SOLVING ORIENTED TANGLE EQUATIONS INVOLVING 4-PLATS

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Revised

ABSTRACT
The system of oriented tangle equations $N(U + \frac{f_1}{g_1}) = K_1$ and $N(U + \frac{f_2}{g_2}) = K_2$ is completely solved for the tangles U and $\frac{f_2}{g_2}$ as a function of $\frac{f_1}{g_1}$ where K_1 and K_2 are 4-plats, and $\frac{f_1}{g_1}$ and $\frac{f_2}{g_2}$ rational tangles such that $|f_1g_2 - g_1f_2| > 1$. As an application, it is completely determined when one 4-plat can be obtained from another 4-plat via a signed crossing change.

1. Introduction

In [5], the system of unoriented tangle equations $N(U + \frac{f_1}{g_1}) = K_1$ and $N(U + \frac{f_2}{g_2}) = K_2$ was completely solved for the tangles U and $\frac{f_2}{g_2}$ as a function of $\frac{f_1}{g_1}$ where K_1 and K_2 are 4-plats, $\frac{f_1}{g_1}$ and $\frac{f_2}{g_2}$ rational tangles, and U is ambient isotopic to a sum of rational tangles. In many biological applications, determining orientation is also very important [3, 1, 11].

In section 2 a brief introduction to oriented tangles is given. Biological motivation and examples are given in section 3. The system of oriented tangle equations $N(U+P) = K_1$ and $N(U+R) = K_2$ is solved in section 4 where P is the zero tangle, R is rational, and K_1 and K_2 are 4-plats and U is ambient isotopic to a sum of rational tangles. These findings are then extended to solve the oriented equations, $N(U + P) = K_1$ and $N(U + R) = K_2$, in section 5 where P and R are arbitrary rational tangles. An application to signed crossing changes is given in section 6.

The main theorems are broken into several cases and are thus rather long. However, since it is necessary to implement these theorems for biological applications, a program which performs these calculations is available at the following URL:

http://www.math.uiowa.edu/~idarcy/PROG/comput.html

This subroutine will also be included in Rob Scharein's KnotPlot which is available at www.KnotPlot.com.

2. Oriented Tangles

A 2-string tangle is a pair (B^3, t) where B^3 is a 3-dimensional ball, $\{x \in \mathbf{R^3} :$

$|x| \leq 1\}$, and t is a pair of arcs and a finite number of circles properly embedded in B^3. Tangles can be added (Fig. 1). A knot or link is formed by taking the numerator closure of a tangle (Fig. 2). The circle product of A and $C = (c_1, ..., c_n)$, is shown in Fig. 3 when n is even and in Fig. 4 when n is odd. A tangle is rational if it is ambient isotopic to the zero tangle where the boundary of B^3 need not be fixed. A generalized Montesinos tangle or generalized M-tangle is a tangle which is ambient isotopic to a sum of rational tangles where the boundary of B^3 need not be fixed. Two tangles are equivalent if they are ambient isotopic keeping the boundary of B^3 fixed.

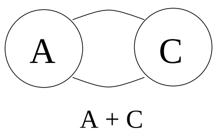

Fig. 1. Adding tangles

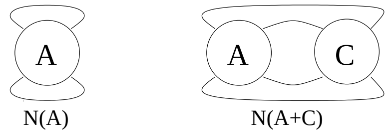

Fig. 2. Numerator closure

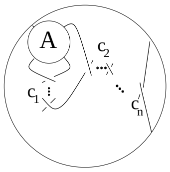
Fig. 3. $A \circ (c_1, ..., c_n)$, n even

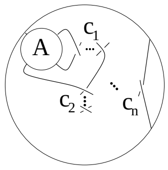
Fig. 4. $A \circ (c_1, ..., c_n)$, n odd

The four endpoints of the arcs will be fixed at $NW = (e^{5i\pi/4}, 0), NE = (e^{i\pi/4}, 0), SW = (e^{-5i\pi/4}, 0), SE = (e^{-i\pi/4}, 0)$. If one of the arcs has endpoints NW and NE, then the tangle is said to have parity zero. A parity one tangle has an arc with endpoints NW and SE, whereas a parity ∞ tangle has an arc with endpoints NW and SW. Tangles can be oriented by orienting each arc and circle. An arrow pointing from $t_i(0)$ to $t_i(1)$ for each arc $t_i : [0,1] \to B^3$ will be used to indicate the orientation (Fig. 5). The orientation of circles will be similarly depicted. A parity zero or a parity one tangle will be called similarly oriented if $t_1^{-1}(NW) = t_2^{-1}(SW)$. Otherwise a parity zero or one tangle will be called oppositely oriented. A parity ∞ tangle will be called similarly oriented if $t_1^{-1}(NE) = t_2^{-1}(NW)$. Otherwise a parity ∞ tangle will be called oppositely oriented. Although four different orientations can be defined, since the knots and links discussed in this paper will be restricted to 4-plats and 4-plats are reversible (equivalent to itself with orientation reversed), these orientations will be grouped into two classes, similarly and oppositely oriented as shown in Fig. 5.

A 4-plat (or 2-bridge or rational knot/link) is a knot or link which can be written as the numerator closure of a rational tangle. Recall two unoriented 4-plats $N(a_1/b_1)$ and $N(a_2/b_2)$, $a_i \geq 0$, are the same if and only if $a_1 = a_2$ and $b_1 b_2^{\pm 1} \cong 1$ (mod a_1) [2]. Since 4-plats are reversible, two oriented 4-plat knots are equivalent if and only if they are equivalent as unoriented knots. $N(a/b)$ is a link if and only if a

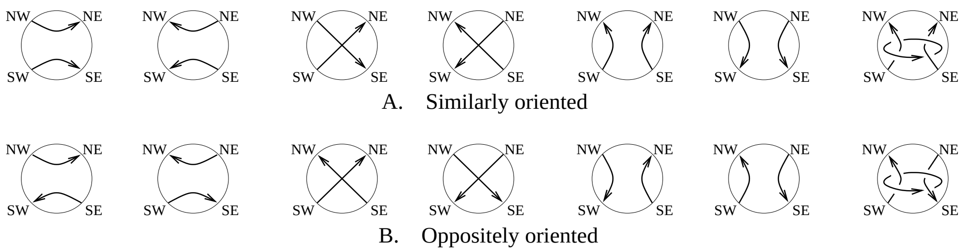

Fig. 5. Orienting tangles A.) Similarly oriented. B.) Oppositely oriented

is even in which case the rational tangle, $\frac{a}{b}$ has parity zero. The oriented 4-plat link, $N(a/b)$, will be oriented so that the tangle a/b is similarly oriented. Equivalence between oriented 4-plat links is determined as follows:

Lemma 1 ([2]). *Two oriented 4-plat links $N(a_1/b_1)$ and $N(a_2/b_2)$, $a_i \geq 0$, are equivalent if and only if $a_1 = a_2$ and $b_1 b_2^{\pm 1} \cong 1$ (mod $2a_1$).*

Note that this means that the numerator closure of the tangle a/b where a/b is oppositely oriented is the oriented 4-plat $N(\frac{a}{b+a})$ (where the tangle $\frac{a}{b+a}$ is similarly oriented by definition).

Since knots/links are now oriented, from now on, two knots/links will be said to be equivalent if and only if they are equivalent as oriented knots/links. In particular, $N(A+C) = K$ if and only if A and C are oriented so that $N(A+C) = K$ as oriented knots/links.

If there exists a solution for U such that $N(U+P) = K_1$ and $N(U+R) = K_2$, then K_2 is said to have been obtained from K_1 by a (P, R) move. Oriented moves will be divided into two types. A (P, R) move will be said to be orientation preserving if the knots/links, $N(U+P) = K_1$ and $N(U+R) = K_2$ are oriented and U has the same orientation in both equations. A (P, R) move will be said to be orientation ignoring if it is not possible to orient K_1 and K_2 so that U has the same orientation in both equations (see for example Fig. 6). In both cases, if the tangle U contains any circles, these circles must have the same orientation in $N(U+P) = K_1$ and $N(U+R) = K_2$.

Definition 1. The system of oriented equations $N(U+P) = N(\frac{a}{b})$ and $N(U+R) = N(\frac{z}{v})$ has a solution if and only if there exists tangles U, P, and R such that

1.) $N(U+P) = N(\frac{a}{b})$ as oriented 4-plats and $N(U+R) = N(\frac{z}{v})$ as oriented 4-plats.

2.) If the tangle U contains any circles, these circles must have the same orientation in $N(U+P) = K_1$ and $N(U+R) = K_2$

3.) One of the following holds:

a.) the (P, R) move is orientation ignoring or

b.) the (P, R) move is orientation preserving (i.e., U has the same orientation in both equations).

