alpha_1+\alpha_2+\alpha_3$, $\alpha_2+2\alpha_3$, $\alpha_2+\alpha_3+\alpha_4$
4	$\alpha_1+\alpha_2+2\alpha_3$
$\vdots$	
12	$2\alpha_1+3\alpha_2+4\alpha_3+3\alpha_4$

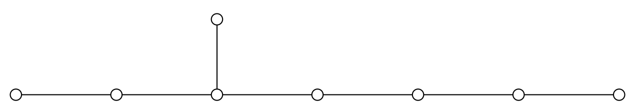
TABLE 3.2. The positive roots of a given height in F_4.

Then $(e_i, e_j) = \delta_{ij}$. Then $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_3 - e_4$, $\alpha_3 = e_4$, and $\alpha_4 = \frac{e_1-e_2-e_3-e_4}{2}$. This gives us

$$\Phi^+ = \{e_i\} \cup \{e_i + e_j : i < j\} \cup \{e_i - e_j : i < j\} \cup \left\{\frac{e_1 - e_2 - e_3 - e_4}{2}\right\}$$

There are similar explicit constructions for G_2, E_6, E_7, and E_8. See [**4**, §21.3] and [**8**, §12].

To make finding roots somewhat easier, one can show that an inclusion of Dynkin diagrams induces an inclusion of root systems. In particular, if we consider E_8,



with simple roots $\{\alpha_1, \ldots, \alpha_8\}$, then $\{\alpha_1, \ldots, \alpha_7\}$ are simple roots (in the space they span) for a root system of the type E_7, and similarly for E_6. One can also find simple roots using the following inclusions of Dynkin diagrams:

- $D_8 \subset E_8$,
- $D_4 \subset F_4$,
- $A_2 \subset G_2$.

Now that we've seen how to recover a root system from a Dynkin diagram, we show how to recover a Lie algebra from a given Dynkin diagram. This exposition will follow [**4**, §21.3] and [**8**, §14].

Suppose $\mathfrak{g}$ is a semisimple Lie algebra with root system Φ. Recall we have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ as vector spaces with

$\dim_{\mathbf{C}}(\mathfrak{g}_\alpha) = 1$ for all α and $\dim_{\mathbf{C}}(\mathfrak{h}) = \mathrm{rank}(\Phi)$. In particular, the root system completely determines the dimension of $\mathfrak{g}$ as $\dim_{\mathbf{C}}(\mathfrak{g}) = \mathrm{rank}(\Phi) + (\#\Phi)$. We also saw that if $\alpha, \beta \in \Phi$ satisfy $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. So if $\Delta = \{\alpha_1, \dots, \alpha_n\}$ is the set of simple roots, then we can generate all of $\mathfrak{g}$ from spaces $\mathfrak{g}_{\alpha_i}$, $\mathfrak{g}_{-\alpha_i}$ and the bracket operation.

To prove this, choose arbitrary $x_i \in \mathfrak{g}_{\alpha_i}$. Then we get $y_i \in \mathfrak{g}_{-\alpha_i}$ satisfying $[x_i, y_i] = h_{\alpha_i} =: h_i$. Since $\{\alpha_1, \dots, \alpha_n\}$ is a basis of $\mathfrak{h}^\vee$, the isomorphism $\mathfrak{h}^\vee \cong \mathfrak{h}$ shows that $\{h_1, \dots, h_n\}$ is a basis of $\mathfrak{h}$. Now given $x \in \mathfrak{g}_\beta$, for $\beta \in \Phi$, we can write $\beta = \alpha_{i_1} + \cdots + \alpha_{i_r}$ such that each partial sum is a root. Then $[x_{i_r}, \cdots, [x_{i_2}, x_{i_1}]] \in \mathfrak{g}_\beta$ is nonzero. Hence we can scale β to get x.

3.14. Example. Let $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{h}$ be the diagonal matrices. Then the roots are $\{\lambda_i - \lambda_j : i \neq j\}$, where $\lambda_i \in \mathfrak{h}^\vee$ is the functional which picks out the ith entry along the diagonal. A choice of simple roots is $\{\lambda_i - \lambda_{i+1} : i = 1, \dots, n-1\}$. Then we get the diagram for A_{n-1}, given by

and take $x_i = e_{i,i+1}$. Then $y_i = e_{i+1,i}$ and $h_i = e_{ii} - e_{i+1,i+1}$.

However, many different $\mathfrak{g}$'s could still give this same diagram.

5. Universal enveloping algebras

Our goal is to describe a Lie algebra using generators and relations (an idea analogous to free groups in group theory).

3.15. Definition. A Lie algebra $\mathfrak{g}$ is free on a set $S \subset \mathfrak{g}$ if it has the following universal property. Given any map of sets $\varphi \colon S \longrightarrow \mathfrak{g}_1$, where $\mathfrak{g}_1$ is any other Lie algebra, there exists a unique Lie algebra homomorphism $\psi\colon \mathfrak{g} \longrightarrow \mathfrak{g}_1$ extending φ. In other words, the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\psi"] & \mathfrak{g}_1 \\
S \arrow[u, hook] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

commutes.

Concretely, we get a free Lie algebra on any set S as follows. If $S = \{s_1, \ldots, s_n\}$, then consider the (associative) $\mathbf{C}$-algebra with vector space basis

$$\left\{s_{i_1} \otimes \cdots \otimes s_{i_r} : r \geqslant 0,\ i_j \in \{1, \ldots, n\}\right\}$$

and multiplication defined by

$$(s_{i_1} \otimes \cdots \otimes s_{i_r})(s_{j_1} \otimes \cdots \otimes s_{j_t}) = s_{i_1} \otimes \cdots \otimes s_{i_r} \otimes s_{j_1} \otimes \cdots \otimes s_{j_t}.$$

One can think of this as a noncommutative polynomial ring. Recall that this is a Lie algebra, so we declare the bracket via $[x, y] = xy - yx$. Then the Lie subalgebra generated by S in here will be free on S.

3.16. Remark. Free objects (or functors) exist in many contexts, and their constructions depend on their context. However, the universal property is always the same.

For any Lie algebra $\mathfrak{g}$, possibly of infinite dimension, there is a closely related associative $\mathbf{C}$-algebra with unity, denoted $U(\mathfrak{g})$, called the ***universal enveloping algebra***. In particular, for any universal enveloping algebra of a Lie group $\mathfrak{g}$, there exists a Lie algebra homomorphism $\iota\colon \mathfrak{g} \longrightarrow U(\mathfrak{g})$ satisfying the following universal property. For any associative $\mathbf{C}$-algebra A with identity and Lie algebra homomorphism $\varphi\colon \mathfrak{g} \longrightarrow A$, there exists a unique $\mathbf{C}$-algebra homomorphism of algebras $\psi\colon U(\mathfrak{g}) \longrightarrow A$ such that

$$\begin{tikzcd} U(\mathfrak{g}) \arrow[rr, "\psi"] & & A \\ \mathfrak{g} \arrow[u, hook, "\iota"] \arrow[urr, "\varphi"'] & & \end{tikzcd}$$

commutes.

Why does such $U(\mathfrak{g})$ exist? Well, we can construct it concretely. Let $T(\mathfrak{g})$ be the tensor algebra on $\mathfrak{g}$ and let I be the two-sided ideal generated by $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{g}$. Then $U(\mathfrak{g}) = T(\mathfrak{g})/I$.

Let's now state some important properties of $U(\mathfrak{g})$.

- $U(\mathfrak{g})$ is functorial in $\mathfrak{g}$. That is, given a Lie algebra homomorphism $\varphi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$, there exists an induced ring homomorphism $\varphi\colon U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$.
- We have an equivalence of categories

$$U(\mathfrak{g})\text{-}\mathsf{mod} \cong \mathfrak{g}\text{-}\mathsf{mod}.$$

In particular, we now have at our disposal all the tools for modules over rings. For example, we can apply homological algebra to the study of Lie algebras using this equivalence. Moreover, this equivalence analogous to the connections between groups and the group ring.

3.17. Example. Suppose $\mathfrak{g}$ is abelian. Then $U(\mathfrak{g})$ is isomorphic to a commutative polynomial ring on a basis of $\mathfrak{g}$. More precisely, $U(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{g})$.

3.18. Theorem (Poincaré-Birkhoff-Witt). *Suppose $\{x_1, x_2, \dots\}$ is a countable, ordered $\mathbf{C}$-basis of $\mathfrak{g}$. The elements*

$$\{1\} \cup \{x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_r}\},$$

where $r \geqslant 1$ and $i_1 \leqslant \cdots \leqslant i_r$, form a $\mathbf{C}$-basis of $U(\mathfrak{g})$. Moreover, the map $\iota\colon \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ is injective.

3.19. Example. Let $\mathfrak{g} = \mathfrak{sl}_2$ with ordered basis $\{x, h, y\}$. We omit the symbol $\otimes$ and write xy when we mean $x \otimes y$. Then $yh = hy - [h, y]$.

Also,

$$hhx = h(xh - [x,h]) = hxh - h[x,h] = xhh - [x,h]h - h[x,h] = \cdots$$

This computation is the spirit of the proof of the spanning set assertion in the PBW theorem. One proceeds by induction on the degree of an element which is not "written in the correct order."

6. Serre's theorem

Let $\mathfrak{g}$ be a semisimple Lie algebra with root system Φ and set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. We saw before that we get a set of generators $\{x_i, h_i, y_i : i = 1, \ldots, n\}$ of $\mathfrak{g}$. Just from our knowledge of $\mathfrak{sl}_2$ and abstract root systems, we have the following relations on $\mathfrak{g}$.

(S1) $[h_i, h_j] = 0$ for all i and j.
(S2) $[x_i, y_j] = \delta_{ij} h_i$.
(S3) $[h_i, x_j] = (\alpha_j, \alpha_i^\vee) x_j = \alpha_j(h_i) x_j$.
(S4) $[h_i, y_j] = -(\alpha_j, \alpha_i^\vee) y_j = -\alpha_j(h_i) y_j$.
(S_{ij}^{+}) $(\operatorname{ad} x_i)^{-(\alpha_j, \alpha_i^\vee)+1} x_j = (\operatorname{ad} x_i)^{1-a_{ij}} x_j = 0$ for $i \neq j$.
(S_{ij}^{-}) $(\operatorname{ad} y_i)^{-(\alpha_j, \alpha_i^\vee)+1} y_j = (\operatorname{ad} y_i)^{1-a_{ij}} y_j = 0$ for $i \neq j$.

The last two relations follow from our knowledge of the α_i-string through α_j.

3.20. Theorem (Serre). *Given an abstract root system Φ of finite type with base Δ, let $\mathfrak{g}(\Phi)$ be the free Lie algebra on $\{x_i, h_i, y_i\}_{i=1}^n$ modulo the relations above. Then $\mathfrak{g}(\Phi)$ is finite-dimensional, semisimple, and the h_i span a maximal toral subalgebra, and this gives back the root system Φ.*

3.21. Corollary. *For each Dynkin diagram of types A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2, there exists a unique semisimple Lie algebra with root system corresponding to that diagram.*

Proof. The existence is immediate from Serre's theorem. To show uniqueness, it suffices to show any semisimple $\mathfrak{g}'$ with root system Φ is isomorphic to $\mathfrak{g}(\Phi)$. We have $h_i' \in \mathfrak{g}'$ corresponding to the roots α_i, as always, and we can select some arbitrary nonzero $x_i' \in \mathfrak{g}'_{\alpha_i}$.

Then $y_i' \in \mathfrak{g}'_{-\alpha_i}$ such that $[x_i', y_i'] = h_i'$. We know $\{x_i', h_i', y_i'\}$ satisfy the Serre relations by definition, so we get a homomorphism of Lie algebras $\mathfrak{g}(\Phi) \twoheadrightarrow \mathfrak{g}'$ by $x_i \mapsto x_i'$, $h_i \mapsto h_i'$, and $y_i \mapsto y_i'$. Moreover, since $\mathfrak{g}(\Phi)$ and $\mathfrak{g}'$ have the same root system, we have

$$\dim_{\mathbf{C}}(\mathfrak{g}(\Phi)) = \operatorname{rank}(\Phi) + (\#\Phi) = \dim_{\mathbf{C}}(\mathfrak{g}').$$

Hence the map above is an isomorphism. ■

7. Representations of semisimple Lie algebras

Let V be an irreducible representation of a complex semisimple Lie algebra $\mathfrak{g}$. Then V has a weight space decomposition with respect to $\mathfrak{h} \subset \mathfrak{g}$ given by $V = \bigoplus_{\lambda \in \mathfrak{h}^\vee} V_\lambda$. That is, $\mathfrak{h}$ acts diagonally on each V_λ with eigenvalue $\lambda(h)$ for any $h \in \mathfrak{h}$. The fundamental calculation showed that the action of a root space $\mathfrak{g}_\beta$, for $\beta \in \Phi$, sends $V_\lambda \longrightarrow V_{\lambda+\beta}$. Hence all weights appearing an irreducible are congruent modulo the root lattice Q (otherwise $V' = \bigoplus_{\alpha \in Q} V_{\lambda+\alpha}$ would be a $\mathfrak{g}$-submodule). This tells us that all weights of an irreducible representation lie in a (possibly translated) $\mathbf{R}$-subspace of $\mathfrak{h}^\vee$ of dimension $\operatorname{rank}(\Phi)$. For each $\alpha \in \Phi$, we get an $\mathfrak{sl}_2$-triple $\mathfrak{s}_\alpha = \mathbf{C}\langle x_\alpha, h_\alpha, y_\alpha \rangle$ which acts on V, so we can apply $\mathfrak{sl}_2$-theory.

