Note the following review problems DO NOT cover all problem types which may appear on the final.

6.3 preliminaries:

1a.) Suppose $f(t) = t^2$, then $f(t-2) = \underline{(t-2)^2}$

1b.) Suppose $f(t) = t^2 + 3t + 4$, then $f(t-2) = \underline{(t-2)^2 + 3(t-2) + 4}$

1c.) Suppose $f(t) = sin(t) + e^{8t}$, then $f(t-2) = \underline{sin(t-2) + e^{8(t-2)}}$

2a.) Suppose $f(t-2) = (t-2)^2$, then $f(t) = \underline{t^2}$

2b.) Suppose $f(t-2) = (t-2)^2 + 3(t-2) + 4$, then $f(t) = \underline{t^2 + 3t + 4}$

2c.) Suppose $f(t-2) = sin(t-2) + e^{8(t-2)}$, then $f(t) = \underline{sin(t) + e^{8t}}$

3a.) Suppose $f(t-2) = t^2 + 2t + 5$, then $f(t) = \underline{t^2 + 6t + 13}$

$t^2 + 2t + 5 = (t-2)^2 + 4t - 4 + 2t + 5 = (t-2)^2 + 6t + 1 = (t-2)^2 + 6(t-2) + 12 + 1$
$= (t-2)^2 + 6(t-2) + 13$

Check: $f(t-2) = (t-2)^2 + 6(t-2) + 13 = t^2 - 4t + 4 + 6t - 12 + 13 = t^2 + 2t + 5$

3b.) Suppose $f(t-2) = 3t^2 + 8t + 1$, then $f(t) = \underline{3t^2 + 20t + 29}$

$3t^2 + 8t + 1 = 3(t-2)^2 - 3(-4t + 4) + 8t + 1 = 3(t-2)^2 + 12t - 12 + 8t + 1$
$= 3(t-2)^2 + 20t - 11 =$
$3(t-2)^2 + 20(t-2) + 40 - 11 = 3(t-2)^2 + 20(t-2) + 29$

Check: $f(t-2) = 3(t-2)^2 + 20(t-2) + 29 = 3(t^2 - 4t + 4) + 20t - 40 + 29 = 3t^2 - 12t + 12 + 20t - 11 =$
$3t^2 + 8t + 1$

3c.) Suppose $f(t-2) = cos(t) + e^{8t}$, then $f(t) = \underline{cos(t+2) + e^{8t+16}}$

$cos(t) + 4^{8t} = cos(t - 2 + 2) + e^{8(t-2)+16}$

Check: $f(t-2) = cos(t-2+2) + e^{8(t-2)+16} = cos(t) + e^{8t-16+16} = cos(t) + e^{8t}$

Chapter 6:

4.) Find the LaPlace transform of the following: (used $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$)

4a.) $\mathcal{L}(u_3(t^2 - 2t + 1)) = \underline{e^{-3s}(\frac{2}{s^3} + 4\frac{1}{s^2} + \frac{4}{s})}$

$\mathcal{L}(u_3(t^2 - 2t + 1)) = \mathcal{L}(u_3((t-3)^2 + 6t - 9 - 2t + 1)) = \mathcal{L}(u_3((t-3)^2 + 4t - 8)) = \mathcal{L}(u_3((t-3)^2 + 4(t-3) + 12 - 8)) = \mathcal{L}(u_3((t-3)^2 + 4(t-3) + 4)) = e^{-3s}\mathcal{L}(t^2 + 4t + 4)) = e^{-3s}(\frac{2}{s^3} + 4\frac{1}{s^2} + \frac{4}{s})$

4b.) $\mathcal{L}(u_4(e^{-8t})) = \underline{e^{-4s-32}\frac{1}{s+8}}$

$\mathcal{L}(u_4e^{-8t}) = \mathcal{L}(u_4e^{-8(t-4)-32}) = e^{-4s}\mathcal{L}(e^{-8t-32}) = e^{-4s}e^{-32}\mathcal{L}(e^{-8t}) = e^{-4s-32}\frac{1}{s+8}$

