Regularity Theory

Lihe Wang
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Chapter 1

Schauder Estimates

The main topic of this chapter is to show that, due to Schauder, if

\[-\Delta u = f \quad \text{in} \quad B_1 \quad (1.1)\]

and if \(f\) is Hölder continuous, then \(u\) is \(C^{2,\alpha}_{\text{loc}}\), i.e. for any \(1 \leq i, j \leq n\), \(u_{ij}\) is Hölder continuous with the same exponent. This is the so-called the Schauder estimates.

We will give three closely related proofs. The most elementary and geometric one is the maximum principle approach, followed by the energy method and the most important, flexible compactness method.

Let us examine Schauder’s theorem more closely. Clearly, it is a perfect theorem: if \(u\) was \(C^{2,\alpha}\), its second order derivatives and hence its Laplacian are \(C^\alpha\).

Next let us examine the estimates with respect to scaling:

\[-\Delta \frac{u(rx)}{r^2} = f(rx)\]

Hence it is expected that

\[\Delta \frac{u(rx) - u(0) - r\nabla u(0) \cdot x}{r^2} \approx f(0)\]

We see that \(\frac{u(rx) - u(0) - r\nabla u(0) \cdot x}{r^2}\) so does the second order derivative satisfies an equation with right hand side close to a constant and the closeness is getting better as how small the \(r\) is. All the proofs are based on this observation.

This leads us to look at the questions how we can say a function is \(C^{2,\alpha}\). Let us recall the definition of continuity: a function is continuous if it is
more and more close to a constant in a smaller and smaller neighborhood, where the closeness is characterized by the fact that the error, which is the popular \( o(1) \), is smaller than any positive constant. Similarly, the definition of differentiability goes like that a function is differentiable if it is more and more close to a linear function in smaller and smaller neighborhoods, where the closeness to a linear function is characterized by saying that the error, which is the popular \( o(|x|) \), between the function and the linear function is smaller than any linear function.

Now let us recall the definition of \( C^\alpha \) continuity. A function \( u \) is \( C^\alpha \) at 0 if
\[
|u(x) - u(0)| \leq C|x|^\alpha.
\]
Here \( C|x|^\alpha \) is of course smaller than any constants. It is little bit more than continuity. It gives a qualitative control of the speed of how the function approaches the constant.

It seems puzzling about why we use \( C^\alpha \) and \( C^{2,\alpha} \). The reason is that like most of mathematics, it is hard to say non qualitative things. Continuity is a very unstable process. However \( C^\alpha \) is a very stable measurement: any uniform limit of functions with the same Hölder continuity is still enjoying the same Hölder continuity.

Similarly we can define \( C^{1,\alpha} \), \( C^{2,\alpha} \). A function is \( C^{2,\alpha} \) at 0, for example, if there is a second order polynomial \( P(x) \) such that
\[
|u(x) - P(x)| \leq C|x|^{2+\alpha}.
\]
It says that \( u \) can be approximated by a second order polynomial and the approximation is getting better and and better as \( x \) goes to 0. \( P \) is also the Taylor polynomial at 0 and the main thing is that the qualitative control of error of the approximation as \( C|x|^{2+\alpha} \).

Now let us examine the difficulty. The main difficulty is that it is hard to get any information about the second order derivatives of the solutions. The equation says and only says that the sum of some second order derivatives is under control. This information certainly is not sufficient for us to determine the second order derivatives. It can be seen later that the second order derivatives actually depend, given the equation, on the boundary values. However, our problem is not to find them. We just want to show that the solution has a second order polynomial approximation and to show the order of approximation is like \( |x|^{2+\alpha} \).

The trivial and psychologically suggestive case is when \( f = 0 \). In this case \( u \) is harmonic and we can take \( P \) as the Taylor polynomial of \( u \). We
have seen from the previous chapter that any order derivatives of a harmonic function are under control.

Now consider the case when $f$ is small. First we need a good guess about the second order polynomial. This leads us to the conclusion if $f$ is small, we should able to say the solution is almost harmonic. That is the solution is close to a harmonic function. Then we have the first approximation, which we simply take as the second order polynomial of the harmonic function.

Here we has not talk about in what sense that $f$ is small and in what sense the solution is close to harmonic. Actually it turns out that no matter how we measure the smallness or closeness, we always obtain the same conclusion at the end. This equivalence is exactly the content of the Morrey or Campanato embedding theorem. All theorems are equivalent.

We will use $L^\infty$ norm to measure the smallness when we use the maximum principle approach, $H^1$ for energy approach and $L^2$ for the compactness.

## 1.1 The Maximum Principle Approach

In this section we give the most elementary proof for Schauder estimates. One of the drawback of this approach is that we have to assume solvability of Dirichlet problem and assume that the solution is $C^2$. However, we actually show that if $u$ is $C^2$ than $u$ is $C^{2,\alpha}$ and moreover that its $C^{2,\alpha}$ norm has a bound which is independent of of its $C^2$ norm. In other words, we obtain a $C^{2,\alpha}$ aprori estimate.

We can overcome this shortcoming, if we use the existence theory of viscosity solutions and the corresponding maximum principle. However, in doing that we have to develop a lot of machinery, which we will postpone to later chapters.

**Lemma 1.1** Suppose $u$ is a $C^2$ solution of $-\Delta u = f(x)$ in $B_1$. Then

$$u(x) \leq \sup_{\partial B_1} u + \frac{1 - |x|^2}{2n} \sup_{B_1} f \text{ for } x \in B_1. \quad (1.2)$$

**Proof.** Apply the maximum principle to the subharmonic function

$$u(x) - \sup_{\partial B_1} u - \frac{1 - |x|^2}{2n} \sup_{B_1} f.$$
Corollary 1.2 Suppose $u$ is a $C^2$ solution of (4.1) in $B_1$. Then
\[
\|u\|_{L^\infty(B_1)} \leq \|u\|_{L^\infty(\partial B_1)} + \frac{1}{2n} \|f\|_{L^\infty(B_1)}.
\] (1.3)

Proof. Apply Lemma 1.1 to $-u$, we obtain,
\[
u(x) \geq \inf_{\partial B_1} u + \frac{1}{2n} \inf_{B_1} f.
\] (1.4)

Clearly (4.3) follows from (4.2) and (4.4).

This Corollary says that the solution is in $L^\infty$ norm is controlled by a multiple of the $L^\infty$ norm of its Laplacian and the $L^\infty$ of its boundary data.

Now we start to prove the $C^{2,\alpha}$ estimates. We will show the discrete version first.

Lemma 1.3 For any $0 < \alpha < 1$, there exist constants $C_0$, $0 < \lambda < 1$, $\varepsilon_0 > 0$ such that for any function $f$ and solution of $\Delta u = f$ in $B_1$ with $|u| \leq 1$ and $\|f\|_{L^\infty(B_1)} \leq \varepsilon_0$, there is a second order harmonic polynomial
\[p(x) = \frac{1}{2} x^T A x + B x + C\]
such that
\[|u(x) - p(x)| \leq \lambda^{2+\alpha} \text{ for } |x| \leq \lambda\]
and
\[|A| + |B| + |C| \leq C_0,\]
where $C_0$ is a universal constant.

Let us first explain why we need the above statement. The essence of the theorem about $C^{2,\alpha}$ regularity of the solution is that the solution is almost a second order polynomial. We want to show that the solution, after subtract a second order polynomial, decays as $C|x|^{2+\alpha}$. These are clearly infinitely many inequalities, for all $x \in B_1$.

Now it comes one of the most important techniques in modern analysis. Instead of showing these uncountable infinitely many inequalities, we show these inequality in countable rings or balls with sizes as a geometric series. We first show that the solution is close to a second polynomial such that the error, measured only in some smaller ball $B_\lambda$, decays as $C|x|^{2+\alpha}$. The beauty lies in that we show this only up to a fixed size of balls, not all the way up to the origin 0. In this way, we reduce the burden of the proof a lot.
1.1. THE MAXIMUM PRINCIPLE APPROACH

Let us explain it further. The following is clearly equivalent:

(1) For all $x \in B_1$,

$$|u(x) - P(x)| \leq C|x|^{2+\alpha}.$$ 

(2) For $k = 0, 1, \ldots$,

$$|u(x) - P(x)| \leq C\lambda^{k(2+\alpha)} \text{ for } |x| \leq \lambda^k.$$ 

The key is to show (2) for $k = 1$ first and reduce the general $k$ to $k = 1$ by scaling. Mathematically (1) means that the second order Taylor polynomial of $u$ is exactly $P$ at 0. It is very rigid. It is very hard to find such $P$ for general $u$. The advantage of (2) is that it behaves well under small errors in $P$, where as (1) is rigid. In practice, we will always to prove (2) in the following form,

(3) For $k = 0, 1, \ldots$,

$$|u(x) - P_k(x)| \leq C\lambda^{k(2+\alpha)} \text{ for } |x| \leq \lambda^k,$$

provided that we could provide the convergence of $P_k$, which in general is the case since the fast convergence of $P_k$ to $u$.

