5.3: Series solutions near an ordinary point, part II

A power series solution exists in a neighborhood of x_0 when the solution is analytic at x_0. I.e, the solution is of the form $y = \Sigma_{n=0}^{\infty} a_n(x - x_0)^n$ where this series has a nonzero radius of convergence about x_0.

That is $f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x - x_0)^n$ for x near x_0.

Thus there are constants $a_n = \frac{f^{(n)}(x_0)}{n!}$ such that,

$$f(x) = \Sigma_{n=0}^{\infty} a_n(x - x_0)^n.$$

When do we know an analytic solution exists? I.e, when is this method guaranteed to work?

Special case: $P(x)y'' + Q(x)y' + R(x)y = 0$

Then $y''(x) = -[\frac{Q}{P}y' + \frac{R}{P}y]$

$y'''(x) = -[(\frac{Q}{P})'y' + \frac{Q}{P}y'' + \frac{R}{P}'y + \frac{R}{P}y']$

If $f(x) = \Sigma_{n=0}^{\infty} a_n(x - x_0)^n$ is a solution where $a_n = \frac{f^{(n)}(x_0)}{n!}$, then $a_0 = f(x_0), a_1 = f'(x_0)$

$2!a_2 = f''(x_0) = -[\frac{Q}{P}f'(x_0) + \frac{R}{P}f(x_0)] = -[\frac{Q}{P}a_1 + \frac{R}{P}a_0]$

$3!a_3 = f'''(x_0) = -[(\frac{Q}{P})'f'(x_0) + \frac{Q}{P}f''(x_0) + \frac{R}{P}'f(x_0) + \frac{R}{P}f'(x_0)]$

To find a_n we could continue taking derivative including derivatives of $\frac{Q}{P}$ and $\frac{R}{P}$ (but much easier to plug series into equation – ie 5.2 method).

Definition: The point x_0 is an *ordinary point* of the ODE,

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

if $\frac{Q}{P}$ and $\frac{R}{P}$ are analytic at x_0. If x_0 is not an ordinary point, then it is a *singular point.*

Theorem 5.3.1: If x_0 is an ordinary point of the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$, then the general solution to this ODE is

$$y = \Sigma_{n=0}^{\infty} a_n(x - x_0)^n = a_0\phi_0(x) + a_1\phi_1(x)$$

where ϕ_i are power series solutions that are analytic at x_0. The solutions ϕ_0, ϕ_1 form a fundamental set of solutions. The radius of convergence for each of these series solutions is at least as large as the minimum radii of convergence of the series for $\frac{Q}{P}$ and $\frac{R}{P}$.

Theorem: If P and Q are polynomial functions with no common factors, then $y = Q(x)/P(x)$ is analytic at x_0 if and only if $P(x_0) \neq 0$. Moreover the radius of convergence of $Q(x)/P(x)$ is

$$min\{||x_0 - x|| \mid x \in \mathbf{C}, P(x) = 0\}$$

where $||x_0 - x||$ = distance from x_0 to x in the complex plane.

Ex: $x(x + 1)y'' + \frac{x^2}{x^2+1}y' + \frac{x}{x-2}y = 0$

$y'' + \frac{x}{(x^2+1)(x+1)}y' + \frac{1}{(x-2)(x+1)}y = 0$

Then $x_0 = -1, 2$ are singular points. All other points are ordinary points.

The zeros of the denominators are $x = \pm i, -1, 2$

Radius of convergence for the series solution to this ODE about the point x_0 if $x_0 \neq -1, 2$ is at least as large as minimum$\{\sqrt{x_0^2 + (\pm 1)^2}, |x_0 - (-1)|, |x_0 - 2|\}$

If $x_0 = 0$, radius of convergence ≥ 1

If $x_0 = -3$, radius of convergence ≥ 2

If $x_0 = 3$, radius of convergence ≥ 1

If $x_0 = \frac{1}{3}$, radius of convergence $\geq \sqrt{(\frac{1}{3})^2 + (\pm 1)^2} = \frac{\sqrt{10}}{3}$

5.4: Euler equation: $x^2y'' + \alpha xy' + \beta y = 0$

Let $L(y) = x^2y'' + \alpha xy' + \beta y$

Recall that L is a linear function and if f is a solution to the euler equation, then $L(f) = 0$.

Note that if $x \neq 0$, then x is an ordinary point and if $x = 0$, then x is a singular point.

Suppose $x > 0$. Claim $L(x^r) = 0$ for some value of r

$y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$x^2y'' + \alpha xy' + \beta y = 0$

$x^2r(r-1)x^{r-2} + \alpha xrx^{r-1} + \beta x^r = 0$

$(r^2 - r)x^r + \alpha rx^r + \beta x^r = 0$

$x^r[r^2 - r + \alpha r + \beta] = 0$

$x^r[r^2 + (\alpha - 1)r + \beta] = 0$

Thus x^r is a solution iff $r^2 + (\alpha - 1)r + \beta = 0$

Thus $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$

Suppose $x < 0$. Claim $L((-x)^r) = 0$ for some value of r

$y = (-x)^r$, $y' = -r(-x)^{r-1}$, $y'' = r(r-1)(-x)^{r-2}$

$x^2y'' + \alpha xy' + \beta y = 0$

$x^2r(r-1)(-x)^{r-2} - \alpha xr(-x)^{r-1} + \beta(-x)^r = 0$

$(r^2 - r)(-x)^r + \alpha r(-x)^r + \beta(-x)^r = 0$

$(-x)^r[r^2 - r + \alpha r + \beta] = 0$

$(-x)^r[r^2 + (\alpha - 1)r + \beta] = 0$

Thus $(-x)^r$ is a solution iff $r^2 + (\alpha - 1)r + \beta = 0$

Thus $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$

Recall $|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$

Thus $|x|^r = \begin{cases} x^r & \text{if } x > 0 \\ (-x)^r & \text{if } x < 0 \end{cases}$

Thus if $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$, then $y = |x|^r$ is a solution to Euler's equation for $x \neq 0$.

Case 1. 2 real distinct roots, r_1, r_2:

General solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$.

Case 2: 2 complex solutions $r_i = \lambda \pm i\mu$:

Convert solution to form without complex numbers.

Note $|x|^{\lambda \pm i\mu} = e^{ln(|x|^{\lambda\pm i\mu})} = e^{(\lambda\pm i\mu)ln|x|} = e^{\lambda ln|x|}e^{i(\pm\mu ln|x|)}$

$= |x|^\lambda[cos(\pm\mu ln|x|) + isin(\pm\mu ln|x|)]$

$= |x|^\lambda[cos(\mu ln|x|) \pm isin(\mu ln|x|)]$

Case 3: 1 repeated root: Find 2nd solution.