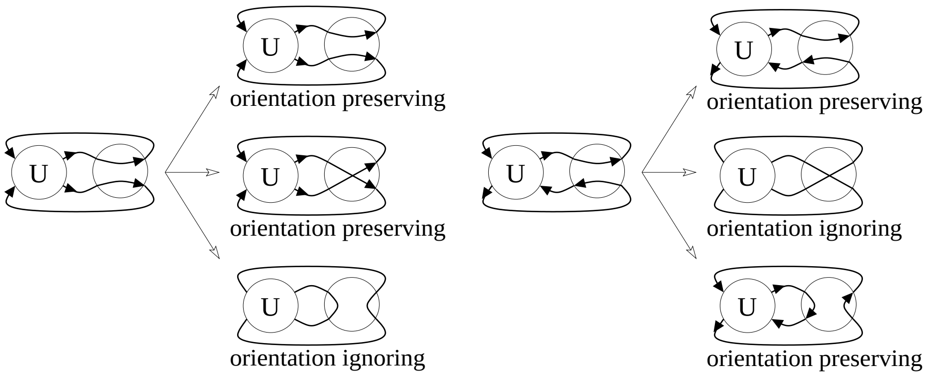

Fig. 6. Orientation ignoring versus orientation preserving

Lemma 2. *If $N(U+\frac{0}{1}) = N(\frac{a}{b})$ and $N(U+\frac{t}{w}) = N(\frac{z}{v})$ as oriented 4-plats and if the tangle $\frac{0}{1}$ is similarly oriented and the tangle $\frac{t}{w}$ has parity ∞, then the $(\frac{0}{1}, \frac{t}{w})$ move is orientation ignoring. If the tangle $\frac{0}{1}$ is oppositely oriented and the tangle $\frac{t}{w}$ has parity one, then the $(\frac{0}{1}, \frac{t}{w})$ move is orientation ignoring. Except for these two cases, a $(\frac{0}{1}, \frac{t}{w})$ is orientation preserving.*

Proof. See Fig. 6 □.

Lemma 3. *If U has parity ∞, then the tangle $\frac{0}{1}$ in $N(U+\frac{0}{1})$ is oppositely oriented. If U has parity one, then the tangle $\frac{0}{1}$ in $N(U+\frac{0}{1})$ is similarly oriented. If U has parity zero, then the tangle $\frac{0}{1}$ could be given either orientation.*

Proof. See Fig. 7 □.

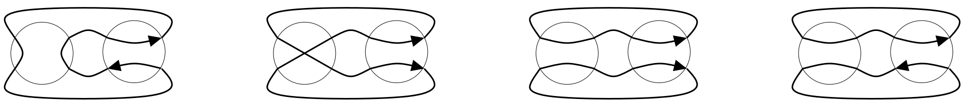

Fig. 7. Parity of U may determine the orientation of the tangle $\frac{0}{1}$ in $N(U+\frac{0}{1})$

In order to determine the parity of U, the Euler bracket function will be used. Let $E[x_1, ..., x_n]$ be the Euler bracket function which equals the sum of products of the x_i's where zero or more disjoint pairs of consecutive x_i's are omitted [12]. If $n = 0$ then $E[x_1, ..., x_n] = E[] = 1$. If $n < 0$ define $E[x_1, ..., x_n] = 0$. Let $[x_n, ..., x_1]$ denote the continued fraction $x_n + \frac{1}{x_{n-1} + \frac{1}{\dots + \frac{1}{x_1}}}$, the fraction corresponding to the tangle $(x_1, ..., x_n)$. The following useful facts for $n \geq 1$ can be found in [12]:

1.) $E[x_1, ..., x_n] = x_1 E[x_2, ..., x_n] + E[x_3, ..., x_n]$.
2.) $[x_n, ..., x_1] = E[x_1, ..., x_n]/E[x_1, ..., x_{n-1}]$.
3.) Let $a = E[x_1, ..., x_n], b = E[x_1, ..., x_{n-1}]$. If $y = (-1)^{n+1} E[x_2, ..., x_{n-1}]$ and $x = (-1)^{n+1} E[x_2, ..., x_n]$, then $bx - ay = 1$.

The following well-known result can be proved using the Euler bracket function.

Lemma 4. *If a is even, then the tangle $\frac{a}{b}$ has parity 0. If a is odd, then if b is odd, the tangle $\frac{a}{b}$ has parity 1 and if b is even, the tangle $\frac{a}{b}$ has parity ∞.*

Proof. Induction on n noting that if $\frac{a}{b} = [x_n, ..., x_1]$, then $a = E[x_1, ..., x_n]$ and $b = E[x_1, ..., x_{n-1}]$ □.

An oriented link is said to be strongly reversible if changing the orientation of any of the components of the link does not change the oriented link type. Since 4-plat knots are reversible, they are strongly reversible. A 4-plat link, $N(\frac{a}{b})$, is strongly reversible if and only if $N(\frac{a}{b}) = N(\frac{a}{b+a})$. Hence the link $N(\frac{a}{b})$ is strongly reversible if and only if $b^2 = 1 + a \bmod 2a$.

In the unoriented case, $N(\frac{j}{p} + \frac{t}{w}) = N(\frac{jw+pt}{dw+qt})$ [7]. This result also holds in the oriented case if $N(\frac{jw+pt}{dw+qt})$ is strongly reversible. Thus, $N(\frac{j}{p} + \frac{t}{w}) = N(\frac{jw+pt}{dw+qt})$ no matter how the tangle $\frac{j}{p}$ is oriented when $N(\frac{jw+pt}{dw+qt})$ is strongly reversible. However, if $N(\frac{j}{p} + \frac{t}{w})$ is a link (i.e. $jw + pt$ is even) which is not strongly reversible, then orientation adds new considerations as illustrated in lemma 9 and the example following lemma 9. The following four lemmas are useful for calculations and are also used to prove lemma 9.

Lemma 5. *$N(A + (c_1, ..., c_n)) = N(A \circ (c_n, ..., c_1))$, for n odd.*

Proof. Induction on n noting that a rational tangle is invariant under a rotation of 180° about the y-axis [8] □.

Lemma 6. *$(c_1, ..., c_n) \circ (d_1, ..., d_m) = (c_1, ..., c_n + d_1, ..., d_m)$, when m is odd.*

Note that even if $(d_1, ..., d_m) = (e_1, ..., e_k)$, it is possible that $\frac{a}{b} \circ (d_1, ..., d_m) \neq \frac{a}{b} \circ (e_1, ..., e_k)$. For example, $(1) = (0, 1, 1)$, but $(3) = (2) \circ (1) \neq (2) \circ (0, 1, 1) = (2, 1, 1)$.

Lemma 7. $[d_1, ..., d_m + c_n, ..., c_1] = \frac{E[d_1,...,d_m]E[c_1,...c_{n-1}]+E[d_1,...,d_{m-1}]E[c_1,...c_n]}{E[d_2,...,d_m]E[c_1,...c_{n-1}]+E[d_2,...,d_{m-1}]E[c_1,...c_n]}$.

Proof. Induction on m. See Roberts [12] □.

Lemma 8. *Let $\{nw, ne, sw, se\}$ denote the endpoints of arcs of the tangle A. Let $\{NW, NE, SW, SE\}$ denote the endpoints of arcs of the tangle $A \circ (d_m, ..., d_1)$, m odd, and let $j = E[d_1, ..., d_m]$, $p = E[d_1, ..., d_{m-1}]$, $d = E[d_2, ..., d_m]$, and $q = E[d_2, ..., d_{m-1}]$. Then there exists arcs connecting the following points:*

(a) if j even, p odd, q even: ne to NE, se to SE.
(b) if j even, p odd, q odd: ne to NE, se to SW.
(c) if j odd, p odd, q even: ne to SE, se to NE.
(d) if j odd, p odd, q odd: ne to SW, se to NE.
(e) if p even, d even: ne to SW, se to SE.
(f) if p even, d odd: ne to SE, se to SW.

Proof. Induction on m □.

Fig. 8 illustrates lemma 8. Note that the terms listed within parenthesis in Fig. 8 do not need to be listed as they are determined by the other terms and the fact that $pd - qj = 1$. For example if j is even, then since $pd - qj$ is odd, p and d must be odd.

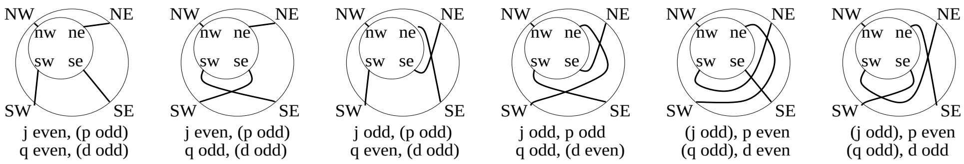

Fig. 8. Determining the parity of $A \circ (d_m, ..., d_1)$.

Lemma 9. *Suppose $N(\frac{j}{p} + \frac{t}{w})$ is a link (i.e. $jw + pt$ is even) which is not strongly reversible. Let q and d be any integers such that $pd - qj = 1$. Then $N(\frac{j}{p} + \frac{t}{w}) = N(\frac{jw+pt}{dw+qt})$ as oriented links (where $\frac{jw+pt}{dw+qt}$ is similarly oriented by definition) if and only if the tangles $\frac{j}{p}$ and $\frac{t}{w}$ are oppositely oriented if pq is odd or if p even, d odd and similarly oriented otherwise.*

Proof. Take a continued fraction expansion of $\frac{d}{q} = [d_m, ..., d_2]$ where m is odd and $d = E[d_2, ..., d_m]$, $q = E[d_2, ..., d_{m-1}]$. Since $pd - qj = 1$, there exists a d_1 such that $j = E[d_1, ..., d_m]$, $p = E[d_1, ..., d_{m-1}]$. Hence $\frac{j}{p} = (d_1, ..., d_m)$, m odd.

$N(\frac{j}{p} + \frac{t}{w}) = N(\frac{t}{w} + \frac{j}{p}) = N(\frac{t}{w} + (d_1, ..., d_m)) = N(\frac{t}{w} \circ (d_m, ..., d_1))$ by lemma 5. If $\frac{t}{w} = (c_1, ..., c_n)$, then $\frac{t}{w} \circ (d_m, ..., d_1) = (c_1, ..., c_n) \circ (d_m, ..., d_1) = (c_1, ..., c_n + d_m, ..., d_1)$ by lemma 6. By lemma 7, $[d_1, ..., d_m + c_n, ..., c_1] = \frac{E[d_1,...,d_m]E[c_1,...c_{n-1}]+E[d_1,...,d_{m-1}]E[c_1,...c_n]}{E[d_2,...,d_m]E[c_1,...c_{n-1}]+E[d_2,...,d_{m-1}]E[c_1,...c_n]} = \frac{jw+pt}{dw+qt}$.

