HW 2 (p 83: 4, 8)

4a.) Since $\emptyset, X \in \mathcal{T}_\alpha$ for all α, $\emptyset, X \in \cap\mathcal{T}_\alpha$

Suppose $U_\beta \in \cap\mathcal{T}_\alpha$ for all $\beta \in B$. Then $U_\beta \in \mathcal{T}_\alpha$ for all α, β. Since $\mathcal{T}_\alpha$ is a topology, $\cup_{\beta\in B} U_\beta \in \mathcal{T}_\alpha$ for all α. Thus $\cup_{\beta\in B} U_\beta \in \cap\mathcal{T}_\alpha$

Suppose $U_i \in \cap\mathcal{T}_\alpha$ for $i = 1, ..., n$. Then $U_i \in \mathcal{T}_\alpha$ for all $\alpha, i = 1, ..., n$. Since $\mathcal{T}_\alpha$ is a topology, $\cap_{i=1}^n U_i \in \mathcal{T}_\alpha$ for all α. Thus $\cap_{i=1}^n U_i \in \cap\mathcal{T}_\alpha$

Let $\mathcal{T}_1 = \{\emptyset, \{a,b\}, \{a,b,c\}\}$ and $\mathcal{T}_2 = \{\emptyset, \{b,c\}, \{a,b,c\}\}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}\}$. $\{a,b\} \cap \{b,c\} = \{b\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$. Thus, $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology.

4b.) Lemma: $\cap\mathcal{T}_\alpha$ is the unique largest topology contained in all the $\mathcal{T}_\alpha$.

$\cap\mathcal{T}_\alpha$ is a topology contained in all the $\mathcal{T}_\alpha$. Suppose $\mathcal{T}$ is a topology contained in all the $\mathcal{T}_\alpha$. Then $\mathcal{T} \subset \mathcal{T}_\alpha$ for all α implies $\mathcal{T} \subset \cap\mathcal{T}_\alpha$. Therefore $\cap\mathcal{T}_\alpha$ is larger than or equal to all other topologies contained in all the $\mathcal{T}_\alpha$ and thus $\cap\mathcal{T}_\alpha$ is the unique largest topology contained in all the $\mathcal{T}_\alpha$.

Lemma: $\cup\mathcal{T}_\alpha$ is a subbasis for the unique smallest topology containing all the $\mathcal{T}_\alpha$.

Since $X \in \mathcal{T}_\alpha$, $\cup_{U_\beta \in \cup\mathcal{T}_\alpha} U_\beta = X$. Thus, $\cup\mathcal{T}_\alpha$ is a subbasis.

Let $\mathcal{T}$ be the topology generated by the subbasis $\cup\mathcal{T}_\alpha$. Suppose that $\mathcal{T}'$ is a topology containing all the $\mathcal{T}_\alpha$. Then $\cup\mathcal{T}_\alpha \subset \mathcal{T}'$. If $U \in \mathcal{T}$, then $U = \cup_{\beta\in B}(\cap_{i=1}^n U_{i,\beta})$ where $U_{i,\beta} \in \cup\mathcal{T}_\alpha \subset \mathcal{T}'$. Hence $U \in \mathcal{T}'$, and thus $\mathcal{T} \subset \mathcal{T}'$. Therefore $\mathcal{T}$ is smaller than all or equal to other topologies containing all the $\mathcal{T}_\alpha$ and thus $\mathcal{T}$ is the unique smallest topology containing all the $\mathcal{T}_\alpha$.

4c.) The largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2 = \{\emptyset, \{a\}, \{a,b,c\}\}$. A subbasis for the largest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2 = \mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{a\}, \{a,b\}, \{b,c\}\{a,b,c\}\}$. Thus, the largest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$.

8a.) Let $\mathcal{B} = \{(a,b) \mid a < b, a$ and b rational$\}$. Since (a,b) is open for every $a < b, a$ and b rational, $\mathcal{B}$ is a collection of open sets of R with the standard topology. Suppose that U is an open set in R and $x \in U$. Since $\mathcal{B}' = \{(a,b) \mid a < b, a$ and b real numbersl$\}$ is a basis for the standard topology and U is open, there exists $a, b \in R$, $a < b$ such that $x \in (a,b) \subset U$. Since the rationals are dense in R, there exists c, d such that $a < c < x < d < b$. Thus $x \in (c,d) \subset U$. Since $(c,d) \in \mathcal{B}$, $\mathcal{B}$ is a basis for the standard topology.

8b.) Let $\mathcal{T}$ be the topology generated by $\mathcal{C}$. $[\pi, 4)$ is open in the lower limit topology since it is a basis element, but $[\pi, 4)$ is not open in $\mathcal{T}$. $\pi \in [\pi, 4)$. If $\pi \in [a,b)$ where a, b are rational, then $a \leq \pi < b$. Since a is rational and π is irrational, $a \neq \pi$. Thus $a < \pi < b$. Hence $\frac{a+\pi}{2} \in [a,b)$, but $\frac{a+\pi}{2} \notin [\pi, 4)$. Thus $[a,b) \not\subset [\pi, 4)$. Hence there does not exists a basis element, $[a,b)$ in $\mathcal{C}$ such that $\pi \in [a,b) \subset [\pi, 4)$. Thus $[\pi, 4)$ is not open in $\mathcal{T}$.