Tuesday, April 6, 2010

Chapter 8

Def: Let M be a Seifert surface for K. Choose bicollar $M^\circ \times [-1,1]$. The Seifert form for K is the function $f: H_1(M^\circ) \times H_1(M^\circ) \to \mathbb{Z}$ defined by

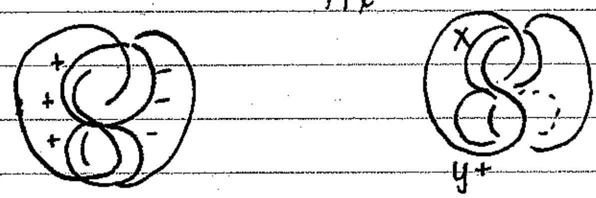
$$f(x,y) = lk(x, y^+)$$

$x = (x, 0)$
$y^+ = (y, 1)$
$y^- = (y, -1)$

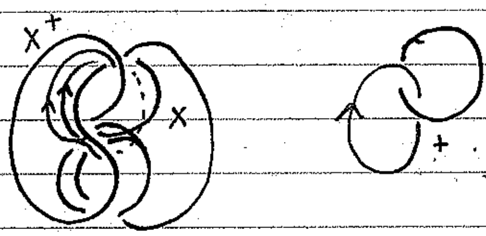
Observe: $lk(x, y^+) = lk(x^-, y) = lk(x^-, y^+)$

$(x,t) \cup (y, (1-t))$
← fix

Ex:



$lk(x, y^+) = 0$



$lk(x, x^+) = 1$
$lk(y, y^+) = 0$
$lk(y, x^+) = -1$

If $e_1, \dots, e_{2g}$ is a basis for $H_1(M^\circ)$ as a $\mathbb{Z}$-module,
← genus

Seifert matrix $= V = (lk(e_i, e_j^+))_{2g \times 2g}$

Ex:
$$\begin{array}{c|cc} & e_1^+ & e_2^+ \\ \hline e_1 & +1 & 0 \\ e_2 & -1 & 0 \end{array}$$

Note: Not an invariant!

Ex 3: f is bilinear

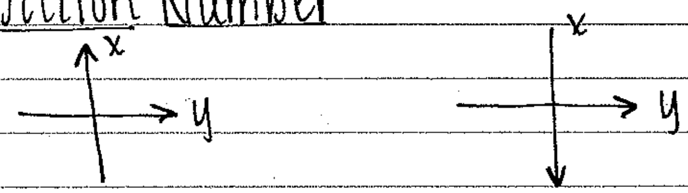
$$lk\left(\sum x_i e_i, \sum y_j e_j^+\right) = \sum\sum x_i y_j \, lk(e_i e_j^+)$$

Let $X = (x_1, \dots, x_{2g})$ & $Y = (y_1, \dots, y_{2g})$

$x = \sum x_i e_i \qquad y = \sum y_j e_j$

$f(x,y) = X \cdot V \cdot Y^T$

Intersection Number



+1   −1

$i(x,y) = \sum$ signed $\cap$

$i : H_1(M^\circ) \times H_1(M^\circ) \longrightarrow \mathbb{Z}$

$\quad (x,y) \longrightarrow i(x,y)$

is the intersection form for $H_1(M^\circ)$

i is bilinear

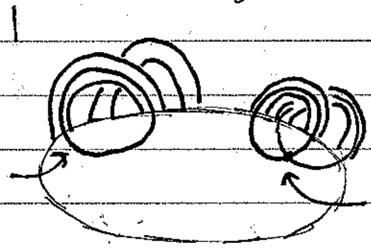
$i(x,y) = X D Y^T$ where $D = (i(e_i, e_j))_{2g \times 2g}$

Consider D^T... $D^T = -D$

$\quad i(y,x) = -i(x,y)$

$\det D = \pm 1$

$0 = i(x,x)$

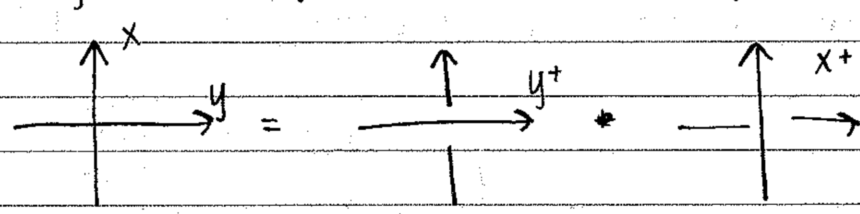


only intersects on main disk, not handles.