If $jw + pt$ is even, the tangle $\frac{jw+pt}{dw+qt}$ has parity zero and by definition is similarly oriented in $N(\frac{jw+pt}{dw+qt})$. Thus, in order for $N(\frac{t}{w} + \frac{j}{p}) = N(\frac{jw+pt}{dw+qt})$ when these 4-plats are oriented, the tangle $\frac{t}{w}$ must be oriented so that $\frac{t}{w} \circ (d_m, ..., d_1) = \frac{jw+pt}{dw+qt}$ is similarly oriented. If $\frac{t}{w}$ has parity zero, then t is even, and thus j is even. Hence p is odd since $pd - qj$ is odd. Thus if q is even, part (a) of lemma 8 holds. Hence $\frac{t}{w}$ is similarly oriented if and only if $\frac{t}{w} \circ (d_m, ..., d_1)$ is similarly oriented (see also Fig. 8). If q is odd, part (b) of lemma 8 holds. Hence $\frac{t}{w}$ is oppositely oriented if and only if $\frac{t}{w} \circ (d_m, ..., d_1)$ is similarly oriented. If $\frac{t}{w}$ has parity one, then t and w are odd, and thus p and j are odd. Thus if q is even, part (c) of lemma 8 holds. Hence $\frac{t}{w}$ is similarly oriented if and only if $\frac{t}{w} \circ (d_m, ..., d_1)$ is similarly oriented. If q is odd, part (d) of lemma 8 holds. Hence $\frac{t}{w}$ is oppositely oriented if and only if $\frac{t}{w} \circ (d_m, ..., d_1)$ is similarly oriented. If $\frac{t}{w}$ has parity ∞, then w is even, and thus p is even. Thus if d is even, part (e) of lemma 8 holds. Hence $\frac{t}{w}$ is similarly oriented if and only if $\frac{t}{w} \circ (d_m, ..., d_1)$ is similarly oriented. If d is odd, part (f) of lemma 8 holds. Hence $\frac{t}{w}$ is oppositely oriented if and only if $\frac{t}{w} \circ (d_m, ..., d_1)$ is similarly oriented. Since $jw + pt$ is even, $\frac{j}{p}$ has the same parity and orientation as $\frac{t}{w}$ $\square$.

Example: Suppose p is odd. Then by lemma 9 $N(\frac{j}{p} + \frac{0}{1}) = N(\frac{j}{d})$ where $\frac{0}{1}$ is oppositely oriented if q is odd and similarly oriented if q is even. Hence $N((2) + \frac{0}{1}) = N(\frac{2}{1})$ where $\frac{0}{1}$ is similarly oriented since $j = E[2] = 2, p = E[] = 1, q = 0, d = E[] = 1$. Note $(2) = (0, -1, 2)$ since $2 + \frac{1}{-1+\frac{1}{0}} = 2$. But if we use $(0, -1, 2)$, then $j = E[0, -1, 2] = 2, p = E[0, -1] = 1, q = E[-1] = -1, d = E[-1, 2] = -1$. Hence by lemma 9, $N((0, -1, 2) + \frac{0}{1}) = N(\frac{2}{-1})$ where $\frac{0}{1}$ is oppositely oriented.

3. Biological Motivation and Example

Protein-bound DNA can be modeled by a tangle where the 3-dimensional ball represents the protein (or protein complex – usually several proteins are involved in any reaction) and the strings represent the segments of DNA bound by protein. Some proteins such as recombinases, will break the DNA, change the DNA configuration bound by protein, and rejoin the ends of the cut DNA. Hence these proteins can produce knotted and linked DNA when acting on circular DNA. This action can be represented by tangle replacement. Different recombinases perform different moves. For example, XER recombinase is believed to perform $(-\frac{1}{3}, -\frac{4}{3})$ moves whereas Flp recombinase has been modeled as performing (0, 1)-moves.

The DNA sequence can be used to orient a DNA knot or link. For example, suppose that the sequence ACGAT occurs exactly once on only one of the strands of a circular DNA molecule Then this sequence can be used to orient the circular DNA molecule as shown in Fig. 9. Usually circular DNA used in an experiment is about 2000 - 5000 base pairs, but for simplicity, the circular DNA shown in Fig. 9 is only 30 base pairs which is not a biologically plausible length for circular DNA.

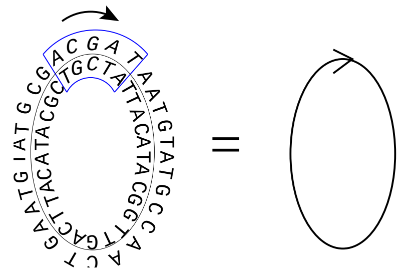

Fig. 9. Oriented DNA

Some recombinases bind to specific sequences. These sequences can also be given an orientation by using a portion of the sequence which occurs exactly once on only one of the strands. If two of these sequences appear in inverted orientation, then recombinases will normally cut both these sequences and interchange the ends, inverting a portion of the DNA sequence (Figs. 10, 12). If two of these sequences appear in direct orientation, then recombinases will normally cut these sequences and interchange the ends, resulting in a change in the number of components (Fig. 11).

Observe that when an orientation ignoring move is performed on a knot, the knot can be written as the union of two segments where an orientation ignoring move inverts the orientation of one segment with respect to the other segment (Fig. 12). Thus inverted repeats usually result in an orientation ignoring move. If a (P, R) move is orientation preserving and if P and R do not have the same parity, then this orientation preserving move changes the number of components. Hence direct repeats usually result in an orientation preserving move.

The orientation of protein binding sequences within protein-bound DNA is also

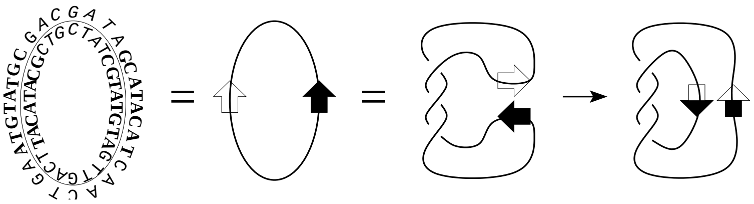

Fig. 10. Recombination: inverted repeats

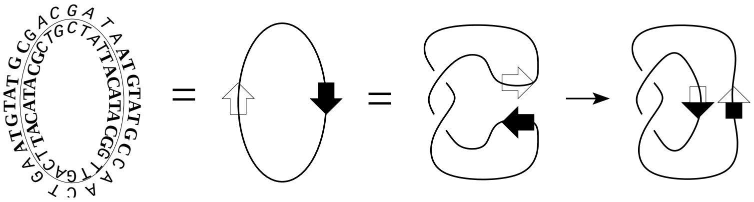

Fig. 11. Recombination: direct repeats

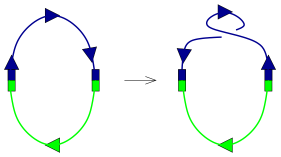

Fig. 12. Orientation ignoring

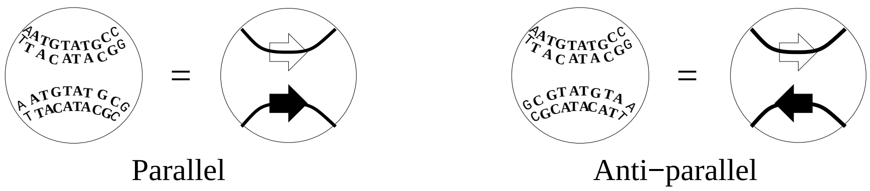

Fig. 13. Parallel versus anti-parallel protein binding sequences

of interest. Suppose that in the tangle equation $N(U + \frac{0}{1}) = N(\frac{a}{b})$, the tangle $\frac{0}{1}$ represents protein bound to two DNA segments. Two protein binding sequences are said to be parallel if they are oriented as shown in Fig. 13 whereas they are said to by anti-parallel if they are oriented as shown in Fig. 13. Note that if the sequences occur in direct orientation, then parallel is equivalent to similarly oriented and anti-parallel is equivalent to oppositely oriented. But if the sequences occur in inverted orientation, then parallel is equivalent to oppositely oriented and anti-parallel is equivalent to similarly oriented.

In the following example, we will solve an oriented system of tangle equations in order to illustrate the aspects of interest.

Example: Solve $N(U + \frac{0}{1}) = N(\frac{4}{-1})$, $N(U + \frac{1}{0}) = N(\frac{8}{-3})$ where U is a generalized Montesinos tangle.

0.) Determine U: If a solution exists, then by theorem 3 in [5], $N(\frac{8}{-3}) = N(\frac{p(-p-4q)\pm 4}{q(-p-4q)\mp 1})$. Hence, $p=2$, $q=-1$ and $h = \frac{-w\pm 1}{t} = 1$. Take $d=0$, $j=1$, then $U = (\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h,0) = (\frac{1}{2}+\frac{1}{2}) \circ (1,0) = (\frac{da-jb}{pb-qa} + \frac{j}{p}) \circ (h,0)$.

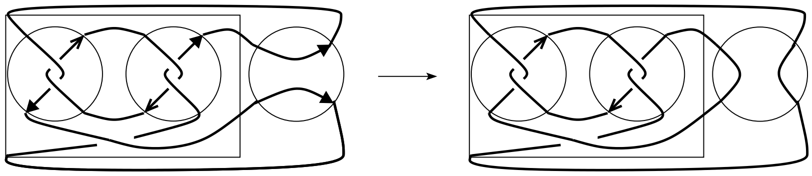

Fig. 14. $N((\frac{1}{2}+\frac{1}{2}) \circ (1,0) = N(\frac{4}{-1})$, $N((\frac{1}{2}+\frac{1}{2}) \circ (1,0) + \frac{1}{0}) = N(\frac{8}{-3})$

1.) Is the tangle $\frac{0}{1}$ similarly oriented or oppositely oriented in the equation $N(U + \frac{0}{1}) = N(\frac{4}{-1})$? Since we know U, we can see from Fig. 14 that the tangle $\frac{0}{1}$ is similarly oriented in the equation $N(U + \frac{0}{1}) = N(\frac{-4}{1})$.

