February 8: Ch 2, 5

Note Title 2/7/2010

2D: Mapping class group of the torus:

Let $*: Aut(T^2) \rightarrow GL(2, \mathbf{Z})$, $*(h) = h_*$

where if h(L) = aL + bM and if h(M) = cL + dM, then

$$h_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Questions: why does $\det(h_*) = \pm 1$ (See Ex 7)

why is * a homomorphism?

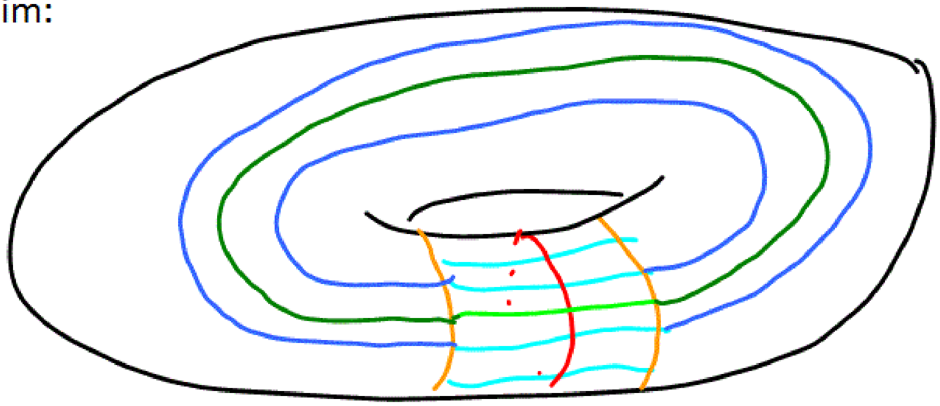
Thm 4: Aut(T2)/ambient isotopies is isomorphic to GL(2, **Z**). Thus two homeomorphisms are ambient isotopic iff they have the same matrix iff they are homotopic as maps.

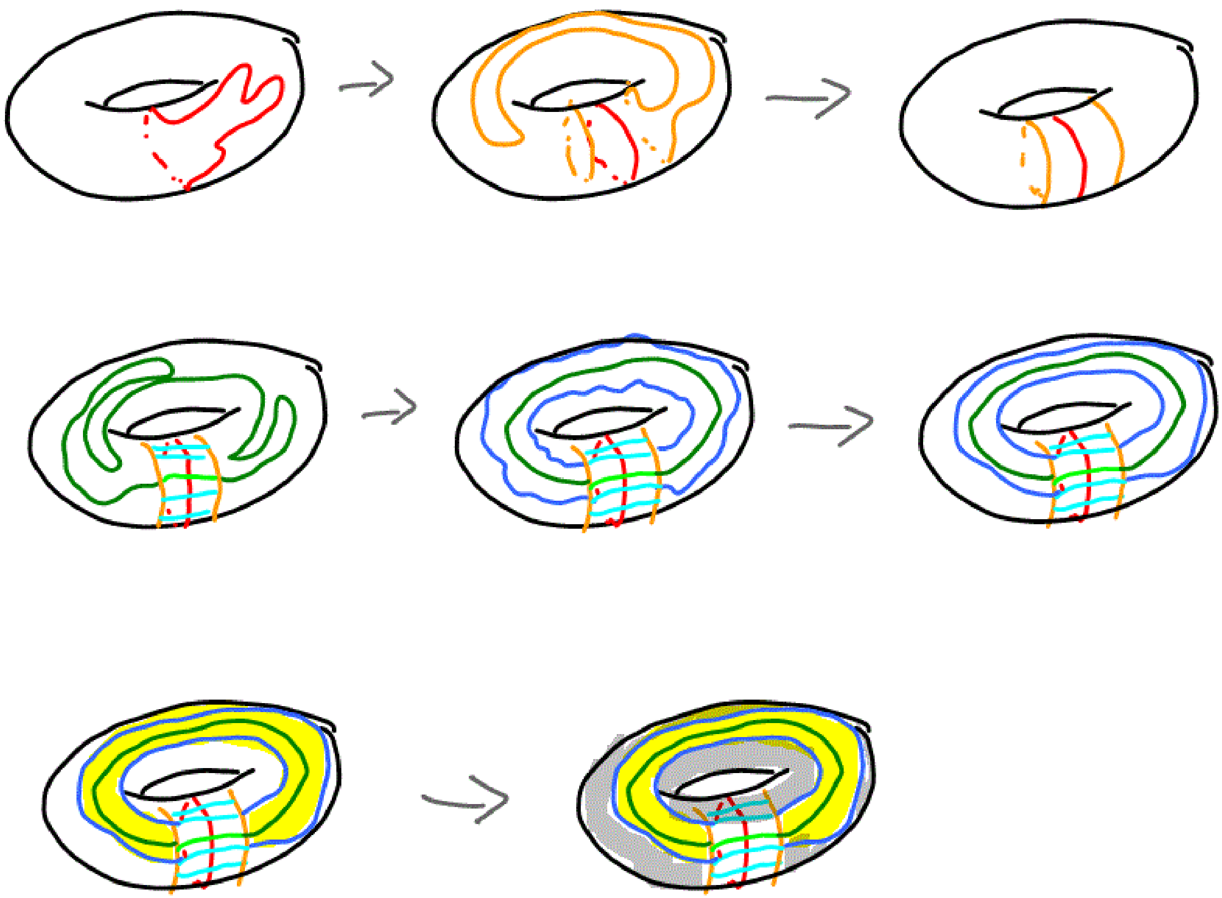
Pf: * is surjective since GL(2, **Z**) is generated by

