# ~~Linear~~ stability conditions & Auslander-Reiten sequences

↳ Joint work with:
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## I. Overview

Stability functions appear across several areas of math:

- representation theory
- algebraic geometry
- cluster theory
- mathematical physics

They began appearing in 1990s in work of Schofield, King, and Rudakov. Later for triangulated categories by Bridgeland.

We will consider today stability functions which make every indecomp. object stable. Previously studied in papers such as:

- Reineke 2003
- Qiu, 2015
- Apruzzese-Igusa, 2019
- Huang-Hu, 2020 (preprint)
- Barnard-Gunawan-Meehan-Schiffler, 2020

## II. Stability Functions

Let $Q$ be a quiver & $\text{rep}^*(Q)$ be the set of isoclasses of nonzero representations of $Q$.

**Def**: A function $\phi : \text{rep}^*(Q) \to \mathbb{R}$ is a stability function if it satisfies the see-saw property, meaning for any short exact sequence (nonzero objects)

$$0 \to A \to B \to C \to 0$$

one of the following occurs:

$\phi(A) < \phi(B) < \phi(C)$ OR $\phi(A) > \phi(B) > \phi(C)$
OR $\phi(A) = \phi(B) = \phi(C)$.

- $V \in \text{rep}^*(Q)$ is $\phi$-stable if $\phi(W) < \phi(V)$ for all $0 < W < V$.

- A stability function is totally stable if every indecomposable representation of $Q$ is $\phi$-stable.

Remark: A $\phi$-stable representation $V$ is indecomposable & $\text{End}_Q(V)$ a division ring.

Proof: from $f \in \text{End}_Q(V)$, get s.e.s.

$$0 \to \ker f \to V \to \operatorname{im} f \to 0.$$

So if $f$ not an isomorphism, by the see-saw property we get $\phi(\ker f) \geq \phi(V)$ or $\phi(\operatorname{im} f) \geq \phi(V)$. So $V$ would not be $\phi$-stable.

So to study totally stable stability

functions, we must restrict to Q Dynkin

## III. Results on Dynkin quivers

Our main result is the following:

**Theorem** (joint work in progress with Yariana Diaz & Cody Gilbert).
Let $\phi$ be a stability function on a Dynkin quiver Q.
Then $\phi$ is totally stable $\Longleftrightarrow$

$\phi(\tau V) < \phi(V)$ for all indecomposable, non-projective $V$ such that the associated Auslander-Reiten sequence

$$(\star) \qquad 0 \to \tau V \to E \to V \to 0$$

has $E$ indecomposable.

**Proof ingredients**

① Elementary considerations with see-saw property give that $V$ is $\phi$-stable

$\Longleftrightarrow$ $\phi(W) < \phi(V)$ for all $0 < W < V$, $W$ indecomp.
$\Longleftrightarrow$ $\phi(V) < \phi(Y)$ for all $V \twoheadrightarrow Y$, $Y$ indecomp.

(2) We can further reduce the problem:
$\phi$ is totally stable
$\Longleftrightarrow$
$\phi(X) < \phi(Y)$ for every irreducible morphism $f: X \to Y$ between indecomposables.

(3) We finish by reducing to short exact sequences (☆) as in the theorem statement by a variety of ad hoc methods with Auslander-Reiten quivers.

IV. Additional results in type A.
[K, 2020 preprint: Total stability functions on type A quivers, arXiv: 2002.12396]

The main result above was proven for type A quivers here.

We focused there on
linear stability conditions:

Let $Z : K_0(Q) \longrightarrow \mathbb{C}$ be an additive group homomorphism, with $Z([S])$ has positive imaginary part for $S$ simple.

$\hookrightarrow \exists\, w \in \mathbb{R}^{Q_0}$, $r \in (\mathbb{R}_{>0})^{Q_0}$ s.t.

$$Z(x) = w \cdot x + (r \cdot x)\, i \quad \text{for } x \in K_0(Q).$$

The associated (linear) stability function is
$$\mu_Z(x) = \frac{w \cdot x}{r \cdot x}, \quad x \in K_0(Q).$$

Additional result in type $A$:
the collection of inequalities in the main theorem are a minimal collection defining the set of linear total stability conditions within the set of all linear stability conditions.

Open Problem (?): Let $Q$ be a Dynkin quiver. Construct a linear total stability condition for $Q$.

↳ Solved in type $A$ in papers mentioned in overview [AI], (HH), [BGMS]. For all other Dynkin types [Qiu] did this for a specific orientation.

In [K, 2020 preprint], I explicitly describe all such linear totall stability conditions in type $A$.

Given any $r \in (\mathbb{R}_{>0})^{Q_0}$, the space of $w \in \mathbb{R}^{Q_0}$ which give totally stable $\mu_z$ is linearly equiv. to

$$\mathbb{R} \times (\mathbb{R}_{>0})^{Q_1}.$$