2.) Is the move orientation preserving or ignoring? Since $\frac{0}{1}$ is similarly oriented and $\frac{1}{0}$ has parity infinity, the move is orientation ignoring.

3.) Can/how are the components oriented? Note that if a is odd in the equation $N(U + \frac{0}{1}) = N(\frac{a}{b})$, it does not matter how U is oriented. But in this case, $a = 4$ is even and hence we must determine how the components are oriented so that $N((\frac{1}{2}+\frac{1}{2}) \circ (1,0) + \frac{0}{1}) = N(\frac{4}{-1})$ as oriented 4-plats. If the tangle $\frac{0}{1}$ involved both components, it would be sufficient to determine if the tangle $\frac{0}{1}$ is similarly oriented or oppositely oriented. But in this example the tangle $\frac{0}{1}$ only involves one component. When $N(\frac{a}{b})$ is a link, the tangles $\frac{j}{p}$ and $\frac{da-jb}{pb-qa}$ will have the same parity and orientation. Thus, determining the orientation of the strings in the tangle $\frac{j}{p} = \frac{1}{2}$ determines the orientation. If the tangle $\frac{j}{p}$ is oppositely oriented, then $N(U + \frac{0}{1}) = N(\frac{4}{-1})$ as oriented 4-plats.

If z is odd in the equation $N(U + \frac{t}{w}) = N(\frac{z}{v})$, it does not matter how U is oriented. But since $z = 8$ is even, orientation must be considered. If the move is orientation preserving, then the orientation of U is obtained from the equation $N(U + \frac{0}{1}) = N(\frac{a}{b})$, and thus one only needs to check if this orientation also gives $N(U + \frac{t}{w}) = N(\frac{z}{v})$ as oriented 4-plats. If it does, then U is a solution to this system of equations. If not, then U is not a solution. If the move is orientation ignoring, then the orientation of U can be chosen so that $N(U + \frac{t}{w}) = N(\frac{z}{v})$ as oriented 4-plats. In this case the tangles $\frac{j}{p}$ and $\frac{da-jb}{pb-qa}$ will have both have parity infinity (see proof of theorem 4) and hence the same orientation. Thus, determining the orientation of the tangle $\frac{j}{p}$ is sufficient. Since $N(\frac{8}{-3}) = N(\frac{8}{5})$ as oriented 4-plats (i.e., $N(\frac{8}{-3})$ is strongly reversible), the tangle $\frac{j}{p} = \frac{1}{2}$ can be either similarly or oppositely oriented so that $N(U + \frac{0}{1}) = N(\frac{8}{-3})$ as oriented 4-plats.

Note, however, that this method only finds solutions for U when U is a generalized Montesinos tangle and in this case, there is no theorem that states that U must be a generalized Montesinos tangle, so there may be other solutions for U.

4. Solving the Oriented Equations $N(U + \frac{0}{1}) = N(\frac{a}{b})$, $N(U + \frac{t}{w}) = N(\frac{z}{v})$

The unoriented system of equations $N(U + \frac{0}{1}) = N(\frac{a}{b})$, $N(U + \frac{t}{w}) = N(\frac{z}{v})$ was solved in [5], theorem 3. For convenience, theorem 3 of [5] is restated below in theorem 1 when U is rational and in theorem 3 when U is a generalized Montesinos tangle. The orientation of these solutions is determined in theorem 2 when U is rational and in theorem 4 when U is a generalized Montesinos tangle.

Theorem 1. *Suppose $w \not\cong \pm 1$ mod t or U is rational. Then $N(U + \frac{0}{1}) = N(\frac{a}{b})$ and $N(U + \frac{t}{w}) = N(\frac{z}{v})$ where $N(\frac{a}{b})$ and $N(\frac{z}{v})$ are unoriented 4-plats if and only if there exists an integer b' such that $b'b^{\pm 1} = 1$ mod a, and for any integers x and y such that $b'x - ay = 1$,*

$$N(\frac{z}{v}) = N(\frac{tb' + wa}{ty + wx}) \tag{4.1}$$

In this case, $U = \frac{a}{b'}$ for all b' satisfying the above.

Theorem 2. *Let $z = tb' + wa$ and $v' = ty + wx$ where $b'x - ay = 1$. $N(\frac{a}{b'} + \frac{0}{1}) = N(\frac{a}{b})$ and $N(\frac{a}{b'} + \frac{t}{w}) = N(\frac{z}{v})$ as oriented 4-plats if and only if one of the following holds:*

(i) If a is even and $b'b^{\pm 1} \cong 1$ mod $2a$ or if a and b' are odd and $b'b^{\pm 1} \cong 1$ mod a, then $v'v^{\pm 1} \cong 1 + yz$ mod Lz, $L = 1$ if z odd, $L = 2$ if z even.

In this case, $U = \frac{a}{b'}$ and $\frac{0}{1}$ are similarly oriented in the equation $N(U + \frac{0}{1}) = N(\frac{a}{b})$. The $(\frac{0}{1}, \frac{t}{w})$ move is orientation ignoring if and only if w is even.

(ii) If a is even and $b'b^{\pm 1} \cong 1 + a$ mod $2a$ or if a is odd, b' is even, and $b'b^{\pm 1} \cong 1$ mod a, then $v'v^{\pm 1} \cong 1 + (x + y)z$ mod Lz, $L = 1$ if z odd, $L = 2$ if z even.

In this case, $U = \frac{a}{b'}$ and $\frac{0}{1}$ are oppositely oriented in the equation $N(U + \frac{0}{1}) = N(\frac{a}{b})$. The $(\frac{0}{1}, \frac{t}{w})$ move is orientation ignoring if and only if tw is odd.

Recall that we know U is rational when $w \not\cong \pm 1$ mod t and thus the above theorem determines all solutions in this case [4, 6]. Other cases in which U must be rational are given in [9, 10].

Proof. Suppose a is odd. Then if $b'b^{\pm 1} \cong 1$ mod a, $N(\frac{a}{b'} + \frac{0}{1}) = N(\frac{a}{b'}) = N(\frac{a}{b})$ as oriented 4-plats. By lemmas 4 and 3, if b' is odd, $\frac{0}{1}$ is similarly oriented and if b' is even, $\frac{0}{1}$ is oppositely oriented.

Suppose a is even. If $b'b^{\pm 1} \cong 1$ mod $2a$, then $N(\frac{a}{b}) = N(\frac{a}{b'}) = N(\frac{a}{b'} + \frac{0}{1})$ as oriented 4-plats if $\frac{a}{b'}$ and thus $\frac{0}{1}$ is similarly oriented by lemma 1. If $b'b^{\pm 1} \cong 1 + a$ mod $2a$, $N(\frac{a}{b}) = N(\frac{a}{a+b'}) = N(\frac{a}{b'} + \frac{0}{1})$ if $\frac{a}{b'}$ and thus $\frac{0}{1}$ is oppositely oriented by lemma 1.

Orientation ignoring versus preserving follows from lemmas 2 and 4.

If z is odd, $N(\frac{a}{b'} + \frac{t}{w}) = N(\frac{tb'+wa}{ty+wx}) = N(\frac{z}{v})$ if $v'v^{\pm 1} \cong 1$ mod z. Suppose z is even. If the tangle $\frac{0}{1}$ is similarly oriented in $N(\frac{a}{b'} + \frac{0}{1})$, then the tangle $\frac{a}{b'}$ is similarly oriented and cannot have parity infinity. Thus b' and w are odd and the move is orientation preserving. Since $\frac{a}{b'}$ is similarly oriented, $N(\frac{a}{b'} + \frac{t}{w}) = N(\frac{z}{v'+yz})$ by lemma 9. If the tangle $\frac{0}{1}$ is oppositely oriented, then the tangle $\frac{a}{b'}$ is oppositely oriented and cannot have parity one. Hence $\frac{t}{w}$ cannot have parity one and the move is orientation preserving. Since $b'x - ay$ is odd, either a is even and $b'$and x are odd OR b' is even and a and y are odd. In this case $N(\frac{a}{b'} + \frac{t}{w}) = N(\frac{z}{v'+(x+y)z})$ by lemma 9 $\square$.

Theorem 3. *Suppose $w \cong \pm 1$ mod t and U is a generalized M-tangle. Then $N(U + \frac{0}{1}) = N(\frac{a}{b})$ and $N(U + \frac{t}{w}) = N(\frac{z}{v})$ where $N(\frac{a}{b})$ and $N(\frac{z}{v})$ are unoriented 4-plats if and only if there exists relatively prime integers, p and q, where p may be chosen to be positive, such that*

$$N(\frac{z}{v}) = N(\frac{tp(pb-qa) \pm a}{tq(pb-qa) \pm b}) \tag{4.2}$$

In this case, the solutions for U are $(\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h, 0)$ and $(\frac{da-jb}{pb-qa} + \frac{j}{p}) \circ (h, 0)$, for all p, q satisfying the above, d and j are any integers such that $pd - qj = 1$, and $h = \frac{-w\pm 1}{t}$ where the $\pm$ sign agrees with that in (4.2) (note, the choice of j and d such that $pd - qj = 1$ has no effect on U).