$P_k$ is usually born to be convergent or as

$$|P_k(x) - P_{k-1}(x)| \leq 2C\lambda^{k(2+\alpha)} \text{ for } |x| \leq \lambda^k.$$ 

We will find $P_k$ is an approximation of $u$ in $B_{\lambda^k}$ where $f$ is very much approximated by a constant. Once $f$ is also like a constant, the solution will close to a harmonic function and (3) is proved by taking $P_k$ as the Taylor polynomial of the harmonic function. This approximation of $P_k$ frees us from the rigidity $P$ and gives us enough freedom so we can find $P_k$. Then the estimates for $B_{\lambda^k}(0)$ gave better and better approximation. These approximations are improved iteratively, which is much easier to an one step shot of (1).

This key lemma gives approximation in $B_{\lambda}$. Later we will repeat the same procedure to estimate $B_{\lambda^2}$, $B_{\lambda^3}$ and so on. This requires countably many steps.

Then there is another beautiful feature of PDEs. They behave nicely under scaling. Scaling, we mean to consider the equation in a blow-up the coordinates

$$v(x) = u(\lambda^k x).$$

In general we can find a similar equation for $v$ as that for $u$. The estimates from $B_{\lambda^k}$ to $B_{\lambda^{k+1}}$ is exactly the same as those from $B_1$ to $B_{\lambda}$. This scaling
consideration reduces the burden of the proof further from countably many steps to one step.

**Proof of Lemma 1.3** Let \( h \) be the harmonic function such that \( h = u \) on \( \partial B_1 \). From Corollary 1.2,

\[
|u(x) - h(x)| \leq \frac{1}{2n} \sup_{B_1} |f(x)|.
\]

Clearly \( |h| \leq 1 \) in \( B_1(0) \) by the maximum principle and \( |u| \leq 1 \). Hence its second order Taylor polynomial at 0

\[
p(x) = \frac{1}{2} x^T A x + B \cdot x + C
\]

has universal bounded coefficients and moreover it is harmonic.

By the mean value theorem, we have for \( x \in B_{\frac{1}{2}} \),

\[
|h(x) - p(x)| \leq \frac{1}{6} |x|^3 \sup_{B_{\frac{1}{2}}(x)} |\nabla^2 h|.
\]

By Theorem 3.18, we have \( \sup_{B_{\frac{1}{2}}(x)} |\nabla^3 h(x)| \leq C \) for some universal constant \( C \). Finally, we get

\[
|u(x) - p(x)| \leq |h(x) - p(x)| + \frac{1}{2n} \sup_{B_1(0)} |f(x)|
\]

\[
\leq \frac{C}{6} |x|^3 + \frac{1}{2n} \sup_{B_1(0)} |f|.
\]

Now, taking \( \lambda \) small enough, such that the first term is less or equal to

\[
\frac{1}{2} \lambda^{2+\alpha} \quad \text{for} \quad |x| \leq \lambda
\]

and then, taking \( \epsilon_0 \) such that

\[
\frac{1}{2n} \sup_{B_1(0)} |f| \leq \frac{1}{2n} \epsilon_0 \leq \frac{\lambda^{2+\alpha}}{2},
\]

the lemma follows.
Theorem 1.4 Suppose $u$ is a $C^2$ solution of (4.1) and suppose $f$ is Hölder continuous at 0, i.e.,

$$[f]_{C^0}(0) = \sup_{|x| \leq 1} \frac{|f(x) - f(0)|}{|x|^\alpha} < +\infty.$$  

Then $u(x)$ is $C^{2,\alpha}$ at 0, i.e. there is a second order polynomial $P(x) = \frac{1}{2} x^T A x + B \cdot x + C$ such that

$$|u(x) - P(x)| \leq D|x|^{2+\alpha}$$

for $|x| \leq 1$, with

$$|D| \leq C_0 \left( [f]_{C^0}(0) + |f(0)| + \|u\|_{L^\infty(B_1)} \right)$$

and

$$|A| + |B| + |C| \leq C_0 \left( [f]_{C^0}(0) + |f(0)| + \|u\|_{L^\infty(B_1)} \right),$$

where $C_0$ is a universal constant depending only on the dimension.

Proof. We first discuss the normalization of the estimates. First, we may assume $f(0) = 0$ otherwise, let $v(x) = u(x) - f(0) 2^n |x|^{\alpha}$. Then

$$\Delta v = f(x) - f(0).$$

Clear the estimates of $v$ will translates to that for $u$. Second, we can assume $\|u\|_{L^\infty(B_1)} \leq 1$ and $[f]_{C^0(0)} \leq \varepsilon_0$, where $\varepsilon_0$ is the constant in Lemma 1.3. There are two ways of realizing this normalization.

The linear method: consider

$$v(x) = \varepsilon_0 \frac{u}{\|u\|_{L^\infty(B_1)} + [f]_{C^0(0)}}.$$  

Clearly we have $\|v\|_{L^\infty(B_1)} \leq 1$ and $[\Delta v]_{C^0(0)} \leq \varepsilon_0$. Again the estimate of $v$ translates to that for $u$.

The nonlinear method: consider

$$v(x) = \frac{u(rx)}{\|u\|_{L^\infty(B_1)} + 1}.$$  

Actually, the division of $\|u\|_{L^\infty(B_1)} + 1$ is also depends on the linear structure. We say this is a nonlinear method because of the scaling $u(rx)$. We will see later that many nonlinear equations behaves well under this scaling.
We see that
\[
\Delta v = r^2 \frac{f(rx)}{\|u\|_{L^\infty(B_1)} + 1}.
\]
Hence
\[
[\Delta v]_{C^\alpha(0)} \leq r^{2-\alpha}[f]_{C^\alpha(0)}
\]
which can be smaller than \(\varepsilon_0\) be taking fixed and small \(r\).

Now, we prove the following inductively: there are harmonic polynomials,
\[
p_k(x) = \frac{1}{2} x^T A_k x + B_k x + C_k
\]
such that
\[
|u(x) - p_k(x)| \leq \lambda^{(2+\alpha)k} \text{ for } |x| \leq \lambda^k
\]
and
\[
|A_k - A_{k+1}| \leq C \lambda^{\alpha k} \\
|B_k - B_{k+1}| \leq C \lambda^{(\alpha+1)k} \\
|C_k - C_{k+1}| \leq C \lambda^{(\alpha+2)k}.
\]

First we see that \(k = 0\) is the normalization condition on \(u\) and \(k = 1\) is exactly the previous lemma.

Let us assume it is true for \(k\). Let
\[
w(x) = \frac{(u - p_k)(\lambda^k x)}{\lambda^{(2+\alpha)k}}.
\]

Then
\[
\Delta w = \frac{f(\lambda^k x)}{\lambda^{\alpha k}} \text{ and } |w| \leq 1 \text{ in } B_1.
\]

Now, we apply the previous lemma on \(w\). There is a harmonic polynomial \(p\) with bounded coefficients, such that
\[
|w(x) - p(x)| \leq \lambda^{2+\alpha} \text{ for } |x| \leq \lambda,
\]
provided
\[
\sup_{|x| \leq 1} \frac{|f(\lambda^k x)|}{\lambda^{\alpha k}} \leq [f]_{C^\alpha(0)} \leq \varepsilon_0.
\]

Now, we scale back,
\[
\left| u(x) - p_k(x) - \lambda^{(2+\alpha)k} p_0\left(\frac{x}{\lambda^k}\right) \right| \leq \lambda^{(k+1)(2+\alpha)}.
\]
1.2. ENERGY METHOD

Clearly we proved the \((k+1)\)-th step by letting

\[ p_{k+1}(x) = p_k(x) + \lambda^{(2+\alpha)k} p_0 \left( \frac{x}{\lambda^k} \right). \]

Finally, it is elementary to see that \(A_k, B_k, C_k\) converge to \(A_\infty, B_\infty, C_\infty\) respectively and the limiting polynomial

\[ p_\infty(x) = \frac{1}{2} x^T A_\infty x + B_\infty x + C_\infty \]

satisfies

\[ |p_k(x) - p_\infty(x)| \leq C\lambda^{k(2+\alpha)} \]

for any \(|x| \leq \lambda^k\). Hence for \(\lambda^{k+1} \leq |x| \leq \lambda^k\),

\[
|u(x) - p_\infty(x)| \leq |u(x) - p_k(x)| + |p_k(x) - p_\infty(x)| \\
\leq \lambda^{k(2+\alpha)} + C|x|^{2+\alpha} \\
= \frac{1}{\lambda^{2+\alpha}} \lambda^{(k+1)(2+\alpha)} + C|x|^{2+\alpha} \\
\leq \frac{1}{\lambda^{2+\alpha}} |x|^{2+\alpha} + C|x|^{2+\alpha} \\
= \left( \frac{1}{\lambda^{2+\alpha}} + C \right) |x|^{2+\alpha}.
\]

1.2 Energy Method

This section is parallel to §4.1. Since we will use energy method, our methods in this section apply to weak solutions. Consequently, this yields a regularity theory for weak solutions, which is stronger than a-priori estimates.

However, it is parallel to the previous section. The only difference between these two sections are the ways we measure the approximations: we measure it by \(L^\infty\) for \(u\) and \(f\) in the maximum principle approach while we use \(H^1\) semi-norm for \(u\) and \(L^2\) norm for \(f\) in this section.