In particular, if $\lambda \in \mathfrak{h}^\vee$ is any weight of any representation of $\mathfrak{g}$, then evaluation of h_α acting on V_λ is $\lambda(h_\alpha) \in \mathbf{Z}$, for all $\alpha \in \Phi$. In terms of the Killing form, $(\lambda, \alpha^\vee) \in \mathbf{Z}$. So we define the ***weight lattice*** of $\mathfrak{g}$ is defined to be

$$P = \{\lambda \in \mathfrak{h}^\vee : (\lambda, \alpha^\vee) \in \mathbf{Z} \text{ for all } \alpha \in \Phi\}.$$

All weights of all representations of $\mathfrak{g}$ lie in P. Note that $\Phi \subset P$, and so $Q \subset P$. In particular, one should check that P is closed under taking $\mathbf{Z}$-linear combinations of its elements.

Recall that in the $\mathfrak{sl}_3$ case, we have the lattice given in Figure 3.3. We know that the eigenvalues of h_α acting on any V are symmetric about the origin (in the lattice). This implies that the weight for a

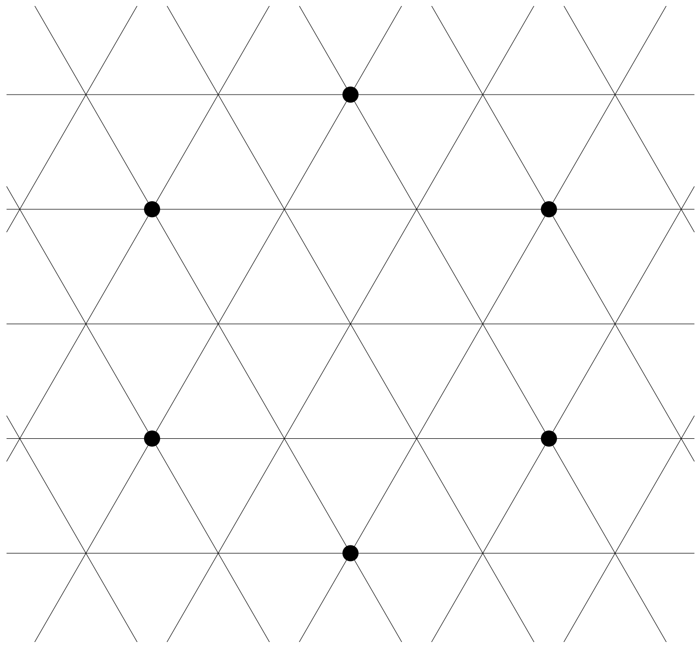


FIGURE 3.3. This is the weight lattice of $\mathfrak{sl}_3$ revisited. Each vertex denotes a weight in the weight lattice, and each $\bullet$ denotes a weight that is also a root of $\mathfrak{sl}_3$.

representation V are symmetric over the hyperplane

$$\Omega_\alpha = \{h \in \mathfrak{h} : \alpha(h) = (h, \alpha^\vee) = 0 \text{ for all } \alpha \in \Phi\}.$$

Thus the set of weights is invariant under the Weyl group W, which is a subgroup of $\mathrm{GL}(\mathfrak{h}^\vee)$. For $\mathfrak{g} = \mathfrak{sl}_3$, we have $W \cong S_3 \cong D_3$. For $\mathfrak{g} = \mathfrak{sp}_4$, then the Weyl group is given by $(\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \rtimes S_2$.[5] [6]

Fix a representation V of a semisimple Lie algebra $\mathfrak{g}$. If β is a weight such that $\beta - \alpha$ is a not a weight for some $\alpha \in \Phi$, we obtain an

[5]This last group is known (or was known) as a "wreath product." It is a hyperoctahedral group.

[6]Of course $S_2 \cong \mathbf{Z}/2\mathbf{Z}$, but we write it as above to allow for generalization to C_n, which has Weyl group $(\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n$.

uninterrupted string

$$\beta, \beta + \alpha, \ldots, \beta + m\alpha,$$

which gives

$$\beta(h_\alpha), \beta(h_\alpha) + 2, \ldots, \beta(h_\alpha) + 2m$$

because $\alpha(h_\alpha) = 2$. Then the symmetry about the origin implies $-\beta(h_\alpha) = \beta(h_\alpha) + 2m$, and hence $m = -\beta(h_\alpha)$. So if we can find a "highest weight" λ (i.e., a weight which is farthest in some direction in the lattice), one can get all weights of V. They are precisely the elements of $\mathfrak{h}^\vee$ satisfying the following two conditions:

(1) They are in the convex hull of $W\lambda$.
(2) They are congruent to λ modulo Q.

Choose a direction $\ell \colon \mathbf{R} \otimes Q \longrightarrow \mathbf{R}$ giving a decomposition $\Phi = \Phi^+ \cup \Phi^-$. Then for a representation V of $\mathfrak{g}$, we want a weight β such that $\beta + \alpha$ is not a weight for $\alpha \in \Phi^+$.

3.22. Proposition. *Every finite-dimensional representation V of a semisimple Lie algebra $\mathfrak{g}$ possesses at least one highest weight β, and thus a highest weight vector v_β. The highest weight vector v_β generates an irreducible submodule of V by applying successive y_i's. Thus an irreducible representation has a unique highest weight and a unique highest weight vector.*

Proof. We find a highest weight β by adding α_i to a given weight until you cannot get a higher weight. To see this in a "fancier" way, as an exercise one can show $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is solvable. Then by Lie's theorem, V has a common eigenvector for the action of $\mathfrak{b}$ which is killed by $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}] = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. This common eigenvector is a highest weight vector.

To show a highest weight vector v_β generates an irreducible submodule, let W_n be the subspace of V spanned by elements of the form $y_1, \ldots, y_n v_\beta$ for any $y_i \in \mathfrak{g}_\alpha$, where $\alpha \in \Phi^-$. By induction and the fundamental calculation show $xW_n \subset W_n$ for each $x \in \mathfrak{g}_\alpha$, where $\alpha \in \Phi^+$.

Then $W = \bigcup_{n=1}^{\infty} W_n$ is stable under $\mathfrak{g}$, so it is an irreducible submodule.[7] Uniqueness is left to the reader. ∎

Summarizing our results, we have a map

$$\begin{array}{ccc} \{\text{irreducible reps. of } \mathfrak{g}\} & \longrightarrow & P \\ V & \mapsto & \text{highest weight.} \end{array}$$

We know this is well-defined because every irreducible representation has a unique highest weight. Moreover, this map is injective by the same argument used for $\mathfrak{sl}_3$. But what is the image?

3.23. Lemma. *If Φ is a root system with simple roots $\Delta \subset \Phi$, then the* **dual** $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ *is a root system with simple roots $\Delta^\vee$.*

The proof is left as an exercise. It is also left as an exercise to show every finite root system is isomorphic to its dual, except for $B_n^\vee = C_n$.

3.24. Proposition. *The weight lattice P has basis given by dual of the h_i corresponding to $\Delta = \{\alpha_1, \ldots, \alpha_n\}$.*

In particular, if we define $w_i \in \mathfrak{h}^\vee$ by $\omega_i(h_j) = \delta_{ij} = (\omega_i, \alpha_j^\vee)$, then the weight lattice is precisely $P = \mathbf{Z}\omega_1 + \cdots + \mathbf{Z}\omega_n$.

Proof. First, we need to show $\omega_i \in P$. This means we need to show $(\omega_i, \alpha^\vee) \in \mathbf{Z}$ for all $\alpha \in \Phi$. Recall that, generally, $(\alpha + \beta)^\vee \neq \alpha^\vee + \beta^\vee$. But by the lemma, $\alpha^\vee$ is a $\mathbf{Z}$-linear combination of $\alpha_i^\vee$, so $\omega_i \in P$.

The set $\{\omega_i\}$ is linearly independent since it is dual to the linearly independent set $\{\alpha_i^\vee\}$. To show $\{\omega_i\}$ spans P, one needs to check $\lambda = \sum_{j=1}^{n} (\lambda, \alpha_j^\vee)\omega_j$ for any $\lambda \in P$. ∎

From these arguments, we see that P lives in the same $\mathbf{R}$-vector space as Q. In particular, we can take the roots themselves

$$\alpha_i = \sum_{j=1}^{n} (\alpha_i, \alpha_j^\vee)\omega_j.$$

[7] It is irreducible because if $W = W' \oplus W''$, then we get $W_\beta = W'_\beta \oplus W''_\beta$ where β is a highest weight. Since W_β is one-dimensional, either $W'_\beta = W_\beta$ or $W'_\beta = W''_\beta$ which contains v_β which generates W. So $W' = W$ or $W'' = W$.

But the coefficients are the entries of the Cartan matrix, so the Cartan matrix is the change of basis matrix from $\{\omega_i\}$ to $\{\alpha_j\}$. Moreover, the determinant of the Cartan matrix, which is an integer, is the order of the finite group P/Q. This group is called the ***fundamental group*** of Φ.

We are ready to answer the question of the image of the map given above. By considering the α-string through a weight β, we can see if $\beta+\alpha$ is not a weight for $\alpha \in \Phi^+$, then $(\beta, \alpha^\vee) \geqslant 0$. So a highest weight satisfies $(\beta, \alpha^\vee) \geqslant 0$ for all $\alpha \in \Phi^+$. Thus β lives in the positive side of all Ω_α for all α, which implies in our basis we get $\beta = \sum a_i \omega_i$ with $a_i \geqslant 0$. Hence all the weights lie in a cone. In fact, every $\sum a_i \omega_i$ with $a_i \geqslant 0$ is a highest weight of some irreducible representation of $\mathfrak{g}$. To see this, consult [**8**, §20.1] or [**4**, §15–20].

APPENDIX A

Solutions and Hints to Homework Assignments

1. Set 1

1.3. You can easily get this from the example in Section 2.1 [**8**] or skip ahead and the matrices are written out in Section 5.1.

2.6. First reduce to showing that any e_{ij} or h_i is in the ideal, similar to the way $\mathfrak{sl}_2$ is treated in Example 1.12. We can even reduce in the same way to being done whenever any linear combination of h_1 and h_2 is in the ideal. Now if $\mathfrak{a}$ is a nonzero ideal, it is stable under ad of anything (in particular, of h_1 and h_2). Since $\mathrm{ad}(h_1)$ and $\mathrm{ad}(h_2)$ are semisimple and commute, they are simultaneously diagonalizable, and so we can find a basis of $\mathfrak{sl}_2$ in which each element is an eigenvector for both operators. (Big surprise! It's the standard basis.)

So $\mathfrak{a}$ has a vector v which is an eigenvector for both operators. We have to be a little careful, though; if two eigenvectors for an operator have the same eigenvalue, then any linear combination is also an eigenvector with that eigenvalue. For example, $e_{13} + e_{21}$ is an eigenvector with eigenvalue 1 for $\mathrm{ad}(h_2)$. Maybe v looks like this, and we didn't reduce to dealing with linear combinations of the e_{ij}. But we can see from the table that its *not* an eigenvector for $\mathrm{ad}(h_1)$, so that's why v does not look like this. The particular way the eigenvalues are distributed can be used with this idea to finish the proof. The vector v

	$\mathrm{ad}(h_1)$	$\mathrm{ad}(h_2)$
h_1	0	0
h_2	0	0
e_{12}	2	-1
e_{13}	1	1
e_{21}	-2	1
e_{23}	-1	2
e_{31}	-1	-1
e_{32}	1	-2

TABLE A.1. The vectors and associated eigenvalues for each operator in $\mathfrak{sl}_2$.

is in the intersection of an eigenspace for $\mathrm{ad}(h_1)$ with an eigenspace for $\mathrm{ad}(h_2)$, which we can see from Table A.1 are spanned by either a single e_{ij} or some linear combination of h_1 and h_2. In either case, we're done by the first reduction.

3.2. If $\mathfrak{g}$ is solvable, the derived series is such a chain of subalgebras; they are even ideals by Example 1.21, and each quotient is abelian since $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is abelian for any Lie algebra (and we can apply this at each step using the definition of the derived series).

Conversely, you can use induction on the length of such a chain. In the base case, $\mathfrak{g}$ is abelian. If the chain has length greater than 1, then by induction $\mathfrak{g}_1$ is solvable of smaller dimension.[1] But then we have a solvable ideal such that the quotient $\mathfrak{g}/\mathfrak{g}_1$ is solvable, so $\mathfrak{g}$ is solvable by Proposition 1.25(1).

3.4. By definition, $\mathrm{ad}(\mathfrak{g})$ is the image of $\mathfrak{g}$ under the Lie algebra homomorphism $\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, which has kernel $Z(\mathfrak{g})$. In other words,

[1]Technically, some of the $\mathfrak{g}_i$ could be equal in the chain, but we can easily dismiss that and work with a shorter chain.

$\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}/Z(\mathfrak{g})$. So we can apply Proposition 1.25(1) (resp. 1.35(2)) to get one direction, and apply the second part to get the other.

2. Set 2

4.1. To see that $R = \mathrm{rad}(\mathfrak{g})$ lies in each maximal solvable subalgebra B: Consider the subspace $R + B \subseteq \mathfrak{g}$, which can easily be checked to be a subalgebra (using that R is an ideal) and R is still a solvable ideal of this subalgebra. Then $(R + B)/R \simeq B/(R \cap B)$ is a quotient of a solvable algebra, so it is solvable. By Proposition 1.25(2), $R + B$ is then solvable, so by maximality of B, we get that $R + B = B$ and so $R \subseteq B$.

Following the hint, we can find a basis of V such that every $x \in R$ is represented by a diagonal element, say x has $(a_1, \ldots, a_n)$ along the diagonal. But R is an ideal, so when we bracket this with some $e_{ij} \in \mathfrak{sl}_n$ $(i \neq j)$ we get

$$[x, e_{ij}] = (a_i - a_j)e_{ij} \in R$$

which implies that $a_i = a_j$ since R consists of diagonal elements. Since every e_{ij} with $i \neq j$ is in $\mathfrak{sl}_n$, we get that $a_i = a_j$ for all i, j, so x is a scalar matrix (multiple of the identity). Scalar matrices are in the center Z of $\mathfrak{sl}_n$, so $R \subseteq Z$. The other inclusion always hold in any Lie algebra, so $R = Z$. Now from Exercise 2.3 [**8**], the center of $\mathfrak{sl}_n$ is 0.