Note that in theorem 4, all $\pm$ signs except those involving exponents are in agreement.

Theorem 4. *Suppose $w = -ht \pm 1$ for some integer h. Let $z = tp(pb - qa) \pm a$, $v' = tq(pb-qa) \pm b$. Let $L = 1$ if z is odd and let $L = 2$ if z is even. $N((\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h, 0) + \frac{0}{1}) = N((\frac{da-jb}{pb-qa} + \frac{j}{p}) \circ (h, 0) + \frac{0}{1}) = N(\frac{a}{b})$ and $N((\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h, 0) + \frac{t}{w}) = N((\frac{da-jb}{pb-qa} + \frac{j}{p}) \circ (h, 0) + \frac{t}{w}) = N(\frac{z}{v})$ as oriented 4-plats with the tangle orientations given below if and only if the following hold:*

i.) $v'v^{\pm 1} \cong 1 + (b + 1)z$ mod Lz if the $(\frac{0}{1}, \frac{t}{w})$ move is orientation preserving or

ii.) $v'v^{\pm 1} \cong 1$ mod z if the move is orientation ignoring.

The $(\frac{0}{1}, \frac{t}{w})$ move is orientation ignoring if and only if $t[q + p(b + 1)]$ is odd.

The tangles have the following orientation:

a.) Orientation of $\frac{0}{1}$ in the equation $N(U + \frac{0}{1}) = N(\frac{a}{b})$: If $h + q + p(b + 1)$ is even, the tangle $\frac{0}{1}$ is similarly oriented. If $h + q + p(b + 1)$ is odd, the tangle $\frac{0}{1}$ is oppositely oriented.

b.) Orientation of $U = (\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h, 0)$ or $U = (\frac{da-jb}{pb-qa} + \frac{j}{p}) \circ (h, 0)$ in the equation $N(U + \frac{0}{1}) = N(\frac{a}{b})$ when a is even: if p is odd, q is even or p is even, d is chosen to be even, then $\frac{j}{p}$ is similarly oriented OR if pq is odd or if p is even and d is chosen to be odd, then $\frac{j}{p}$ is oppositely oriented.

c.) Orientation of $U = (\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h, 0)$ or $U = (\frac{da-jb}{pb-qa} + \frac{j}{p}) \circ (h, 0)$ in the equation $N(U + \frac{0}{1}) = N(\frac{z}{v})$ when z is even and the move is orientation ignoring: the move involves only component and if $v'v^{\pm 1} \cong 1 + dz$ mod $2z$, then $\frac{j}{p}$ is similarly oriented OR if $v'v^{\pm 1} \cong 1 + (d + 1)z$ mod $2z$, then $\frac{j}{p}$ is oppositely oriented.

Recall that theorem 3 and hence theorem 4 includes the U rational case when $w \cong \pm 1$ mod t. Hence if $|t| > 1$, then the above list of solutions to the system of equations, $N(U + \frac{0}{1}) = N(\frac{a}{b})$ and $N(U + \frac{t}{w}) = N(\frac{z}{v})$, is complete [4, 6].

Proof. Let $U_1 = \frac{j}{p}$, $U_2 = \frac{da-jb}{pb-qa}$. $N((U_2+U_1)\circ(h,0)+\frac{0}{1}) = N(U_2+U_1) = N(U_1+U_2) = N((U_1+U_2)\circ(h,0)+\frac{0}{1})$ where the orientation of the U_i's and $\frac{0}{1}$ are not affected. Note that similarly versus oppositely oriented is invariant under a 180° rotation about the x-axis. Hence if $h = \frac{-w\pm1}{t}$, $N((U_2+U_1)\circ(h,0)+\frac{t}{w}) = N(U_2+U_1\pm t) = N(U_2\pm t+U_1) = N(U_1+U_2\pm t) = N((U_1+U_2)\circ(h,0)+\frac{t}{w})$ where the orientation of the U_i's and $\frac{t}{w}$ are not affected.

If $p/q = [d_1,...,d_{m-1}]$ where m is odd and $p = E[d_1,...,d_{m-1}]$, $q = E[d_2,...,d_{m-1}]$, let $j = [d_1,...,d_m]$, $d = [d_2,...,d_m]$ for any integer d_m. Since d_m is arbitrary, any j and d such that $pd - qj = 1$ can be chosen. $N((\frac{j}{p}+\frac{da-jb}{pb-qa})\circ(h,0)+\frac{0}{1}) = N(\frac{a}{b})$ if a is odd.

Suppose a is even. If $N(\frac{a}{b})$ is not strongly reversible, by lemma 9, $N((\frac{j}{p}+\frac{da-jb}{pb-qa})\circ(h,0)+\frac{0}{1}) = N(\frac{j}{p}+\frac{da-jb}{pb-qa}) = N(\frac{j(pb-qa)+p(da-jb)}{d(pb-qa)+q(da-jb)}) = N(\frac{a}{b})$ if and only if U_1 is similarly oriented and p odd, q even or p even, d even OR if U_1 is oppositely oriented and pq odd or p even, d odd. If $N(\frac{a}{b})$ is strongly reversible then U_1 can be given any orientation. However, we can assume the orientation in part b of theorem 4 as the other orientation is redundant (using different values for p and q). In particular, if p, q are solutions to the unoriented equation $N(\frac{z}{v}) = N(\frac{tp(pb-qa)\pm a}{tq(pb-qa)\pm b})$ and if $b^2 = 1+ka$ where k is odd, then $P = pb-qa$, $Q = pk-qb$ is also a solution to the unoriented equation $N(\frac{z}{v}) = N(\frac{tp(pb-qa)\pm a}{tq(pb-qa)\pm b})$. Let $J = da - jb$ and $D = db - jk$. Then $PD - QJ = 1$ if and only if $pd - qj = 1$. $\frac{j}{p} = \frac{Da-Jb}{Pb-Qa}$ and $\frac{da-jb}{pb-qa} = \frac{J}{P}$. Since a is even, $\frac{j}{p}$ and $\frac{da-jb}{pb-qa}$ have the same parity. Thus, $U = \frac{j}{p}+\frac{da-jb}{pb-qa} = \frac{Da-Jb}{Pb-Qa}+\frac{J}{P}$ as oriented tangles if $\frac{j}{p}$ and $\frac{J}{P}$ have the same orientations (i.e., both similarly or both oppositely oriented). P is odd and Q is even if and only if pq is odd. P is even and D is even if and only if p is even and d is odd. Hence if $U = (\frac{Da-Jb}{Pb-Qa}+\frac{J}{P})\circ(h,0)$ has the orientation given in part b of theorem 4, then U can also be written as $U = (\frac{j}{p}+\frac{da-jb}{pb-qa})$ where the U's have the same orientation if $\frac{j}{p}$ is oppositely oriented if p odd, q even OR p even, d even or $\frac{j}{p}$ is similarly oriented if pq odd or if p even, d odd. Thus we can assume the orientation in part b of theorem 4 as the other orientation is obtained using the orientation in $4b$, but with different values for p and q.

If p is even then U_1 has parity ∞ and q is odd. Thus the tangle $\frac{0}{1}$ is oppositely oriented if h is even and therefore $h+q+p(b+1)$ is odd and similarly oriented if h is odd and therefore $h+q+p(b+1)$ is even. If p is odd, j can be taken to be even. If a is even, then b is odd and U_1+U_2 has parity zero and is similarly oriented if q even and oppositely oriented if q odd. Thus the tangle $\frac{0}{1}$ is oppositely oriented if $h+q$ is odd and therefore $h+q+p(b+1)$ is odd and similarly oriented if $h+q$ is even and therefore $h+q+p(b+1)$ is even. If p and a are odd, U_1+U_2 has parity ∞ if b and q are both even or both odd and parity one otherwise. Thus the tangle $\frac{0}{1}$ is oppositely oriented if $h+q+p(b+1)$ is odd and similarly oriented if $h+q+p(b+1)$ is even.

By lemmas 2 and 4, the $(\frac{0}{1},\frac{t}{w})$ move is orientation ignoring if and only if t is odd and either w is even and the tangle $\frac{0}{1}$ is similarly oriented or w is odd and the tangle $\frac{0}{1}$ is oppositely oriented. Since $ht + w = \pm 1$, this is equivalent to t odd and if h is odd (i.e. w even), $h + q + p(b+1)$ is even or if h is even (i.e. w odd), then $h + q + p(b+1)$ is odd. Thus a move is orientation ignoring if and only if $t(q + p(b+1))$ is odd.

$N((\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h,0) + \frac{t}{w}) = N(\frac{z}{v'})$ if z odd. Suppose $z = tp(pb - qa) \pm a$ is even. $N((\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h,0) + \frac{t}{w}) = N((\frac{j}{p} + \frac{da-jb}{pb-qa} \pm t) = N(\frac{j}{p} + \frac{da-jb\pm(pb-qa)t}{pb-qa}) = N(\frac{j(pb-qa)+p[da-jb\pm(pb-qa)t]}{d(pb-qa)+q[da-jb\pm(pb-qa)t]}) = N(\frac{z}{v'})$ if U_1 is similarly oriented and p odd, q even or if p even and d even OR if U_1 is oppositely oriented and pq odd or if p even and d odd. If t is even, then a is even and the move is orientation preserving. Thus, U_1 is similarly oriented if p odd, q even or if p even and d even OR U_1 is oppositely oriented if pq odd or if p even and d odd. Hence $N((\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h,0) + \frac{t}{w}) = N(\frac{z}{v'}) = N(\frac{z}{v'+(b+1)z})$ since a even implies $b+1$ is even. If t and p are odd, then z even implies $pb - qa$ and a are odd. Since p, a, and $pb - qa$ are odd, b is odd if and only if q is even. Hence, $q + p(b+1)$ is even and the move is orientation preserving. If h is even, the tangle $\frac{0}{1}$ in $N((U_1 + U_2) \circ (h,0) + \frac{0}{1})$ is similarly oriented. If h is odd, the tangle $\frac{0}{1}$ in $N((U_1 + U_2) \circ (h,0) + \frac{0}{1})$ is oppositely oriented. Since p is odd, U_1 has parity one or zero. Thus, in both cases, U_1 is similarly oriented and $N((\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h,0) + \frac{t}{w}) = N(\frac{z}{v'+qz}) = N(\frac{z}{v'+(b+1)z})$. If t is odd and p is even, then q is odd and the $(\frac{0}{1}, \frac{t}{w})$ move is orientation ignoring in which case, U_1 can be given either orientation. Also, since p is even, U_1 has parity ∞ and the move involves only one component. If U_1 is similarly oriented, $N((\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h,0) + \frac{t}{w}) = N(\frac{z}{v'+dz})$. If U_1 is oppositely oriented, $N((\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h,0) + \frac{t}{w}) = N(\frac{z}{v'+(d+1)z})$ $\square$.

