Math 34 Differential Equations Exam #1
September 26, 2008

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[18] 1.) Solve the differential equation $2y' + \frac{y}{t} = t^2$

$$y' + \frac{y}{2t} = \frac{t^2}{2}$$

$$u(t) = e^{\int \frac{1}{2t}} = e^{\frac{1}{2}\ln|t|} = e^{\ln|t|^{1/2}} = |t|^{1/2}$$

$$y' t^{1/2} + \frac{y}{2t^{1/2}} = \frac{t^{5/2}}{2}$$

$$\int (y t^{1/2})' = \int \frac{t^{5/2}}{2}$$

$$y t^{1/2} = \frac{\cancel{2} t^{7/2}}{7 \cdot \cancel{2}} + C$$

$$\boxed{y = \frac{1}{7} t^3 + C t^{-1/2}}$$

Answer 1.) ______

[24] 2.) Solve the differential equation $y'' - 2y' + y = t$, $y(0) = 3$, $y'(0) = 4$.

homog : $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0 \Rightarrow r = 1$

homog sol'n : $y = c_1 e^t + c_2 t e^t$

Non homog guess : $y = At + B$

$y' = A$

$y'' = 0$

$y'' - 2y' + y = t$

$0 - 2A + At + B = t \Rightarrow$ $At = t \Rightarrow A = 1$

$-2A + B = 0 \Rightarrow B = 2A = 2$

General sol'n : $y = c_1 e^t + c_2 t e^t + t + 2$

IVP : $y(0) = 3$: $3 = c_1 + 0 + 0 + 2 \Rightarrow c_1 = 1$

$y' = c_1 e^t + c_2 e^t + c_2 t e^t + 1$

$y'(0) = 4$: $4 = c_1 + c_2 + 0 + 1$

$c_1 = 1$: $4 = 1 + c_2 + 1 = 2 + c_2 \Rightarrow c_2 = 2$

Answer 2.) $y = e^t + 2te^t + t + 2$

[18] 3.) Solve the differential equation $(t^2+t-2)y'y''=1$ for y' in terms of t.

Let $v = y'$, $v' = y''$

$(t^2+t-2)\,v\,v' = 1$

$(t^2+t-2)\,v\,\frac{dv}{dt} = 1$

$v\,dv = \frac{dt}{t^2+t-2}$

$\int v\,dv = \int \frac{-dt}{3(t+2)} + \int \frac{dt}{3(t-1)}$

$\frac{1}{2}v^2 = -\frac{1}{3}\ln|t+2| + \frac{1}{3}\ln|t-1| + C$

$v^2 = \ln|t-1|^{2/3} - \ln|t+2|^{2/3} + C$

$v^2 = \ln\left|\frac{t-1}{t+2}\right|^{2/3} + C$

$(y')^2 = \ln\left|\frac{t-1}{t+2}\right|^{2/3} + C$

$y' = \pm\sqrt{\ln\left|\frac{t-1}{t+2}\right|^{2/3} + C}$

$\frac{1}{t^2+t-2} = \frac{1}{(t+2)(t-1)}$

$= \frac{A}{t+2} + \frac{B}{t-1}$

$\Rightarrow$

$1 = A(t-1) + B(t+2)$

$1 = At - A + Bt + 2B$

$1 = (A+B)t - A + 2B$

$A + B = 0 \Rightarrow B = -A$

$\Rightarrow A + B$

$\Rightarrow 1 = -A + 2B = 3B$

$\Rightarrow B = 1/3,\ A = -1/3$

Answer 3.) $y' = \pm\left(\ln\left|\frac{t-1}{t+2}\right|^{2/3} + C\right)^{1/2}$

