Real-valued function:

If $f : \mathbf{R}^n \to \mathbf{R}$, then $Df = \nabla f = (\frac{\partial f}{\partial x_1}....\frac{\partial f}{\partial x_n})$

Ch 3: Path in $\mathbf{R}^n$:

If $\mathbf{x} : I \subset \mathbf{R} \to \mathbf{R}^n$, then $D\mathbf{x} = [\mathbf{x}'(t)]^T$

If $\mathbf{x}$ represents position,

$\mathbf{v}(t) = \mathbf{x}'(t) =$ velocity vector and

$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{x}''(t) =$ acceleration vector.

Speed of path $= ||\mathbf{x}'(t)||$

If $x : [a, b] \to \mathbf{R}^n$, then

length of path $= L(\mathbf{x}) = \int_a^b ||\mathbf{x}'(t)||dt$

Arclength parameter $= s : [a, b] \to [0, L(\mathbf{x})]$,

$$s(t) = \int_a^t ||\mathbf{x}'(u)||du$$

Example: $\mathbf{x} : [10, 100] \to \mathbf{R}^2$, $\mathbf{x}(t) = (t, 4t)$

velocity vector $= \mathbf{v}(t) = \mathbf{x}'(t) = (1, 4)$ and

acceleration vector $= \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{x}''(t) = (0, 0)$

Speed $= ||\mathbf{x}'(t)|| = ||(1, 4)|| = \sqrt{1^2 + 4^2} = \sqrt{17}$

length of path $= L(\mathbf{x}) = \int_{10}^{100} ||\mathbf{x}'(t)||dt = \int_{10}^{100} \sqrt{17}dt$

$$= \sqrt{17}t|_{10}^{100} = \sqrt{17}(100 - 10) = 90\sqrt{17}$$

Arclength parameter $= s : [10, 100] \to [0, 90\sqrt{17}]$,

$$s(t) = \int_{10}^t \sqrt{17}du = \sqrt{17}u|_{10}^t = \sqrt{17}(t - 10)$$

Thus $s : [10, 100] \to [0, 90\sqrt{17}]$,
$s(t) = t\sqrt{17} - 10\sqrt{17}$

Change of parametrization:

Solve for t to find s^{-1}: $t = \frac{s+10\sqrt{17}}{\sqrt{17}}$.

Let the function s^{-1} be denoted by t

Hence $t : [0, 90\sqrt{17}] \to [10, 100]$, $t(s) = \frac{s+10\sqrt{17}}{\sqrt{17}}$

$\mathbf{y} : [0, 90\sqrt{17}] \to \mathbf{R}^2$, $\mathbf{y}(s) = (\mathbf{x} \circ t)(s)$

$\mathbf{y}(s) = \mathbf{x}(t(s)) = \mathbf{x}(\frac{s+10\sqrt{17}}{\sqrt{17}}) = (\frac{s+10\sqrt{17}}{\sqrt{17}}, 4(\frac{s+10\sqrt{17}}{\sqrt{17}}))$

Thus the reparametrization of the path $\mathbf{x}$ using the arclength parameter s is

$\mathbf{y} : [0, 90\sqrt{17}] \to \mathbf{R}^2$, $\mathbf{y}(s) = (\frac{s+10\sqrt{17}}{\sqrt{17}}, 4(\frac{s+10\sqrt{17}}{\sqrt{17}}))$

Observe $||\mathbf{y}'(s)|| = 1$ and the length of the path traveled by time s is $\int_0^s ||\mathbf{y}'(u)||du = s$

Unit tangent vector: $\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{||\mathbf{x}'(t)||}$

Note $\mathbf{T}(t)$ is perpendicular to $\mathbf{T}'(t)$

outline of proof: Use $\frac{d}{dt}||\mathbf{T}(t)||^2 = \frac{d}{dt}(1)$ to show $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$

Let unit normal vector $= \mathbf{N} = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$

Let unit binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

Note $\mathbf{B}$ is a unit vector.

$\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$ is a set of mutually orthogonal unit vectors called the moving frame (or Frenet frame).

Curvature $\kappa = ||\frac{d\mathbf{T}}{ds}|| = \frac{||\frac{d\mathbf{T}}{ds}||}{\frac{ds}{dt}}$

Torsion τ: $||\frac{d\mathbf{B}}{ds}|| = \frac{||\frac{d\mathbf{B}}{ds}||}{\frac{ds}{dt}} = \tau\mathbf{N}$

Intrinsic quantity: does **NOT** depend on parametrization:

Ex: $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ, τ, length of path.

Extrinsic quantity: depends on parametrization:

Ex: speed, velocity, acceleration