We see that, by Theorem 1.12 and 1.13, the Hölder continuity does not depend on the way we measure the approximation of \(u\) from a second order polynomial once we prove the estimates in a domain.

Lemma 1.5 Suppose \(u\) is a \(H^1\) weak solution of

\[
\begin{aligned}
-\Delta u &= f \quad \text{in } B_1(0) \\
u &= 0 \quad \text{on } \partial B_1(0).
\end{aligned}
\]
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Then

\[ \int_{B_1} |\nabla u|^2 \leq C \int_{B_1} f^2. \]

Lemma 1.6 For any \(0 < \alpha < 1\), there exist \(C_0, 0 < \lambda < 1\) and \(\varepsilon_0 > 0\) such that for \(\int_{B_1} |\nabla u|^2 \leq 1\) and \(u\) is a solution of (4.1), there is a second order harmonic polynomial

\[ p(x) = \frac{1}{2} x^T Ax + B \cdot x + C \]

such that

\[ \lambda^2 \int_{B_\lambda(0)} |\nabla (u - p)|^2 \leq \lambda^{2(2+\alpha)} \]

and \(|A| + |B| \leq C_0\) a universal constant, provided

\[ \int_{B_1} f^2 \leq \varepsilon_0^2. \]

Proof. Let \(v\) be the solution of

\[ \begin{cases} 
\Delta v = f \\
v = 0 \quad \text{on } \partial B_1(0). 
\end{cases} \]

By the previous lemma, we have

\[ \int_{B_1} |\nabla v|^2 \leq C \int_{B_1} |f|^2. \]

Let \(h = u - v\). Then \(h\) is harmonic. Clearly, we have

\[ \int_{B_1} |\nabla h|^2 \leq 2 \int_{B_1} |\nabla u|^2 + 2 \int_{B_1} |\nabla v|^2 \leq 2B_1, \]

and consequently,

\[ |\nabla h(x)|^2 + |\nabla^3 h(x)|^2 \leq C \int_{B_1} |\nabla h|^2 \leq C \]

for \(|x| \leq \frac{1}{2}\) and

\[ |\nabla^2 p| + |\nabla p|(0) \leq C_0 \]

for some universal constant \(C_0\). Now, let \(p(x)\) be the second order Taylor polynomial of \(h\) at 0. We have

\[ |\nabla (h - p)(x)| \leq C_0 |x| \quad \text{for} \quad |x| \leq \frac{1}{2}. \]
Therefore, we obtain,
\[ \lambda^2 \int_{B_\lambda} |\nabla (u - p(x))|^2 \, dx \leq 2\lambda^2 \int_{B_\lambda} |\nabla (u - h)|^2 \, dx + 2\lambda^2 \int_{B_\lambda} |\nabla (h - p)|^2 \, dx \]
\[ \leq 2\lambda^{2-\alpha} C_0^2 \int_{B_1} f^2 \, dx + 2\lambda^2 C \lambda^4 |B_\lambda| \, . \]

Now, taking \( \lambda \) small such that the second term is less than or equal to
\[ \frac{1}{2} \lambda^{2(2+\alpha)} |B_\lambda| \, , \]
and then taking \( \varepsilon_0 \) such that the first term is less than or equal to
\[ \frac{1}{2} \lambda^{2(2+\alpha)} |B_\lambda| \, , \]
the lemma follows.

**Theorem 1.7** Suppose \( u \) is a weak solution of (4.1) and suppose \( f \) is Hölder continuous at 0 in the \( L^2 \) sense, i.e.
\[
[f]_{L^{2,\alpha}(0)} =: \sup_{0 < r \leq 1} \frac{1}{r^\alpha} \sqrt{\int_{B_r} |f(x) - f(0)|^2} < +\infty \, ,
\]
where \( f(0) \) is defined as \( f(0) =: \lim_{r \to 0} \int_{B_r} f \). Then \( u(x) \) is \( C^{2,\alpha} \) at 0 in \( H^1 \) semi-norm, i.e.,
\[
[u]_{H^{1,2+\alpha}_0(0)} = \inf_{P_2} \sup_{0 < r \leq 1} r^{-1-\alpha} \sqrt{\int_{B_r} |\nabla (u(x) - P_2(x))|^2} < +\infty, \]
where \( P_2 \) is taken over the set of second order polynomials. Moreover the coefficients of \( P_2 \) and \( [u]_{H^{1,2+\alpha}_0(0)} \) are bounded by \( [f]_{L^{2,\alpha}(0)} \) and \( \int_{B_1} |\nabla u|^2 \, dx \) as
\[
[u]_{H^{1,2+\alpha}_0(0)} \leq C([f]_{L^{2,\alpha}(0)} + (\int_{B_1} |\nabla u|^2 \, dx)^{\frac{1}{2}}). \]

**Proof.** It is exactly the translation of the proof of Theorem 1.4, while we have to use Lemma 1.6 repeatedly. See also the proof of Theorem 1.11.

We can do the normalization as before so that \( [f]_{L^{2,\alpha}(0)} \) is small and \( \int_{B_1} |\nabla u|^2 \, dx \leq 1 \). Now, we prove the following inductively: there are harmonic polynomials,
\[
p_k(x) = \frac{1}{2} x^T A_k x + B_k x + C_k \]
such that
\[ \lambda^{2k} \int_{B_{\lambda}} |\nabla(u - p_k)|^2 \leq \lambda^{2(k+\alpha)} \]
and
\[
\begin{align*}
|A_k - A_{k+1}| &\leq C \lambda^{\alpha k} \\
|B_k - B_{k+1}| &\leq C \lambda^{(\alpha+1)k}.
\end{align*}
\]

\(k = 0\) is the condition on \(u\).
\(k = 1\) is exactly the previous lemma.

Let us assume it is true for \(k\). Let

\[ w(x) = \frac{(u - p_k)(\lambda^k x)}{\lambda^{(2+\alpha)k}}. \]

Then
\[
\Delta w = \frac{f(\lambda^k x)}{\lambda^{2k}} \text{ and } \int_{B_1} |\nabla w|^2 dx \leq 1
\]

Now, we apply the previous lemma on \(w\). There is a harmonic polynomial \(p\) with bounded derivatives, such that
\[ \lambda^{2} \int_{B_{\lambda}} |\nabla(w - p)|^2 \leq \lambda^{2(2+\alpha)} \]
provided
\[ \int_{B_1} \frac{|f(\lambda^k x)|^2}{\lambda^{2k}} \leq (|f|_{L^{2,\alpha}(0)})^2 \leq \varepsilon_0^2. \]

Now, we scale back,
\[ \lambda^{2(k+1)} \int_{B_{\lambda^{k+1}}} |\nabla(u(x) - p_k(x) - \lambda^{(2+\alpha)k}p_0\left(\frac{x}{\lambda^k}\right))|^2 \leq \lambda^{2(k+1)(2+\alpha)}. \]

Clearly we proved the \((k+1)\)-th step by letting
\[ p_{k+1}(x) = p_k(x) + \lambda^{(2+\alpha)k}p_0\left(\frac{x}{\lambda^k}\right). \]

Finally, it is elementary to see that \(A_k, B_k\) converge to \(A_\infty, B_\infty\) respectively and the limiting polynomial
\[ p_\infty(x) = \frac{1}{2}x^TA_\infty x + B_\infty x. \]
1.3. COMPACTNESS METHOD

In this section, we discuss the most important method in nonlinear analysis — the compactness method. It has wide applications. It is an extremely powerful tool for the regularity theory. This method was discovered by DeGiorgi in the 60’s when he studied Plateau problem. The method presented here applies to weak solutions and hence gives a regularity theory, which is somewhat better than a priori estimates.

The big advantage of compactness method is its potential application to nonlinear equations. The maximum principle and energy methods have their limitations, namely we have to use the solvability of Dirichlet problems and more seriously, the equation of the difference of the solution and its approximation. We will come back to this point when we deal with nonlinear equations.

Lemma 1.8 Suppose \(-\Delta u = f\) in \(B_1\). Then for any smooth function \(\eta\) with compact support in \(B_1\), we have

\[
\int_{B_1} \eta^2 |\nabla u|^2 dx \leq \int_{B_1} \eta^2 f^2 dx + \int_{B_1} \left(4|\nabla \eta|^2 + \eta^2\right) u^2 dx .
\]

Proof. This is Caccioppoli type inequality. Multiplying the equation by \(\eta^2 u\), we have

\[
\int \eta^2 u(-\Delta u) = \int \eta^2 u f dx .
\]
As before, integrating by parts, we have

\[ \int \eta^2 |\nabla u|^2 = \int \eta^2 uf - 2 \int \eta \nabla \eta \cdot u \nabla u \leq \frac{1}{2} \int \eta^2 f^2 + \frac{1}{2} \int \eta^2 u^2 + \frac{1}{2} \int \eta^2 |\nabla u|^2 + 2 \int |\nabla \eta|^2 u^2. \]

Cancelling \( \frac{1}{2} \int \eta^2 |\nabla u|^2 \), we obtain the inequality immediately.

The following theorem uses the so-called compactness argument. It is also called the indirect method or blow-up method. The major advantage of this method compared with the maximum principle method and the energy method is that it requires no solvability of the Dirichlet problem.