Note: This argument goes through for the other classical algebras using the explicit bases given in Section 1.2 of [**8**]. We will use this to solve Exercise 6.5(b) below.

Note: The computation in the second paragraph can be avoided (so that we don't have to do the classical algebras case-by-case) using the following argument: if $x \in R$ is diagonal, then multiplying xy for any $y \in \mathfrak{g}$ just multiplies the diagonal entries of y by those of x. Similarly for yx, so the diagonal of $[x, y]$ will be 0. But R is an ideal, so $[x, y] \in R$ must be diagonal and thus entirely 0. So x is in the center of $\mathfrak{g}$. Thanks to Andrew for pointing this out.

4.5. If x, y commute, then so do their semisimple and nilpotent parts by Proposition 1.48(2). Commuting semisimple endomorphisms are simultaneously diagonalizable, so $x_s + y_s$ will be diagonal in this basis also and is this thus semisimple. The binomial formula shows that when x_n, y_n commute, their sum $x_n + y_n$ is also also nilpotent. So we can write $x + y = (x_s + y_s) + (x_n + y_n)$ as a sum of semisimple and nilpotent elements that commute. By uniqueness of the Jordan decomposition, these must be the semisimple and nilpotent parts of $x + y$.

5.5. If $\{v_1, \ldots, v_n\}$ is a basis of a vector space V and β a nondegenerate symmetric bilinear form on V, the dual basis for V under β is defined as having the property that $\beta(v_i, w_j) = \delta_{ij}$ for all i, j. The matrix of the Killing form is given in [**8**, p. 22], from which you can guess that the dual basis to x, y, h must be $\tilde{x} = \frac{y}{4}$, $\tilde{y} = \frac{x}{4}$, and $\tilde{h} = \frac{h}{8}$.

6.1. To compute the Casimir element of a representation, we need the dual basis associated to the trace form. For the adjoint representation this is just the Killing form and we computed the dual basis in the last exercise. Accordingly,

$$\begin{aligned} c_{\mathrm{ad}} &= \operatorname{ad} x \operatorname{ad} \tilde{x} + \operatorname{ad} h \operatorname{ad} \tilde{h} + \operatorname{ad} y \operatorname{ad} \tilde{y} \\ &= \frac{1}{4} \operatorname{ad} x \operatorname{ad} y + \frac{1}{8} (\operatorname{ad} h)^2 + \frac{1}{4} \operatorname{ad} y \operatorname{ad} x \end{aligned}$$

which we have explicit matrices for from previous homework. Multiplying it all out gives $c_{\mathrm{ad}} = \mathrm{id}_{\mathfrak{g}}$ (this is what we expected since $V = \mathfrak{g}$).

The usual representation of $\mathfrak{sl}_3$ is just the defining one on $\mathbf{C}^3$, so the the trace form is $\beta(x, y) = \operatorname{Tr}(xy)$ where x, y are matrices in $\mathfrak{sl}_3$. Using the $\mathfrak{sl}_2$ case as a guide, we can find that the dual basis to the standard basis of $\mathfrak{sl}_3$, relative to β, is

$$\tilde{e}_{ij} = e_{ji}, i \neq j, \qquad \tilde{h}_1 = \frac{2}{3} h_1 + \frac{1}{3} h_2, \quad \tilde{h}_2 = \frac{1}{3} h_1 + \frac{2}{3} h_2.$$

Plug-and-chug to get the Casimir operator.

6.5. (a) If $\operatorname{rad}(\mathfrak{g}) = Z(\mathfrak{g})$, then $\operatorname{ad}(\mathfrak{g}) \cong \mathfrak{g}/Z(\mathfrak{g})$ is semisimple and so any representation is completely reducible, in particular its natural representation on $\mathfrak{g}$. *Note: this is the definition of "reductive" in most texts.*

There are a number of ways to see that $\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ now: the most concrete is to see that Theorem 1.56 holds for reductive Lie algebras (decomposition as a direct sum of minimal simple ideals) if we now just allow that some of the ideals be $\mathbf{C}$ with the trivial bracket. Then $Z(\mathfrak{g})$ is the sum of the minimal abelian ideals, and $[\mathfrak{g}, \mathfrak{g}]$ is the sum of the rest of them.

(b) We are assuming that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ from the referenced exercise. We just need to show that $\mathrm{rad}(\mathfrak{g}) = Z(\mathfrak{g})$, then apply part a). But that follows from the first problem of this set, Exercise 4.1.

(c) This is the same argument as part (a). If $\mathfrak{g}$ is completely reducible as an $\mathrm{ad}(\mathfrak{g})$-module, then it decomposes as a direct sum of irreducible submodules. But submodules under the adjoint action are ideals, so it decomposes as a sum of simple and abelian ideals. Then $Z(\mathfrak{g})$ is the sum of the minimal abelian ideals, and $[\mathfrak{g}, \mathfrak{g}]$ is the sum of the rest of them. $\mathrm{rad}(\mathfrak{g})$ certainly contains $Z(\mathfrak{g})$, but if it contained any of the simple ideals I with $[I, I] = I$ then it couldn't be solvable (this summand would keep showing up in every step of the derived series). So $\mathrm{rad}(\mathfrak{g})$ is just exactly $Z(\mathfrak{g})$.

(d) *Note: I ended up writing a long example into this solution, don't let the length intimidate you.* Let V be a representation of $\mathfrak{g}$ on which every element of $Z = Z(\mathfrak{g})$ acts by a semisimple endomorphism. A reasonable first guess would be to consider this as a representation of $[\mathfrak{g}, \mathfrak{g}]$ (via restriction), which we know to be semisimple from (a), and get a decomposition $V = \bigoplus V_i$ into irreducible $[\mathfrak{g}, \mathfrak{g}]$ modules. But the problem is that these don't have to be stable under the action of Z, so they might not actually be $\mathfrak{g}$ modules. The following example captures the essence of what can go wrong here.

Let $\mathfrak{g} = \mathbf{C} \oplus \mathfrak{sl}_2$, where $\mathbf{C}$ has the trivial bracket and is thus $Z(\mathfrak{g})$ here (and $\mathfrak{g}$ is reductive). Let V be the representation $\mathbf{C}^2 \oplus \mathbf{C}^2$, where $\mathfrak{sl}_2$ acts diagonally by the usual representation on each $\mathbf{C}^2$, and Z acts by interchanging the copies of $\mathbf{C}^2$ while multiplying by scalars. Concretely, a matrix $M \in \mathfrak{sl}_2$ and complex number $t \in Z$ act by the block form matrices

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \qquad \begin{pmatrix} 0 & t\,\mathrm{id}_{\mathbf{C}^2} \\ t\,\mathrm{id}_{\mathbf{C}^2} & 0 \end{pmatrix}.$$

Note that although the image of Z doesn't consist of diagonal matrices, they are all semisimple because they would be diagonal in another basis (find the basis of eigenvectors explicitly if this isn't clear to you). Of course, we have to check that this is a representation of $\mathfrak{g}$ by checking that the bracket of two element of $\mathfrak{g}$ will be sent to the bracket of their corresponding matrices; this is easily checked to be the case. So what goes wrong here? If we just look at V as an $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}_2$-module, the two copies of $\mathbf{C}^2$ we used to define the representation are irreducible submodules and thus give a decomposition of V into irreducible $[\mathfrak{g}, \mathfrak{g}]$-submodules. But they aren't stable under the action of Z, since any $t \in Z$ interchanges the two subspaces $\mathbf{C}^2$. In other words, we can't conclude in our general setup that the V_i are irreducible $\mathfrak{g}$ modules also.

On the other hand, if we first look at the action of Z on V, things work out, and in fact illustrate some techniques that we will be applying in the next few weeks. By hypothesis, Z acts by semisimple matrices on V, and since Z is abelian these matrices all commute. One of our basic linear algebra facts (see the review sheet) was that any collection of *commuting* semisimple operators can be simultaneously diagonalized. This means that we can find a basis of V consisting of eigenvectors for the action of every element of Z. Of course, different elements of Z can act with different eigenvalues on the same vector (e.g., the eigenvalues of the action of $2z$ will be twice the eigenvalues of z for any $z \in Z$), but linearity of the action guarantees that we can find linear functions $\lambda \colon Z \longrightarrow \mathbf{C}$ (i.e., $\lambda \in Z^\vee$) which give these eigenvalues for any given eigenvector. So by defining (for each $\lambda \in Z^\vee$)

$$V_\lambda = \{v \in V : z \cdot v = \lambda(z)v\},$$

we get a decomposition $V = \bigoplus_\lambda V_\lambda$ (look familiar?). Now, using that $\mathfrak{g} = Z \oplus [\mathfrak{g}, \mathfrak{g}]$, check that each V_λ is stable under the action of $[\mathfrak{g}, \mathfrak{g}]$, and is thus an $[\mathfrak{g}, \mathfrak{g}]$-module. Since $[\mathfrak{g}, \mathfrak{g}]$ is semisimple, each V_λ decomposes as a sum of irreducible $[\mathfrak{g}, \mathfrak{g}]$-modules: $V_\lambda = \bigoplus_i V_{\lambda,i}$. But now these are exactly rigged, by considering the Z action *first*, to be stable under Z also. So the situation of the above example can't happen, and each $V_{\lambda,i}$ is an $\mathfrak{g}$-module. Being irreducible $[\mathfrak{g}, \mathfrak{g}]$-modules means they have no proper nonzero subspace stable under the action of $[\mathfrak{g}, \mathfrak{g}]$, so they can't have any proper nonzero subspace stable under the action of the even bigger algebra $\mathfrak{g}$. So they are irreducible as $\mathfrak{g}$-modules, and we have a decomposition of V into irreducible $\mathfrak{g}$-modules.

APPENDIX B

Kac-Moody Algebras

1. History and background

Let G be a complex (simply connected) Lie group; i.e., $\mathrm{GL}_n(\mathbf{C})$, $\mathrm{SL}_n(\mathbf{C})$, $\mathrm{Sp}_{2n}(\mathbf{C})$, or $\mathrm{SO}_n(\mathbf{C})$. The tangent space at the identity of G is a complex Lie algebra, denoted $\mathfrak{g}$. The Lie algebras corresponding to the Lie groups given above are $\mathfrak{gl}_n(\mathbf{C})$, $\mathfrak{sl}_n(\mathbf{C})$, $\mathfrak{sp}_{2n}(\mathbf{C})$, and $\mathfrak{so}_n(\mathbf{C})$, respectively.

Given any Lie algebra $\mathfrak{g}$, we can extract the essential data in the form of the root system. In particular, a Lie algebra is completely determined by its Dynkin diagram, which has a corresponding Cartan matrix. In summary, this yields a construction of a Cartan matrix from a complex Lie group. Can we start with start with a Cartan matrix and construct a complex Lie group? In other words, can we take the path

$$A \iff \Phi \rightsquigarrow \mathfrak{g} \rightsquigarrow G?$$

The answer is yes! This construction was done by Chevalley, and uses generators and relations.

Precisely, let A be a Cartan matrix; i.e., a matrix $A = (a_{ij})$ such that $a_{ii} = 2$ for all i, $a_{ij} \leqslant 0$ for $i \neq j$, $a_{ij} = 0$ if and only if $a_{ji} = 0$ for $i \neq j$, and A is positive definite (which follows from the nondegeneracy of the Killing form).

In 1968, V. Kac and R. Moody independently studied what happens when the positive definiteness of A is omitted. As a result, new classes of Lie algebras were obtained, called ***Kac-Moody algebras***.

What new properties do these new Lie algebras have?

- They are infinite-dimensional.
- (Sensational) applications all over mathematics.

B.1. Example. Kac generalized the Weyl character formula to the so-called Weyl-Kac character formula. In a special case, one can obtain Macdonald identities. This observation started to spark interest in this new class of Lie algebras. Moreover, Kac proved that this formula is related to modular forms in number theory, which initiated a boom in the study of Kac-Moody algebras.

2. Definition of Kac-Moody algebras

Let I be any finite index set, and consider a generalized Cartan matrix (GCM) $A = (a_{ij})$ satisfying the conditions above (with positive definiteness omitted). Let $P^\vee$ be the free abelian group of rank $\#I + (\#I - \mathrm{rank}(A))$. We define $\mathfrak{h} = \mathbf{C} \otimes_{\mathbf{Z}} P^\vee$. For convenience, fix a basis $\{h_i : i \in I\} \cup \{d_s\}$ for $P^\vee$. Define $P = \{\lambda \in \mathfrak{h}^\vee : \lambda(P^\vee) \subset \mathbf{Z}\}$. Then P is called the ***weight lattice***. Moreover, define $\Pi = \{h_i : i \in I\}$ and $\Pi^\vee = \{\alpha_i : i \in I\} \subset \mathfrak{h}^\vee$ subject to the following conditions:

- Π is linearly independent.
- $\alpha_j(h_i) = a_{ij}$.

So given a GCM, we can choose $\Pi^\vee$ and Π appropriately.

B.2. Definition. The Kac-Moody algebra $\mathfrak{g}$ associated with the datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the Lie algebra generated by e_i, f_i, for $i \in I$, and $h \in P^\vee$ subject to the following defining relations.

(1) $[h, h'] = 0$ for all $h, h' \in P^\vee$.

(2) $[e_i, f_j] = \delta_{ij} h_i$.
(3) $[h, e_i] = \alpha_i(h) e_i$ for $h \in P^\vee$.
(4) $[h, f_i] = -\alpha_i(h) f_i$ for $h \in P^\vee$.
(5) $\mathrm{ad}(e_i)^{1-a_{ij}} e_j = 0$ for $i \neq j$.
(6) $\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0$ for $i \neq j$.