$$h_{L*} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad h_{M*} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{inversion} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Claim: The kernel of (*) is the subgroup of homeomorphisms of T^2 which are isotopic to the identity.

Proof of Claim:





2E: Solid tori.

Let h: $S^1 \times D^2 \rightarrow V$ be a homeomorphism. Then h is called a *framing* of the solid torus V.

Ex 1: If J is a s.c.c. in ∂V which is essential in ∂V, then TFAE:

a) J is homologically trivial in V
b) J is homotopically trivial in V
c) J bounds a disk
d) for some framing h: $S^1 \times D^2 \rightarrow V$, $J = h(1 \times \partial D^2)$

Defn: THE *meridian* of V is a s.c.c satisfying the above conditions. A *longitude* of V is ANY s.c.c of the form $h(\partial D^2 \times 1)$ for SOME framing h.

Ex 3: If K is a s.c.c. in ∂V, then TFAE:

a.) K is a longitude of V,
b.) K represents a generator of $H_1(V) = \pi_1(V) = \mathbf{Z}$
c.) K intersects some meridian of V (transversely) in a single pt.

Meridian of V: Unique up to ambient isotopy: $\langle 0, \pm 1 \rangle$.

Longitude of V: Unique up to homeomorphism. Infinitely many ambient isotopy classes of longitudes: $\langle \pm 1, b \rangle$.

Ex 5: A homeomorphism f: $\partial V \rightarrow \partial V$ extends to a self-homeo of V if and only if f(M) = M' for some meridians M, M' in V.

Corollary: Suppose $\partial N = T^2$ and h: $\partial V \rightarrow \partial N$ is a homeomorphism, then up to homeomorphism, $N \cup_h V$ is determined by h(M).