**Theorem 1.9** For any \( \varepsilon > 0 \), there exists a \( \delta = \delta(\varepsilon) > 0 \) such that for any weak solution of \(-\Delta u = f \) in \( B_1 \) with \( \int_{B_1} u^2 dx \leq 1 \) and \( \int_{B_1} f^2 \leq \delta^2 \), there exists a harmonic function \( h \) such that

\[ \int_{B_{1/2}} |u - h|^2 \leq \varepsilon^2. \]

**Remark** Later it will be clear that the argument of this lemma does not depend on the linear structure of the equation. It can be generalized to solutions of different equations if the two equations are close.

**Proof.** We prove it by contradiction. Suppose there exist \( \varepsilon_0 > 0 \), \( u_n \) and \( f_n \) with

\[ -\Delta u_n = f_n, \quad \int_{B_1} u_n^2 \leq 1, \quad \int_{B_1} |f_n|^2 \leq \frac{1}{n}, \]

so that for any harmonic function \( h \) in \( B_{1/2} \),

\[ \int_{B_{1/2}} |u_n - h|^2 dx \geq \varepsilon_0^2. \]

By the previous lemma,

\[ \int_{B_{1/2}} |\nabla u_n|^2 dx \leq C. \]
1.3. COMPACTNESS METHOD

Hence \( \{u_n\} \) has a subsequence, which we still denoted as \( u_n \), such that

\[
\begin{align*}
  u_n &\to u \text{ weakly in } H^1(B_{\frac{3}{4}}) \text{ and } \\
  u_n &\to u \text{ strongly in } L^2(B_{\frac{3}{2}}).
\end{align*}
\]

We will show that \( u \) itself is harmonic in \( B_{\frac{3}{4}} \), which is a contradiction. Actually, for any test function \( \varphi \in C_0^\infty(B_{\frac{3}{4}}) \),

\[
\int \nabla \varphi \nabla u_n = \int \varphi f_n dx.
\]

Letting \( n \) go to infinity, we have

\[
\int \nabla \varphi \nabla u = 0.
\]

Hence \( u \) is harmonic!

The rest of the proof for \( C^{2,\alpha} \) is almost the same as that of the previous section. We state it as the following.

**Lemma 1.10** For any \( 0 < \alpha < 1 \), there exist constants \( C_0, 0 < \lambda < 1 \), \( \delta_0 > 0 \) such that for any function \( f \) and solution of \( \Delta u = f \) in \( B_1 \) with \( \int_{B_1} u^2 dx \leq 1 \) and \( \int_{B_1} f^2 dx \leq \delta_0^2 \), there is a second order harmonic polynomial

\[
p(x) = \frac{1}{2} x^T A x + B x + C
\]

such that

\[
\int_{B_\lambda} |u - p(x)|^2 \leq \lambda^{2(2+\alpha)}
\]

and

\[
|A| + |B| + |C| \leq C_0,
\]

where \( C_0 \) is a universal constant.

**Proof.** The proof is the same as that of Lemma 1.6. Let \( h \) be the harmonic function of the previous lemma with some \( \epsilon < 1 \) to be determined. Hence we have

\[
\int_{B_{\frac{1}{2}}} |h|^2 \leq 2 \int_{B_1} |u|^2 + 2 \int_{B_{\frac{1}{2}}} |u - h|^2 \leq 2 + 2 \epsilon^2 \leq 4.
\]

By the estimates of harmonic functions, we have

\[
|\nabla^3 h(x)|^2 \leq C \int_{B_1} |u|^2 \leq C \quad \text{for } |x| \leq \frac{1}{4}.
\]
Now, let \( p(x) \) be the second order Taylor polynomial of \( h \) at 0. We have
\[
| (h - p)(x) | \leq C_0 |x|^3 \quad \text{for} \quad |x| \leq \frac{1}{4}.
\]
Therefore for each \( 0 < \lambda < \frac{1}{4} \), we have,
\[
\int_{B_\lambda} |u - p|^2 \, dx \leq 2 \int_{B_\lambda} |u - h|^2 \, dx + 2 \int_{B_\lambda} |h - p|^2 \, dx \leq \frac{2\varepsilon^2}{|B_\lambda|} + 2C_0^2 \lambda^6.
\]
Now, taking \( \lambda \) small such that the second term is less than or equal to
\[
\frac{1}{2} \lambda^{2(2+\alpha)},
\]
and then taking \( \delta = \delta_0 \) so that \( \varepsilon \) so small that the first term is less than or equal to
\[
\frac{1}{2} \lambda^{2(2+\alpha)},
\]
and the lemma follows.

**Theorem 1.11** For each \( 0 < \alpha < 1 \) and the dimension \( n \), there is a constant \( C_0 \) so that for all weak solution \( u \) of \(-\Delta u = f \) in \( B_1 \) for \( \|f\|_{L^{2,\alpha}}(0) < \infty \) there exists a second order polynomial \( P(x) = \frac{1}{2} x^T Ax + Bx + C \) such that
\[
\int_{B_r} |u - P(x)|^2 \, dx \leq C_1 r^{2(2+\alpha)}
\]
where the constant \( C_1 \) depends on \( \int_{B_1} u^2 \, dx \) and \( \|f\|_{L^{2,\alpha}}(0) \). Moreover
\[-\Delta P = f(0)\]
and
\[
C_1^2 + |A|^2 + |B|^2 + |C|^2 \leq C_0 \left( \|f\|_{L^{2,\alpha}}(0)^2 + |f(0)|^2 + \int_{B_1} u^2 \, dx \right),
\]
where \( f(0) \) is defined as before.

**Proof.** We can do the normalization as before so that \( f(0) = 0 \) and \( \|f\|_{L^{2,\alpha}}(0) \leq \delta_0 \) and \( \int_{B_1} u^2 \, dx \leq 1 \). Now, we prove the following inductively: there are harmonic polynomials,
\[
p_k(x) = \frac{1}{2} x^T A_k x + B_k x + C_k
\]
such that
\[
\int_{B_{\lambda k}} |u - p_k|^2 \leq \lambda^{2k(2+\alpha)},
\]
and
\[
\begin{align*}
|A_k - A_{k+1}| &\leq C \lambda^{\alpha k} \\
|B_k - B_{k+1}| &\leq C \lambda^{(\alpha+1)k} \\
|C_k - C_{k+1}| &\leq C \lambda^{(\alpha+2)k}.
\end{align*}
\]

\( k = 0 \) is the condition on \( u \).
\( k = 1 \) is exactly the previous lemma.

Let us assume it is true for \( k \). Let
\[
w(x) = \frac{(u - p_k)(\lambda^k x)}{\lambda^{(2+\alpha)k}}.
\]
Then
\[
-\Delta w = \frac{f(\lambda^k x)}{\lambda^{\alpha k}} \quad \text{and} \quad \int_{B_1} w^2 \, dx \leq 1
\]
Now, we apply the previous lemma. There is a harmonic polynomial \( p \) with bounded coefficients, such that
\[
\int_{B_\lambda} |u - p|^2 \leq \lambda^{2(2+\alpha)}
\]
provided
\[
\int_{B_1} \frac{|f(\lambda^k x)|^2}{\lambda^{2\alpha k}} \leq [f]_{C^{2,\alpha}}(0)^2 \leq \delta_0^2.
\]
Now, we scale back,
\[
\int_{B_{\lambda^{k+1}}} |u(x) - p_k(x) - \lambda^{(2+\alpha)k} p \left( \frac{x}{\lambda^k} \right) |^2 \leq \lambda^{(k+1)(2+\alpha)}.
\]
Clearly we proved the \((k+1)\)-th step by letting
\[
p_{k+1}(x) = p_k(x) + \lambda^{(2+\alpha)k} p_0 \left( \frac{x}{\lambda^k} \right).
\]
Finally, it is elementary to see that \( A_k, B_k, C_k \) converge to \( A_\infty, B_\infty, C_\infty \) respectively and the limiting polynomial
\[
p_\infty(x) = \frac{1}{2} x^T A_\infty x + B_\infty x + C_\infty.
\]
satisfies
\[ |p_k(x) - p_\infty(x)| \leq C\lambda^{k(2+\alpha)} \]
for any \( |x| \leq \lambda^k \). Hence for \( \lambda^{k+1} \leq r \leq \lambda^k \),
\[ \int_{B_r} |u(x) - p_\infty(x)|^2 \leq 2 \int_{B_r} |u(x) - p_k(x)|^2 + |P_k(x) - p_\infty(x)|^2 \]
\[ \leq 2 \int_{B_{\lambda^k}} |u(x) - P_k(x)|^2 + |P_k(x) - p(x)|^2 \]
\[ \leq (1 + C)\lambda^{2k(2+\alpha)} \]
\[ \leq \frac{1 + C}{\lambda^{2(2+\alpha)}} r^{2(2+\alpha)}. \]

We remark that sometimes the following definition is also convenient.

**Definition 1**
\[ [f]_{L^2;\alpha}(0) = \sup_{0 < r \leq 1} \frac{1}{r^\alpha} \left( \int_{B_r} |f - \overline{f}_{B_r}|^2 \right)^{\frac{1}{2}}, \]
where
\[ \overline{f}_{B_r} = \int_{B_r} f. \]

Now we have showed \( C^{2,\alpha} \) estimates by three methods all of which are the second order polynomial approximation.