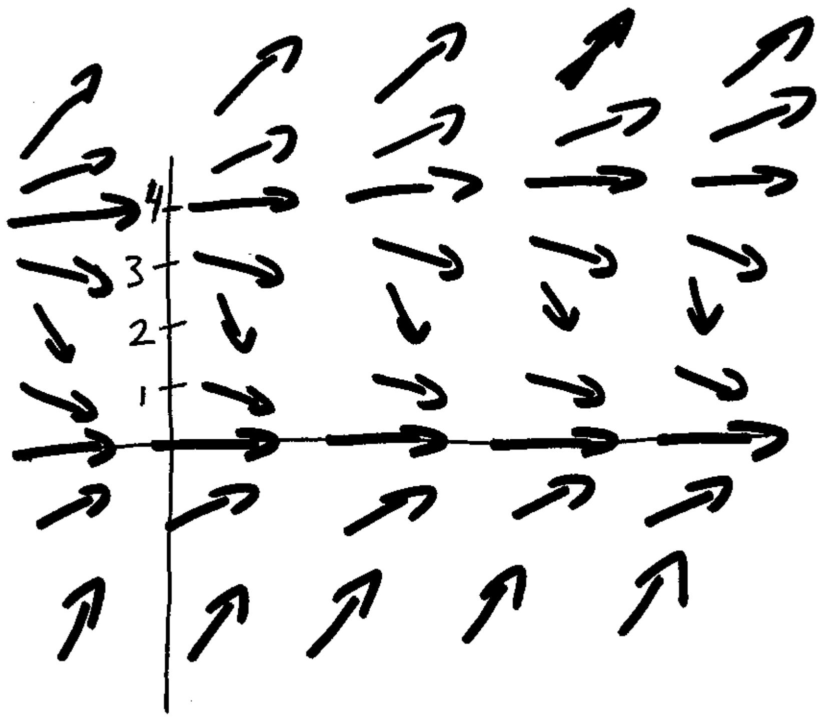
[14] 4.) Draw the direction field for $y' = y(y-4)$. Find the equilibrium solution(s) and determine if asymtptotically stable, semistable, or unstable.

asymptotically unstable

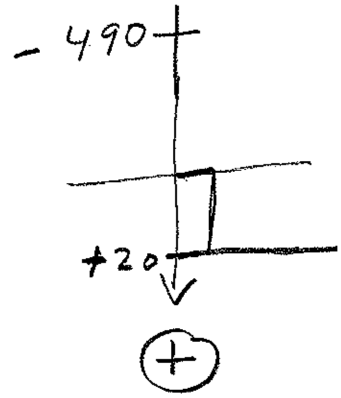
$\boxed{y = 4}$

$\boxed{y = 0}$

asymptotically stable



[14] 5.) A ball with mass 0.3kg is thrown upward with an initial velocity of 98 m/sec from the roof of a building 20m high. If there is no air resistance, find the maximum height above the ground that the ball reaches.



$$F = ma = mg$$

$$\cancel{m}\frac{dv}{dt} = \cancel{m}g$$

$$\Rightarrow v = gt + c$$

$$t=0,\ v=-98:\ -98 = 9.8(0) + c$$

$$\Rightarrow c = -98$$

$$\frac{dx}{dt} = v = 9.8t - 98$$

$$x = 4.9t^2 - 98t + c$$

$$t=0,\ x=0\ ;\quad 0 = 0 + c \Rightarrow c = 0$$

max height: $v = 0 = 9.8t - 98 \Rightarrow t = 10$

$$x = 4.9(100) - 98(10)$$

$$= 490 - 980 = -490$$

$$490 + 20 = 510$$

Answer 5.) 510 m

[14] 6.) Suppose $y = f(t)$ is a solution to $y' + p(t)y = q(t)$. Show that $y = 3f(t)$ is a solution to $y' + p(t)y = 3q(t)$.

$y = f(t)$ solution to

$$y' + p(t)y = q(t)$$

$$\Rightarrow f'(t) + p(t)f(t) = q(t)$$

$$\Rightarrow 3f'(t) + p(t)[3f(t)] = 3q(t)$$

If $y = 3f(t) \Rightarrow y' = 3f'(t)$

Thus $y = 3f(t)$ is a sol'n to $y' + p(t)y = 3q(t)$

Alternate proof:

Define $L(g) = g' + p(t)g$

Note L is a linear function

$y = f(t)$ is a sol'n to $y' + p(t)y = q(t)$

$$\Rightarrow L(f) = q(t)$$

$$\Rightarrow L(3f) = 3L(f) = 3q(t)$$

$\Rightarrow y = 3f$ is a sol'n to $y' + p(t)y = 3q(t)$