The first four relations are known as the ***Weyl relations***, but the last two are known as the ***Serre relations***. We summarize the definition in this way: A Kac-Moody algebra $\mathfrak{g}$ is the free Lie algebra generated by the e_i's, f_i's, and h's modulo the relations given. Note also that any finite-dimensional simple Lie algebra is also a Kac-Moody algebra.

B.3. Proposition. *Let $\mathfrak{g}$ be a Kac-Moody algebra. Then*

(1) $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$,
(2) $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$, *where Q is the root lattice, and* $\dim(\mathfrak{g}_\alpha) < \infty$.
(3) $\mathfrak{g}$ *is "almost simple" if A is indecomposable.*

In general, $\dim(\mathfrak{g}) = \infty$, and $\dim(\mathfrak{g}) \geqslant 1$.

3. Classification of generalized Cartan matrices

Let $u = (u_1, \ldots, u_n)^\top$ be a column vector in $\mathbf{R}^n$. We say $u > 0$ if $u_i > 0$ for all $i = 1, \ldots, n$.

B.4. Theorem. *Let A be an indecomposable GCM. Then exactly one of the following three possibilities holds for A and $A^\top$.*

(Fin) *There exists $u > 0$ such that $Au > 0$;* $\det(A) \neq 0$.
(Aff) *There exists $u > 0$ such that $Au = 0$;* $\operatorname{corank}(A) = 1$.
(Ind) *There exists $u > 0$ such that $Au < 0$.*

Here, Fin, Aff, and Ind correspond to finite, affine and indefinite types, respectively.

B.5. Example. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Then A is of finite type, of type A_1 with Dynkin diagram

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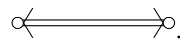
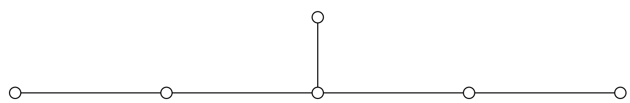
Let $A = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. Then A is of affine type, called type $A_1^{(1)}$ with Dynkin diagram

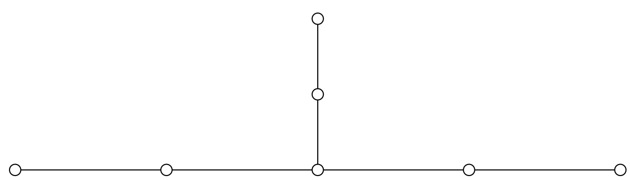


If $A = \left(\begin{smallmatrix} 2 & -3 \\ -3 & 2 \end{smallmatrix}\right)$, then A is of indefinite type with Dynkin diagram

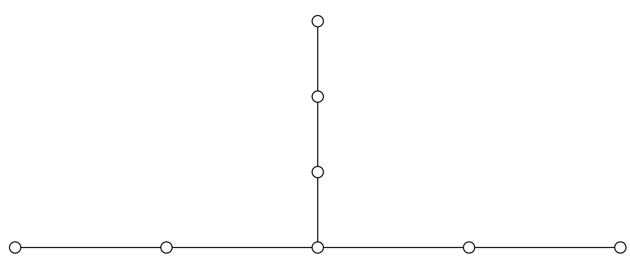
B.6. Example. Consider the finite-dimensional exception Lie algebra E_6. This has Dynkin diagram



If we add one more node,

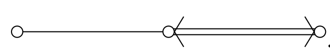


we get the affine Lie algebra $E_6^{(1)}$. Adding one more node



gives a Lie algebra of indefinite type, denoted $T_{4,3,3}$.

B.7. Example. Another example of indefinite type is the Lie algebra with Dynkin diagram



One way to tell whether or not a diagram is of affine type is the following. A diagram is of affine type if, whenever any one node is removed, the remaining diagram is of finite type. In this case, removing the right-most node leaves A_1, but removal of the left-most node leaves $A_1^{(1)}$, which is of affine type.

4. The Weyl group

For each $i \in I$, we define a $r_i \in \mathrm{GL}(\mathfrak{h}^\vee)$ by

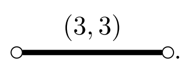
$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i.$$

The subgroup $W \subset \mathrm{GL}(\mathfrak{h}^\vee)$ generated by r_i is called the ***Weyl group***.

B.8. Example. In type A_2, the Weyl group is $W \cong \mathfrak{S}_3$.

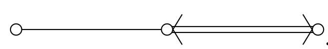
B.9. Example. For type $A_1^{(1)}$, the Weyl group is $\mathfrak{S}_2 \ltimes \mathbf{Z}$.

B.10. Example. Consider the Kac-Moody algebra associated to



Then $W \cong \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$. In this group, $s^2 = t^2 = 1$. This is an infinite group.

B.11. Example. The Weyl group of the Kac-Moody algebra associated to



is $W \cong \mathrm{PGL}_2(\mathbf{Z})$. This group is very important in number theory.

5. Representations of Kac-Moody algebras

Recall for a Lie algebra $\mathfrak{g}$, we obtain the universal enveloping algebra $U(\mathfrak{g})$. The same is true in the Kac-Moody algebra case. Let 1 denote the unital element in $U(\mathfrak{g})$. Fix $\lambda \in \mathfrak{h}^\vee$ and let $J(\lambda)$ be the left ideal in $U(\mathfrak{g})$ generated by all e_i, for $i \in I$, and $h - \lambda(h)1$, for $h \in \mathfrak{h}$. We define

$$M(\lambda) = U(\mathfrak{g})/J(\lambda).$$

Then $e_i 1 = 0$ and $h \cdot 1 = \lambda(h) \cdot 1$ in $M(\lambda)$. Then $M(\lambda)$ is a $U(\mathfrak{g})$-module, called the ***Verma module***. Then $M(\lambda)$ is the largest module satisfying these properties.

B.12. Proposition. *Let $\mathfrak{g}$ be a Kac-Moody algebra.*

(1) *$M(\lambda)$ is a highest weight $\mathfrak{g}$-module with highest weight λ and highest weight vector $v_\lambda = 1 + J(\lambda)$.*

(2) *Every highest weight $\mathfrak{g}$-module with highest weight λ is a homomorphic image of $M(\lambda)$.*

(3) *$M(\lambda)$ has a unique maximal submodule $N(\lambda)$. Thus we obtain the irreducible highest weight module $V(\lambda) = M(\lambda)/N(\lambda)$.*

Let A be a GCM of finite type. Then $\dim(V(\lambda)) < \infty$ if and only if $\lambda(h_i) \in \mathbf{Z}_{\geqslant 0}$ for all $i \in I$.

B.13. Definition. The $\mathfrak{g}$-module $V(\lambda)$ is called ***integrable*** if for each $v \in V(\lambda)$, there exists $N > 0$ such that $e_i^N v = 0$ and $f_i^N v = 0$ for all $i \in I$.

B.14. Proposition. *The $\mathfrak{g}$-module $V(\lambda)$ is integrable if and only if $\lambda(h_i) \in \mathbf{Z}_{\geqslant 0}$ for all $i \in I$.*

Note that the dimension of $V(\lambda)$ is infinite, in general. So this is the analogue of finite-dimensionality in the Kac-Moody case.

We have a weight space decomposition

$$V(\lambda) = \bigoplus_{\mu \in \mathfrak{h}^\vee} V_\mu,$$

where

$$V_\mu = \{v \in V : hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}.$$

We define the ***character*** of $V(\lambda)$ to be the following formal sum:

$$\mathrm{ch}(V(\lambda)) = \sum_{\mu \in \mathfrak{h}^\vee} \dim(V_\mu) e^\mu,$$

where e^μ are formal basis elements of the group algebra $\mathbf{C}\mathfrak{h}^\vee$ with $e^\lambda e^\mu = e^{\lambda+\mu}$.

Why is the character important? The character of $V(\lambda)$ is invariant of $V(\lambda)$. That is, if two modules have different characters, then the two modules in question are not isomorphic. Moreover, for any $w \in W$, we have $we^\mu = e^{w\mu}$. Hence, from $\dim(V_\mu) = \dim(V_{w\mu})$, we have

$$w\,\mathrm{ch}(V(\lambda)) = \mathrm{ch}(V(\lambda)).$$

Thus the character is a symmetric function! This is a very concrete object, which is important in combinatorics and other areas of mathematics.

B.15. Theorem (Weyl-Kac). *Assume* $\lambda(h_i) \in \mathbf{Z}_{\geqslant 0}$ *for all* $i \in I$. *Then we have*

$$\operatorname{ch}(V(\lambda)) = \frac{\displaystyle\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\displaystyle\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{\dim(\mathfrak{g}_\alpha)}},$$

where $\ell(w)$ *denotes the length of* w *and* $\rho(h_i) = 1$ *for all* $i \in I$.

Note if $W = \mathfrak{S}_n$, then $(-1)^{\ell(w)} = \operatorname{sgn}(w)$.

B.16. Corollary. *We have*

$$\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{\dim(\mathfrak{g}_\alpha)} = \sum_{w \in W} (-1)^{\ell(w)} e^{w\rho - \rho}.$$

If $\mathfrak{g} = \mathfrak{sl}_n$, then we have $\frac{1}{2}n(n+1)$ terms in the product and $n!$ terms in the sum on the right. One can check the identity in this case using a lot of high school algebra. If $\mathfrak{g}$ is a Kac-Moody algebra, then both sides have infinitely many terms.

B.17. Example. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{C})$. Let $\lambda = m\Lambda_1$. Then $\dim(V(\lambda)) = m + 1$. Then the vectors in the representation are $\{1, f_1, f_1^2, \ldots, f_1^m\}$, and the corresponding weights are $\{\lambda, \lambda - \alpha_1, \lambda - 2\alpha_1, \ldots, \lambda - m\alpha_1\}$. Then

$$\operatorname{ch}(V(\lambda)) = e^\lambda + e^{\lambda - \alpha_1} + \cdots + e^{\lambda - m\alpha_1}.$$

Using the change of variables $e^{\Lambda_1} = X$, we have $e^{\alpha_1} = X^2$ and

$$\operatorname{ch}(V(\lambda)) = X^m + X^{m-2} + \cdots + X^{-m+2} + X^{-m}.$$

B.18. Example. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbf{C})$ and $\lambda = \rho = \Lambda_1 + \Lambda_2$. Then $\dim(V(\rho)) = 8$ and

$$\begin{aligned}\operatorname{ch}(V(\rho)) = e^\lambda + e^{\lambda-\alpha_1} + e^{\lambda-\alpha_2} + 2e^{\lambda-\alpha_1-\alpha_2} \\ + e^{\lambda-2\alpha_1-\alpha_2} + e^{\lambda-\alpha_1-2\alpha_2} + e^{\lambda-2\alpha_1-2\alpha_2}.\end{aligned}$$

Using the change of variables $e^{\Lambda_1} = X$ and $e^{\Lambda_2} = Y$, we have

$$\operatorname{ch}(V(\rho)) = XY + X^{-1}Y^2 + X^2Y^{-1} + 2 \\ + X^{-2}Y + XY^{-2} + X^{-1}Y^{-1}.$$

Recall that this is invariant under the action of $W = \mathfrak{S}_3$. Using the character formula, we have

$$\operatorname{ch}(V(\rho)) = \frac{1 - e^{-\alpha_1+\alpha_2} + e^{-3\alpha_1-\alpha_2} + e^{-\alpha_1-3\alpha_2} + 3e^{-3\alpha_1-3\alpha_2}}{(1-e^{-\alpha_1})(1-e^{-\alpha_2})(1-e^{-\alpha_1-\alpha_2})}.$$

B.19. Example. Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_2(\mathbf{C})$ and $\lambda = 0$. Then the denominator becomes

$$\prod_{n=1}^{\infty}(1-p^nq^n)(1-p^{n-1}q^n)(1-p^nq^{n-1}) = \sum_{k\in\mathbf{Z}}(-1)^k p^{\frac{k(k-1)}{2}} q^{\frac{k(k+1)}{2}}.$$

B.20. Example. Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_2(\mathbf{C})$ and $\lambda = \Lambda_0$. Then

$$F(a) = \left(q^{-4\rho} \middle| \frac{S(a)}{\operatorname{ch}(V(\lambda))} \right).$$

This equation was taken from a mathematical physics book. We don't know what it means, but the point is that this is useful in mathematical physics.

APPENDIX C

Presentations

1. A correspondence between subfields and Lie algebras of derivations

BY ANDREW PHILLIPS

We will describe a Galois-type correspondence between subfields of a field of characteristic p and certain subalgebras of the Lie algebra of derivations of the field. This result is important because it can be applied to field extensions where there is no Galois theory available (our example below is an inseparable extension). Let F be a field of characteristic p. A *derivation* of F is an additive map $F \longrightarrow F$ satisfying the product rule. That is, $\delta\colon F \longrightarrow F$ is a derivation if

$$\delta(\alpha+\beta) = \delta(\alpha) + \delta(\beta), \quad \delta(\alpha\beta) = \alpha\delta(\beta) + \delta(\alpha)\beta$$

for all $\alpha, \beta \in F$. Let $\mathfrak{g}$ be the set of all derivations of F. Then $\mathfrak{g}$ is a (left) vector space over F with addition and scaling given by

$$(\delta+\delta')(\alpha) = \delta(\alpha) + \delta'(\alpha), \quad (a\delta)(\alpha) = a(\delta(\alpha))$$

for all $\delta, \delta' \in \mathfrak{g}$ and $a, \alpha \in F$. Also, if $\delta, \delta' \in \mathfrak{g}$ then $[\delta, \delta'] = \delta\delta' - \delta'\delta$ is in $\mathfrak{g}$. However, $\mathfrak{g}$ is not a Lie algebra over F since $[\cdot, \cdot]$ is not F-bilinear:

$$\begin{aligned}
[a\delta, \delta'](\alpha) &= ((a\delta)\delta' - \delta'(a\delta))(\alpha) \\
&= (a\delta)(\delta'(\alpha)) - \delta'((a\delta)(\alpha)) \\
&= a(\delta\delta'(\alpha)) - \delta'(a(\delta(\alpha))) \\
&= a(\delta\delta'(\alpha)) - (a\delta'(\delta(\alpha)) + \delta'(a)\delta(\alpha)) \\
&= (a[\delta, \delta'])(\alpha) - \delta'(a)\delta(\alpha)
\end{aligned}$$

for any $a, \alpha \in F$ and $\delta, \delta' \in \mathfrak{g}$. However, $\mathfrak{g}$ is a Lie algebra over F^p: if $a \in F^p$ then $\delta'(a) = 0$, so $[a\delta, \delta'](\alpha) = (a[\delta, \delta'])(\alpha)$.