We can use the same method to prove \( C^\alpha, C^{1,\alpha} \) and any \( C^{k,\alpha} \) estimates for the solution if we replace the second order polynomial approximation to constant, linear and \( k \)-th order polynomial approximations.

**Definition 2**
\[ [f]_{L^2;-1+\alpha}(0) = \sup_{0 < r \leq 1} \frac{1}{r^{-1+\alpha}} \left( \int_{B_r} |f|^2 \right)^{\frac{1}{2}}. \]

We remark that we don’t have to normalized by averages or \( f(0) \) in our definition. Such average doesn’t converge when \( [f]_{L^2;-1+\alpha}(0) < \infty \). We also remark that from the Hölder inequality we have that if \( f \) is in \( L^p \) for \( p > n \), then
\[ [f]_{L^2;-1+\alpha}(0) \leq \|f\|_{L^p(B_1)}, \]
for \( \alpha = 1 - \frac{n}{p} \).

Let me state two of the theorem that our methods apply.
1.4. BOUNDARY ESTIMATES ON FLAT BOUNDARIES

**Theorem 1.12** Suppose $u$ is a weak solution of $-\Delta u = f$ in $B_1$ and $[f]_{L^{2,-1+\alpha}}(0) < \infty$, then there exists a linear function $L(x) = Bx + C$ such that

$$\int_{B_r} |u - L(x)|^2 \leq C_0 r^{2(1+\alpha)}$$

where the constant $C_0$ depends on $\int_{B_1} u^2 \, dx$ and $[f]_{L^{2,-1+\alpha}}(0)$. Moreover

$$C_0 + |B|^2 + |C|^2 \leq C \left( [f]_{L^{2,-1+\alpha}}^2 + \int_{B_1} u^2 \, dx \right).$$

**Definition 3**

$$[f]_{L^{2,-2+\alpha}}(0) = \sup_{0<r\leq 1} \frac{1}{r^{2+\alpha}} \left( \int_{B_r} |f|^2 \right)^{\frac{1}{2}}.$$

We remark that if $f$ is in $L^p$ for $n > p > \frac{n}{2}$, then

$$[f]_{L^{2,-2+\alpha}}(0) \leq \|f\|_{L^p(B_1)},$$

for $\alpha = 2 - \frac{n}{p}$.

**Theorem 1.13** Suppose $u$ is a weak solution of $-\Delta u = f$ in $B_1$ then there exists a constant $A$ such that

$$\int_{B_r} |u - A|^2 \leq C_0^2 r^{2\alpha}$$

where the constant $C_0$ depends on $\int_{B_1} u^2 \, dx$ and $[f]_{L^{2,-2+\alpha}}(0)$. Moreover

$$C_0^2 + |A|^2 \leq C \left( [f]_{L^{2,-2+\alpha}}^2 + \int_{B_1} u^2 \, dx \right).$$

We also remark that we can prove similar theorems by using $[f]_{L^{p,-2+\alpha}}(0)$ for any $1 \leq p \leq \infty$. In order to carry out the argument the only estimates that we need is an estimate of $u$ by the $L^p$ norm of $\Delta u$.

1.4 Boundary Estimates on Flat Boundaries

Let us start with an example of boundary estimates when the boundary is flat. We will use the $L^\infty$ approach since it has the best geometric intuitions.
We will denote $x = (x',x_n)$ and $\mathbf{R}_+^n = \{ x : x_n \geq 0 \}$, 

\[
T_r = \{ x' \in \mathbf{R}^{n-1} : |x'| < r \}
\]

\[
T_r(x'_0) = T_r + x'_0 \quad \text{for} \quad x'_0 \in \mathbf{R}^{n-1}
\]

\[
B^+_r = B_1(0) \cap \mathbf{R}_+^n
\]

\[
B^+_r(x_0) = B_r + (x_0,0).
\]

**Lemma 1.14** Suppose $u$ in weak solution of $\Delta u = 0$ in $B^+_1$ and $u = 0$ on $T_1$ and $|u(x)| \leq 1$ in $C^+_1$. Then $u$ is smooth in $B^+_2$. Moreover

\[
|D^\alpha u(x)| \leq C_\alpha
\]

for $x \in B^+_2$.

**Proof.** The conclusion is trivially proved by the odd extension, which is also weakly and therefore classically harmonic in $B_1$.

**Lemma 1.15** Suppose

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in} \quad B^+_1 \\
|u|_{L^\infty(B^+_1)} &\leq 1 \\
u(x',0) &= \phi(x') \quad \text{on} \quad T_1.
\end{aligned}
\]

(1.5)

Then there exist constants $a$ and a universal constant $C$ such that

\[
-C|x|^2 + \inf_{T_1} \phi \leq u(x) - ax_n \leq C|x|^2 + \sup_{T_1} \phi.
\]

**Proof.** Let $v$ be the harmonic function (provided by the Poisson integral formula)

\[
\begin{aligned}
-\Delta v &= 0 \\
v \big|_{T_1} &= \sup_{T_1} \phi \\
v \big|_{\partial+(B^+_1)} &= u
\end{aligned}
\]

Clearly

\[
v \geq u
\]

and

\[
u \geq v - \sup_{T_1} \phi + \inf_{T_1} \phi.
\]
The lemma follows by applying the previous lemma to the function $v - \sup_{T_1} \varphi$ and the fact that
\[
\nabla v(0) = (0, \ldots, 0, \partial_n v(0)) =: (0, \ldots, 0, a).
\]

**Corollary 1.16** For any $0 < \alpha < 1$ there are universal positive constants $\varepsilon_0$ and $\lambda < 1$ so that for any solution of
\[
\begin{align*}
\Delta u &= 0 & & \text{in } B_1^+, \\
|u|_{L^\infty(B_1^+)} &\leq 1 \\
u(x', 0) &= \varphi(x') & & \text{on } T_1,
\end{align*}
\]
there are constants $a$ and $b$ such that
\[
|u(x) - ax_n - b|_{L^\infty(B_1^+)} \leq \lambda^{1+\alpha}
\]
provided $\text{osc}_{T_1} \varphi \leq \varepsilon_0$.

An iteration of this corollary proves the following theorem.

**Theorem 1.17** Suppose $-\Delta u = 0$ in $B_1^+(0)$ and $\varphi(x') = u(x', 0)$ is $C^{1,\alpha}$ at 0. Then $u(x)$ is $C^{1,\alpha}$ at 0. Moreover
\[
[u]_{C^{1,\alpha}(0)} \leq C \left( [\varphi]_{C^{1,\alpha}(0)} + \|u\|_{L^\infty(B_1)} \right).
\]

We can similarly prove $C^{k,\alpha}$, for $k > 1$, estimates at the boundary.

### 1.5 Estimates Near the Boundary

In this section, we introduce the basic tools for boundary estimates. We will also prove $C^\alpha$ boundary estimates on Lipschitz domain, which is a very important class of domain since they are invariant under scaling.

#### 1.5.1 Barrier Functions

Let us start out this section with some discussion on the most important function for elliptic equations: the Newtonian potential,
\[
p(x) = \begin{cases} 
\frac{1}{|x|^{n-2}} & x \neq 0 \ (n \geq 3) \\
-ln|x| & x \neq 0 \ n = 2.
\end{cases}
\]
As seen in previous chapter, 
\[ \Delta p(x) = -C_n \delta_0(x) , \]
where \( \delta_0(x) \) is the delta function with pole at 0.

We will use the following function repeatedly: 
\[ 0 < r, R < \infty . \]

\[ h_{r,R}(x) = \frac{p(x) - p(R)}{p(r) - p(R)} . \]

Clearly, \( h_{r,R} \) is harmonic in between \( B_R \) and \( B_r \) and
\[ h_{r,R} = 0 \text{ on } \partial B_R \text{ and } \quad h_{r,R} = 1 \text{ on } \partial B_r . \]

(1.6) \hspace{1cm} (1.7)

It is also clear that, if \( r < R \), \( h_{r,R} \) is convex in the radial direction and
\[ \frac{\partial h_{r,R}}{\partial n} \leq C_{r,R} < 0 . \]

(1.8)

As showed in the previous section that harmonic functions takes boundary data as regularly as the boundary data.

### 1.5.2 Hopf Lemma and Slope Estimates

The following important theorem (according to tradition we call it Lemma) says that the normal derivative of a harmonic function depends on the values of \( u \) far away in a qualitative way. This also tells us the information between the values of a harmonic function is infinitely fast and the first order derivative of a harmonic function is lifted if we lift the function a little bit somewhere else.