Example: Use theorems 3 and 4 to solve $N(U + \frac{0}{1}) = N(\frac{4}{-1})$, $N(U + \frac{1}{0}) = N(\frac{8}{-3})$ where U is a generalized Montesinos tangle.

Solving the unoriented equation $N(\frac{8}{-3}) = N(\frac{p(-p-4q)\pm 8}{q(-p-4q)\mp 1})$ results in $(p,q) = (2,-1)$. Thus $z = tp(pb - qa) + a = 8$ and $h = \frac{0+1}{2} = 1$. Taking $d = 0$, $j = 1$, $U = (\frac{1}{2} + \frac{1}{2}) \circ (1,0)$ is the solution to the unoriented system of equations. $t[q + p(b+1)] = 1[-1 + 2(-1+1)]$ is odd, so the move is orientation ignoring. Hence the tangles can be oriented so that U is a solution to the oriented system of equations.

a.) $h + q + p(b+1) = 1 + -1 + 2(-1+1)$ is even. Hence the tangle $\frac{0}{1}$ in $N(U + \frac{0}{1}) = N(\frac{4}{-1})$ is similarly oriented.

b.) Orientation of $U = (\frac{1}{2} + \frac{1}{2}) \circ (1,0)$ in the equation $N(U + \frac{0}{1}) = N(\frac{a}{b})$ when a is even: Since p and d are even, $\frac{1}{2}$ is similarly oriented.

c.) Orientation of $U = (\frac{1}{2} + \frac{1}{2}) \circ (1,0)$ in the equation $N(U + \frac{0}{1}) = N(\frac{z}{v})$ when z is even and the move is orientation ignoring: $v' = q(-p-4q) \mp 1 = -3 = v + 0z$. Hence $\frac{1}{2}$ can be similarly oriented. $v'v = (-3)(-3) = 9 = 1 + 8 = 1 + (0+1)8$ mod 16. Hence $\frac{1}{2}$ can also be oppositely oriented.

Example: Use theorems 3 and 4 to solve $N(U+\frac{0}{1}) = N(\frac{8}{-3})$, $N(U+\frac{2}{1}) = N(\frac{64}{25})$.

By [4, 6] U is a generalized Montesinos tangle. Thus we can find all solutions for U.

Solving the unoriented equation $N(\frac{64}{25}) = N(\frac{2p(-3p-8q)\pm 8}{2q(-3p-8q)\mp 3})$ results in $(p,q) = (2,1), (14,-5)$. In both cases for this example, $z = tp(pb-qa) - a = -64$ (and hence $v = -25$) and $h = \frac{-1-1}{2} = -1$.

Case 1: $(p,q) = (2,1)$. Taking $d = 1$, $j = 1$, $U = (\frac{1}{2} + \frac{11}{-14}) \circ (-1,0)$ and $U = (\frac{11}{-14} + \frac{1}{2}) \circ (-1,0)$ are both solutions to the unoriented system of equations. $t[q + p(b+1)] = 2[q + p(b+1)]$ is even, so the move is orientation preserving. Hence, we need $v'v^{\pm 1} \cong 1 + (b+1)z = 1 + (-3+1)(-64) = 1 \bmod 2(-64)$. This holds since $v = -25$ and $v' = tq(pb-qa) - b = 2(1)[2(-3) - (1)8] - (-3) = -25$. Thus U can be given the orientation in theorem 4b so that it is a solution to the oriented system of equations.

a.) $h + q + p(b+1) = -1 + 1 + 2(-3+1)$ is even. Hence the tangle $\frac{0}{1}$ in $N(U+\frac{0}{1}) = N(\frac{8}{-3})$ is similarly oriented.

b.) Orientation of U in the equation $N(U+\frac{0}{1}) = N(\frac{a}{b})$ when a is even: Since p is even and d is odd, $\frac{1}{2}$ and $\frac{11}{-14}$ are oppositely oriented. Hence $U = (\frac{1}{2} + \frac{11}{-14}) \circ (-1,0)$ and $U = (\frac{11}{-14} + \frac{1}{2}) \circ (-1,0)$ where $\frac{1}{2}$ and $\frac{11}{-14}$ are oppositely oriented are both solutions to the oriented system of equations.

Case 2: $(p,q) = (14,-5)$. Taking $d = 4$, $j = -11$, $U = (\frac{-11}{14} + \frac{1}{2}) \circ (-1,0)$ and $U = (\frac{1}{2} + \frac{-11}{14}) \circ (-1,0)$ are both solutions to the unoriented system of equations. $t[q+p(b+1)] = 2[q+p(b+1)]$ is even, so the move is orientation preserving. Hence, we need $v'v^{\pm 1} \cong 1 + (b+1)z = 1 + (-3+1)(-64) = 1 \bmod 2z$. In this case, $v = -25$ and $v' = tq(pb-qa) \pm b = 2(-5)[14(-3) - (-5)8] - (-3) = 23$. Since $v'v^{\pm 1} \not\cong 1 \bmod 2z$, U with the orientation given in theorem 4b (where $\frac{1}{2}$ and $\frac{11}{-14}$ are similarly oriented) is NOT a solution to the oriented system of equations.

$N(\frac{8}{-3})$ is strongly reversible. Hence by the proof of theorem 4, we expected two solutions for (p,q) would both result in the same U, but with different orientations. Since $N(\frac{64}{25})$ is not reversible and the move is orientation preserving, only one of these solutions would satisfy the oriented system of equations.

In some cases, lemma 10 can be a quick method to use instead of the above theorems to determine the parity of U. By lemma 3, it is then possible to determine the orientation of the tangle $\frac{0}{1}$. Note that the formulas given in Theorem 1 and 3 can be combined into $N(\frac{z}{v}) = N(\frac{tp(pb-qa)+(w+ht)a}{tp(py-qx)+(w+ht)x})$ or $N(\frac{tp(pb-qa)+(w+ht)a}{tq(pb-qa)+(w+ht)b})$ and $U = (\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h,0)$ or $U = (\frac{da-jb}{pb-qa} + \frac{j}{p}) \circ (h,0)$ as follows: If $w \not\cong \pm 1 \bmod t$, then $h = 0$, $p = 1$, $q = k$ in $N(\frac{tp(pb-qa)+(w+ht)a}{tp(py-qx)+(w+ht)x}) = N(\frac{t(b-ak)+wa}{t(y-xk)+wx})$ and $h = 0$, $p = x$, $q = y$ in $N(\frac{tp(pb-qa)+(w+ht)a}{tq(pb-qa)+(w+ht)b}) = N(\frac{tx+wa}{ty+wb})$. If $w \cong \pm 1 \bmod t$, then $(p,q) = 1$ and $N(\frac{tp(pb-qa)+(w+ht)a}{tp(py-qx)+(w+ht)x}) = N(\frac{tp(pb-qa)+(w+ht)a}{tq(pb-qa)+(w+ht)b})$ as unoriented 4-plats. Although in practice it is generally easier to use the formulas given in the above theorems, for conciseness, this more compact form will be used in lemma 10.

Lemma 10. *Suppose* $N(U+\frac{t}{w}) = N(\frac{z}{v})$ *where* $U = (U_1+U_2)\circ(h,0)$ *or* $(U_2+U_1)\circ(h,0)$, $U_1 = \frac{j}{p}$, $U_2 = \frac{-bj+ad}{pb-qa}$,$\pm z = z' = tp(pb-qa)+(w+ht)a$, *and if* $w \cong \pm 1$ *mod* t,$(p,q)=1$, $h = \frac{-w\pm 1}{t}$ *or if* $w \ncong \pm 1$ *mod* t, $h=0$. *Then*

i.) U has parity ∞ *if* $\frac{z'-(w+ht)a}{t}$ *and h are even OR if* $\frac{z'-(w+ht)a}{t}$, *h, a are odd.*

ii.) U has parity one if $\frac{z'-(w+ht)a}{t}$ *is even and h is odd OR if* $\frac{z'-(w+ht)a}{t}$ *and a are odd and h is even.*

iii.) U has parity zero if $\frac{z'-(w+ht)a}{t}$ *is odd and a is even.*

Proof. Note $\frac{z'-(w+ht)a}{t} = p(pb-qa)$ and $U = (U_1+U_2)\circ(h,0)$ or $(U_2+U_1)\circ(h,0)$ where $U_1 = \frac{j}{p}$ and $U_2 = \frac{-bj+ad}{pb-qa}$. Note that U_2+U_1 has the same parity as U_1+U_2. If $p(pb-qa)$ is even, then p is even or $pb-qa$ is even, and thus U_1+U_2 has parity infinity. If h and $p(pb-qa)$ are both even, then U has parity infinity. If h is odd and $p(pb-qa)$ is even, then U has parity one.

