Exam 1 March 5, 2008 SHOW ALL WORK

Math 28 Calculus III Either circle your answers or place on answer line.

[12] 1.) Let $f(x) = (x^2, ln(5-x^2), 2)$. Let $g(x, y, z) = xyz$. Use the chain rule to calculate $D(f \circ g)(1,2,3)$ and $D(g \circ f)(2)$

$$Df(x) = \begin{pmatrix} 2x \\ \frac{-2x}{5-x^2} \\ 0 \end{pmatrix} \qquad Dg(x,y,z) = \begin{pmatrix} yz & xz & xy \end{pmatrix}$$

Note $f\circ g : \mathbb{R}^3 \to \mathbb{R}^3 \Rightarrow 3\times 3$ matrix

Note $g \circ f : \mathbb{R} \to \mathbb{R} \Rightarrow 1\times 1$ matrix

$$D(f\circ g)\big|_{(1,2,3)} = Df\big|_{g(1,2,3)} \, Dg\big|_{(1,2,3)}$$

$$= Df\big|_{x=6} \, Dg\big|_{(x,y,z)=(1,2,3)}$$

$$= \begin{pmatrix} 12 \\ -12/5-36 \\ 0 \end{pmatrix} \begin{pmatrix} 6 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 12 \\ +12/+31 \\ 0 \end{pmatrix} \begin{pmatrix} 6 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 72 & 36 & 24 \\ +72/31 & 36/31 & 24/31 \\ 0 & 0 & 0 \end{pmatrix} = D(f\circ g)\big|_{(1,2,3)}$$

$$D(g\circ f)\big|_{x=2} = Dg\big|_{f(2)} \, Df\big|_{x=2}$$

$$= Dg\big|_{(4,0,2)} \, Df\big|_{2}$$

$$= \begin{pmatrix} 0 & 8 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix} = 0 - 32 + 0$$

$D(f \circ g)(1,2,3) =$ ____________ $D(g \circ f)(2) =$ -32

2.) Suppose $f(x,y) = ln(xy)$. $Df = \left(\frac{y}{xy} \quad \frac{x}{xy}\right) = \left(\frac{1}{x} \quad \frac{1}{y}\right)$

[10] a.) Find an equation for the tangent plane to the graph of f at the point $(2, \frac{1}{2}, 0)$

x y z

$z = f(\vec{a}) + Df|_a (\vec{x} - \vec{a})$

$z = 0 + \left(\frac{1}{2}, 2\right)\begin{pmatrix} x-2 \\ y - 1/2 \end{pmatrix}$

$$\boxed{z = \frac{1}{2}(x-2) + 2(y - 1/2)}$$

[4] b.) Approximate $f(1.8, 0.6) = \boxed{0.1}$

$h(x,y) = \frac{1}{2}(x-2) + 2(y - 1/2)$ approximates f near $(x,y) = (2, \frac{1}{2})$

$h(1.8, 0.6) = \frac{1}{2}(-0.2) + 2(0.1)$

$= -0.1 + 0.2 = 0.1$

[3] c.) A vector normal to this tangent plane is $\boxed{\left(\frac{1}{2}, 2, -1\right)}$

$0 = \frac{1}{2}x + 2y - z - 2$

[4] 3a.) $proj_{(3,2)}(5,9) = \boxed{\left(\frac{99}{13}, \frac{66}{13}\right)}$

$proj_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a} = \frac{(3,2)\cdot(5,9)}{3^2 + 2^2}(3,2)$

$= \frac{15+18}{9+4}(3,2) = \frac{33}{13}(3,2)$

[5] 3b.) Find a unit vector perpendicular to the vectors (3, 2, 4) and (-1, 5, 0)

$\begin{vmatrix} i & j & k \\ 3 & 2 & 4 \\ -1 & 5 & 0 \end{vmatrix} = (-20, -4, 17)$

unit vector : $\boxed{\left(\frac{-20}{\sqrt{400+16+17^2}}, \frac{-4}{\sqrt{416+17^2}}, \frac{17}{\sqrt{416+17^2}}\right)}$

$$\nabla h = (8x \quad -2y)$$

[27] 4.) Suppose the elevation of is given by $h(x, y) = 4x^2 - y^2$. Suppose you are at the point $(x, y) = (1, 2)$.

[3] a.) $\nabla h(1, 2) =$ $(8 \quad -4)$

[3] b.) What is the direction of steepest ascent? $\dfrac{(8, -4)}{\sqrt{64+16}}$

[3] c.) What is the rate of increase in the direction of steepest ascent? $\sqrt{80}$

$$\sqrt{64+16} = \sqrt{80}$$

[3] d.) what is the direction of steepest descent? $(-8, 4)$

[3] e.) What is the rate of decrease in the direction of steepest descent? $-\sqrt{80}$

[3] f.) What is the direction where there is no change in elevation?