4c.) $\mathcal{L}(u_2(t^2e^{3t})) = \underline{e^{-2s+6}(\frac{2}{(s-3)^3} + \frac{4}{(s-3)^2} + \frac{4}{(s-3)})}$

$\mathcal{L}(u_2(t^2e^{3t})) = \mathcal{L}(u_2([(t-2)^2+4t-4]e^{3(t-2)+6})) = \mathcal{L}(u_2([(t-2)^2+4(t-2)+8-4]e^{3(t-2)+6})) = \mathcal{L}(u_2([(t-2)^2+4(t-2)+4]e^{3(t-2)+6})) = e^{-2s}\mathcal{L}([t^2+4t+4]e^{3t+6})) = e^{-2s}e^6\mathcal{L}([t^2+4t+4]e^{3t})) = e^{-2s}e^6\mathcal{L}(t^2e^{3t} + 4te^{3t} + 4e^{3t})) = e^{-2s+6}(\frac{2}{(s-3)^3} + 4\frac{1}{(s-3)^2} + \frac{4}{(s-3)})$

5.) Find the inverse LaPlace transform of the following: (usually used $u_c(t)f(t-c) = \mathcal{L}^{-1}(e^{-cs}\mathcal{L}(f(t)))$

5a.) $\mathcal{L}^{-1}(e^{-8s}\frac{1}{s-3}) = \underline{u_8(t)e^{3(t-8)}}$

$\mathcal{L}^{-1}(e^{-8s}\frac{1}{s-3}) = u_8f(t-8)$ where

$\mathcal{L}(f(t)) = \frac{1}{s-3})$. Hence $f(t) = \mathcal{L}^{-1}(\frac{1}{s-3}) = e^{3t}$

5b.) $\mathcal{L}^{-1}(e^{4s}\frac{1}{s^2-3}) = \underline{u_{-4}(t)\frac{1}{\sqrt{3}}\ sinh(\sqrt{3}(t+4))}$

$\mathcal{L}^{-1}(e^{4s}\frac{1}{s^2-3}) = u_{-4}(t)f(t+4)$ where

$\mathcal{L}(f(t)) = \frac{1}{s^2-3})$. Hence $f(t) = \frac{1}{\sqrt{3}}\mathcal{L}^{-1}(\frac{\sqrt{(3)}}{s^2-3}) = \frac{1}{\sqrt{3}}sinh(\sqrt{3}t)$

5c.) $\mathcal{L}^{-1}(e^s\frac{1}{(s-3)^2+4}) = \underline{\frac{1}{2}u_{-1}(t)e^{3(t+1)}sin(2(t+1))}$

$\mathcal{L}^{-1}(e^s\frac{1}{(s-3)^2+4}) = u_{-1}(t)f(t+1)$ where

$\mathcal{L}(f(t)) = \frac{1}{(s-3)^2+4})$. Hence $f(t) = \frac{1}{2}\mathcal{L}^{-1}(\frac{2}{(s-3)^2+4}) = \frac{1}{2}e^{3t}sin(2t)$

5d.) $\mathcal{L}^{-1}(e^{-s}\frac{5}{(s-3)^4}) = \underline{u_1(t)\frac{5}{6}(t-1)^3e^{3(t-1)}}$

$\mathcal{L}^{-1}(e^{-s}\frac{5}{(s-3)^4}) = u_1(t)f(t-1)$ where

$\mathcal{L}(f(t)) = \frac{5}{(s-3)^4})$. Hence $f(t) = \frac{5}{6}\mathcal{L}^{-1}(\frac{6}{(s-3)^4}) = \frac{5}{6}t^3e^{3t}$

5e.) $\frac{\mathcal{L}^{-1}(e^s}{4s)=\underline{\frac{1}{4}u_{-1}(t)}}$

$\frac{\mathcal{L}^{-1}(e^s}{4s)}\ \frac{=\frac{1}{4}\mathcal{L}^{-1}(e^s}{s)} = \frac{1}{4}u_{-1}(t)f(t+1)$ where

$\mathcal{L}(f(t)) = \frac{1}{s}$. Hence $f(t) = 1$. Thus $f(t+1) = 1$

5f.) $\mathcal{L}^{-1}(e^s) = \underline{\delta(t+1)}$

6.) Use the definition and not the table to find the LaPlace transform of the following:

6a.) $\mathcal{L}(t^2) = \underline{\frac{2}{s^3}}$

$\int_0^\infty e^{-st}t^2dt = t^2\frac{e^{-st}}{-s}|_0^\infty - \int_0^\infty 2t\frac{e^{-st}}{-s} = lim_{t\to\infty}t^2\frac{e^{-st}}{-s} - 0^2\frac{e^0}{-s} - [2t\frac{e^{-st}}{s^2}|_0^\infty - \int_0^\infty 2\frac{e^{-st}}{s^2}]$
$= 0 - 0 - [lim_{t\to\infty}2t\frac{e^{-st}}{s^2} - 2(0)\frac{e^0}{s^2} - 2\frac{e^{-st}}{-s^3}|_0^\infty] = -[0 - 0 - (lim_{t\to\infty}2\frac{e^{-st}}{-s^3} - 2\frac{e^0}{-s^3})] = (0 - \frac{2}{-s^3})]$
$= \frac{2}{s^3}$

Let $u = t^2,\ dv = e^{-st}$

$du = 2t,\ v = \frac{e^{-st}}{-s}$

$d^2u = 2,\ \int v = \frac{e^{-st}}{s^2}$

6b.) $\mathcal{L}(cos(t)) = \underline{\frac{s}{1+s^2}}$

$\int_0^\infty e^{-st}cos(t)dt = e^{-st}sin(t)|_0^\infty - \int_0^\infty -se^{-st}sin(t)dt$

$= lim_{t\to\infty}e^{-st}sin(t) - e^0sin(0) - [se^{-st}cos(t)|_0^\infty - \int_0^\infty -s^2e^{-st}cos(t)dt]$

$= 0 - 0 - [lim_{t\to\infty}se^{-st}cos(t) - se^0cos(0) + s^2\int_0^\infty e^{-st}cos(t)dt]$

$= -[0 - s + s^2\int_0^\infty e^{-st}cos(t)dt]$

$= s - s^2\int_0^\infty e^{-st}cos(t)dt$

Hence $\int_0^\infty e^{-st}cos(t)dt = s - s^2\int_0^\infty e^{-st}cos(t)dt$

Thus $\int_0^\infty e^{-st}cos(t)dt + s^2\int_0^\infty e^{-st}cos(t)dt = s$

Thus $(1+s^2)\int_0^\infty e^{-st}cos(t)dt = s$

Thus $\int_0^\infty e^{-st}cos(t)dt = \frac{s}{1+s^2}$

Let $u = e^{-st},\ dv = cos(t)$

$du = -se^{-st},\ v = sin(t)$

$d^2u = s^2e^{-st}, \int v = -cos(t)$

7.) Find the inverse LaPlace transform of the following. Leave your answer in terms of a convolution integral:

7a.) $\mathcal{L}^{-1}(\frac{1}{(s-2)(s^2+4)}) = \underline{\frac{1}{2}\int_0^t e^{2(t-s)}sin(2s)ds}$

$\mathcal{L}^{-1}(\frac{1}{(s-2)(s^2+4)}) = \mathcal{L}^{-1}(\frac{1}{(s-2)}) * \mathcal{L}^{-1}(\frac{1}{(s^2+4)}) = e^{2t} * \frac{1}{2}sin(2t) = \frac{1}{2}\int_0^t e^{2(t-s)}sin(2s)ds$

7b.) $\mathcal{L}^{-1}(\frac{1}{(s-2)(s^2-4s+5)}) = \underline{\int_0^t e^{2(t-s}e^{2s}sin(s)ds}$

$\mathcal{L}^{-1}(\frac{1}{(s-2)}) * \mathcal{L}^{-1}(\frac{1}{(s^2-4s+5)}) = \mathcal{L}^{-1}(\frac{1}{(s-2)}) * \mathcal{L}^{-1}(\frac{1}{(s-2)^2+1)}) = e^{2t} * e^{2t}sin(t) = \int_0^t e^{2(t-s)}e^{2s}sin(s)ds$

Note we can easily calculate this integral:

$= \int_0^t e^{2t}e^{-2s}e^{2s}sin(s)ds = \int_0^t e^{2t}sin(s)ds = e^{2t}\int_0^t sin(s)ds = -e^{2t}cos(s)|_0^t = -e^{2t}cos(t) + e^{2t}$

7c.) $\mathcal{L}^{-1}(\frac{2s}{(s-2)(s^2-4s+5)}) = \underline{= 2\int_0^t e^{2(t-s)}(e^{2s}cos(s) + 2e^{2s}sin(s))ds}$

$\mathcal{L}^{-1}(\frac{2s}{(s-2)(s^2-4s+5)}) = 2\mathcal{L}^{-1}(\frac{1}{(s-2)} \cdot \frac{s-2+2}{((s-2)^2+1)}) = 2\mathcal{L}^{-1}(\frac{1}{(s-2)} \cdot [\frac{s-2}{((s-2)^2+1)} + \frac{2}{((s-2)^2+1)}]) = 2e^{2t} * (e^{2t}cos(t) + 2e^{2t}sin(t)) = 2\int_0^t e^{2(t-s)}(e^{2s}cos(s) + 2e^{2s}sin(s))ds$

Note we can easily calculate this integral:

$= 2e^{2t}\int_0^t e^{-2s}(e^{2s}cos(s) + 2e^{2s}sin(s))ds = 2e^{2t}\int_0^t cos(s) + 2sin(s))ds = 2e^{2t}[sin(s) - 2cos(s)]|_0^t = 2e^{2t}[sin(t) - 2cos(t) - (0-2)] = 2e^{2t}[sin(t) - 2cos(t) + 2]$

8.) Find $f * g$

8a.) $4t * 5t^4 = \underline{\frac{2}{3}t^6}$

$\int_0^t 4(t-s)5s^4ds = \int_0^t 20(ts^4 - s^5)ds = 4ts^5 - \frac{20}{6}s^6|_0^t = 4t^6 - \frac{10}{3}t^6 = \frac{2}{3}t^6$

8b.) $5t^4 * 4t = \underline{\frac{2}{3}t^6}$

$5t^4 * 4t = 4t * 5t^4 = \frac{2}{3}t^6$

8c.) $sin(t) * e^t = \underline{\frac{1}{2}(-sin(t) - cos(t) + e^t)}$

$\int_0^t e^{t-s}sin(s)ds = \int_0^t e^te^{-s}sin(s)ds = e^t\int_0^t e^{-s}sin(s)ds = e^t[-e^{-s}sin(s)|_0^t - \int_0^t -e^{-s}cos(s)ds]$

$= e^t[-e^{-t}sin(t) - -e^0sin(0) - \{e^{-s}cos(s)|_0^t - \int_0^t -e^{-s}sin(s)ds\}]$

$= e^t[-e^{-t}sin(t) - \{(e^{-t}cos(t) - e^0cos(0)) - \int_0^t -e^{-s}sin(s)ds\}]$

$= e^t[-e^{-t}sin(t) - e^{-t}cos(t) + 1 - \int_0^t e^{-s}sin(s)ds]$

$= -sin(t) - cos(t) + e^t - e^t\int_0^t e^{-s}sin(s)ds$

Hence $e^t\int_0^t e^{-s}sin(s)ds = -sin(t) - cos(t) + e^t - e^t\int_0^t e^{-s}sin(s)ds$

Hence $2e^t\int_0^t e^{-s}sin(s)ds = -sin(t) - cos(t) + e^t$

Hence $e^t\int_0^t e^{-s}sin(s)ds = \frac{1}{2}(-sin(t) - cos(t) + e^t)$

Let $u = sin(s),\ dv = e^{-s}$

$du = cos(s),\ v = -e^{-s}$

$d^2u = -sin(s),\ \int v = e^{-s}$

Make sure you can also solve a quick differential equation using the LaPlace transform and use any of the formulas on p. 304.