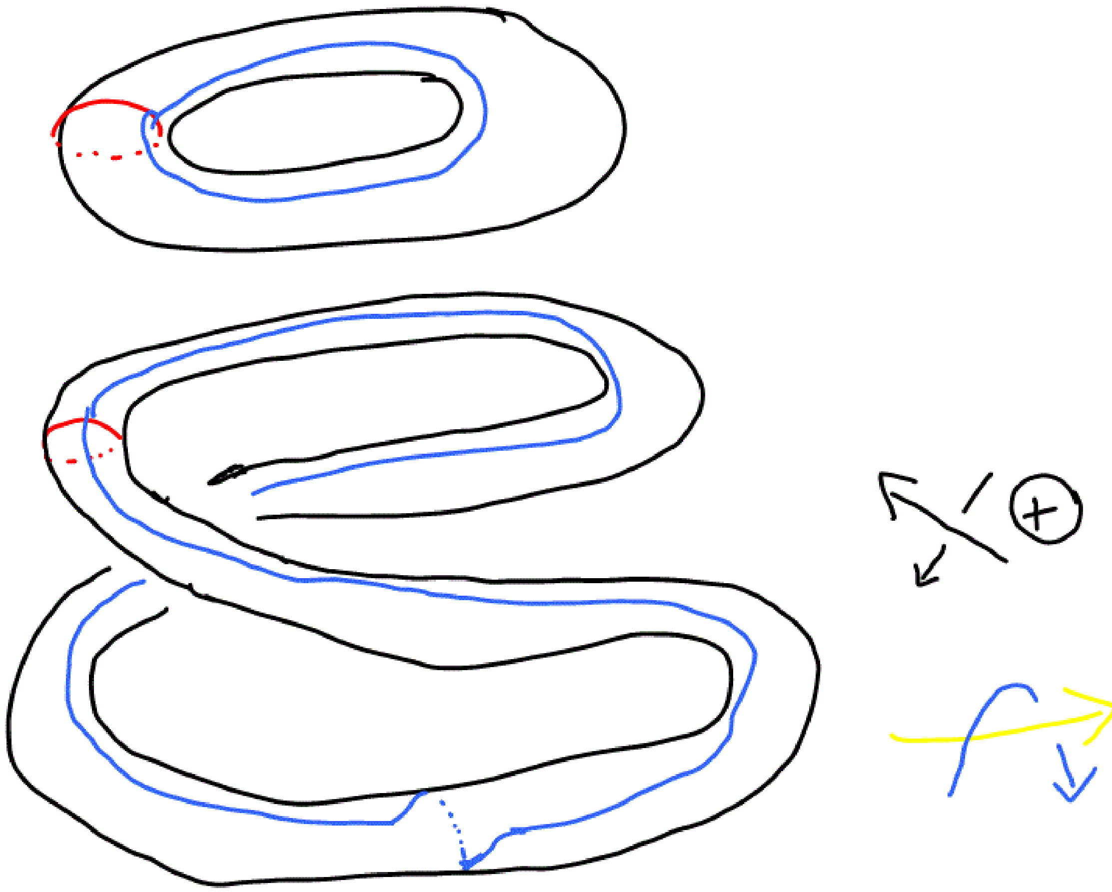
Pf: h is determined by h(M) and h(L), but by ex 5, we only need to know h(M).

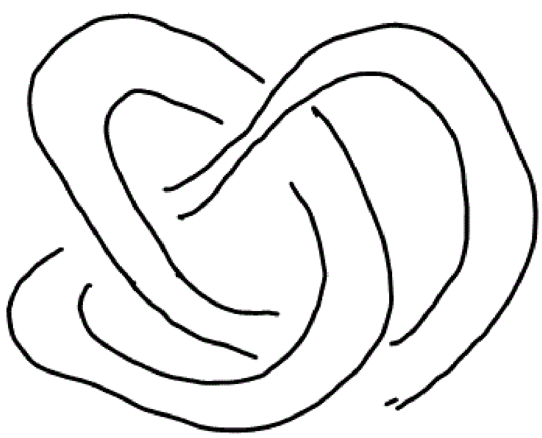
Ex 6: $H_1(S^3 - V) = \begin{cases} Z, & i = 0, 1 \\ 0, & i \geq 2 \end{cases}$

Moreover, a generator of $H_1(S^3 - V)$ is a meridian of V

Ex 7: Up to ambient isotopy of V, there is a unique longitude which is homologically trivial in $S^3 - V$.

Defn: The preferred framing of V is the homeomorphism h: $S^1 \times D^2 \rightarrow V$ such that $h(S^1 \times 1)$ is homologically trivial.