**Lemma 1.18 (Hopf boundary point lemma)** Suppose \( \Delta u = 0 \text{ in } B_1 \), \( u \geq 0 \text{ in } B_1 \). If \( u(x_0) = 0 \text{ for } x_0 \in \partial B_1 \text{ then } u(x) \geq C|x - x_0|u(0) \text{ along the radial direction of } x_0 \text{ for some } C \text{ depending only on the dimension.}

**Proof.** From Harnack inequality, we have 
\[ u(x) \geq cu(0) \text{ for } |x| \leq \frac{1}{2} . \]

Hence, we have
\[ u(x) \geq cu(0) h_{\frac{1}{2},1}(x) \]

by applying the maximum principle to the annulus \( \{ \frac{1}{2} \leq |x| \leq 1 \} \). The lemma follows immediately from the property of \( h_{\frac{1}{2},1}(x) \).
Lemma 1.19 Suppose $\Omega$ is a domain such that $B_1 \cap \Omega = \emptyset$ and $x_0 \in \partial B_1 \cap \partial \Omega$. Let $\Delta u = 0$ in $\Omega$ with $u \leq 1$ in $\Omega$ with $u \leq 0$ on $\partial \Omega \cap B_2$. Then $u(x) \leq C|x - x_0|$ for $|x| \leq 1$, where $C$ is a constant depending only on the dimension.

Proof. Applying the maximum principle to the function

$$u(x) - h_{2,1}(x)$$

on the domain $\Omega \cap B_2$, we have

$$u(x) \leq h_{2,1}(x) \text{ in } \Omega \cap B_2.$$ 

The lemma follows immediately since $h_{2,1}(x)$ is a smooth function.

Lemma 1.18 and 1.19 the control of the behavior of the harmonic function near the boundary. One immediate consequence of Lemma 1.19 is the following gradient estimates.

Lemma 1.20 Suppose $\Omega$ is a domain such that $B_1 \cap \Omega = \emptyset$ and $x_0 \in \partial B_1 \cap \partial \Omega$. Let $\Delta u = 0$ in $\Omega$ with $|u| \leq 1$ in $\Omega$ with $u = 0$ on $\partial \Omega \cap B_2$. Then if $u$ is differentiable at $x_0$, $|\nabla u(x_0)| \leq C$.

Proof. We apply Lemma 1.19 to $u$ and $-u$ to obtain

$$|u(x)| \leq C|x - x_0| \text{ for } x \in B_2 \cap \Omega.$$ 

The estimates follow.

A nice application of Lemma 1.20 is that it can easier adapted to situations that a domain is a small perturbation from a hyperplane and the boundary is also approximately constant.

Lemma 1.21 Suppose $u \leq 1$ and subharmonic in $B_3 \cap \Omega$. If the domain is flat as

$$B_3 \cap \{x_n \geq \epsilon\} \subset B_3 \cap \Omega \subset B_3^\pm,$$

then there is a universal constant $C$ so that

$$u(x) \leq \sup_{B_3 \cap \partial \Omega} u + C(\sqrt{\epsilon} + x_n).$$
Proof. Without lose of generality, we may assume that \( \sup_{B_3 \cap \partial \Omega} = 0 \). For each \( y \in T_1 = B_1 \cap \{ x_n = 0 \} \), we see that \( B_1(y, -1) \) is outside \( \Omega \). Now let \( t \) be the maximum of \( 0 \leq t \leq 1 \) so that the balls \( B_1(y, t) \) is outside \( \Omega \). From the flatness condition we see that \( 0 \leq t \leq \epsilon \) and \( \overline{B}_1(y, t) \cap \partial \Omega \) contains at least one point, say \( y_0 \) with \( |y_0 - y| \leq C \sqrt{\epsilon} \). Now we apply Lemma 1.20 (with \( B_1 \) replaced by \( B_1(y, t) \)) to obtain
\[
    u(x) \leq C|x - y_0|.
\]
Therefore \( u(0, x_n) \leq C(|x_n - (y_0)_n| + |y'_0|) \leq C(x_n + \sqrt{\epsilon}) \).

**Exercise** Consider the domain
\[
    C_\alpha = \{(x, y) : \text{angle}((x, y), (1, 0)) < \frac{\pi}{\alpha}\}.
\]
Suppose
\[
    \begin{cases}
        \Delta u(x) = 0 & \text{in } C_\alpha \cap B_1 \\
        u(x) = 0 & \text{on } (\partial C_\alpha) \cap B_1.
    \end{cases}
\]
Prove that
1. \( u \) is Lipschitz at 0 if and only if \( \alpha \geq 1 \).
2. \( u \) is \( C^k(0) \) if and only if \( \alpha \) is an integer or \( \alpha \geq k \).

### 1.5.3 \( C^\alpha \) Estimates on Lipschitz Domains

**Theorem 1.22** Suppose \( \Omega \) is a Lipschitz domain. Let \( 0 \in \partial \Omega \). Suppose \( \varphi(x) \) is \( C^\alpha \) at 0:
\[
    [\varphi]_{C^\alpha}(0) = \sup_{0 < |x| \leq 1} \frac{|\varphi(x) - \varphi(0)|}{|x|^\alpha} < +\infty.
\]
Then any solution of
\[
    \begin{cases}
        \Delta u = 0 & \text{in } \Omega \\
        u = \varphi(x) & \text{on } (\partial \Omega) \cap B_1
    \end{cases}
\]
is Hölder continuous at 0 with exponent \( \beta = \beta_0 \wedge \alpha \) for some \( \beta_0 \) depending on the Lipschitz character of the domain.

The only thing we need to show for \( C^\alpha \) regularity is that the oscillation of the solution is smaller if we restrict the solution to a smaller domain.

**Proof.** Without loss of generality, we may suppose \( \partial \Omega \) is a Lipschitz graph in the \( x_n \) direction and \( \Omega \) is in the \( x^+_n \) direction. Suppose its Lipschitz
1.5. ESTIMATES NEAR THE BOUNDARY

constant is \( K \). From trigonometry,

\[
B_{r_0}(0, -1/3) \subset \Omega^c \quad \text{for} \quad r_0 = \frac{1}{3\sqrt{K^2 + 1}}.
\]

Next, as usual, we may suppose, \( u(0) = 0, [\varphi]_{C^\alpha(\partial\Omega)}(0) \leq \varepsilon_0 \) and \( |u(x)| \leq 1 \) in \( B_1(0) \). Applying the maximum principle to the function

\[
u(x) - \sup_{(\partial\Omega) \cap B_1} u - h_{r_0, 2/3}(x - (0, -1/3))\]

on the domain

\[
\Omega \cap B_{2/3}((0, -1/3)),
\]

we obtain

\[
u(x) \leq \sup_{\partial\Omega \cap B_1} u + h_{r_0, 2/3}(x - (0, -1/3)) .
\]

By a symmetric argument, we can obtain

\[
u(x) \geq \inf_{\partial\Omega \cap B_1} u - h_{r_0, 2/3}(x, (0 - 1/3)) .
\]

Hence

\[
|\nu(x)| \leq \text{Osc}[\varphi] + \gamma \text{ for } |x| \leq \frac{1}{6},
\]

where

\[
\gamma = \sup_{|x| \leq \frac{1}{6}} h_{r_0, 2/3}(x) < 1.
\]

If \( \text{Osc}[\varphi] \leq \frac{1-\gamma}{2} \), then

\[
|\nu(x)| \leq \frac{1+\gamma}{2} =: \gamma_1 < 1
\]

Now, letting \( \gamma_2 = \max(\frac{1}{6\alpha}, \gamma_1) \) we will prove

\[
\sup_{B_{1/6}^1(0)} |u| \leq \gamma_2^k
\]

and the theorem follows.

We will prove it inductively as

\[
\sup_{B_{1/6}^1(0)} |u| \leq \text{osc} \sup_{B_{1/6}^{1/2}(0)} [\varphi] + \gamma \sup_{B_{1/6}^{1/2}(0)} |u|
\]

\[
\leq [\varphi]_{C^\alpha(0)} \frac{1}{6^{\alpha(k-1)}} + \gamma \gamma_2^{k-1}
\]

\[
\leq ([\varphi]_{C^\alpha(0)} + \gamma) \gamma_2^{k-1} .
\]

It follows immediately by if \( [\varphi]_{C^\alpha(0)} \) is small enough.
1.6 Boundary Estimates on Curved Domains

A combination of the methods in §1.1 to §1.6 provides different proof for the following theorems.

**Theorem 1.23** Suppose \( \Omega \) is a Lipschitz domain, \( 0 \in \partial \Omega \). \( \varphi \) is \( C^{1,\alpha} \) at \( 0 \) and \( \partial \Omega \) is \( C^{1,\alpha} \) at \( 0 \). Let \( u \) be a solution of

\[
\begin{cases}
\Delta u = f(x) & \text{in } B_1 \cap \Omega \\
u = \varphi & \text{on } (\partial \Omega) \cap B_1.
\end{cases}
\]

Then \( u \) is \( C^{1,\alpha} \) at \( 0 \) if \( [f]_{C^{-1,\alpha}(\partial \Omega)} < \infty \). Moreover

\[
[u]_{C^{1,\alpha}(0)} \leq C ([f]_{C^{-1,\alpha}(0)} + [\varphi]_{C^{1,\alpha}(0)} + \|u\|_{L^\infty(B_1 \cap \Omega)}),
\]

for some \( C \) depending only on \([\partial \Omega]_{C^{1,\alpha}}\) and the dimension.