By induction on n and the product rule,

$$\delta^n(\alpha\beta) = \sum_{i=0}^{n} \binom{n}{i} \delta^i(\alpha)\delta^{n-i}(\beta)$$

for all $\alpha, \beta \in F$, $\delta \in \mathfrak{g}$, and $n \geqslant 1$, where $\delta^n = \delta \circ \delta \circ \cdots \circ \delta$ (n times). In particular,

$$\delta^p(\alpha\beta) = \alpha\delta^p(\beta) + \sum_{i=1}^{p-1} \binom{p}{i} \delta^i(\alpha)\delta^{p-i}(\beta) + \delta^p(\alpha)\beta$$

and since $\binom{p}{i}$ is divisible by p for $1 \leqslant i \leqslant p-1$ and F has characteristic p, we have $\delta^p(\alpha\beta) = \alpha\delta^p(\beta) + \delta^p(\alpha)\beta$, which shows $\delta^p \in \mathfrak{g}$.

Below we will give a correspondence between intermediate fields $F^p \subset K \subset F$ and certain Lie subalgebras of $\mathfrak{g}$. First we establish some notation. Let $\mathscr{C}$ be the collection of subfields $K \subset F$ such that $F^p \subset K \subset F$ and $[F : K]$ is finite. For each $K \in \mathscr{C}$, let

$$\mathfrak{g}(K) = \{\delta \in \mathfrak{g} : \delta(\alpha) = 0 \text{ for all } \alpha \in K\}.$$

Let $\mathscr{L}$ be the collection of F-subspaces $\mathfrak{h} \subset \mathfrak{g}$ of finite dimension over F such that $[\delta, \delta'] \in \mathfrak{h}$ and $\delta^p \in \mathfrak{h}$ for all $\delta, \delta' \in \mathfrak{h}$. In particular, $\mathfrak{h}$ is an F^p-subalgebra of $\mathfrak{g}$. For each $\mathfrak{h} \in \mathscr{L}$, let

$$I(\mathfrak{h}) = \{\alpha \in F : \delta(\alpha) = 0 \text{ for all } \delta \in \mathfrak{h}\}.$$

C.1. Theorem (Jacobson). *With notation as above, the mappings $K \mapsto \mathfrak{g}(K)$ and $\mathfrak{h} \mapsto I(\mathfrak{h})$ are bijections of $\mathscr{C}$ onto $\mathscr{L}$ and of $\mathscr{L}$ onto*

$\mathscr{C}$, respectively, which are inverses of each other. If $K \in \mathscr{C}$ and $\mathfrak{h} \in \mathscr{L}$ correspond, then $[F : K] = p^{\dim_F(\mathfrak{h})}$.

These bijections are inclusion reversing: if $K \subset K'$, K corresponds to $\mathfrak{h}$, and K' corresponds to $\mathfrak{h}'$, then $\mathfrak{h}' \subset \mathfrak{h}$. Conversely, if $\mathfrak{h} \subset \mathfrak{h}'$ then $K' \subset K$.

C.2. Example. Let $F = \mathbf{F}_p(x, y)$ be the field of rational functions in the variables x and y over the field $\mathbf{F}_p$ with p elements, where p is a prime. It can be shown that $F^p = \mathbf{F}_p(x^p, y^p)$, which we write as L. Also, $[F : L] = p^2$ with $\{x^i y^j : 0 \leqslant i, j \leqslant p-1\}$ being an L-basis of F. Hence if K is a field such that $L \subsetneqq K \subsetneqq F$, then $[F : K] = [K : L] = p$. As before let $\mathfrak{g}$ be the set of all derivations of F. If $\delta \in \mathfrak{g}$, it is easily verified by induction on n that $\delta(f^n) = nf^{n-1}\delta(f)$ for all $f \in F$ and $n \geqslant 1$. In particular, $\delta(c^p) = 0$ for all $c \in \mathbf{F}_p$. Since $\mathbf{F}_p^p = \mathbf{F}_p$, we have $\delta(c) = 0$ for all $c \in \mathbf{F}_p$.

Before considering derivations of the field $\mathbf{F}_p(x, y)$, we start with derivations $\mathbf{F}_p[x, y] \longrightarrow \mathbf{F}_p(x, y)$. Define the formal partial derivatives $\partial_x, \partial_y \colon \mathbf{F}_p[x, y] \longrightarrow \mathbf{F}_p(x, y)$ by

$$\partial_x \colon \sum_{i,j} c_{ij} x^i y^j \mapsto \sum_{i,j} i c_{ij} x^{i-1} y^j,$$
$$\partial_y \colon \sum_{i,j} c_{ij} x^i y^j \mapsto \sum_{i,j} j c_{ij} x^i y^{j-1}.$$

It is simple to verify these are derivations. We claim that any derivation $\delta \colon \mathbf{F}_p[x, y] \longrightarrow \mathbf{F}_p(x, y)$ has the form $\delta = f\partial_x + g\partial_y$ for some $f, g \in \mathbf{F}_p(x, y)$. Since $\delta(c) = 0$ for all $c \in \mathbf{F}_p$,

$$\delta(ch) = c\delta(h) + \delta(c)h = c\delta(h)$$

for all $h \in \mathbf{F}_p[x, y]$, so δ is $\mathbf{F}_p$-linear. Hence δ is determined by its values on the $\mathbf{F}_p$-basis $\{x^i y^j : i, j \geqslant 0\}$ of $\mathbf{F}_p[x, y]$. By the linearity of δ and the fact that $\delta(x^n) = nx^{n-1}\delta(x)$ (and similarly for $\delta(y^n)$), for $h(x, y) = \sum_{i,j} c_{ij} x^i y^j \in \mathbf{F}_p[x, y]$ we have

$$\delta(h) = \sum_{i,j} c_{ij} \delta(x^i y^j)$$

$$= \sum_{i,j} c_{ij}(x^i\delta(y^j) + \delta(x^i)y^j)$$
$$= \sum_{i,j} c_{ij}(ix^{i-1}y^j\delta(x) + jx^iy^{j-1}\delta(y))$$
$$= \delta(x)\partial_x(h) + \delta(y)\partial_y(h).$$

Setting $f(x,y) = \delta(x)$ and $g(x,y) = \delta(y)$ shows $\delta = f\partial_x + g\partial_y$.

Since $\mathbf{F}_p(x,y)$ is the fraction field of $\mathbf{F}_p[x,y]$, it can be shown that any derivation $\delta\colon \mathbf{F}_p[x,y] \longrightarrow \mathbf{F}_p(x,y)$ uniquely extends to a derivation $\widetilde{\delta}\colon \mathbf{F}_p(x,y) \longrightarrow \mathbf{F}_p(x,y)$ given by the quotient rule:

$$\widetilde{\delta}(f/g) = \frac{g\delta(f) - \delta(g)f}{g^2}.$$

Therefore every derivation of $F = \mathbf{F}_p(x,y)$ has the form $f\partial_x + g\partial_y$ for some $f, g \in F$. Also, ∂_x and ∂_y are F-linearly independent: if $f\partial_x + g\partial_y = 0$, evaluating both sides at x and y gives $f = g = 0$. This shows $\mathfrak{g}$ is a (left) F-vector space with basis $\{\partial_x, \partial_y\}$:

$$\mathfrak{g} = F\partial_x \oplus F\partial_y.$$

If $\mathfrak{h} \subset \mathfrak{g}$ is the subalgebra corresponding to a subfield $K \subset F$ as in the theorem, then since $p = [F : K] = p^{\dim_F(\mathfrak{h})}$, we have $\dim_F(\mathfrak{h}) = 1$. Thus $\mathfrak{h}$ is a 1-dimensional F-subspace of $\mathfrak{g}$ satisfying $[\delta, \delta'] \in \mathfrak{h}$ and $\delta^p \in \mathfrak{h}$ for all $\delta, \delta' \in \mathfrak{h}$. Before giving specific examples of this, let us first consider the p-th power of elements of $\mathfrak{g}$. We will show for any $\delta = a\partial_x + b\partial_y$ in $\mathfrak{g}$ that $\delta^p = A\partial_x + B\partial_y$ for some A and B that are polynomials in a, b, and their derivatives. We know $\delta^p \in \mathfrak{g}$, so $\delta^p = A\partial_x + B\partial_y$ for some $A, B \in F$. Next,

$$(A\partial_x + B\partial_y)(x) = A\partial_x(x) + B\partial_y(x) = A$$

and

$$\delta^p(x) = \delta^{p-1}(\delta(x)) = \delta^{p-1}(a\partial_x(x) + b\partial_y(x))$$
$$= \delta^{p-1}(a) = (a\partial_x + b\partial_y)^{p-1}(a).$$

Thus $A = (a\partial_x + b\partial_y)^{p-1}(a)$. Similarly, $B = (a\partial_x + b\partial_y)^{p-1}(b)$. Any x or y partial derivative of a polynomial expression in a, b, and their iterated partial derivatives is also such an expression, so A and B are polynomials in a, b, and their iterated partial derivatives.

As an example, let $\mathfrak{h} = F\partial_x$. Let us first check that $\mathfrak{h}$ is closed under the bracket: let $\delta = f\partial_x$ and $\delta' = g\partial_x$ be in $\mathfrak{h}$. Then by the product rule,

$$\begin{aligned}[\delta, \delta'] &= (f\partial_x)(g\partial_x) - (g\partial_x)(f\partial_x) \\ &= f(g\partial_{xx} + g_x\partial_x) - g(f\partial_{xx} + f_x\partial_x) \\ &= fg_x\partial_x - gf_x\partial_x \\ &= (fg_x - gf_x)\partial_x,\end{aligned}$$

which is in $\mathfrak{h}$. For any $\delta = f\partial_x \in \mathfrak{h}$, by what we showed above, $\delta^p = A\partial_x$ for some $A \in F$, so $\delta^p \in \mathfrak{h}$. By the theorem, the subfield of F corresponding to $\mathfrak{h}$ is

$$K = I(\mathfrak{h}) = \{\alpha \in F : \delta(\alpha) = 0 \text{ for all } \delta \in \mathfrak{h}\}.$$

Clearly $K \supset L(y)$. Since $L \subsetneqq L(y) \subset K$ and $[K : L] = p$, we must have $K = L(y)$. Similarly the subfield of F corresponding to the subalgebra $F\partial_y$ is $L(x)$.

Now we start with a subfield of F. We want to construct an infinite number of intermediate fields $L \subset K \subset F$ and we will use the theorem to show the fields are distinct. For each $n \geqslant 0$ let $K_n = L(x + y^{p^n}y)$. Since $L \subset K_n$ and $[F : K_n]$ is finite, by the theorem, the subalgebra of $\mathfrak{g}$ corresponding to K_n is

$$\mathfrak{h}_n = \mathfrak{g}(K_n) = \{\delta \in \mathfrak{g} : \delta(\alpha) = 0 \text{ for all } \alpha \in K_n\}.$$

Any $\delta \in \mathfrak{h}_n$ is determined by the condition $\delta(x + y^{p^n}y) = 0$ because if this holds then $\delta((x + y^{p^n}y)^k) = 0$ for all $k \geqslant 1$ by the power rule. To find a derivation satisfying this, write $\delta = f\partial_x + g\partial_y$ for some $f, g \in F$. Then

$$0 = \delta(x + y^{p^n}y) = f + y^{p^n}g,$$

so $f = -y^{p^n}g$. Thus the derivation $\delta = -y^{p^n}\partial_x + \partial_y$ satisfies the desired condition, so $\mathfrak{h}_n = F(-y^{p^n}\partial_x + \partial_y)$. To show the fields K_n are distinct, first we will show the subalgebras $\mathfrak{h}_n$ are distinct. Let us check for

$m \neq n$ that $-y^{p^m}\partial_x + \partial_y$ and $-y^{p^n}\partial_x + \partial_y$ are F-linearly independent. Suppose

$$a(-y^{p^m}\partial_x + \partial_y) + b(-y^{p^n}\partial_x + \partial_y) = 0$$

for some $a, b \in F$. Then since $\{\partial_x, \partial_y\}$ is an F-basis of $\mathfrak{g}$, $a + b = 0$. Hence $-ay^{p^m} + ay^{p^n} = 0$, which forces $a = 0$ because $y^{p^m} \neq y^{p^n}$, so $b = 0$. Thus for $m \neq n$, $\mathfrak{h}_m = F(-y^{p^m}\partial_x + \partial_y)$ and $\mathfrak{h}_n = F(-y^{p^n}\partial_x + \partial_y)$ are distinct subalgebras. Since the subalgebras $\mathfrak{h}_n$ are in bijection with the subfields K_n, for $m \neq n$ we have $K_m \neq K_n$. Therefore we have constructed an infinite number of fields K_n (one for each $n \geqslant 0$) such that $L \subset K_n \subset F$. A similar construction can be carried out with the fields $L(x^{p^n}x + y)$.

See Chapter V, Section 13 of [**3**] or Section 8.16 of [**9**] for more details.