Chapter 3:

9.) Solve the following initial problems:

9a.) $y'' + 6y' + 8y = 0,\ y(0) = 0,\ y'(0) = 0$

Suppose $y = e^{rt}$. Then $y' = re^{rt},\ y'' = r^2e^{rt}$

$r^2e^{rt} + 6re^{rt} + 8e^{rt} = 0$ Hence $r^2 + 6r + 8 = 0$. Thus, (r + 2)(r + 4) = $0.Hence$r = -2, -4

Hence general solution is $y(t) = c_1e^{-2t} + c_2e^{-4t}$

$y(0) = 0 : 0 = c_1 + c_2$. Thus $c_2 = -c_1$

$y'(0) = 0 : y' = -2c_1e^{-2t} - 4c_2e^{-4t}$

$0 = -2c_1 - 4c_2 = 2c_2 - 4c_2 = -2c_2$ Thus $c_2 = 0, c_1 = 0$

Thus, $y(t) = 0$ is the solution to the initial value problem.

9b.) $y'' + 6y' + 9y = 0,\ y(0) = 0,\ y'(0) = 0$

$r^2 + 6r + 9 = (r + 3)(r + 3) = 0$

Hence general solution is $y(t) = c_1e^{-3t} + c_2te^{-3t}$

$y(0) = 0 : 0 = c_1 + c_2(0)$. Thus $c_1 = 0$

$y = c_2te^{-3t}$

$y' = c_2[e^{-3t} - 3te^{-3t}]$

$y'(0) = 0 : 0 = c_2[e^0 - 3(0)e^0]$. Thus $c_2 = 0$.

Thus, $y(t) = 0$ is the solution to the initial value problem.

9c.) $y'' + 6y' + 10y = 0,\ y(0) = 0,\ y'(0) = 0$

$r^2 + 6r + 10 = 0,\ r = \frac{-6\pm\sqrt{36-4(1)(10)}}{2} = \frac{-6\pm 2i}{2} = -3 \pm i$

Hence general solution is $y(t) = c_1e^{-3t}sin(t) + c_2e^{-3t}cos(t)$

$y(0) = 0 : 0 = c_1(0) + c_2(1)$. Thus $c_2 = 0$

$y = c_1e^{-3t}sin(t)$

$y' = c_1[-3e^{-3t}sin(t) + e^{-3t}cos(t)]$

$y'(0) = 0 : 0 = c_1[0 + 1]$. Thus $c_1 = 0$.

Thus, $y(t) = 0$ is the solution to the initial value problem.

Quicker method to solve 9a, b, c:

Note that $y = 0$ was the obvious solution to the initial value problems in 9a, b, c.

Clearly the constant function $y(t) = 0$ satisfies $ay'' + by' + cy = 0$ for any $a,\ b,\ c$. It also satisfies $y(0) = 0,\ y'(0) = 0$. Thus this constant function is a solution to IVP of this type. Since IVP of this type have unique solutions, $y(t) = 0$ is the only solution.

9d.) $y'' + 6y' + 8y = cos(t),\ y(0) = 0,\ y'(0) = 0$

Guess $y = Acos(t) + Bsin(t)$ is a solution.

Then $y' = -Asin(t) + Bcos(t)$ and $y'' = -Acos(t) - Bsin(t)$

$-Acos(t) - Bsin(t) + 6(-Asin(t) + Bcos(t)) + 8(Acos(t) + Bsin(t)) = cos(t)$

$(-A + 6B + 8A)cos(t) + (-B - 6A + 8B)sin(t) = cos(t)$

$(6B + 7A)cos(t) + (-6A + 7B)sin(t) = cos(t) + 0sin(t)$

$cos(t)$ and $sin(t)$ are linearly independent functions,

Hence $(-6A + 7B)sin(t) = 0sin(t)$. Thus $-6A + 7B = 0$ so $A = \frac{7}{6}B$

And $(6B + 7A)cos(t) = cos(t)$. Thus $6B + 7A = 1$, so $6B + 7(\frac{7}{6}B) = 1$. $\frac{36+49}{6}B = \frac{85}{6}B = 1$

Thus $B = \frac{6}{85}$ and $A = \frac{7}{85}$.