**Theorem 1.24** Suppose \( \Omega \) is a Lipschitz domain, \( 0 \in \partial \Omega \). \( \varphi \) is \( C^{2,\alpha} \) at \( 0 \) and \( \partial \Omega \) is \( C^{2,\alpha} \) at \( 0 \). Let \( u \) be a solution of

\[
\begin{cases}
\Delta u = f(x) & \text{in } B_1 \cap \Omega \\
u = \varphi & \text{on } (\partial \Omega) \cap B_1.
\end{cases}
\]

Then \( u \) is \( C^{2,\alpha} \) at \( 0 \) if \( [f]_{C^\alpha(0)} < \infty \). Moreover

\[
[u]_{C^{2,\alpha}(0)} \leq C ([f]_{C^\alpha(0)} + [\varphi]_{C^{2,\alpha}(0)} + [u]_{L^\infty(B_1 \cap \Omega)}),
\]

for some \( C \) depends only on \([\partial \Omega]_{C^{2,\alpha}}\) and the dimension.

Here we will outline the proof using the compactness method.

**Lemma 1.25** If \( u \) is a weak solution of

\[
\begin{cases}
\Delta u = f(x) & \text{in } B_1 \cap \Omega \\
u = \varphi & \text{on } (\partial \Omega) \cap B_1.
\end{cases}
\]

Then for any \( \eta \in C_0^\infty(B_1) \) we have

\[
\int_{B_1 \cap \Omega} \eta^2 |\nabla u|^2 \leq \int_{B_1 \cap \Omega} \eta^2 f^2 + C \int_{B_1 \cap \Omega} (|\nabla \eta|^2 + \eta^2) |u|^2 + C \|\varphi\|^2_{H^2(\partial \Omega \cap B_1)}.
\]
**Lemma 1.26** For any $\epsilon > 0$ there is a $\delta > 0$ so that if $u$ is a solution of
\[
\begin{cases}
\Delta u = f(x) & \text{in } B_1 \cap \Omega \\
u = \varphi & \text{on } (\partial \Omega) \cap B_1.
\end{cases}
\]
then there is a harmonic function $h$ in $B_{\frac{1}{2}}^+$ with $h(x',0) = 0$ so that
\[
\int_{B_{\frac{1}{2}}^+ \cap \Omega} |u - h|^2 \leq \epsilon^2,
\]
provided the following set of smallness condition
\[
\int_{B_1 \cap \Omega} f^2 \leq \delta^2, \|\varphi\|^2_{H^{\frac{1}{2}}(\partial \Omega \cap B_1)} \leq \delta,
\]
and a flatness condition on the boundary as
\[
B_1 \cap \Omega \subset B_1^+ \text{ and } \{ x \in B_1 : x_n \geq \delta \} \subset \Omega.
\]
The proof of this is an application of the compactness argument. $C^{2,\alpha}$ and $C^{1,\alpha}$ estimates will follow by approximations as before.

One can also use the maximum principle approach by the estimates in Lemma 1.21. We prove the key approximation.

**Lemma 1.27** Suppose $u$ is a solution of
\[
\begin{cases}
-\Delta u = f(x) & \text{in } B_3 \cap \Omega \\
u = \varphi & \text{on } (\partial \Omega) \cap B_3.
\end{cases}
\]
with $|u| \leq 1$ and $\|f\| \leq \delta$ and the domain is flat as
\[
B_3 \cap \Omega \subset B_3^+ \text{ and } \{ x \in B_3 : x_n \geq \epsilon \} \subset \Omega.
\]
then there is a harmonic function $h$ in $B_1^+$ with $h(x',0) = 0$ so that
\[
|u - h| \leq C(\sqrt{\epsilon} + \delta) \text{ in } B_1 \cap \Omega.
\]

**Proof.** From Lemma 1.21 applied to $u \pm \frac{\delta}{2\pi}|x|^2$, we see that
\[
|u| \leq C(x_n + \sqrt{\epsilon} + \delta).
\]
We notice that this inequality will be enough for $C^\alpha$ estimates. In order to improve the estimates, we consider the harmonic $h$ with boundary conditions as

$$
\begin{align*}
-\Delta h &= 0 \text{ in } B_1 \\
h(x) &= C(\sqrt{\epsilon} + \delta) \text{ for } x \in \partial B_1^+ \setminus \Omega \\
h(x) &= u \text{ for } x \in \partial B_1 \cap \Omega,
\end{align*}
$$

and by the maximum principle and with estimate 1.12, we have

$$
h \geq -C_1 h_1 \frac{1}{2} (x - (y', \frac{1}{2})) - C(\sqrt{\epsilon} + \delta) \text{ in } B_1(y', -\frac{1}{2}) \cap B_1^+; \tag{1.13}
$$

for some constant $C_1$ and each $|y'| \leq 1$.

Therefore $h \geq -C\sqrt{\epsilon}$ for $x_n = \epsilon$. We apply the maximum principle to $u - h$ in the domain $B_1 \cap \{x_n \geq \epsilon\}$ to yield the estimates

$$
u \geq h - 2C(\sqrt{\epsilon} + \delta),
$$

for $x_n \geq \epsilon$.

### 1.7 Patching of Interior Estimates with Boundary Estimates

We only discuss estimates in $L^\infty$-norm. For $L^2$ and $H^1$ norms, there are corresponding patching theorems.

**Theorem 1.28** Let $S$ be a set of functions defined on a closed bounded set $\Omega$. Suppose the following hold:

1. **(Boundary Estimates)** For any $y \in \partial \Omega$, there exists a polynomial $P_y$ of degree $k$ such that
   $$
   |u(z) - P_y(z)| \leq E|z - y|^{k+\alpha}, z \in \overline{\Omega}.
   $$
   with
   $$
   \sum_{|\sigma| \leq k} |D^\sigma P(z)| \leq E.
   $$

2. **(Interior Estimates)** For any $x \in \Omega^0$, letting $d_x = d(x, \partial \Omega)$, then there exists a polynomial $P$ of degree $k$ such that

$$
|D^\sigma p(x)| \leq Cd_x^{|\sigma|}\|u\|_{L^\infty(B_2)} + Bd_x^{|\sigma|-2+\alpha} \tag{1.13}
$$

$$
|u - p_x(z)| \leq \left( A\frac{\|u\|_{L^\infty(B_2)}}{d_x^{k+\alpha}} + D\right)|z - x|^{k+\alpha} \text{ for } |z - x| \leq \frac{d_x}{2} \tag{1.14}
$$
(3) (Invariance) For any $u \in S$, $P_x$ in (1), $v = u - P$ also satisfy the estimates of (2).

Then $S \subset C^{k,\alpha}(\Omega)$. In fact $S \subset B_M \subset C^{k+\alpha}(\Omega)$ for some $M$ depending on $A$, $B$, $C$, $D$ and $E$.

**Proof.** For any $x \in \Omega^0$, there exists an $y \in \partial\Omega$ such that
\[ d(x, y) = d_x. \]

Use (2) for $y$, we have
\[ |u(z) - P_y(z)| \leq E|z - y|^{k+\alpha} \]
for $z \in \overline{\Omega}$.

Applying (1) to $u - P_y(z)$, we have
\[ |u(z) - P_y(z) - P_x(z)| \leq (AE + D)|z - x|^{k+\alpha} \]
for $|z - x| < \frac{d_x}{2}$. Clearly, for $|z - x| \geq \frac{d_x}{2}$,
\[ |u(z) - P_y(z) - P_x(z)| \leq E|y - z|^{k+\alpha} + |P_x(z)| \leq C_1(dx)^{k+\alpha} \leq C_1|z - x|^{k+\alpha}. \]

### 1.8 Examples

Let us review our technique in this section.

**Example 1** Let $u$ be a solution of
\[
\begin{cases}
\Delta u = f(x) & \text{in } \Omega \\
 u |_{\partial\Omega} = \varphi(x) & \text{on } \partial\Omega.
\end{cases}
\]

1. If $\varphi \in C^\alpha(\partial \Omega)$, $\partial \Omega \in C^1$, $\|f\|_{L^\infty} < +\infty$, then $u \in C^\alpha(\overline{\Omega})$.
2. If $\varphi \in C^{1,\alpha}(\partial \Omega)$, $\partial \Omega \in C^{1,\alpha}$, $f \in L^\infty(\Omega)$, then $u \in C^{1,\alpha}(\overline{\Omega})$. 
3. If \( \varphi \in C^{2,\alpha}(\partial \Omega) \), \( \partial \Omega \in C^{2,\alpha} \), \( f \in C^{2,\alpha}(\overline{\Omega}) \), then \( u \in C^{2,\alpha}(\overline{\Omega}) \).

**Proof.** These are immediate consequences of scaled interior estimates, boundary estimates and the technique of patching.

**Example 2** Let \( u \) be a solution of

\[
\begin{aligned}
\Delta u &= f(u) \\
u \mid_{\partial \Omega} &= \varphi(x)
\end{aligned}
\]

if \( \|u\|_{L^\infty(\Omega)} \leq C \) and \( \varphi, f \in C^\infty \) then \( u \in C^\infty(\overline{\Omega}) \).

**Proof.** Homework.

**Exercise** Suppose

\[
\begin{aligned}
\Delta u &= f(x) \in L^\infty(B_1) \\
\|u\|_{L^\infty(B_1)} &< +\infty
\end{aligned}
\]

Show

\[
\|\nabla u\|_{L^\infty(B_r)} \leq C \left( \frac{1}{1-r}\|u\|_{L^\infty(B_1)} + r\|f\|_{L^\infty} \right)
\]

for any \( 0 \leq r \leq 1 \).