$(4, 8)$ or $(-4, -8)$ or $\left(\dfrac{4}{\sqrt{80}}, \dfrac{8}{\sqrt{80}}\right)$ etc

[3] g.) What is the rate of increase if you travel in the direction (3, 4)? $8/5$

$\dfrac{(3,4)}{\sqrt{9+16}} \to \left(\frac{3}{5}, \frac{4}{5}\right)$ $D_{\left(\frac{3}{5}, \frac{4}{5}\right)} h(1,2) = (8, -4) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \dfrac{24-16}{5}$

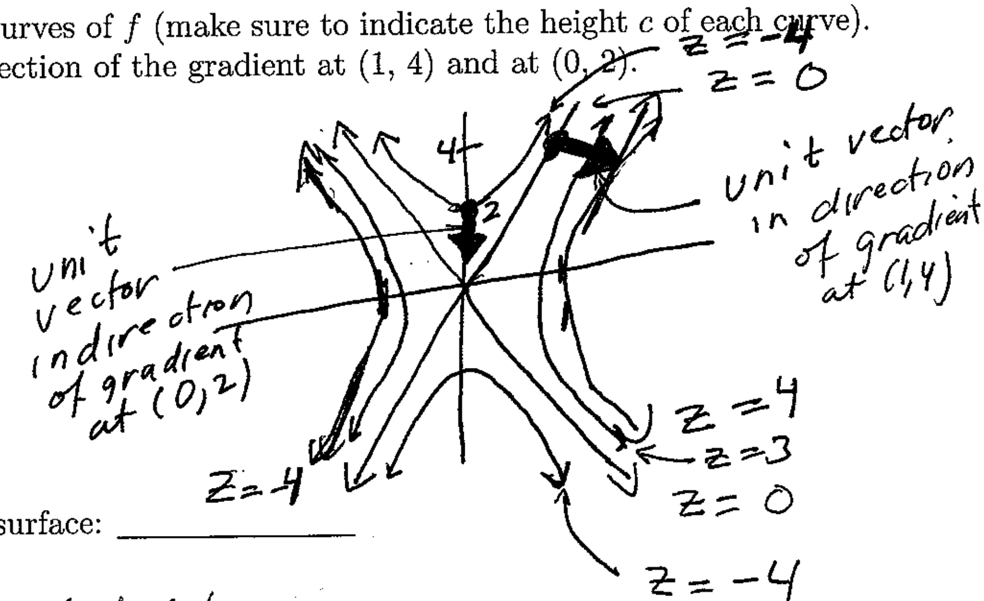
[4] h.) Graph several level curves of f (make sure to indicate the height c of each curve). Draw unit vectors in the direction of the gradient at (1, 4) and at (0, 2).

x	y	$h(x,y)$
1	4	$4-4=0$
0	2	-4
		0

$z = 0$

$4x^2 = y^2$

$y = \pm 2x$



[2] i.) Identify the quadric surface:

Hyperbolic paraboloid

[9] 5.) Find the following limit if it exists. If it doesn't exist, state why you know it doesn't exist.

$$lim_{(x,y)\to(1,0)} = \frac{y(x+3)}{x+y-1}$$

$$\lim_{(x,0)\to(1,0)} \frac{y(x+3)}{x+y-1} = \lim_{x\to 1} \frac{0}{x-1} = 0$$

$$\lim_{(1,y)\to(1,0)} \frac{y(x+3)}{x+y-1} = \lim_{y\to 0} \frac{4y}{y} = 4$$

$0 \neq 4$ so limit doesn't exist since $f(\vec{x})$ needs to approach same value no matter how approach $(x,y)=(1,0)$

[3] 6a.) State the definition of $lim_{\mathbf{x}\to\mathbf{a}} f(x) = \mathbf{L}$.

For all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\vec{x}-\vec{a}\| < \delta \Rightarrow \|f(\vec{x}) - L\| < \varepsilon$

[7] 6b.) Suppose $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x,y) = 2y$. Prove that the $lim_{(x,y)\to(3,4)} f(x,y) = 8$

Let $\varepsilon > 0$. Choose $\delta = \underline{\varepsilon/2}$

Suppose $\|(x,y) - (3,4)\| < \delta$

$\Rightarrow \|(x-3, y-4)\| < \delta = \sqrt{(x-3)^2 + (y-4)^2} < \delta$

$\Rightarrow |x-3| < \delta$ and $|y-4| < \delta$

$\Rightarrow 2|y-4| < 2\delta = 2(\varepsilon/2)$

$\Rightarrow |2y-8| < \varepsilon$

7.) Circle T for True and F for False:

[3] a.) Suppose $f : \mathbf{R}^n \to \mathbf{R}$. If $\frac{\partial f}{\partial x_i}(\mathbf{a})$ exists for $i = 1, ..., n$, then f is differentiable at $\mathbf{a}$. T F

[3] b.) Suppose $f : \mathbf{R}^n \to \mathbf{R}$. If $\frac{\partial f}{\partial x_i}(\mathbf{a})$ exists for $i = 1, ..., n$, then $D_{\mathbf{v}}(f) = \nabla f \cdot \mathbf{v}$. T F

[3] c.) Suppose $f : \mathbf{R}^n \to \mathbf{R}$. If f is differentiable at $\mathbf{a}$, then $D_{\mathbf{v}}(f) = \nabla f \cdot \mathbf{v}$. T F

[3] d.) Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is smooth. Then $\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$ T F

[3] e.) Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable. Then $\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$ T F

[3] f.) If f is differentiable, then f is continuous. T F

[3] g.) If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbf{R}^n$, then $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$. T F

[3] h.) If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbf{R}^n$, then $\mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{v}$. T F