Check: $-\frac{7}{85}cos(t) - \frac{6}{85}sin(t) + 6(-\frac{7}{85}sin(t) + \frac{6}{85}cos(t)) + 8(\frac{7}{85}cos(t) + \frac{6}{85}sin(t)) = (-\frac{7}{85} + \frac{36}{85} + \frac{56}{85})cos(t) + (-\frac{6}{85} - \frac{42}{85} + \frac{48}{85})sin(t) = cos(t)$

Thus general solution is $y(t) = c_1e^{-2t} + c_2e^{-4t} + \frac{7}{85}cos(t) + \frac{6}{85}sin(t)$

Use initial values to find $c_1,\ c_2$:

$0 = c_1 + c_2 + \frac{7}{85} + 0$. Hence $c_1 = -c_2 - \frac{7}{85}$

$y'(t) = -2c_1e^{-2t} + -4c_2e^{-4t} - \frac{7}{85}sin(t) + \frac{6}{85}cos(t)$

$0 = -2c_1 - 4c_2 - 0 + \frac{6}{85} = -2(-c_2 - \frac{7}{85}) - 4c_2 + \frac{6}{85} = 2c_2 + \frac{14}{85} - 4c_2 + \frac{6}{85} = -2c_2 + \frac{20}{85}$

Thus $2c_2 = \frac{20}{85}$ or $c_2 = \frac{10}{85}$ and $c_1 = -\frac{10}{85} - \frac{7}{85} = -\frac{17}{85}$

Note these are NOT the same values for c_1 and c_2 for the homogeneous case.

The the solution to the initial value problem is $y(t) = -\frac{17}{85}e^{-2t} + \frac{10}{85}e^{-4t} + \frac{7}{85}cos(t) + \frac{6}{85}sin(t)$

9e.) $y'' + 6y' + 9y = cos(t),\ y(0) = 0,\ y'(0) = 0$

Guess $y = Acos(t) + Bsin(t)$ is a solution.

Then $y' = -Asin(t) + Bcos(t)$ and $y'' = -Acos(t) - Bsin(t)$

$-Acos(t) - Bsin(t) + 6(-Asin(t) + Bcos(t)) + 9(Acos(t) + Bsin(t)) = cos(t)$

$(-A + 6B + 9A)cos(t) + (-B - 6A + 9B)sin(t) = cos(t)$

$(6B + 8A)cos(t) + (-6A + 8B)sin(t) = cos(t) + 0sin(t)$

$cos(t)$ and $sin(t)$ are linearly independent functions,

Hence $(-6A + 8B)sin(t) = 0sin(t)$. Thus $-6A + 8B = 0$ so $A = \frac{8}{6}B = \frac{4}{3}B$

And $(6B + 8A)cos(t) = cos(t)$. Thus $6B + 8A = 1$, so $6B + 8(\frac{4}{3}B) = 1$. $\frac{18+32}{3}B = \frac{50}{3}B = 1$

Thus $B = \frac{3}{50}$ and $A = \frac{8}{100} = \frac{2}{25}$.

Thus general solution is $y(t) = c_1e^{-3t} + c_2te^{-3t} + \frac{2}{25}cos(t) + \frac{3}{50}sin(t)$

Use initial values to find $c_1,\ c_2$:

$0 = c_1 + c_2(0) + \frac{2}{25} + 0$. Hence $c_1 = -\frac{2}{25}$

$y'(t) = -3c_1e^{-3t} + c_2e^{-3t} - 3c_2te^{-3t} - \frac{2}{25}sin(t) + \frac{3}{50}cos(t)$

$0 = -3c_1 + c_2 - 0 - 0 + \frac{3}{50} = -3(-\frac{2}{25}) + c_2 + \frac{3}{50} = \frac{6}{25} + c_2 + \frac{3}{50} = c_2 + \frac{15}{50}$

Thus $c_2 = -\frac{15}{50}$ and $c_1 = -\frac{2}{25}$

Note these are NOT the same values for c_1 and c_2 for the homogeneous case.

