New inequalities for subspace arrangements

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Outline

Subspace arrangements

Basic inequalities

Beyond basic inequalities

Concluding remarks

Subspace arrangements

Notation/Definition. A subspace arrangement is:

- a finite-dimensional vector space V over a field K,
- with a collection of subspaces $V_1, V_2, \ldots, V_n \subseteq V$.

(through the whole talk)

Examples. $V = \mathbb{R}^3$, with standard basis $\{e_1, e_2, e_3\}$,
(a) $V_1 = \langle e_1 \rangle,\ V_2 = \langle e_2 \rangle,\ V_3 = \langle e_3 \rangle$
(b) $V_1 = \langle e_1 \rangle,\ V_2 = \langle e_2 \rangle,\ V_3 = \langle e_1 + e_2 \rangle$

What is different between (a) and (b)?

Rank functions

Notation/Definition. A rank function is any function

$$r\colon \mathrm{Subsets}(\{1,2,\ldots,n\}) \to \mathbb{Z}.$$

A subspace arrangement determines a rank function:
for $I \subseteq \{1,2,\ldots,n\}$, define

$$V_I = \sum_{i\in I} V_i, \qquad r(I) = \dim V_I.$$

Examples continued. ($V = \mathbb{R}^3$, with standard basis $\{e_1, e_2, e_3\}$)
(a) $V_1 = \langle e_1 \rangle,\ V_2 = \langle e_2 \rangle,\ V_3 = \langle e_3 \rangle$

$$\begin{aligned} r(1) = r(2) = r(3) &= 1, \\ r(12) = r(13) = r(23) &= 2, \\ r(123) &= 3 \end{aligned}$$

(Notice $\{,\}$ are omitted to make notation clearer.)

(b) $V_1 = \langle e_1 \rangle,\ V_2 = \langle e_2 \rangle,\ V_3 = \langle e_1 + e_2 \rangle$
The rank function is the same as (a) EXCEPT $r(123) = 2$.

Observations

- Many different arrangements give the same rank function.
- Some rank functions cannot come from a subspace arrangement.
  For example:

  $$r(1) = 1, \quad r(2) = 1, \quad r(12) = 3$$

  is not possible!

Basic inequalities

A rank function from a subspace arrangement always satisfies the 3 basic inequalities:

- *(Non-negative)* $r(I) \geq 0$ for all I,
- *(Increasing)* $r(I) \leq r(J)$ when $I \subseteq J$,
- *(Convex)* $r(I \cap J) + r(I \cup J) \leq r(I) + r(J)$ for all I, J (follows from "sum-intersection formula".)

For example, if $I = \{1, 2\}$ and $J = \{2, 3\}$, then

$$\begin{aligned}\dim(V_{12} + V_{23}) &= \dim V_{12} + \dim V_{23} - \dim(V_{12} \cap V_{23}) \\ &\leq \dim V_{12} + \dim V_{23} - \dim V_2.\end{aligned}$$

Converse?

Question: If r is a rank function satisfying the 3 basic inequalities, does it come from a subspace arrangement?

–*No!*

Definition. A rank function which comes from a subspace arrangement is called (linearly) representable.

Remarks

- A rank function satisfying the 3 basic inequalities is called a "polymatroid". The concept was introduced by H. Whitney (1935). "Matroid theory" is a major branch of combinatorics.
- There exist rank functions which are only realizable by subspace arrangements over fields of certain characteristics. *For a rank function to be representable, we just require the existence of a subspace arrangement over one field.*
- Everything can be translated into the language of projective geometry (S. Maclane, 1936). Classical examples/theorems of projective geometry give interesting rank functions and subspace arrangements.

Beyond basic inequalities

A long-standing open problem is to determine which rank functions are linearly representable.

~~**Conjecture.**~~ **Theorem.** [Mayhew, Newman, Whittle 2014]
There is no finite set of axioms that can characterize when a rank function is linearly representable.

We can try to give partial answers, like many necessary conditions. For example, we already know a rank function must satisfy the basic inequalities.

<u>Question:</u> Are there more inequalities always satisfied by the rank function of a subspace arrangement?

–*Yes!*

Theorem. [Ingleton, 1969] For any arrangement of 4 subspaces, the rank function satisfies

$$r(3) + r(4) + r(12) + r(134) + r(234) \\ \leq r(13) + r(14) + r(23) + r(24) + r(34).$$

Furthermore, this inequality is not a sum of basic inequalities.

In 2000, Hammer, Romashchenko, Shen, and Vereshchagin proved that the basic inequalities and Ingleton inequality are a complete list for the case of 4 subspaces.

Ingleton asked whether there are more inequalities that are always satisfied. You can probably guess the answer from the title of the presentation...

–*Yes! In fact, infinitely many more.*

Theorem. [K, 2009] For any arrangement of n subspaces, the rank function satisfies

$$r(12) + r(13n) + r(3) + \sum_{i=4}^{n} r(i) + r(2, i-1, i)$$
$$\leq r(13) + r(1n) + r(23) + \sum_{i=4}^{n} r(2, i) + r(i-1, i).$$

Furthermore, for each n, the inequality is not a sum of inequalities that hold for fewer than n subspaces.

Are there still more?
–*Yes! Many, many more were discovered by Dougherty, Freiling, and Zeger (2010).*

DFZ prove:

- an explicit list of 24 new inequalities in the case of 5 subspaces; the list is complete.
- 3490 new inequalities for 6 subspaces; the list was not complete.
- an infinite list of new inequalities for higher numbers of subspaces.

Connections to other fields

(1) Entropies of a collection of jointly distributed random variables gives a rank function, which satisfies the basic inequalities, but does not necessarily satisfies other subspace arrangement inequalities.
It is very difficult to determine which rank functions can come from entropy of random variables. The subspace arrangement problem gives an "inner bound".

(2) A collection of subgroups of a finite group gives a rank function. These rank functions are closely related to rank functions from entropy of random variables, giving constraints on subgroup lattices of finite groups.

Open question

The following fundamental question is open, as far as I know.

Question. [DFZ] For fixed n, are there always *finitely many* inequalities which can be used to derive all others?

In other words, is the cone cut out by subspace arrangement inequalities *polyhedral*?