M is homeo to ↺

$$D = \left[\begin{array}{cc:cc} 0 & 1 & \multicolumn{2}{c}{0} \\ -1 & 0 & & \\ \hdashline \multicolumn{2}{c:}{0} & 0 & 1 \\ & & -1 & 0 \end{array}\right] \dots$$

$i(X,y) = lk(x, y^+) - lk(y, x^+) = f(x,y) - f(y,x) = V_{ij} - V_{ji}$

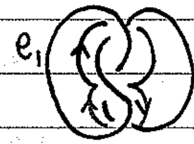


+1   subtract   −1

$\frac{1-(-1) = 2}{2} = 1$

$D = V - V^T$

<u>Corollary</u>: $\det(V - V^T) = \det(D) = \pm 1$



$$D = \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} ?$$

<u>Algebra Presentation Matrices</u>

Let A be a commutative ring with unit.

<u>Ex</u>: $\Lambda = \mathbb{Z}[t^{-1}, t]$ (Laurent polynomials)

Let M be a finitely presented module over A.

<u>Ex</u>: $H_1(\tilde{X})$

$\tilde{X} = S^3 - K$, K a tame knot.

$M = (m_1, \ldots, m_c \mid r_1, \ldots, r_s)$

$r_i = \sum_{j=1} a_{ij} m_j \quad a_{ij} \in A$

Presentation Matrix $P = (a_{ij})_{\substack{i=1,\ldots,s \\ j=1,\ldots,c}}$

↳ determines M ↳ $s \times c$ matrix

We can perform the following operations on P w/o changing M

1. row $i \leftrightarrow$ row j : reordering relators
   col $i \leftrightarrow$ col j : reordering generators
2. row i + a row $j \longrightarrow$ row i : $a \in A$, $i \neq j$: moding relators
3. col i + a col $j \longrightarrow$ col i : $a \in A$, $i \neq j$: change of basis
4. $u \cdot$ row $i \leftrightarrow$ row i ⎫ u a unit in A
   $u \cdot$ col $i \leftrightarrow$ col i ⎭

(5)/(6) $P \leftrightarrow \begin{bmatrix} 1 & * * * \cdots * \\ 0 & \\ \vdots & P \\ 0 & \end{bmatrix}$ ← new relator which can be used to remove new generator.

↳ new generator

(7)/(8) $P \leftrightarrow \begin{bmatrix} P \\ \sum \text{row}_i \end{bmatrix} \longleftrightarrow \begin{bmatrix} P \\ 0 \end{bmatrix}$

<u>Proposition 1</u>: $M_1 \cong M_2$ as A-modules

$\Leftrightarrow P_1 \longleftrightarrow P_2$

$\Leftarrow$ op (1) – (8)

$$\cdots \longrightarrow A^s \xrightarrow{P} \underbrace{A \times \cdots \times A}_{c \text{ copies}} = A^c \xrightarrow[\text{onto}]{\pi} M \longrightarrow 0$$

If P is 1-1, sequence is short exact

$\ker \pi = (a_{i1}, \ldots, a_{ic}) \longrightarrow \sum_{i=1}^{c} a_{ii} m_i = 0$ in M

↑ row of P

$\ker \pi$ generated by $\{row_1, \ldots, row_s\}$ of P

$A^s \xrightarrow{P} A^c$

$(x_1, \ldots, x_s) = (x_1, \ldots, x_s)P$

$(1, 0, \ldots, 0) \rightarrow (1, 0, \ldots, 0)P = row_1$

$\operatorname{Im} P = \ker \pi$

<u>Def</u>: The order ideal of M (module) is the ideal of A generated by all $c \times c$ minors of P, $s \geq c$. If $s < c$, then the order ideal is 0.

<u>Note</u>: If $s = c$, order ideal = det P.

<u>Section 8.C</u>: <u>Alexander Invariant</u> = $H_1(\tilde{X})$ (invariant)

Alexander Matrix = P = presentation matrix (inv. up to ops. (1) – (8))

Alexander Ideal = order ideal $\subset \Lambda$ (invariant)

Alexander Polynomial = a generator of order ideal IF principal (inv. up to mult. by units)

<u>Theorem 3</u>: K tame (always assume)

V = Seifert Matrix

$\Rightarrow P = V^T - tV$

or $V - tV^T$

are both presentation matrices for $H_1(\tilde{X})$

<u>Corollary</u>: $\exists$ square $P \Rightarrow$ Alex. ideal is principal generated by $\Delta(t) = \det(V^T - tV)$