Lecture 1: What is a knot?

Note Title 12/23/2009

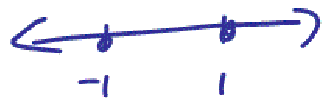
$$S^n = \{x \in R^{n+1} : \ ||x|| = 1\}$$

$$= R^n \cup \{\infty\}$$

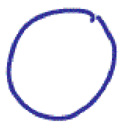
$$= D^n \cup D^n \quad \text{where } D^n = \{x \in R^n : \ ||x|| \leq 1\}$$

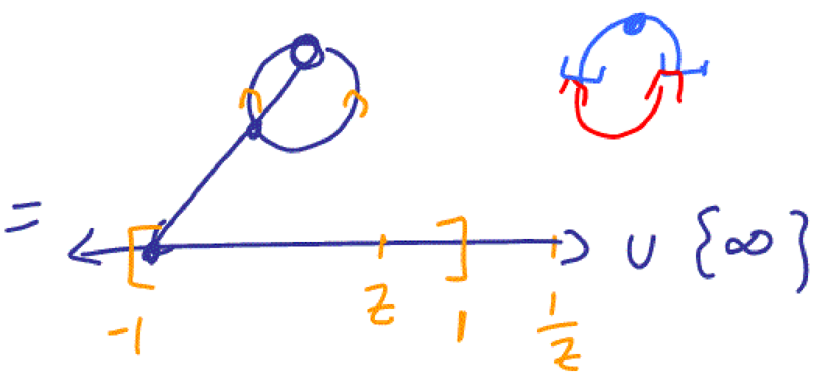
$$= \partial D^{n+1}$$

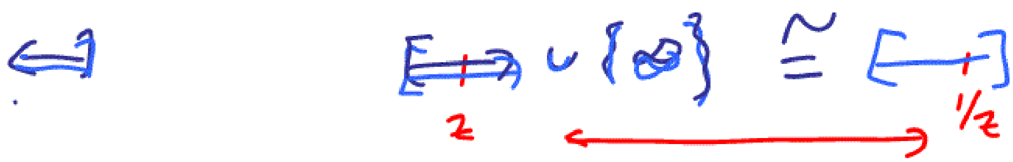
$$S^0 = \{-1, 1\}$$

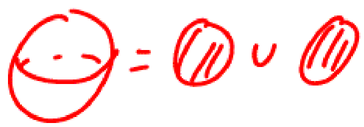


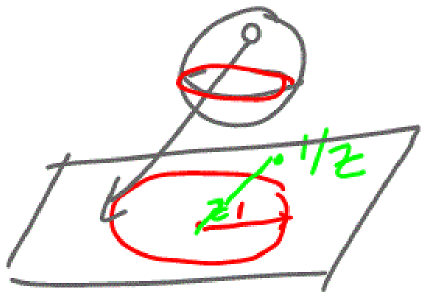
$$S^1 =$$







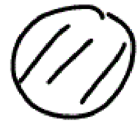
$S^2 =$ 



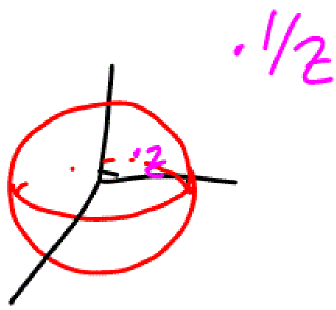
$\cup \{\infty\}$

$\cup \{\infty\}$

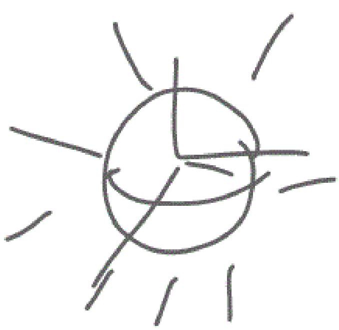
$\| \}$ $z \downarrow 1/z$



$S^3 =$ 



$\cup \{\infty\}$



$\cup \{\infty\} \cong$ 

K is an n-dimensional *knot* if K is homeomorphic to S^n

Alternative definition:

A knot is an *embedding* f: $S^n \rightarrow M$

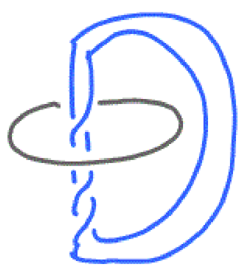
$f(S^n)$

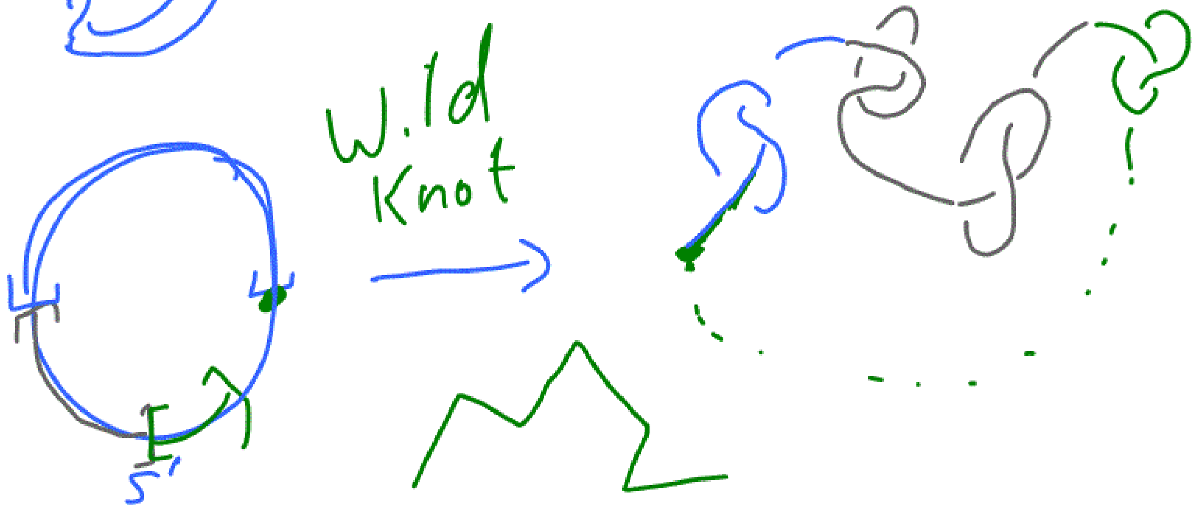
Often $M = S^{n+2}$.

Our main focus will be 1-dimensional knots embedded in S^3, $K: S^1 \rightarrow S^3$

K is a *link* if K is homeomorphic to a disjoint union

$S^n \quad S^n \quad \dots \quad S^n$





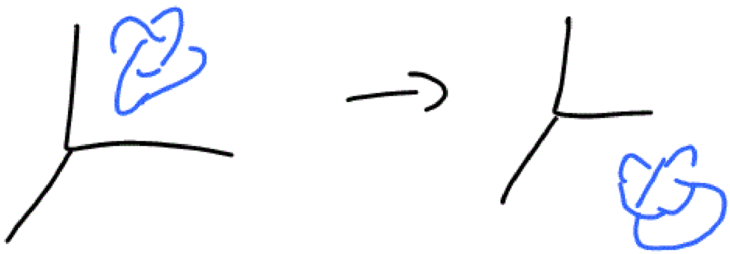
We will work in the Piecewise-Linear (PL) category

(In S^3, PL = smooth) NO WILD KNOTS

Suppose K_1, K_2 are knots in M.

K_1, K_2 are equivalent if there exists a homeomorphism of pairs $h: (M, K_1) \to (M, K_2)$

That is $h: M \to M$ and $h(K_1) = K_2$



$$h(x, y, z) = h(x, y, -z)$$

Mirror image

$$3_1 = 3_1^*$$

Map equivalence: equivalence plus require

$$h \circ K_1 = K_2$$

Oriented equivalence: equivalence plus require

h preserve orientation

$$3_1 \neq 3_1^*$$

Ambient isotopy: time

K_1 and K_2 are ambient isotopic in M if there exists a map h: M x [0, 1] → M such that

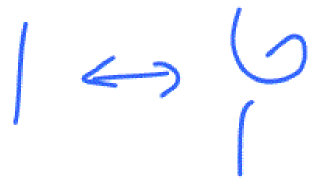
0.) h_t is a homeomorphism for all t in [0, 1] where h_t: M → M, $h_t(x) = h(x, t)$.

1.) h_0 = identity

2.) $h_1(K_1) = K_2$

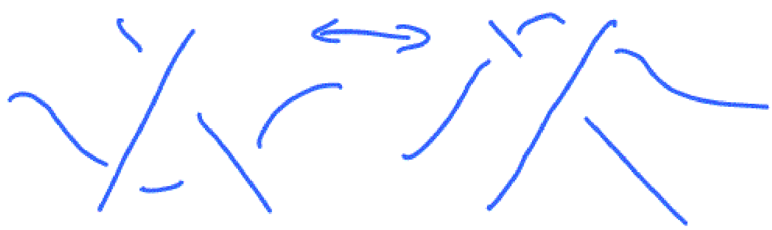
Two knots in S^3 are ambient isotopic iff one can be obtained from the other via a sequence of Reidemeister moves:

R1



R2

R3



A function f: set of knots → X is a knot invariant if

$$f(K_1) = f(K_2) \text{ whenever } K_1 = K_2$$

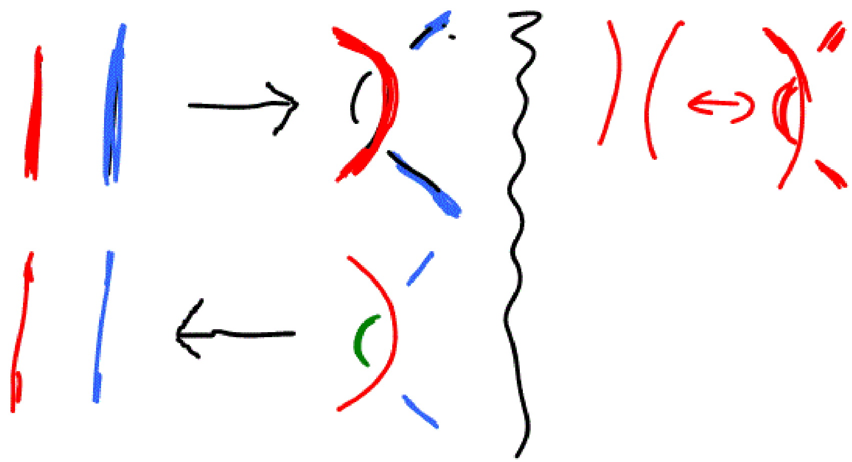
Ex: f: set of knots →{3-colored, not 3 colored}

where f(K) = 3-colored iff the arcs of a diagram of K can be colored with exactly 3 colors such that at each crossing either all three colors appear or only one color appears

Check if c-coloring is a knot invariant

= Check RI, R2, R3

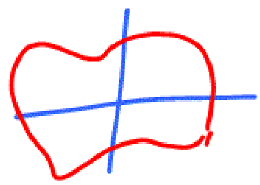
R2:

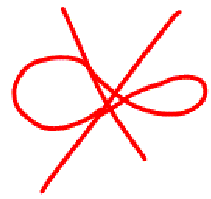


Chapter 2

Note Title 1/19/2010

Jordan Curve Thm: If J is a simple closed curve in R^2, then $R^2 - J$ has two components and J is the boundary of each.





Schonflies Thm: If J is a simple closed curve in R^2, then one of the components of $R^2 - J$ is homeomorphic to the unit disk D^2.

$\cong D^2$

HW 3: Show that $S^2 = D^2 \cup D^2$ where $D^2 \cap D^2 = J$

Lemma (Alexander): Suppose A and B are homeomorphic to D^n. Any homeomorphism $h: \partial A \to \partial B$ extends to a homeomorphism $h: A \to B$.

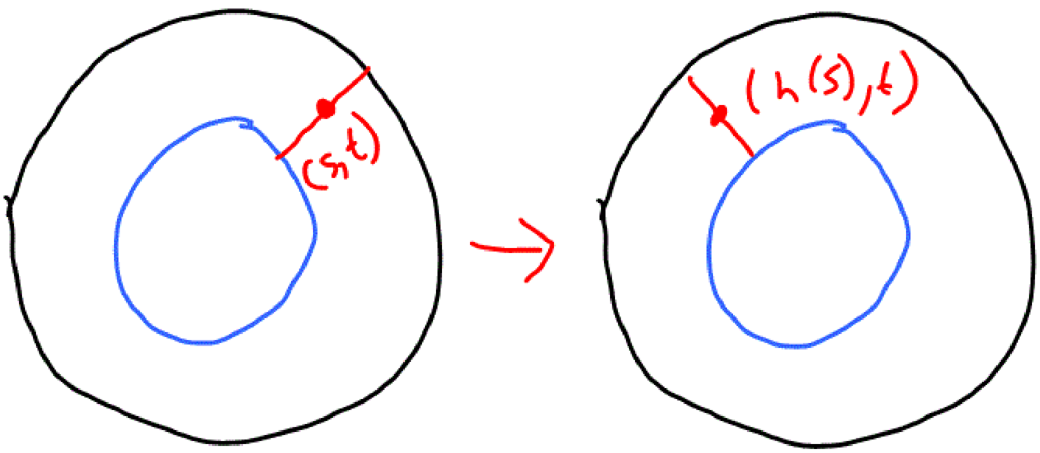
Cor: Any two S^1 knots in S^2 are equivalent.

Annulus Thm: The closure of the region between two disjoint simple closed curves in S^2 is an annulus ($S^1 \times [0, 1]$).

Lemma: Any homeomorphism

h: $S^1 \times 0 \rightarrow S^1 \times 0$ extends to a homeomorphism

h: $S^1 \times [0, 1] \rightarrow S^1 \times [0, 1]$ $(s,t) \rightarrow (h(s),t)$



Cor: Any two links in S^2 are equivalent.

Cor: Any two knots in S^2 are ambient isotopic.