Chapter 5: Seifert surfaces

Classification of 2-manifolds: Every closed orientable 2-manifold is homeomorphic to S^2 or to a connected sum of tori.

manifold	S^2	T^2	$T^2 \# T^2$	…	$T^2 \# \ldots \# T^2$
genus	0	1	2	…	g
χ	2	0	-2	…	2-2g

$$\chi(M) = \Sigma(-1)^i \text{ (rank of } H_i(M)) = \Sigma(-1)^i \text{ (\# of i-simplices)}$$

Defn: If M is an orientable manifold, $g(M) = g(\hat{M})$ where $\hat{M}$ is its associated closed surface.

Ex 5A2: $g(M) = 1 - [\chi(M) + b]/2$ where b = # of ∂ components.

From **Livingston's Knot Theory**:

Thm 1, 2: If 2 connected orientable surfaces, S, S' intersect in a arc contained in their boundary, then

$$\chi(S \cup S') = \chi(S) + \chi(S') - 1$$
$$g(S \cup S') = g(S) + g(S')$$

Thm 3: Let S = a connected orientable surface formed by attaching bands to a collection of disks. Then

$$\chi(S) = \#\text{disks} - \#\text{bands}$$
$$g(S) = [2 - \#\text{disks} + \#\text{bands} - \#\text{boundary components}]/2$$

Thm 5: Every connected surface with boundary is homeomorphic to a surface constructed by attaching bands to a disk.

Thm 6: Two disks with bands attached are homeomorphic if and only if the following conditions are met:

(1) they have the same number of bands,
(2) they have the same number of boundary components,
(3) both are orientable or both are nonorientable.

Suppose $X \subset Y$. X is *bicollared* in Y if there exists an embedding b: $X \times [-1, 1] \rightarrow Y$ such that $b(x, 0) = x$ for all x in X.

A *tubular neighborhood* of a 1-manifold X in a 3-manifold M is an embedding n: $X \times B^2 \rightarrow M$ such that $n(x, 0) = x$ for all x in X.

A *Seifert surface* for a link $K \subset S^3$ is a connected, bicollared, compact manifold $M^2 \subset S^3$ with $\partial M = K$.

Thm 5A4: Every knot has a Seifert surface.

Defn: If K is a knot in S^3, then

$$g(K) = \min\{g(M) \mid M \text{ is a Seifert surface of } K\}$$

Thm 5A14: $g(K_1 \# K_2) = g(K_1)\#g(K_2)$

Cor 5A17: If $K_1 \# K_2$ = unknot, then K_1 = unknot = K_2.

Seifert Van Kampen Thm:
Suppose $X = A \cup B$ and $Y = A \cap B$, where A, B, Y are open, nonempty and path-connected. Suppose

$$\pi_1(A) = \{ a_1, \ldots, a_h \mid r_1, \ldots, r_m \}$$
$$\pi_1(B) = \{ b_1, \ldots, b_k \mid s_1, \ldots, s_n \}$$
$$\pi_1(Y) = \{ y_1, \ldots, y_p \mid t_1, \ldots, t_q \}$$

Then,

$$\pi_1(X) = \{ a_1, \ldots, a_h, b_1, \ldots, b_k \mid r_1, \ldots, r_m, s_1, \ldots, s_n, \quad i(y_1) = j(y_1), \ldots, i(y_p) = j(y_p)\}$$

where $i: A \to X$, $j: B \to X$ are inclusion maps.

$-L = L^r$ = the *reverse* of L = L with the orientation of all its components reversed. L is *invertible* or *reversible* if L = -L.

L* = the mirror image of L with orientation induced by the orientation reversing homeomorphism of S^3.

L is *achiral* if L = L*. L is *chiral* if $L \neq L^*$.

L is (+) *amphichiral* if L = L*. L is (-) *amphichiral* if L = -L*.

The *inverse* of L = -L*

Thm: $\pi_1(K \# K') = \pi_1(-K^* \# -K')$

Thm: If K, K' are prime knots, then $\pi_1(K) = \pi_1(K')$ implies K' = K or K*

For links, $S^3 - L = S^3 - L'$ does NOT imply L = L', BUT see **Kawauchi's A survey of knot theory**.