22M:100 (MATH:3600:0001) Exam 2 [April 27, 2013] Answers

1.) Circle T for True and F for false.

[4] 1a.) If $y = f(t)$ and $y = g(t)$ are solutions to $y'' + p(t)y' + q(t)y = 0$, then $y = c_1 f(t) + c_2 g(t)$ is also a solution to this differential equation for any constants c_1 and c_2.

T

[4] 1b.) If $\mathbf{x} = \mathbf{f}(t)$ and $\mathbf{x} = \mathbf{g}(t)$ are solutions to $\mathbf{x}' = A\mathbf{x}$, then $\mathbf{x} = c_1\mathbf{f}(t) + c_2\mathbf{g}(t)$ is also a solution to this system of differential equation for any constants c_1 and c_2.

T

[12] 2.) The phase portrait for $\frac{dx}{dt} = x(y-2)$ and $\frac{dy}{dt} = x + 3$ is drawn below. Find all equilibrium solutions and determine whether the critical point is asymptotically stable, stable, or unstable. Also classify it as to type (nodal source, nodal sink, saddle point, spiral source, spiral sink, center).

equilibrium solutions: $\frac{dx}{dt} = x(y-2) = 0$ and $\frac{dy}{dt} = x + 3 = 0$

Hence $[x = 0$ or $y = 2]$ and $[x = -3]$. Thus $(x, y) = (-3, 2)$ is the only solution satisfying both equations,

Equilibrium solution: $\underline{\quad (x, y) = (-3, 2) \quad}$

Stability : $\underline{\quad \textit{stable} \quad}$ Type: $\underline{\quad \textit{center} \quad}$

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[20] 3.) Find a recursive formula for the constants of the series solution to $y'' + 2y = 0$ near $x_0 = 0$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + 2a_n]x^n = 0$

Thus $(n+2)(n+1)a_{n+2} + 2a_n = 0$

Hence $a_{n+2} = \frac{-2a_n}{(n+2)(n+1)}$

Answer: $\underline{\quad a_{n+2} = \frac{-2a_n}{(n+2)(n+1)} \quad}$

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[20] 4.) Suppose that $a_{2k} = \dfrac{-a_{2k-2}}{4k^2}$. Use induction to prove that $a_{2k} = \dfrac{(-1)^k a_0}{4^k (k!)^2}$ for all $k \geq 0$.

For $k = 0$, $\frac{(-1)^0 a_0}{4^0(0!)^2} = a_0 = a_{2(0)}$

Induction hypothesis: Suppose that for $k = n$, $a_{2n} = \frac{(-1)^n a_0}{4^n (n!)^2}$

Claim: for $k = n + 1$: $a_{2(n+1)} = \frac{(-1)^{n+1} a_0}{4^{n+1}((n+1)!)^2}$

$a_{2(n+1)} = a_{2n+2} = \frac{-a_{2n}}{4(n+1)^2}$ by hypothesis.

$= \frac{-(-1)^n a_0}{[4^n (n!)^2][4(n+1)^2]}$ by the induction hypothesis.

$= \frac{(-1)^{n+1} a_0}{4^{n+1}((n+1)!)^2}$

[20] 5.) Solve $\mathbf{x}' = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \mathbf{x}$

Eigenvalues: $det(A - rI) = \begin{vmatrix} -r & 3 \\ -3 & -r \end{vmatrix} = r^2 + 9 = 0$. Thus $r^2 = -9$ and $r = \pm 3i$

Eigenvectors: $A - rI = \begin{pmatrix} \mp 3i & 3 \\ -3 & \mp 3i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Thus $\mp 3ix_1 + 3x_2 = 0$. Let $x_1 = \pm 1, x_2 = i$. Note these values also satisfy the second equation, $-3x_1 \mp 3ix_2$.

Thus an e. vector corresponding to e. value $\pm 3i$ is any nonzero multiple of $\begin{pmatrix} \pm 1 \\ i \end{pmatrix} = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

If you forgot the formula, you can use Euler's formula to derive it:

General solution: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{3it} \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{-3it} \begin{pmatrix} -1 \\ i \end{pmatrix}$

$= c_1(cos(3t) + isin(3t) \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2(cos(-3t) + isin(-3t)) \begin{pmatrix} -1 \\ i \end{pmatrix}$

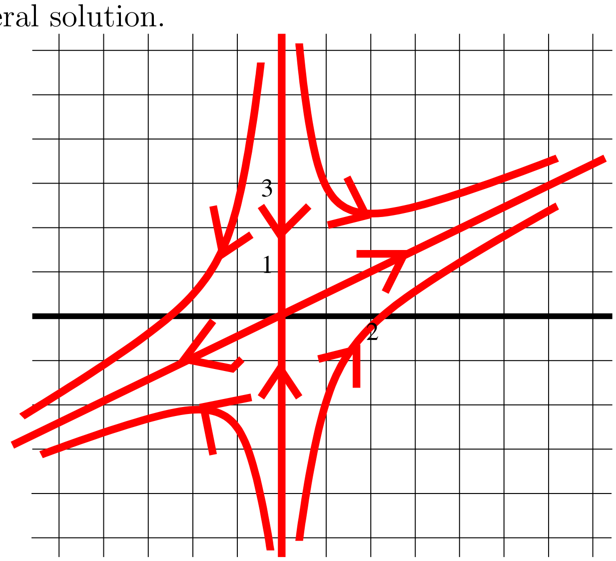
$= c_1(cos(3t) + isin(3t)) \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2(cos(3t) - isin(3t)) \begin{pmatrix} -1 \\ i \end{pmatrix}$

$= c_1 \begin{pmatrix} cos(3t) + isin(3t) \\ icos(3t) - sin(3t) \end{pmatrix} + c_2 \begin{pmatrix} -cos(3t) + isin(3t) \\ icos(3t) + sin(3t) \end{pmatrix} = \begin{pmatrix} (c_1 - c_2)cos(3t) + i(c_1 + c_2)sin(3t) \\ i(c_1 + c_2)cos(3t) - (c_1 - c_2)sin(3t) \end{pmatrix}$

$= \begin{pmatrix} k_1 cos(3t) + k_2 sin(3t) \\ k_2 cos(3t) - k_1 sin(3t) \end{pmatrix} = k_1 \begin{pmatrix} cos(3t) \\ -sin(3t) \end{pmatrix} + k_2 \begin{pmatrix} sin(3t) \\ cos(3t) \end{pmatrix}$

Answer: $\underline{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} cos(3t) \\ -sin(3t) \end{pmatrix} + c_2 \begin{pmatrix} sin(3t) \\ cos(3t) \end{pmatrix}}$

[20] 6.) Suppose the matrix A has eigenvalue -1 with eigenvector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and eigenvalue 4 with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Draw the phase portrait for this system of differential equations. Identify the equilibrium solution. Also state the general solution.



Equilibrium solution is $\underline{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$

General solution is $\underline{\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 0 \\ 3 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$