2. Connection between Lie groups and Lie algebras

BY JAKOB LISS

C.3. Definition. A Lie group G is a differentiable manifold such that the map $G \times G \longrightarrow G$ by $(g, a) \mapsto ga^{-1}$ is differentiable.

Using this definition, one can show for $h \in G$ that the following maps are diffeomorphisms:

- (left translation) $L_h\colon G \longrightarrow G$ by $x \mapsto hx$,
- (right translation) $R_h\colon G \longrightarrow G$ by $x \mapsto xh$,
- (inner automorphism) $\alpha_h = L_h \circ R_{h^{-1}}$ by $x \mapsto hxh^{-1}$.

C.4. Definition. Let $M \subset \mathbf{R}^n$ be an m-dimensional smooth manifold. For $p \in M$, we define the tangent space of M at p to be

$$T_pM = \left\{\gamma'(0) : {\scriptstyle \text{there exists } \varepsilon > 0 \text{ such that} \atop \gamma \in C^1((-\varepsilon,\varepsilon),\mathbf{R}^n),\ \gamma(-\varepsilon,\varepsilon)\subset M,\ \gamma(0)=p}\right\}.$$

We call

$$TM = \{(p, v) : p \in M,\ v \in T_pM\}$$

the tangent bundle of M.

C.5. Definition. A smooth vector field on a smooth manifold M is a smooth map $F\colon M \longrightarrow TM$ such that $F(x) \in T_pM$. The set of vector fields on M is denote $\tau(M)$.

C.6. Definition. Let $f\colon M \longrightarrow N$ be a diffeomorphism between smooth manifolds M and N. Then for $F \in \tau(N)$, we define the pullback $f^*(F) \in \tau(M)$ by

$$f^*(F)|_p(f^*w) = F|_{f(p)}(w)$$

for all $w \in (T_{f(p)}N)^\vee$, where $f^*w(v) = w(\mathrm{d}f|_p(v))$ for all $v \in T_pM$.

Similarly, we define the pushforward $f_*(F) \in \tau(N)$ by

$$f_*(F)|_p(f_*w) = F|_{f^{-1}(p)}(w)$$

for all $w \in (T_{f^{-1}(p)}M)^\vee$, where $f_*w(v) = w((\mathrm{d}f|_p)^{-1}(v))$ for all $v \in T_pN$.

C.7. Definition. Let

$$\mathfrak{g} = \{X \in \tau(G) : X \text{ is left invariant}\},$$

where $X \in \tau(G)$ is left invariant if

$$(L_h)_*(X)|_p = \mathrm{d}L_h|_{L_h^{-1}} X|_{L_h^{-1}(p)} = X|_p$$

for all $h \in G$. Then $\mathfrak{g}$ together with the vector field commutator $[X,Y] = X \circ Y - Y \circ X$ is called the Lie algebra of the Lie group G.

Notice that $X \in \mathfrak{g}$ is already determined by $X|_{1_G} \in T_{1_G}G$ as

$$X|_h = (L_h)_* X|_h = \mathrm{d}L_h|_{1_G} X|_{1_G}.$$

C.8. Definition. A map $\Psi\colon G_1 \longrightarrow G_2$ is a homomorphism of Lie groups G_1 and G_2 if it is a differentiable group homomorphism.

C.9. Proposition. *Let $\Psi\colon G_1 \longrightarrow G_2$ be a Lie group homomorphism, and let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be the Lie algebras of G_1 and G_2, respectively. Then $\Psi_*\colon \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ by $X \mapsto \Psi_* X = \mathrm{d}\Psi|_{1_G}(X|_{1_G})$ is a Lie algebra homomorphism.*

C.10. Proposition. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g}$, let $\varphi_X \colon I_{\max} \longrightarrow G$ denote the maximal integral curve of X through $1 \in G$; that is, $I_{\max} \subset \mathbf{R}$ is the maximum interval such that $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_X = X(\varphi_X)$ and $\varphi_X(0) = 1_G$ exists. Then*

(1) $I_{\max} = \mathbf{R}$,
(2) $\varphi_X \colon \mathbf{R} \longrightarrow G$ *is a Lie group homomorphism,*
(3) $\varphi_{sX}(t) = \varphi_X(st)$ *for all* $s, t \in \mathbf{R}$.

C.11. Proposition. *The map* $\exp\colon \mathfrak{g} \longrightarrow G$ *by* $X \mapsto \varphi_X(1)$ *is called the exponential map. It is a local diffeomorphism about* $0 \in \mathfrak{g}$ *with the following properties.*

(1) $\exp(0) = 1_G$.
(2) $\exp(-X) = (\exp(X))^{-1}$.
(3) $\exp((t+s)X) = \exp(sX)\exp(tX)$ *for all* $s, t \in \mathbf{R}$.
(4) *If* $\Psi\colon \mathbf{R} \longrightarrow G$ *is a continuous group homomorphism to a Lie group* G*, then there exists* $X \in \mathfrak{g}$ *such that* $\Psi(t) = \exp(tX)$.

C.12. Proposition. *Every continuous Lie group homomorphism is smooth.*

C.13. Proposition. *Let* $\Psi\colon G_1 \longrightarrow G_2$ *be a Lie group homomorphism. Then* $\Psi(\exp(X)) = \exp(\Psi_* X)$.

C.14. Proposition. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let H be a closed subgroup of G. Then H is a Lie subgroup of G with Lie algebra*

$$\mathfrak{h} = \{X \in \mathfrak{g} : \exp(tX) \in H \text{ for all } t \in \mathbf{R}\}.$$

C.15. Example. Let $\mathrm{GL}_n(\mathbf{R})$ be a Lie group identified as an open subset of $\mathbf{R}^{n^2}$. Then $T_{I_n}\,\mathrm{GL}_n(\mathbf{R}) = \mathbf{R}^{n^2} \equiv \mathfrak{gl}_n(\mathbf{R})$. Let $X \in T_{I_n}\,\mathrm{GL}_n(\mathbf{R})$. Then we find the left invariant vector field $\widetilde{X} \in \mathfrak{gl}_n(\mathbf{R})$ determined by X as

$$\widetilde{X}(A) = \mathrm{d}L_A|_{I_n} X = \frac{\mathrm{d}}{\mathrm{d}t}(L_A(I_n + tX))|_{t=0} = AX$$

for all $A \in \mathrm{GL}_n(\mathbf{R})$.

Thus, for the maximal integral curve, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{\widetilde{X}} = \widetilde{X}(\varphi_{\widetilde{X}}) = \varphi_{\widetilde{X}} \circ \widetilde{X},$$

which, by $\widetilde{X}(I_n) = X$, induces a maximal integral curve φ_X for X such that $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_X = \varphi_X \circ X$ and $\varphi_X(0) = I_n$. But uniqueness of solutions to ordinary differential equations gives

$$\varphi_X(t) = e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!} \in \mathrm{GL}_n(\mathbf{R}).$$

Therefore, $\exp(X) = \varphi_X(1) = e^X$.

Now consider the closed subgroup

$$\mathrm{SL}_n(\mathbf{R}) = \{A \in \mathrm{GL}_n(\mathbf{R}) : \det(A) = 1\}.$$

From the last proposition, we know

$$\mathfrak{sl}_n(\mathbf{R}) = \{A \in \mathfrak{gl}_n(\mathbf{R}) : \det(\exp(tA)) = 1\}.$$

The identity $1 = \det(\exp(tA)) = \exp(\mathrm{Tr}(A))$ now gives the known form $\mathfrak{sl}_n(\mathbf{R}) = \{A \in \mathfrak{gl}_n(\mathbf{R}) : \mathrm{Tr}(A) = 0\}$.

C.16. Definition. Let V be a real or complex vector space and let G be a Lie group G. Then a representation of G is a Lie group homomorphism $\rho\colon G \longrightarrow \mathrm{GL}(V)$.

C.17. Proposition. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. If $\rho\colon G \longrightarrow \mathrm{GL}(V)$ is a representation G, then $\rho_*\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$.*

C.18. Theorem. *Let G be a Lie group. For $h \in G$, let $\alpha_h = L_h \circ R_{h^{-1}}\colon G \longrightarrow G$ be the inner automorphism and let $(\alpha_h)_*\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ be the associated isomorphism of the Lie algebra $\mathfrak{g}$.*

- *The map $\mathrm{Ad}\colon G \longrightarrow \mathrm{GL}(\mathfrak{g})$ by $h \mapsto (\alpha_h)_*$ is a representation of the Lie group G.*
- *The differential $\mathrm{ad} = (\mathrm{Ad})_*\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is defined by*

$$\mathrm{ad}(X)(Y) = [X, Y].$$

3. Quivers, reflection functors, and Gabriel's theorem

BY LUCAS DAVID-ROESLER

In the study of the representation theory of complex semi-simple Lie algebras and the study of root systems Dynkin diagrams play an important role in classifying different algebras. The importance of Dynkin diagrams is not restricted to the representation theory of Lie algebras. The Dynkin diagrams of A, D and E type appear in many classification theorems, such as [**6**]:

(a) the classification of platonic solids,
(b) classification of the singularities of algebraic hypersurfaces, with a definite intersection form of the nieghboring smooth fibre,
(c) classification of the critical points of functions having no moduli.

In 1972, Gabriel [**5**] showed that these diagrams also play an important role in the study of quivers and the representation theory of associative algebras. In particular, the Dynkin diagrams of A, D and E type classify quivers of finite representation type.

In this article we give an introduction to the concept of quiver representations and refection functors which was developed by Bernsein, Gelfand, and Ponomarev [**2**] and provides a connection between Lie theory and the representation theory of quivers. We rely heavily on [**1**].

3.1. Quivers. A ***quiver*** Q is a directed graph denoted by the quintuple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices in Q, Q_1 the set of arrows, $s\colon Q_1 \longrightarrow Q_0$ defines the source of an arrow, and $t\colon Q_1 \longrightarrow Q_0$ the target of an arrow.

C.19. EXAMPLE. Some example quivers:

(1) Let Q be the quiver $5 \longrightarrow 4 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1$. This quiver has no multiple edges or loops. Such quivers are referred to as ***acyclic***.
(2) An example which is not acyclic is Q' given by $\begin{tikzcd} 3 \arrow[loop below] \arrow[r] & 2 \arrow[r, shift left] & 1 \arrow[l, shift left] \end{tikzcd}$.

(3) A quiver need not have finitely many vertices:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots .$$

However, in this discussion we will restrict ourselves to quivers with only finitely many vertices and arrows.

In general, there is no restriction on how to number the vertices of a quiver. But to properly define reflection functors we will require an ***admissible numbering*** of the vertices of Q. Throughout we let $\#Q_0 = n$. Then a numbering is admissible if whenever $j \to i$ in Q, then $j > i$. If Q is finite and acyclic there is always a non-unique admissible numbering. Given a finite acyclic quiver Q we can construct an admissible numbering as follows. Let 1 be any sink; a vertex which no arrow leaves. Then define $Q(1)$ to be the quiver by removing the vertex 1 and all arrows which entered 1. We then define 2 to be any sink of $Q(1)$ and continue by induction.

To any quiver Q we can associate a K-algebra $A = KQ$ called the ***path algebra***. The path algebra is defined to be the K-algebra with basis the set of all paths in Q and multiplication defined by concatenation of paths. It is not difficult to see that KQ is finite-dimensional if and only if Q is finite and acyclic. Moreover, given any algebra A we can construct the ***ordinary quiver*** Q_A, see [**1**]. However, we are not guaranteed that $A \cong KQ_A$. It turns out that there are distinct algebras which can have the same quiver and there are finite dimensional algebras which many have quivers which are not acyclic. But we do have the following theorem:

C.20. THEOREM ([**1**]). *Let A be a basic and connected finite dimensional K-algebra. Then there exists an ideal $\mathcal{I}$ of KQ_A such that $A \cong KQ_A/\mathcal{I}$.*

Because of this theorem we have a nice concrete presentation for many algebras. This presentation of algebras is particularly useful in studying the modules over an algebra A. In particular, when studying the representation theory of algebras one is often trying to describe

$\mathsf{mod}\,A$, the category of finitely generated modules over A. Quivers give us a tool for thoroughly describing $\mathsf{mod}\,A$.

A ***representation*** of Q is a collection of K-vector spaces and linear maps, usually written as $M = (M_a, f_\alpha)$ where $a \in Q_0$ and $\alpha \in Q_1$, which is compatible with Q. That is if α is an arrow from a to b in Q, then $f_\alpha \colon M_a \longrightarrow M_b$ in M.

C.21. Example. Using the quivers we have already seen, we have the following quiver representations:

(1) $K^3 \xrightarrow{(1\ 1\ 1)} K \xrightarrow{1} K \xrightarrow{1} K \xrightarrow{\binom{1}{1}} K^2,$

(2)
$$\begin{tikzcd} K \arrow[loop below, "1"] \arrow[r, "\binom{1}{1}"] & K^2 \arrow[r, "(0\ 1)"] & K^2 \arrow[l, bend left, "(1\ 0)"] \end{tikzcd},$$

where each linear map is represented by a matrix.

The representation M is called ***finite-dimensional*** if each M_a is finite dimensional. We write $\mathsf{rep}\,Q$ for the category of finite dimensional representations of Q. If M is finite-dimensional, then the ***dimension vector*** of M, denoted $\mathbf{dim}\,M$, is the vector $\mathbf{v} \in \mathbf{Z}^n$ with $\mathbf{v} = (\dim_K M_i)$ where $n = \#Q_0$. Dimension vectors are one way to describe the representations of a quiver. However, they may not completely describe the representations of Q. For example, let M_μ be the representation

$$\begin{tikzcd} K \arrow[r, shift left, "1"] \arrow[r, shift right, "\mu"'] & K \end{tikzcd}.$$

Then $M_\mu \neq M_\lambda$ for $\mu \neq \lambda$ but we have $\mathbf{dim}\,M_\mu = \mathbf{dim}\,M_\lambda = \binom{1}{1}$. However, we will see with Gabriel's theorem that there are situations in which the dimension vectors are sufficient information.

