6.6: The Convolution Integral

Defn: The convolution of f and g is the function $f * g$ defined by

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds = \int_0^t f(x)g(t-x)dx$$

Note $*$ is

1.) commutative: $f * g = g * f$

2.) associative: $(f * g) * h = f * (g * h)$

3.) distributive w.r.t +: $f * (g_1 + g_2) = f * g_1 + f * g_2$

4.) $f * 0 = 0 * f = 0$

Example: $cos(t) * 1 =$

Example: $sin(t) * sin(t) \not\geq 0$

Thm: $\mathcal{L}((f * g)(t)) = \mathcal{L}(f(t))\mathcal{L}(g(t))$

Proof:

$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \int_0^\infty e^{-sy} f(y)dy \int_0^\infty e^{-sx} g(x)dx$

$= \int_0^\infty [\int_0^\infty e^{-sy} f(y)dy] e^{-sx} g(x)dx$

$= \int_0^\infty [\int_0^\infty e^{-sy} f(y) e^{-sx} g(x)dy]dx$

$= \int_0^\infty [\int_0^\infty e^{-s(y+x)} f(y)g(x)dy]dx$

$= \int_0^\infty [\int_0^\infty e^{-s(y+x)} f(y)g(x)dx]dy$

Let $t = x + y$, $dt = dx$

$= \int_0^\infty [\int_y^\infty e^{-s(y+t-y)} f(y)g(t-y)dt]dy$

$= \int_0^\infty [\int_y^\infty e^{-st} f(y)g(t-y)dt]dy$

$= \int_0^\infty [\int_0^t e^{-st} f(y)g(t-y)dy]dt$

$= \int_0^\infty e^{-st} [\int_0^t f(y)g(t-y)dy]dt$

$= \int_0^\infty e^{-st} (f * g)(t)dt$

$= \mathcal{L}(f * g)$

Example: $\mathcal{L}^{-1}(\frac{1}{s(s-a)}) =$