Let $f(x)$ be a given function, and assume it has derivatives around some point $x = a$ (with as many derivatives as we find necessary). For the error in the Taylor polynomial $p_n(x)$, we have the formulas

\[
f(x) - p_n(x) = \frac{1}{n!} \int_a^x (x - t)^n f^{(n+1)}(t) \, dt = \frac{1}{(n+1)!} (x - a)^{n+1} f^{(n+1)}(c_x)
\]

The point $c_x$ is restricted to the interval bounded by $x$ and $a$, and otherwise $c_x$ is unknown. We will use the second form of this error formula, although the first is more precise in that you do not need to deal with the unknown point $c_x$. 
Consider the special case of $n = 0$. Then the Taylor polynomial is the constant function:

$$f(x) \approx p_0(x) = f(a)$$

The first form of the error formula becomes

$$f(x) - p_0(x) = f(x) - f(a) = (x - a) f'(c_x)$$

with $c_x$ between $a$ and $x$. You have seen this in your beginning calculus course, and it is called the **mean-value theorem**. The error formula

$$f(x) - p_n(x) = \frac{1}{(n + 1)!} (x - a)^{n+1} f^{(n+1)}(c_x)$$

can be considered a generalization of the mean-value theorem.
EXAMPLE: \( f(x) = e^x \)

For general \( n \geq 0 \), and expanding \( e^x \) about \( x = 0 \), we have that the degree \( n \) Taylor polynomial approximation is given by

\[
p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n
\]

For the derivatives of \( f(x) = e^x \), we have

\[
f^{(k)}(x) = e^x, \quad f^{(k)}(0) = 1, \quad k = 0, 1, 2, \ldots
\]

For the error,

\[
e^x - p_n(x) = \frac{1}{(n + 1)!}x^{n+1}e^{c_x}
\]

with \( c_x \) located between 0 and \( x \). Note that for \( x \approx 0 \), we must have \( c_x \approx 0 \) and

\[
e^x - p_n(x) \approx \frac{1}{(n + 1)!}x^{n+1}
\]

This last term is also the final term in \( p_{n+1}(x) \), and thus

\[
e^x - p_n(x) \approx p_{n+1}(x) - p_n(x)
\]
Consider calculating an approximation to \( e \). Then let \( x = 1 \) in the earlier formulas to get

\[
p_n(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}
\]

For the error,

\[
e - p_n(1) = \frac{1}{(n + 1)!} e^{c_x}, \quad 0 \leq c_x \leq 1
\]

To bound the error, we have

\[
e^0 \leq e^{c_x} \leq e^1, \quad \frac{1}{(n + 1)!} \leq e - p_n(1) \leq \frac{e}{(n + 1)!}
\]

To have an approximation accurate to within \( 10^{-5} \), choose \( n \) s.t.

\[
\frac{e}{(n + 1)!} \leq 10^{-5}
\]

which is true if \( n \geq 8 \). In fact,

\[
e - p_8(1) \leq \frac{e}{9!} \doteq 7.5 \times 10^{-6}
\]

Then calculate \( p_8(1) \doteq 2.71827877 \), and \( e - p_8(1) \doteq 3.06 \times 10^{-6} \).
\[
\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x}
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!}
\]

\[+ (-1)^{m+1} \frac{x^{2m+2}}{(2m+2)!} \cos c_x \]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{m-1} \frac{x^{2m-1}}{(2m-1)!}
\]

\[+ (-1)^m \frac{x^{2m+1}}{(2m+1)!} \cos c_x \]

with \(c_x\) between 0 and \(x\).
Most Taylor polynomials are found by means other than using the formula
\[ p_n(x) = f(a) + (x - a)f'(a) + \frac{1}{2!}(x - a)^2f''(a) + \cdots + \frac{1}{n!}(x - a)^n f^{(n)}(a) \]
because of the difficulty of obtaining the derivatives \( f^{(k)}(x) \) for larger values of \( k \). Actually, this is now much easier, as we can use Maple or Mathematica. Nonetheless, most formulas have been obtained by manipulating standard formulas; and examples of this are given in the text.
For example, use

\[ e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots + \frac{1}{n!}t^n + \frac{1}{(n+1)!}t^{n+1}e^{ct} \]

in which \( ct \) is between 0 and \( t \). Let \( t = -x^2 \) to obtain

\[ e^{-x^2} = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \cdots + \frac{(-1)^n}{n!}x^{2n} + \frac{(-1)^{n+1}}{(n+1)!}x^{2n+2}e^{-\xi x} \]

Because \( ct \) must be between 0 and \( -x^2 \), we have it must be negative. Thus we let \( ct = -\xi x \) in the error term, with \( 0 \leq \xi x \leq x^2 \).