Section 5.4 continued

Solve $x^2y'' - 2xy' = 0$ (*).

We could solve by letting $v = y'$, but we will instead use 5.4 methods

Note x is an ordinary point iff $x \neq 0$ $\quad (y'' - \frac{2}{x}y' = 0.)$
$x = 0$ is a singular point.

Note $x^2x^{r-2}r(r-1) - 2xx^{r-1}r = 0$ implies $r^2 - r - 2r = 0$ and recall $y = (-x)^r$ gives same equation for r as $y = x^r$.

Thus $y = |x|^r$ implies $r^2 + (\alpha - 1)r + \beta = r^2 - 3r + 0 = r(r-3) = 0$

Thus $r = 0, 3$. Thus $y = |x|^0 = 1$ and $y = |x|^3$ are solutions to (*)

Since (*) is a linear equation, the general solution is $y = c_1 + c_2|x|^3$.

Note an equivalent general solution is $y = k_1 + k_2x^3$.

Both forms are valid for all x.

When is a unique solution to the following initial value problem guaranteed?

$$x^2y'' - 2xy' = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1 \text{ (**)}$$

$$y'' - \frac{2}{x}y' = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

Since $\frac{2}{x}$ and the zero constant function are continuous on $(-\infty, 0) \cup (0, \infty)$,

(**) has a unique solution for $t_0 < 0$ and this solution exists on $(-\infty, 0)$.

(**) has a unique solution for $t_0 > 0$ and this solution exists on $(0, \infty)$.

There are an infinite number of solutions for $y(0) = a$, $y'(0) = 0$.

How is x^r defined:

If n is a positive integer: $x^n = x \cdot x \cdot ... \cdot x$

If m is a positive integer: If $f(x) = x^m$, then $f^{-1}(x) = x^{\frac{1}{m}}$ and $x^{\frac{n}{m}} = (x^n)^{\frac{1}{m}}$

Let $r \geq 0$. Let r_n be any sequence consisting of positive rational numbers such that $lim_{n\to\infty} r_n = r$. Then

$$x^r = lim_{n\to\infty} x^{r_n}.$$

See more advanced class for why the above is well-defined.

If $r < 0$, then $x^r = x^{-r}$.

If x is a real number, when is x^r a real number?

$x^n = x \cdot x \cdot ... \cdot x$ is a real number when n is a positive integer.

If $f(x) = x^n$, then the image of $f = \begin{cases} \text{real numbers} & n \text{ odd} \\ [0, \infty) & n \text{ even} \end{cases}$

Thus if $f^{-1}(x) = x^{\frac{1}{n}}$ is real-valued, then

$$\text{the domain of } f^{-1} \text{ is } \begin{cases} \text{real numbers} & n \text{ odd} \\ [0, \infty) & n \text{ even} \end{cases}$$

In complex analysis, $\left(\frac{1+i\sqrt{3}}{2}\right)^3 = -1$, $(-1)^3 = -1$, $\left(\frac{1-i\sqrt{3}}{2}\right)^3 = -1$

Recall $\left(e^{\frac{i\pi}{3}}\right)^3 = (cos\frac{\pi}{3} + isin\frac{\pi}{3})^3 = -1$

Complex numbers are also roots of unity:

$$\left(e^{\frac{2i\pi}{3}}\right)^3 = 1 \quad \left(e^{\frac{-2i\pi}{3}}\right)^3 = 1, \quad (1)^3 = 1$$

Solve $x^2y'' + \alpha xy' + \beta y = 0$.   Let $y = x^r$,
$y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$ (case when $y = (-x)^r$ is similar).

$x^2x^{r-2}r(r-1) + \alpha x x^{r-1}r + \beta x^r = 0$

$x^r[r^2 - r + \alpha r + \beta] = 0$ for all x implies $r^2 + (\alpha - 1)r + \beta = 0$

Thus x^r is a solution iff $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$

Case 1: Two real roots, r_1, r_2.

General solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$

Case 2: Two complex roots, $r_i = \lambda \pm i\mu$:

Convert solution to form without complex numbers.