If $p(pb-qa)$ is odd, then p is odd and $pb-qa$ is odd, and thus U_1+U_2 has parity zero if a is even or parity one if a is odd. If h is even and $p(pb-qa)$ is odd, then U has parity zero if a is even or parity one if a is odd. If both h and $p(pb-qa)$ are odd, then U has parity zero if a is even and parity ∞ if a is odd $\square$.

Example: Suppose it is known that $N(U+\frac{0}{1}) = N(\frac{1}{0})$ and $N(U+\frac{4}{1}) = N(\frac{3}{1})$. Since $h = \frac{-w\pm 1}{t} = \frac{-1\pm 1}{4} = 0, -\frac{1}{2}$, and h is an integer, $h=0$. $z' = \pm 3$, and $\frac{z'-(w+ht)a}{t} = \frac{\pm 3-1}{4}$ = -1 ($\frac{1}{2}$ is not an integer). Therefore, U has parity one. Hence, $\frac{0}{1}$ is similarly oriented. Thus by lemma 10 and [4, 6], it is not possible to convert the unknot into the knot $N(\frac{3}{1})$ via a $(\frac{0}{1}, \frac{4}{1})$ move where $\frac{0}{1}$ is oppositely oriented.

Lemma 10 can be a quick method to determine if certain oriented moves are possible. However if $\frac{z-(w+ht)a}{t}$ and $\frac{-z-(w+ht)a}{t}$ are both integers, one of which is even and the other odd, then lemma 10 gives no information unless one knows whether $z = +z'$ or if $z = -z'$.

5. Solving $N(U+\frac{f_1}{g_1}) = N(\frac{a}{b}), N(U+\frac{f_2}{g_2}) = N(\frac{z}{v})$

Theorem 5 in [5], relates $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ moves to $(\frac{0}{1}, \frac{t}{w})$ moves. Theorem 5 below relates the orientation of $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ moves to $(\frac{0}{1}, \frac{t}{w})$ moves. Theorems 6 and 7 summarize solving the system of oriented tangle equations $N(U+\frac{f_1}{g_1}) = N(\frac{a}{b})$ and $N(U+\frac{f_2}{g_2}) = N(\frac{z}{v})$.

Theorem 5. *Suppose* $f_1/g_1 = (c_1, ..., c_n)$, *n odd, where* $f_1 = E[c_1, ..., c_n]$ *and* $g_1 = E[c_1, ..., c_{n-1}]$. *Let* $e_1 = E[c_2, ..., c_n], i_1 = E[c_2, ..., c_{n-1}]$, *then*

$$N(U+\frac{f_1}{g_1}) = K_1 \qquad N(U+\frac{f_2}{g_2}) = K_2$$

if and only if

$$N(U'+\frac{0}{1}) = K_1 \qquad N(U'+\frac{t}{w}) = K_2$$

where if $\frac{f_2}{g_2} = (a_1, ..., a_m)$, *then* $\frac{t}{w} = \frac{g_1 f_2 - g_2 f_1}{e_1 g_2 - i_1 f_2} = (a_1, ..., a_m - c_n, -c_{n-1}, ..., -c_1)$ *and* $U' = U \circ (c_n, ..., c_1)$.

Or equivalently, if $\frac{t}{w} = (b_1, ..., b_k)$, *then* $\frac{f_2}{g_2} = \frac{te_1 + wf_1}{ti_1 + wg_1} = (b_1, ..., b_k + c_1, ..., c_n)$ *and* $U = U' \circ (-c_1, ..., -c_n)$.

The tangles $\frac{0}{1}$ *and* $\frac{f_1}{g_1}$ *are either both similarly oriented or both oppositely oriented if* e_1 *odd and* i_1 *even or both* e_1 *and* g_1 *are even. If* $e_1 i_1$ *odd or* e_1 *even,* g_1 *odd, then one of the tangles* $\frac{0}{1}$ *and* $\frac{f_1}{g_1}$ *is similarly oriented if and only if the other is oppositely oriented.*

The $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ *move is orientation ignoring if and only if the* $(\frac{0}{1}, \frac{t}{w})$ *move is orientation ignoring.*

Proof. The orientation of $\frac{f_1}{g_1}$ versus $\frac{0}{1}$ follows from lemma 8 (see also Figs. 15 and 16). Since the $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ move is equivalent to the $(\frac{0}{1}, \frac{t}{w})$ move, the $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ move will be orientation ignoring if and only if the $(\frac{0}{1}, \frac{t}{w})$ move is orientation ignoring □.

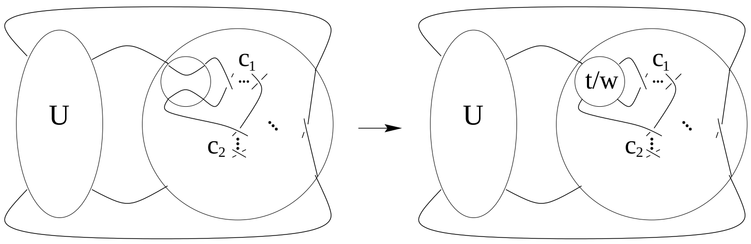

Fig. 15. $N(U + \frac{f_1}{g_1}) \to N(U + \frac{f_2}{g_2})$.

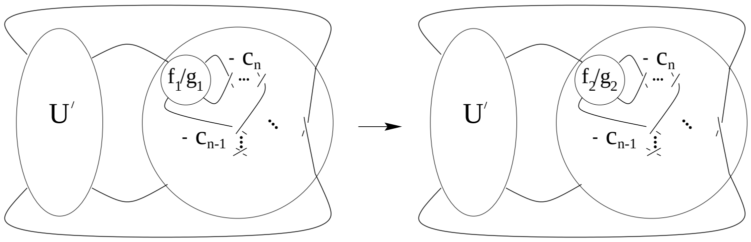

Fig. 16. $N(U' + \frac{0}{1}) \to N(U' + \frac{t}{w})$.

To solve a system of equations involving $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ moves, one can first solve $N(U + \frac{0}{1}) = N(\frac{a}{b})$ and $N(U + \frac{t}{w}) = N(\frac{z}{v})$ using theorems 1 and 2 (when $w \not\cong \pm 1$ mod t) or theorems 3 and 4 (when $w \cong \pm 1$ mod t) and then use theorem 5 to solve $N(U + \frac{f_1}{g_1}) = N(\frac{a}{b})$ and $N(U + \frac{f_2}{g_2}) = N(\frac{z}{v})$. However, these results are summarized in theorems 6 and 7 and hence these next two theorems can be used to directly solve the oriented system of equations, $N(U + \frac{f_1}{g_1}) = N(\frac{a}{b})$ and $N(U + \frac{f_2}{g_2}) = N(\frac{z}{v})$ or to double check calculations if theorems 1 - 5 are used instead.

Theorems 6 and 7 follow directly from theorems 1 - 5. Also note that in these theorems, all ± signs except those involving exponents are in agreement.

Theorem 6. *Let* $\frac{f_1}{g_1} = (c_1, ..., c_n), n$ *odd,* $f_1 = E[c_1, ..., c_n], g_1 = E[c_1, ..., c_{n-1}]$, $e_1 = E[c_2, ..., c_n], i_1 = E[c_2, ..., c_{n-1}]$, $t = g_1 f_2 - g_2 f_1$.

Suppose $w = e_1 g_2 - i_1 f_2 \not\cong \pm 1 \bmod t$ *or* U *is rational. Then* $N(U + \frac{f_1}{g_1}) = N(\frac{a}{b})$ *and* $N(U + \frac{f_2}{g_2}) = N(\frac{z}{v})$ *as oriented 4-plats if and only if there exists an integer* b' *such that* $|z| = |z'|$ *where* $z' = tb' + wa$, $\sigma = \frac{z'}{z}$, *and if* $v' = \sigma(ty + wx)$ *for any choice of* x, y *such that* $b'x - ay = 1$, *then one of the following holds:*

(i) If a *is even and* $b'b^{\pm 1} \cong 1 \bmod 2a$ *or if* a *and* b' *are odd and* $b'b^{\pm 1} \cong 1 \bmod a$, *then* $v'v^{\pm 1} \cong 1 + yz \bmod Lz$, $L = 1$ *if* z *odd,* $L = 2$ *if* z *even.*

In this case, the tangle $\frac{f_1}{g_1}$ *in the equation* $N(U + \frac{f_1}{g_1}) = N(\frac{a}{b})$ *is similarly oriented if* e_1 *odd and* i_1 *even or both* e_1 *and* g_1 *are even and is oppositely oriented if* $e_1 i_1$ *odd or* e_1 *even* g_1 *odd. The* $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ *move is orientation ignoring if and only if* w *is even.*

(ii) If a *is even and* $b'b^{\pm 1} \cong 1 + a \bmod 2a$ *or if* a *is odd,* b' *is even, and* $b'b^{\pm 1} \cong 1 \bmod a$, *then* $v'v^{\pm 1} \cong 1 + (x + y)z \bmod Lz$, $L = 1$ *if* z *odd,* $L = 2$ *if* z *even.*

In this case, the tangle $\frac{f_1}{g_1}$ *in the equation* $N(U + \frac{f_1}{g_1}) = N(\frac{a}{b})$ *is oppositely oriented if* e_1 *odd and* i_1 *even or both* e_1 *and* g_1 *are even and is similarly oriented if* $e_1 i_1$ *odd or* e_1 *even* g_1 *odd. The* $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ *move is orientation ignoring if and only if* tw *is odd.*

In either case, $U = \frac{a}{b'} \circ (-c_1, ..., -c_n) = \frac{-f_1 b' + e_1 a}{g_1 b' - i_1 a}$.