It turns out that quiver representations are the correct way to present modules over algebras.

C.22. Theorem. *There exists an equivalence of categories*

$$\mathsf{mod}\,KQ/\mathcal{I} \cong_K \mathsf{rep}(Q, \mathcal{I}),$$

where $\mathsf{rep}(Q, \mathcal{I})$ *is the category of finite-dimensional representations of* Q *which are bounded by* $\mathcal{I}$. *If* ρ *is a path in* $\mathcal{I}$, *then* $f_\rho = 0$ (*the composition of the maps along the path* ρ).

With Theorem C.20 and Theorem C.22 we can focus on quiver representations of Q_A instead of studying $\mathsf{mod}\,A$ directly. One of the first question one should ask is: given a quiver Q, what are the building blocks of the representations of Q and how many of these do I need to study in order to completely understand $\mathsf{rep}(Q)$? Because $\mathsf{mod}\,A$ is an abelian category, it suffices to study only the ***indecomposable representations*** of Q. These are representations which can not be written as the direct sum of two sub-representations. We say that an algebra is ***representation finite*** if it has only finitely many indecomposable representations up to isomorphism. With this language in place, we are now able to state and understand Gabriel's theorem, which will tell us when we may hope to be able to completely understand the structure of $\mathsf{rep}(Q)$.

C.23. Theorem (Gabriel). *The path algebra of a connected quiver* Q *is representation-finite if and only if the underlying graph of the quiver is of Dynkin type* A, D, *or* E. *Further, there is a bijection between indecomposable representations of* Q *and their dimension vectors.*

For $n \geq 4$ we have the following Dynkin diagrams

$$A_n\colon\ 1 \text{ —— } 2 \text{ —— } 3 \text{ —— } \cdots \text{ —— } n-1 \text{ —— } n\ ,$$

$$\begin{tikzcd}[arrows=dash] & & & & & n-1 \\ D_n\colon\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n-2 \arrow[ur] \arrow[dr] & & , \\ & & & & & n \end{tikzcd}$$

$$\begin{tikzcd}[arrows=dash] & & 6 \arrow[d] & & \\ E_6\colon\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & 5\ , \end{tikzcd}$$

$$
\begin{array}{lccccccccccc}
 & & & & & 7 & & & & & & \\
 & & & & & | & & & & & & \\
E_7\colon & 1 & — & 2 & — & 3 & — & 4 & — & 5 & — & 6\ ,
\end{array}
$$

$$
\begin{array}{lccccccccccccc}
 & & & & & 8 & & & & & & & & \\
 & & & & & | & & & & & & & & \\
E_8\colon & 1 & — & 2 & — & 3 & — & 4 & — & 5 & — & 6 & — & 7\ .
\end{array}
$$

Quivers with one of these underlying shapes are the nicest quivers to deal with.

We will only sketch an outline of the proof of Theorem C.23. Notice that every sub-diagram of a Dynkin diagram is also Dynkin, though not necessarily of the same type. Thus there are minimal non-Dynkin diagrams called ***Euclidean*** or ***affine Dynkin*** diagrams of type $\widetilde{A}$, $\widetilde{D}$, and $\widetilde{E}$. Further, every non-Dynkin diagram must contain at least one Euclidean diagram as a subdiagram. Hence it suffices to show that each Euclidean quiver Q has an infinite family of non-isomorphic representations of Q. Then this family extends by 0 to any quiver containing Q. For an explicit construction of these families of representations see [**13, 1**].

At this point we have shown that the only possible quivers of finite-representation type must be Dynkin. But the question remains: are all quivers of Dynkin type representation finite? The answer requires reflection functors.

3.2. Reflection Functors.

3.2.1. *Reflections via quadratic forms.* Given a quiver Q, we can associate a quadratic form on $\mathbf{Z}^n$ defined by

$$q_Q(\mathbf{x}) = \sum_{i\in Q_0} x_i^2 - \sum_{\alpha\in Q_1} x_{s(\alpha)}x_{t(\alpha)}.$$

We say that a vector $\mathbf{x}$ is a ***root*** of q_Q if $q_Q(\mathbf{x}) = 1$. A vector $\mathbf{x}$ is called a ***positive root*** if $\mathbf{x} \neq 0$ and $x_j \geq 0$ for all j. One should note that $q_Q(\mathbf{e}_i) = 1$ for all i, where $\{\mathbf{e}_i\}$ is the standard basis of $\mathbf{Z}^n$. Similarly, one can define the symmetric bilinear form of Q by

$$(\mathbf{x},\mathbf{y})_Q = \sum_{i\in Q_0} x_i y_i - \frac{1}{2}\sum_{\alpha\in Q_1} x_{s(\alpha)}y_{t(\alpha)} + x_{t(\alpha)}y_{s(\alpha)}.$$

Notice that $(\mathbf{x},\mathbf{y})_Q = \dfrac{1}{2}[q_Q(\mathbf{x}+\mathbf{y}) - q_Q(\mathbf{x}) - q_Q(\mathbf{y})]$, the symmetrized version of q_Q.

C.24. Example. If Q is given by $2 \to 1$, then $q_Q(\mathbf{x})$ and $(\mathbf{x},\mathbf{y})$ are given by

$$(\mathbf{x},\mathbf{y})_Q = \mathbf{x}^t \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \mathbf{y} \qquad \text{and} \qquad q_Q(\mathbf{x}) = (\mathbf{x},\mathbf{x})_Q.$$

C.25. Remark. From the previous example one should notice that the matrix of $(-,-)_Q$ is exactly $\frac{1}{2}$ the Cartan matrix for the root system A_2. This is always true. This follows from the fact that that $(\mathbf{e}_i,\mathbf{e}_j)_Q$ counts (upto a scalar of $\frac{1}{2}$) the number of edges between the vertices i and j. This corresponds exactly to the entries in the Cartan matrix of a Dynkin diagram and these correspond to the values of bilinear form associated with the root system, $(\alpha_i, \alpha_j^\vee)$. Hence, this bilinear form we have defined corresponds up to a scalar multiple of the bilinear form of the root system when Q is Dynkin.

We are now ready to define the three types of reflection functors needed to finish the proof of Gabriel's theorem. The first is on vectors in $\mathbf{Z}^n$. Fix a quiver Q with $n = \#Q_0$ and let $(-,-) := (-,-)_Q$. Then we define the ***simple reflection*** in the hyperplane orthogonal to $\mathbf{e}_i$ by $s_i(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x},\mathbf{e}_i)\mathbf{e}_i$. One can verify that $\mathbf{y} = s_i(\mathbf{x})$ has the coordinates

$$y_j = \begin{cases} x_j & \text{if } j \neq i, \\ -x_i + \sum_{k-i} x_k & \text{if } j = i, \end{cases}$$

where the sum is over edges of Q which touch the vertex i. More specifically, for the basis elements $\{\mathbf{e}_j\}$ we have the following computation:

$$s_i(\mathbf{e}_j) = \mathbf{e}_j - 2(\mathbf{e}_j, \mathbf{e}_i)\mathbf{e}_i = \begin{cases} -\mathbf{e}_i & \text{if } i = j, \\ \mathbf{e}_j + \mathbf{e}_i & \text{if } i, j \text{ are connected by an edge,} \\ \mathbf{e}_j & \text{otherwise.} \end{cases}$$

We fix an admissible numbering of Q, then we define the ***Coxeter transformation*** as $c := s_n s_{n-1} \cdots s_1$. Obviously, $c^{-1} = s_1 \cdots s_n$. We also define special elements of $\mathbf{Z}^n$ by $\mathbf{p}_i = s_1 \cdots s_{i-1}\mathbf{e}_i$.

C.26. Proposition ([**1**]). *Let Q be a quiver whose underlying graph is Dynkin and c be the Coxeter transformation of Q. If m_i is the least integer such that $c^{-m_i-1}\mathbf{p}_i \not> 0$, then the set*

$$\Phi^+ = \{c^{-s}\mathbf{p}_i : 1 \leq i \leq n, 0 \leq s \leq m_i\}$$

equals the set of all positive roots of q_Q.

We have chosen a very suggestive and correct notation for the positive roots of q_Q. When Q is of Dynkin type then these positive roots correspond to the positive roots of the corresponding root system.

One can show (see [**?, 1**]) that the following are equivalent:

(1) the form $(-,-)_Q$ is positive definite,
(2) the group $W = \langle s_1, \dots, s_n\rangle$ generated by the simple reflections is finite,
(3) Q is Dynkin.

Notice that if W is finite, then we must have that Φ^+ is also finite. We will develop the concept of reflections on quivers and quiver representations to show that the dimension vectors of a indecomposable representations are in bijection with Φ^+, completing our outline of the proof of Gabriel's theorem.

3.2.2. *Reflections on quivers and representations.* Let Q be a fixed finite acyclic quiver. We define a reflection of Q by $\sigma_a \colon Q \longrightarrow Q'$ a map between quivers where all the arrows of Q having a as a source or as target are reversed, all other arrows remain unchanged. Using this quiver reflection we define a reflection on representations. When a is sink, we define reflection functor

$$\mathcal{S}_a^+\colon \mathsf{rep}_K(Q) \longrightarrow \mathsf{rep}_K(\sigma_a Q).$$

When a is a source, we define the reflection functor

$$\mathcal{S}_a^-\colon \mathsf{rep}_K(\sigma_a Q) \longrightarrow \mathsf{rep}_K(Q).$$

When applied to a representation M, these functors only affect the vector space at a and the linear maps entering/leaving a. These functors are characterized by the following exact sequences:

$$M_a \longrightarrow \bigoplus_{s\alpha=a} M_{t\alpha} \longrightarrow \mathcal{S}_a^- M_a \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{S}^+ M_a \longrightarrow \bigoplus_{t\alpha=a} M_{s\alpha} \longrightarrow M_a,$$

which gives induced maps over the reversed arrows. We are concerned with the following properties of the reflection functors:

C.27. Proposition ([**1**]). *Let $M \in \mathsf{rep}_K(Q)$.*

(1) $\mathcal{S}^-(M)$ *is indecomposable if M is indecomposable and not simple.*
(2) $\mathcal{S}^-\mathcal{S}^+(M) \cong M$.
(3) $\mathbf{dim}\,\mathcal{S}_i^\pm M = s_i(\mathbf{dim}\,M)$.

The proof of Gabriel's theorem now hinges on showing that the map $\mathbf{dim}\colon M \mapsto \mathbf{dim}\,M$ is a bijection between the indecomposable representations of Q and Φ^+. For the details of this proof we direct the reader to [**1**]. We present two important corollaries to this proof.

C.28. Corollary. *For any indecomposable module M there exists integers $t \geq 0$ and i with $0 \leq i \leq n-1$ (depending only on the vector $\mathbf{dim}\,M$) such that*

$$M \cong C^{-t}\mathcal{S}_1^- \cdots \mathcal{S}_i^- S(i+1).$$

Where $C^{-t} = (\mathcal{S}_1^- \cdots \mathcal{S}_n^-)^t$.

C.29. Corollary. *If $S(i)$ is the simple module at i for $\sigma_i \cdots \sigma_n Q$, then*

$$P(i) \cong \mathcal{S}_1^- \cdots \mathcal{S}_{i-1}^- S(i),$$

where $P(i)$ is the projective module associated to vertex i.

Using these corollaries we see how the simple roots generate the indecomposable modules of Q via reflection functors. This gives us a computational method of generating all the indecomposable representations starting with only the simple representations.

C.30. Example. Let Q be the quiver $2 \longrightarrow 1$. This is of type A_2, so the indecomposable representations are in bijection with the positive roots of $\mathfrak{sl}_3$. The simple representations are

$$S(1) : 0 \longrightarrow K,$$
$$S(2) : K \longrightarrow 0.$$

Applying Corollary C.29, we have $P(1) = S(1)$ in $\sigma_1\sigma_2 Q = Q$, hence $P(1) = S(1)$. Similarly $P(2) = \mathcal{S}_1^- S(2)$ in $\sigma_2 Q$. That is $P(2)$ is the reflection at vertex 1 of

$$K \longleftarrow 0.$$

The definition of the reflection tells us that the vector space at vertex 2 will remain K. At vertex 1 we place the vector space such that the sequence

$$0 \longrightarrow K \longrightarrow (\mathcal{S}^- S(2))_1 \longrightarrow 0,$$

is exact. Hence $(\mathcal{S}^- S(2))_1 = K$. So $P(2)$ is given by $K \longrightarrow K$. One can easily see that this exhausts all the possibilities. Hence the positive roots of A_2 are $\alpha_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $\alpha_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$,$\alpha_1 + \alpha_2 = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$. This can be verified in [**?**].

We provide one last example which demonstrates that vector spaces with dimension greater than 2 can occur.

C.31. Example. Consider the quiver of type D_4 given by

$$\begin{tikzcd}
 & & 3 \arrow[dl] \\
2 \arrow[r] & 1 & \\
 & & 4 \arrow[ul]
\end{tikzcd}$$

Here are the steps to construct $\mathcal{S}_1^-\mathcal{S}_2^-\mathcal{S}_3^-\mathcal{S}_4^-(S(1))$.

$$\begin{tikzcd}
 & & & 0 \arrow[dl] \\
\mathcal{S}_4^-(S(1)): & 0 \arrow[r] & K \arrow[dr] & \\
 & & & K
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & K \\
\mathcal{S}_3^-\mathcal{S}_4^-(S(1)): & 0 \arrow[r] & K \arrow[ur] \arrow[dr] & \\
 & & & K
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & K \\
\mathcal{S}_2^-\mathcal{S}_3^-\mathcal{S}_4^-(S(1)): & K & K \arrow[l] \arrow[ur] \arrow[dr] & \\
 & & & K
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & K \arrow[dl] \\
\mathcal{S}_1^-\mathcal{S}_2^-\mathcal{S}_3^-\mathcal{S}_4^-(S(1)): & K \arrow[r] & K^2 & \\
 & & & K \arrow[ul]
\end{tikzcd}$$

From the theorem and corollaries we know that this representation gives one of the positive roots of D_4. We also know that the representation is indecomposable because it is the reflection of a simple representation.