The the solution to the initial value problem is

$y(t) = -\frac{2}{25}e^{-3t} - \frac{15}{50}te^{-3t} + \frac{2}{25}cos(t) + \frac{3}{50}sin(t)$

Check: $y(0) = -\frac{2}{25} + 0 + \frac{2}{25} = 0$

$y'(t) = \frac{6}{25}e^{-3t} + \frac{45}{50}te^{-3t} - \frac{15}{50}e^{-3t} - \frac{2}{25}sin(t) + \frac{3}{50}cos(t)$

$y'(0) = \frac{6}{25} + 0 - \frac{15}{50} - 0 + \frac{3}{50} = \frac{12-15+3}{50} = 0$

Thus initial conditions are satisfied. To fully check that this is the solution, should check if $y'' + 6y' + 9y = cos(t)$

9f.) $y'' + 6y' + 10y = cos(t),\ y(0) = 0,\ y'(0) = 0$

Hence general solution to the homogeneous equation is $y(t) = c_1 e^{-3t} sin(t) + c_2 e^{-3t} cos(t)$

Thus $y = Acos(t) + Bsin(t)$ is not a solution to the homogeneous equation.

Guess $y = Acos(t) + Bsin(t)$ is a solution.

Then $y' = -Asin(t) + Bcos(t)$ and $y'' = -Acos(t) - Bsin(t)$

$-Acos(t) - Bsin(t) + 6(-Asin(t) + Bcos(t)) + 10(Acos(t) + Bsin(t)) = cos(t)$

$(-A + 6B + 10A)cos(t) + (-B - 6A + 10B)sin(t) = cos(t)$

$(6B + 9A)cos(t) + (-6A + 9B)sin(t) = cos(t) + 0sin(t)$

$cos(t)$ and $sin(t)$ are linearly independent functions,

Hence $(-6A + 9B)sin(t) = 0sin(t)$. Thus $-6A + 9B = 0$ so $A = \frac{9}{6}B = \frac{3}{2}B$

And $(6B + 9A)cos(t) = cos(t)$. Thus $6B + 9A = 1$, so $6B + 9(\frac{3}{2}B) = 1$. $\frac{12+27}{2}B = \frac{39}{2}B = 1$

Thus $B = \frac{2}{39}$ and $A = \frac{3}{39} = \frac{1}{13}$.

Thus general solution is $y(t) = c_1 e^{-3t} sin(t) + c_2 e^{-3t} cos(t) + \frac{2}{39} cos(t) + \frac{3}{39} sin(t)$

Use initial values to find $c_1,\ c_2$:

$0 = c_1(0) + c_2(1) + \frac{2}{39} + 0$. Hence $c_2 = -\frac{2}{39}$

$y'(t) = -3c_1 e^{-3t} sin(t) + c_1 e^{-3t} cos(t) - 3c_2 e^{-3t} cos(t) - c_2 e^{-3t} sin(t) - \frac{2}{39} sin(t) + \frac{3}{39} cos(t)$

$0 = 0 + c_1 - 3c_2 + 0 - 0 + \frac{3}{39}$ Thus $c_1 = 3c_2 - \frac{3}{39} = 3(-\frac{2}{39}) - \frac{3}{39} = \frac{-12}{39} \frac{-4}{13}$

Note these are NOT the same values for c_1 and c_2 for the homogeneous case.

The the solution to the initial value problem is

$y(t) = \frac{-4}{13} e^{-3t} sin(t) + -\frac{2}{39} e^{-3t} cos(t) + \frac{2}{39} cos(t) + \frac{3}{39} sin(t)$

Check: $y(0) = 0 - \frac{2}{39} + \frac{2}{39} + 0 = 0$

To fully check that this is the solution, should check if $y'' + 6y' + 9y = cos(t)$ and if $y'(0) = 0$.

3.8: 1-5, 7, 11, 14, 3.9: 1 - 8

Make sure you understand sections 3.8, 3.9

10.) Solve the following initial problems:

10a.) $y' + 3y + 1 = 0,\ y(0) = 0$

Method 1: separate variable

$\frac{dy}{dx} = -3y - 1$

$\frac{dy}{-3y-1} = dx$

$\int \frac{dy}{-3y-1} = \int dx$

Let $u = -3y - 1$, $du = -3dy$

$-\frac{1}{3} \int \frac{du}{u} = \int dx$

$-\frac{1}{3} ln(u) = x + C$

$ln(-3y - 1) = -3x + C_1$

$-3y - 1 = e^{-3x+C_1}$

$-3y = e^{-3x}e^{C_1} + 1$

$y = \frac{1}{3}e^{C_1}e^{-3x} - \frac{1}{3}$

$y = Ke^{-3x} - \frac{1}{3}$

$y(0) = 0 : 0 = K - \frac{1}{3}$. Hence $K = \frac{1}{3}$

Thus $y = \frac{1}{3}e^{-3x} - \frac{1}{3}$

Method 2: integrating factor:

$y' + 3y = -1$

Let $u = e^{\int 3dx} = e^{3x}$

$e^{3x}y' + 3e^{3x}y = -e^{3x}$

$(e^{3x}y)' = -e^{3x}$

$\int (e^{3x}y)' = -\int e^{3x}$

$e^{3x}y = -\frac{1}{3}e^{3x} + C$

$y = -\frac{1}{3} + Ce^{-3x}$

$y(0) = 0 : 0 = -\frac{1}{3} + C$. Hence $C = \frac{1}{3}$

Thus $y = \frac{1}{3}e^{-3x} - \frac{1}{3}$

Method 3: LaPlace transform:

$\mathcal{L}(y') + 3\mathcal{L}(y) = -\mathcal{L}(1)$

$s\mathcal{L}(y) + y(0) + 3\mathcal{L}(y) = -\frac{1}{s}$

$(s+3)\mathcal{L}(y) = -\frac{1}{s}$

$\mathcal{L}(y) = -\frac{1}{s(s+3)} = -\frac{1}{3}[\frac{1}{s} - \frac{1}{s+3}]$

$y = -\frac{1}{3}[1 - e^{-3x}]$

10b.) , $y(0) = 0$

*10c.) $cos(t)y' - sin(t)y = \frac{1}{t^2}$, $y(0) = 1$

Note $cos(t)y' - sin(t)y = (cos(t)y)'$

$\int (cos(t)y)' = \int \frac{1}{t^2}dt$

$cos(t)y = -t^{-1} + C = \frac{-1+Ct}{t}$

$y(0) = 0$: $0 = -1 + C$. Hence $C = 1$.

$y = \frac{-1+Ct}{tcos(t)}$

10d.) $y' = \frac{3x^2-2}{xy-xy^2}$, $y(e) = 0$

$\frac{dy}{dx} = \frac{3x^2-2}{x(y-y^2)}$

$(y - y^2)dy = \frac{3x^2-2}{x}dx$

$int(y - y^2)dy = \int[3x - \frac{2}{x}]dx$

$\frac{1}{2}y^2 - \frac{1}{3}y^3 = \frac{3}{2}x^2 - 2ln(x) + C$

$y(0) = 0$: $0 = \frac{3}{2}e^2 - 2ln(e) + C$

$C = 2ln(e) - \frac{3}{2}e^2 = 2 - \frac{3}{2}e^2$

Answer: $\frac{1}{2}y^2 - \frac{1}{3}y^3 = \frac{3}{2}x^2 - 2ln(x) + 2 - \frac{3}{2}e^2$

Chapter 1:

11.) For each of the following, draw the direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = o$, describe this dependency.

11a.) $y' = y$

11a.) $y' = 1$

11a.) $y' = y(y + 4)$

Chapter 7:

12.) Transform the given equation into a system of first order equations:

12a.) $x''' - 2x'' + 3x' - 4x = t^2$

$x_1 = x$

$x_1' = x_2 = x'$

$x_2' = x_3 = x''$

$x_3' = x'''$

Answer: $x_1' = x_2$, $x_2' = x_3$, $x_3' - 2x_3 + 3x_2 - 4x_1 = t^2$

12b.) $x'''' - 2x'' + 3x' - 4x = t^2$

$x_1 = x$

$x_1' = x_2 = x'$

$x_2' = x_3 = x''$

$x_3' = x_4 = x'''$

$x_4' = x''''$

Answer: $x_1' = x_2$, $x_2' = x_3$, $x_3' = x_4$, $x_4' - 2x_3 + 3x_2 - 4x_1 = t^2$

Make sure you also study exam 1 and 2 as well as everything else. Remember the above list is INCOMPLETE.

* means optional type problem. If a problem like 9c appeared on the final, it would be in the "choose" section.