Recall that we know U is rational when $w = e_1 g_2 - i_1 f_2 \not\cong \pm 1 \bmod t$ and thus the above theorem determines all solutions in this case. Other cases in which U must be rational are given in [9, 10].

Theorem 7. *Let* $\frac{f_1}{g_1} = (c_1, ..., c_n), n$ *odd,* $f_1 = E[c_1, ..., c_n], g_1 = E[c_1, ..., c_{n-1}]$, $e_1 = E[c_2, ..., c_n], i_1 = E[c_2, ..., c_{n-1}]$, $t = g_1 f_2 - g_2 f_1$.

Suppose $w = e_1 g_2 - i_1 f_2 \cong \pm 1 \bmod t$, $L = 1$ *if* z *is odd and* $L = 2$ *if* z *is even.* $N(U + \frac{f_1}{g_1}) = N(\frac{a}{b})$ *and* $N(U + \frac{f_2}{g_2}) = N(\frac{z}{v})$ *as oriented 4-plats where* U *is a generalized M-tangle with the tangle orientations given below if and only if there exists relatively prime integers,* p *and* q, *where* p *may be chosen to be positive, such that* $|z| = |z'|$ *where* $z' = tp(pb - qa) \pm a$ *and if* $\sigma = z/z'$, $v' = \sigma(tq(pb - qa) \pm b)$ *and*

i.) $v'v^{\pm 1} \cong 1 + (b+1)z \bmod Lz$ *if the* $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ *move is orientation preserving or*

ii.) $v'v^{\pm 1} \cong 1 \bmod z$ *if the move is orientation ignoring.*

The $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ *move is orientation ignoring if and only if* $t[q + p(b + 1)]$ *is odd.*

In this case, $U = (U_1 + U_2) \circ (h, -c_1, ..., -c_n))$ *and* $U = (U_2 + U_1) \circ (h, -c_1, ..., -c_n)$ *where* $U_1 = \frac{j}{p}$, $U_2 = \frac{da - jb}{pb - qa}$ *are both solutions for* U, *for all* p, q *satisfying the above, and* d *and* j *are any integers such that* $pd - qj = 1$, *and* $h = \frac{-w \pm 1}{t}$ *(note, the choice of* d *and* j *such that* $pd - qj = 1$ *has no effect on* U*).*

The tangles have the following orientation:

a.) Orientation of $\frac{f_1}{g_1}$ *in the equation* $N(U + \frac{f_1}{g_1}) = N(\frac{a}{b})$:

The tangle $\frac{f_1}{g_1}$ *is similarly oriented if* $h + q + p(b + 1)$ *is even and* e_1 *odd and* i_1 *even or both* e_1 *and* g_1 *are even OR* $h + q + p(b + 1)$ *is odd and* $e_1 i_1$ *odd or* e_1 *even*

g_1 odd.

The tangle $\frac{f_1}{g_1}$ is oppositely oriented if $h+q+p(b+1)$ is even and $e_1 i_1$ odd or e_1 even and g_1 odd OR $h+q+p(b+1)$ is odd and e_1 odd and i_1 even or both e_1 and g_1 are even.

b.) Orientation of U in the equation $N(U+\frac{f_1}{g_1}) = N(\frac{a}{b})$ when a is even: if p is odd, q even or p is even, d is chosen to be even, then $U_1 = \frac{j}{p}$ is similarly oriented OR if pq odd or if p is even and d is chosen to be odd, then $U_1 = \frac{j}{p}$ oppositely oriented.

c.) Orientation of U in the equation $N(U+\frac{f_2}{g_2}) = N(\frac{z}{v})$ when z is even and the move is orientation ignoring: the move involves only one component, and if $v'v^{\pm 1} \cong 1+dz$ mod $2z$, then $U_1 = \frac{j}{p}$ similarly oriented OR if $v'v^{\pm 1} \cong 1+(d+1)z$ mod $2z$, then $U_1 = \frac{j}{p}$ oppositely oriented.

Recall that if $|f_1g_2 - f_2g_1| > 1$, then the above list of solutions to the system of equations, $N(U+\frac{f_1}{g_1}) = N(\frac{a}{b})$ and $N(U+\frac{f_2}{g_2}) = N(\frac{z}{v})$, is complete.

6. Signed Crossing Changes

The signed crossing change distance between two knots/links, $d_{-+}(K_1, K_2)$, is the minimum number of negative crossings that need to be changed to a positive crossing to change K_1 into K_2 without changing any positive crossing to a negative crossing. If K_2 cannot be obtained from K_1 by only changing negative crossings, then the distance between these two knots is said to be ∞. Changing a negative crossing to a positive crossing is equivalent to an $(\frac{f_1}{g_1}, \frac{f_2}{g_2}) = (-1,+1)$ move where both tangles are similarly oriented. Corollary 1 classifies when it is possible to convert one 4-plat into another 4-plat by changing exactly one negative crossing into a positive crossing. Again note that in the following corollary, all $\pm$ signs are in agreement.

Corollary 1. *$N(U+-1) = N(\frac{a}{b})$ and $N(U+1) = N(\frac{z}{v})$ as oriented 4-plats where the tangle -1 is similarly oriented if and only if there exists relatively prime integers, p and q, where p may be chosen to be positive, such that $N(\frac{z}{v}) = N(\frac{2p(pb-qa)\pm a}{2q(pb-qa)\pm b})$ and $h+q+p(b+1)$ is even where $h = \frac{-1\pm 1}{2}$.*

In this case, $U = (U_1+U_2)\circ(h,1)$ and $U = (U_2+U_1)\circ(h,1)$ where $U_1 = \frac{j}{p}$, $U_2 = \frac{da-jb}{pb-qa}$, are both solutions for U, for all p, q satisfying the above, and d and j are any integers such that $pd-qj=1$, (note, the choice of j and d such that $pd-qj=1$ has no effect on U). If a is even, $U_1 = \frac{j}{p}$ in the equation $N(U+\frac{f_1}{g_1}) = N(\frac{a}{b})$ is similarly oriented if p is odd, q even or p is even, d is chosen to be even OR U_1 oppositely oriented if pq odd or if p is even and d is chosen to be odd.

Proof. Apply theorem 7: Since $\frac{f_1}{g_1} = (-1)$, $f_1 = E[-1] = -1$, $g_1 = E[] = 1$, $e_1 = E[] = 1$, $i_1 = 0$. Thus this move is equivalent to a $(\frac{0}{1}, \frac{t}{w})$ move where $\frac{t}{w} = \frac{g_1f_2 - g_2f_1}{e_1g_2 - i_1f_2} = +2$ and the tangle $\frac{0}{1}$ is similarly oriented since $\frac{f_1}{g_1} = (-1)$ is similarly oriented, e_1 is odd, and i_1 is even. The move is orientation preserving since t is even.

Since $|t| > 1$, U is a generalized M-tangle. Thus, $N(U + -1) = N(\frac{a}{b})$ and $N(U+1) = N(\frac{z}{v})$ as oriented 4-plats if and only if there exists relatively prime integers, p and q where p may be chosen to be positive, such that $|z| = |z'|$ where $z' = 2p(pb - qa) \pm a$ and if $\sigma = z/z'$, $v' = \sigma(2q(pb-qa) \pm b)$ and $v'v^{\pm 1} \cong 1 + (b+1)z$ mod Lz, $L = 1$ if z odd, $L = 2$ if z even, since the $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ move is orientation preserving. If z is even, then a is even and hence $b+1$ is even. Thus, $N(\frac{z}{v}) = N(\frac{2p(pb-qa)\pm a}{2q(pb-qa)\pm b})$. The form of U where $h = \frac{-1\pm 1}{2}$ follows from theorem 7. Since tangle $\frac{f_1}{g_1}$ is similarly oriented, e_1 odd, and i_1 even, $h + q + p(b+1)$ must be even $\square$.

Example 1: $d_{+-}(N(\frac{1}{0}), N(\frac{z}{v})) = 1$ if and only if $N(\frac{z}{v}) = N(\frac{2pq\mp 1}{2q^2})$ where p and q are relatively prime integers, $p > 0$, and $h + q + p$ is even where $h = \frac{-1\pm 1}{2}$.

Example 2: $d_{+-}(N(\frac{0}{1}), N(\frac{z}{v})) = 1$ if and only if $N(\frac{z}{v}) = N(\frac{2p^2}{2pq\pm 1})$ where p and q are relatively prime integers, $p > 0$, and $h + q$ is even where $h = \frac{-1\pm 1}{2}$.

Example 3: Solving N(U + -1) = N(5/-1), N(U + 1) = N(7/-3) where the tangle -1 is similarly oriented results in exactly two solutions for U: $U = (\frac{1}{2} + \frac{1}{3}) \circ (-1, 1)$ and $U = (\frac{1}{3} + \frac{1}{2}) \circ (-1, 1)$.

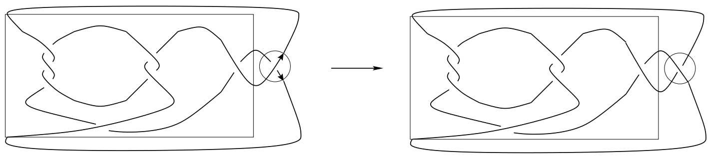

Fig. 17. $N((\frac{1}{3} + \frac{1}{2}) \circ (-1,1) + -1) = N(5/-1) \rightarrow N((\frac{1}{3} + \frac{1}{2}) \circ (-1,1) + 1) = N(7/-3)$.

Acknowledgments

This work was supported by a grant from the Joint DMS/NIGMS Initiative to Support Research in the Area of Mathematical Biology (NIH GM76242).

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