4. Decomposing tensor products of irreducible representations using crystals

BY BEN SALISBURY

Throughout, let $\mathfrak{g} = \mathfrak{sl}_n$ with weight lattice $P \cong \mathbf{Z}^n$. Let λ and μ be dominant integral weights. By Weyl's theorem, every representation of $\mathfrak{sl}_n$ has a decomposition into a direct sum of irreducible factors. Hence, the tensor product $V(\lambda) \otimes V(\mu)$ has a decomposition into irreducible factors. But how? In general, this problem is known as the Clebsch-Gordan problem. One solution is to use a result by Nakashima [**12**], which uses crystals and Young tableaux.

Therefore, to apply Nakashima's result, we must first define Young tableaux, and then define the $\mathfrak{sl}_n$-highest weight crystal, which consists of maps between Young tableaux corresponding to the action of $\mathfrak{sl}_n$ the highest weight module.

C.32. DEFINITION. A Young diagram is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row. A semistandard Young tableau is a Young diagram filled with numbers from $\{1, \ldots, n\}$ such that entries in each row are weakly increasing and entries in each column are strictly increasing.

To each dominant integral weight in P, we can associate a Young diagram. The correspondence is given by associating a dominant integral weight λ to the Young diagram with shape λ. In particular, if $\lambda = a_1\Lambda_1 + \cdots a_{n-1}\Lambda_{n-1}$, then the partition associated to λ is the partition with a_j columns of height j, juxtaposed left to right in columns of weakly decreasing height.

C.33. EXAMPLE. For $n \geqslant 3$, let $\lambda = \Lambda_1 + \Lambda_2$ and $\mu = \Lambda_2$. Then the corresponding Young diagrams are $\ydiagram{2,1}$ and $\ydiagram{1,1}$, which we will denote $Y(\lambda)$ and $Y(\mu)$, respectively.

C.34. DEFINITION. Let $\lambda \in P^+$. A highest weight $\mathfrak{sl}_n$-crystal corresponding to the highest weight representation $V(\lambda)$, denoted $B(\lambda)$, is the set of semistandard tableaux with shape λ, together with maps

$$\widetilde{e}_i, \widetilde{f}_i \colon B(\lambda) \longrightarrow B(\lambda) \cup \{0\}$$
$$\mathrm{wt} \colon B(\lambda) \longrightarrow \mathbf{Z}^n,$$

subject to certain conditions. The most important of these conditions is if $\widetilde{f}_i b = b'$, then $\widetilde{e}_i b' = b$, where $b, b' \in B(\lambda)$. Thus we can attach an i-colored edge connecting b and b', which will create a graph structure on $B(\lambda)$.

We will now work to define the operators $\widetilde{e}_i$ and $\widetilde{f}_i$, which reflects the action of $\mathfrak{sl}_n$ on $V(\lambda)$. The operators are intimately related to the fundamental representation of $\mathfrak{sl}_n$, so we will take the crystal of the fundamental representation as definition.

C.35. Definition. The crystal of the fundamental representation of $\mathfrak{sl}_n$ is given by

$$B(\Lambda_1) : \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n}.$$

Here, the i-colored arrows represent the action of $\widetilde{f}_i$ on an element in the graph. In particular,

$$\widetilde{f}_i \boxed{j} = \begin{cases} \boxed{j+1} & \text{if } i = j \in \{1, \dots, n-1\} \\ 0 & \text{otherwise.} \end{cases}$$

For $i = 1, \dots, n$, define maps $\varepsilon_i, \varphi_i \colon B(\Lambda_1) \longrightarrow \mathbf{Z}$ by

$$\varepsilon_i(\boxed{j}) = \max\{k \geqslant 0 : \widetilde{e}_i^k \boxed{j} \in B(\Lambda_1)\}$$
$$\varphi_i(\boxed{j}) = \max\{k \geqslant 0 : \widetilde{f}_i^k \boxed{j} \in B(\Lambda_1)\}.$$

Now the operations on the crystal are carried out by embedding the tableaux, which has say N boxes, into the tensor product $B(\Lambda_1)^{\otimes N}$, and using the so-called tensor product rule to calculate the action of $\widetilde{f}_i$.

C.36. Example. Let $n = 3$ and consider $b = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} \in B(\Lambda_1 + \Lambda_2)$. Hence $N = 3$. To embed b into $B(\Lambda_1)^{\otimes 3}$, we peel off columns of b from right to left and then take off boxes from each column starting at the top. In our case,

$$b = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} = \boxed{1} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} = \boxed{1} \otimes \boxed{1} \otimes \boxed{2} \in B(\Lambda_1)^{\otimes 3}.$$

Now using the crystal graph $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$, we compute ε_1, φ_1, ε_2, and φ_2 on each element of the tensor product.

	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$
ε_1	0	0	1
φ_1	1	1	0
ε_2	1	1	0
φ_2	0	0	1

Then we write an i-signature, which is a sequence of $-$'s and $+$'s using the rule

$$i\text{-sgn}(b) = (\underbrace{-\cdots-}_{\varepsilon_i(b_1)}, \underbrace{+\cdots+}_{\varphi_i(b_1)}, \ldots\ldots, \underbrace{-\cdots-}_{\varepsilon_i(b_r)}, \underbrace{+\cdots+}_{\varphi_i(b_r)}).$$

Then we delete any $(+-)$-pairs in the signature, to create a sequence of $-$'s followed by $+$'s. Then $\widetilde{f}_i$ will act on the component in b corresponding to the left-most $+$ in the signature. In our case, we have

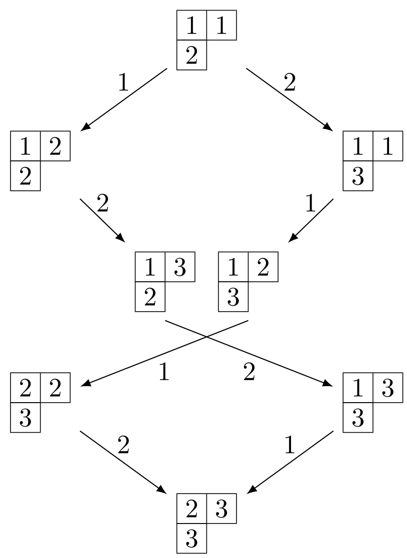
$$1\text{-sgn}(b) = (+,+,-) = (+,\cdot,\cdot), \qquad 2\text{-sgn}(b) = (\cdot,\cdot,+)$$

Hence

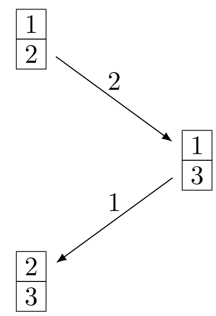
$$\widetilde{f}_1 b = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \qquad \widetilde{f}_2 b = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

We can then continue this calculation on all resulting tableaux to obtain a graph. In type A_{n-1}, the highest weight vector in $B(\lambda)$ is given by the filling the first row with ith row with i's. For example, $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$ is the highest weight vector of $B(\Lambda_1 + \Lambda_2)$.

C.37. Example. Let $\mathfrak{g} = \mathfrak{sl}_3$. Then $B(\Lambda_1 + \Lambda_2)$ and $B(\Lambda_2)$ are given by



$B(\Lambda_1 + \Lambda_2)$



$B(\Lambda_2)$

One of the main motivations behind the study of crystals is their remarkably nice behavior with respect to tensor products. In parricular, Nakashima [**12**] proved the following decomposition tensor products of highest weight representations.

C.38. Theorem ([**12**, Thm. 6.3.1]). *Let $Y(\lambda) \leftarrow i$ be the diagram obtained from $Y(\lambda)$ by adding one block to the ith row. Then for $\lambda, \mu \in P^+$,*

$$V(\lambda) \otimes V(\mu) \cong \bigoplus_{\boxed{b_1}\otimes\cdots\boxed{b_N}\in B(\mu)} V((\cdots(Y(\mu) \leftarrow b_1) \leftarrow \cdots) \leftarrow b_N),$$

such that $Y(\lambda) \leftarrow b_1$, $(Y(\lambda) \leftarrow b_1) \leftarrow b_2, \ldots$ are all Young diagrams.

C.39. Example. Let $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = \Lambda_2$ and $\mu = \Lambda_1 + \Lambda_2$. Then

$$B(\mu) = \left\{ \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array} \right\}.$$

Since λ corresponds to $\ydiagram{1,1}$, applying the theorem gives

$$\ydiagram{1,1} \leftarrow 1 \leftarrow 1 \leftarrow 2 = \ydiagram{2,1} \leftarrow 1 \leftarrow 2 = \ydiagram{3,1} \leftarrow 2 = \ydiagram{3,2}$$

$$\ydiagram{1,1} \leftarrow 2 \leftarrow 1 \leftarrow 2 = \ydiagram{1,2} \leftarrow 1 \leftarrow 2 = \varnothing$$

$$\ydiagram{1,1} \leftarrow 3 \leftarrow 1 \leftarrow 2 = \ydiagram{1,1,1} \leftarrow 1 \leftarrow 2 = \ydiagram{2,1,1} \leftarrow 2 = \ydiagram{2,2,1}$$

$$\ydiagram{1,1} \leftarrow 1 \leftarrow 1 \leftarrow 3 = \ydiagram{2,1} \leftarrow 1 \leftarrow 3 = \ydiagram{3,1} \leftarrow 3 = \ydiagram{3,1,1}$$

$$\ydiagram{1,1} \leftarrow 2 \leftarrow 1 \leftarrow 3 = \ydiagram{1,2} \leftarrow 1 \leftarrow 3 = \varnothing$$

$$\ydiagram{1,1} \leftarrow 3 \leftarrow 1 \leftarrow 3 = \ydiagram{1,1,1} \leftarrow 1 \leftarrow 3 = \ydiagram{2,1,1} \leftarrow 3 = \ydiagram{2,2,2} = \varnothing$$

$$\ydiagram{1,1} \leftarrow 2 \leftarrow 2 \leftarrow 3 = \ydiagram{1,2} \leftarrow 2 \leftarrow 3 = \varnothing$$

$$\ydiagram{1,1} \leftarrow 3 \leftarrow 2 \leftarrow 3 = \ydiagram{1,1,1} \leftarrow 2 \leftarrow 3 = \ydiagram{1,2,1} \leftarrow 3 = \varnothing.$$

Thus

$$B(\lambda) \otimes B(\mu) \cong B\left(\ydiagram{3,2}\right) \oplus B\left(\ydiagram{2,2,1}\right) \oplus B\left(\ydiagram{3,1,1}\right).$$

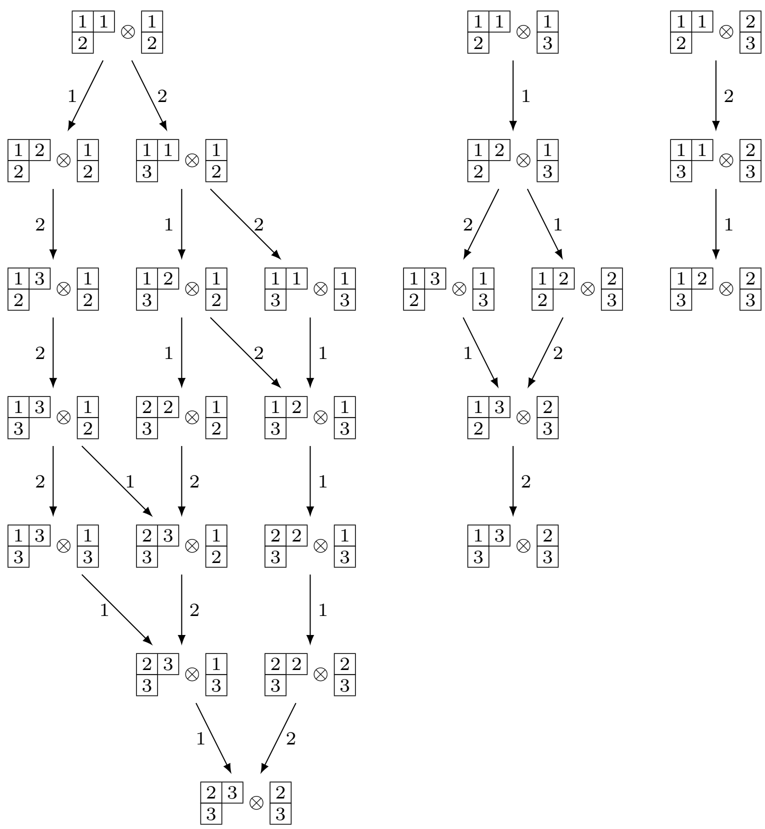
We are in type A_2, so there is no Λ_3. In this case, we view $\Lambda_3 = 0$, so we get the decomposition

$$B(\lambda) \otimes B(\mu) \cong B\left(\ydiagram{3,2}\right) \oplus B\left(\ydiagram{1,1}\right) \oplus B\left(\ydiagram{2}\right).$$

In terms of representations, we have

$$V(\Lambda_2) \otimes V(\Lambda_1 + \Lambda_2) \cong V(\Lambda_1 + 2\Lambda_2) \oplus V(\Lambda_2) \oplus V(2\Lambda_1).$$

As a check, we draw the crystal graph for this tensor product. Note that each summand represents a connected component of the graph.



For more information about crystals in general, see [**7**] or [**11**]. For more information about this tensor product decomposition, see [**12**].

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