Note $|x|^{\pm i\mu} = e^{ln(|x|^{\pm i\mu})} = e^{(\pm i\mu)ln|x|} = e^{i(\pm\mu ln|x|)}$

$$= cos(\pm\mu ln|x|) + isin(\pm\mu ln|x|)$$
$$= cos(\mu ln|x|) \pm isin(\mu ln|x|)$$

General solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2} = c_1|x|^{\lambda+i\mu} + c_2|x|^{\lambda-i\mu}$

$= |x|^\lambda(c_1|x|^{i\mu} + c_2|x|^{-i\mu})$

$= |x|^\lambda(c_1[cos(\mu ln|x|) + isin(\mu ln|x|)] + c_2[cos(\mu ln|x|) - isin(\mu ln|x|)])$

$= |x|^\lambda([c_1 + c_2]cos(\mu ln|x|) + i[c_1 - c_2]sin(\mu ln|x|))$

$= |x|^\lambda(k_1 cos(\mu ln|x|) + k_2 sin(\mu ln|x|))$

$= k_1|x|^\lambda cos(\mu ln|x|) + k_2|x|^\lambda sin(\mu ln|x|)$

Case 3: one repeated root, $r_1 = \frac{-(\alpha-1)}{2}$. (i.e., $\sqrt{(\alpha-1)^2 - 4\beta = 0}$): $\blacksquare$

Thus $|x|^{r_1}$ is a solution. Find 2nd solution.

Method 1. Reduction of order: Suppose $y = u(x)|x|^{r_1}$ is a solution to $x^2y'' + \alpha xy' + \beta y = 0$. Plug in and determine $u(x)$

Method 2: Let $L(y) = x^2y'' + \alpha xy' + \beta y$ where $y' = \frac{dy}{dx}$.
$L(|x|^r) = |x|^r(r - r_1)^2$

$\frac{\partial}{\partial r}[L(|x|^r)] = \frac{\partial}{\partial r}[|x|^r(r - r_1)^2] = (|x|^r)'(r - r_1)^2 + 2|x|^r(r - r_1) = 0$
if $r = r_1$.

Suppose x is constant with respect to r and all the partial derivatives are continuous. Then

$$\frac{\partial}{\partial r}[L(y)] = \frac{\partial}{\partial r}[x^2y'' + \alpha xy' + \beta y] = x^2\frac{\partial y''}{\partial r} + \alpha x\frac{\partial y'}{\partial r} + \beta\frac{\partial y}{\partial r}$$
$$= x^2\frac{\partial}{\partial r}[\frac{\partial^2 y}{\partial x^2}] + \alpha x\frac{\partial}{\partial r}[\frac{\partial y}{\partial x}] + \beta\frac{\partial y}{\partial r}$$
$$= x^2\frac{\partial^2}{\partial x^2}[\frac{\partial y}{\partial r}] + \alpha x\frac{\partial}{\partial x}[\frac{\partial y}{\partial r}] + \beta\frac{\partial y}{\partial r}$$
$$= L(\frac{\partial y}{\partial r}) \text{ for all } r$$

$L(\frac{\partial|x|^r}{\partial r}) = \frac{\partial}{\partial r}[L(|x|^r)] = 0$ for $r = r_1$.

$\frac{\partial|x|^r}{\partial r} = \frac{\partial e^{ln|x|^r}}{\partial r}\ \frac{\partial e^{rln|x|}}{\partial r} = (e^{rln|x|})ln|x| = |x|^r ln|x|$

Thus $|x|^{r_1}ln|x|$ is a solution.

Thus general solution is $y = c_1|x|^{r_1} + c_2|x|^{r_1}ln|x|$

since by the Wronskian, $|x|^{r_1}$ and $|x|^{r_1}ln|x|$ are linearly independent. Suppose $x > 0$ and $r_1 \neq 0$.

$$\begin{vmatrix} x^{r_1} & x^{r_1}ln|x| \\ r_1x^{r_1-1} & r_1x^{r_1-1}ln|x| + x^{r_1-1} \end{vmatrix}$$

$$= x^{r_1}(r_1x^{r_1-1}ln|x| + x^{r_1-1}) - x^{r_1}ln|x|r_1x^{r_1-1}$$
$$= x^{2r_1-1}[r_1ln|x| + 1 - ln|x|r_1] = x^{2r_1-1} \neq 0 \text{ for } x \neq 0$$

Other cases for Wronskian are similar.