**Exercise** (Hu and Wang) Let \( \Omega \subset B_1 \) be a \( C^{1,\alpha} \) domain \( \chi_\Omega \) be the characteristic function of \( \Omega \). Suppose

\[
\begin{aligned}
-\Delta u &= \chi_\Omega \text{ in } B_1 \\
\|u\|_{L^\infty(B_1)} &< +\infty
\end{aligned}
\]

Then \( u \in C^{2,\alpha}(\overline{\Omega}) \) and

\( u \in C^{2,\alpha}_{\text{loc}}(\overline{B_1-\Omega}) \).

**Hint** Using

\[
\|u\|_{L^1(B_1)} \leq C\|\Delta u\|_{L^1(B_1)}
\]

if \( u \mid_{\partial B_1} = 0 \).

Most of the techniques for the Dirichlet problem can be extended to Neumann boundary value problems. However, the following problem is a
negative result for mixed value problems.

**Exercise** Find a function $u$ in $B_1^+$ such that

$$
\begin{cases}
    \Delta u = 0 & \text{in } B_1^+ \\
    u = 0 & \text{on } T_1 \cap \{x_{n-1} \leq 0\} \\
    \frac{\partial u}{\partial n} = 0 & \text{on } T_1 \cap \{x_{n-1} > 0\}
\end{cases}
$$

and $u \in C^{1\frac{1}{2}}(B_1^+)$ but not in $C^{1\frac{1}{2}+\varepsilon}(B_1^+)$ for any $\varepsilon > 0$. 
Chapter 2

$L^p$ Estimates

In this chapter we will show a proof of the classical Calderón–Zygmund estimates established in [2] 1952. The estimates are among the fundamental estimates for elliptic equations.

2.1 Introduction

In this paper, we will mainly discuss the theorem,

**Theorem 2.1 (Calderón–Zygmund)** If $u$ is a solution of

$$\nabla u = f \quad \text{in } B_2$$

then

$$\int_{B_1} |D^2 u|^p \leq C \left( \int_{B_2} |f|^p + \int_{B_2} |u|^p \right) \quad \text{for any } 1 < p < +\infty. \quad (2.2)$$

The analytical tools in this chapter are the maximal function, energy estimates and Vitali Covering Lemma.

We recall the basic notations that we use. $B_r = \{ x \in \mathbb{R}^n : |x| < r \}$, $Q_r = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : -r < x_i < r \}$ and $B_r(x) = B_r + x$, $Q_r(x) = Q_r + x$. For any measurable set $A$, $|A|$ is its measure. For any integrable function $u$, we denote the average of $u$ as

$$\bar{u}_A = \int_A u = \frac{1}{|A|} \int_A u.$$

The classical proof of Calderón–Zygmund estimates, uses the singular integrals

$$\frac{\partial^2 u}{\partial x_i \partial x_j} (x) = \int_{\mathbb{R}^n} w_{ij} (y) f (x - y) \, dy \quad (2.3)$$
where \( w_{ij} \) is a homogeneous function of degree \(-n\) with cancellation conditions. The approach involves an \( L^2-L^2 \) estimate and an \( L^1 \) to weak-\( L^1 \) estimate. See details in the book of Stein [?].

Our approach is more elementary. It gives an unified proof for elliptic, parabolic and subelliptic operators. Our proof is built upon geometrical intuitions. Our basic tools in this approach are the standard estimates for the equations, the Vitali covering lemma and Hardy–Littlewood maximal function.

Our approach is very much influenced by [4] and the early works in [3] and [29], in which the Calderón–Zygmund decompositions were used. Here we will use the Vitali covering lemma. Analytically the difference between the Calderón–Zygmund decomposition and Vitali covering lemma is not quite essential but subtle. One is on cubes and the latter is on balls. However we hope that Vitali covering lemma can easily adapted to more complicated situations where balls can be easily defined.

2.2 The geometry of functions and sets

2.2.1 Geometry of Hölder spaces

We should start out with a geometric description of Hölder space which is the key to visualize the estimates.

First of all, the geometry of \( \|u\|_{L^\infty(B_1)} \leq 1 \) is that the graph of \( u \) is in the box \( B_1 \times [-1, 1] \). This gives a very mild control of \( u \).

The Hölder norm of \( u \) is actually very geometrical. Let us recall that \( u \) is Hölder with \( [u]_{C^{\alpha}(B_1)} \leq 1 \) if

\[
|u(x) - u(y)| \leq |x - y|^{\alpha} \text{ for all } x, y \text{ in } B_1. \tag{2.4}
\]

Geometrically, the graph of \( u \) is not a box anymore rather than a surface which is away from spikes: \(|x|^{\alpha}\). That is, if \((x_0, y_0)\) is on the graph of \( u \), then all the points \((x, y)\) with \( y - y_0 > |x - x_0|^{\alpha} \) or \( y - y_0 < |x - x_0|^{\alpha} \) are not on the graph.

This actually leads us to say that if \( u \) is \( C^{\alpha} \) at \( x_0 \) if

\[
|u(x) - u(x_0)| \leq C|x - x_0|^{\alpha}.
\]

2.2.2 \( L^p \) spaces

The information carried by the \( L^p \) norm of a function is not as local as by the Hölder norms. In contrast to Hölder spaces, there is no way to say a function is in \( L^p \) function at a point.
2.2. THE GEOMETRY OF FUNCTIONS AND SETS

In order to examine the information carried by $L^p$ norm of a function, let us recall the formula,

$$\int_{\Omega} |u|^p \, dx = p \int_0^\infty \lambda^{p-1} \, |\{x \in \Omega : |u| > \lambda\}| \, d\lambda. \quad (2.5)$$

If $\int_{\Omega} |u|^p \, dx = 1$, one will have that

$$|\{x \in \Omega : |u| > \lambda\}| \leq \frac{1}{\lambda^p}, \quad (2.6)$$

i.e., the measure

$$|\{x : \in \Omega : |u| > \lambda\}|$$

decays for $\lambda$ large. This tells us that if we randomly choose a point $x$, then the probability for $|u(x)| > \lambda$ is small for $\lambda$ large. The identity (2.5) shows the decay of $|\{x : \in \Omega : |u| > \lambda\}|$ in a precise way and this decay is the only information carried by the $L^p$ norm. We also observe that the faster this probability decays the bigger the $p$ is.

Now let us discuss how we can show a function is in $L^p$.

First we see that one has to prove the decay of $|\{|u| > \lambda\}|$. As in the Hölder estimates, we should prove this decay inductively. A reasonable expectation of this sort is to prove:

$$|\{|u| > \lambda_0\}| \leq \varepsilon |\{|u| > 1\}|. \quad (2.7)$$

The smaller $\varepsilon$ or $\lambda_0 - 1$ is, the faster the decay is. Here one should realize that this estimate should be scaled to

$$|\{|u| > \lambda_0 \lambda\}| \leq \varepsilon |\{|u| > \lambda\}| \quad (2.8)$$

with proper conditions on the data. As in the Hölder space case, one should expect that an inductive argument proves the decay.

The $W^{2,p}$ theory of (2.1) says that $D^2 u$ is in $L^p$ if $\Delta u$ is. Hence a reasonable expectation of an inductive estimate could be

$$\left|\{|D^2 u| > \lambda_0\}\right| \leq \varepsilon \left(\left|\{|D^2 u| > 1\}\right| + \left|\{|f| > \delta_0\}\right|\right). \quad (2.9)$$

Here we can scale (2.9) to

$$\left|\{|D^2 u| > \lambda_0 \lambda\}\right| \leq \varepsilon \left(\left|\{|D^2 u| > \lambda\}\right| + \left|\{|f| \geq \delta_0 \lambda\}\right|\right) \quad (2.10)$$
which is the so-called good-$\lambda$ inequality. It is elementary to see that (2.9) implies the estimate. This expectation (2.9), however, is not true. One reason for the failure of (2.9) is that the condition
\[ |D^2 u(x_0)| \leq 1 \]  
(2.11)
is unstable in the setting of $W^{2,p}$ theory.

Although (2.9) is not true, its modification (2.15) is true.

The key modification is provided by one of the treasures in analysis: the Hardy–Littlewood maximal function.

For a locally integrable function $v$ defined in $\mathbb{R}^n$, its maximal function is defined as
\[ Mv(x) = \sup_{r>0} \int_{B_r(x)} |v| \, d\mathcal{L}^n. \]  
(2.12)

We also use
\[ M\Omega v(x) = M(v\chi_\Omega)(x), \]
if $v$ is not defined outside $\Omega$ or equivalently we replace or extend $v$ by 0 outside $\Omega$. We will drop the index $\Omega$ if $\Omega$ is understood clearly in the context. We can also define the maximal function by taking the supremum in cubes.

\[ \tilde{M}v(x) = \sup_{Q_r(x)} \int_{Q_r(x)} |v| \, d\mathcal{L}^n. \]  
(2.13)

It is clear that,
\[ Mv \leq C\tilde{M}v \leq CMv. \]

We will use the maximal function $Mv$ defined in (2.12) on balls in this paper.

The basic theorem for Hardy–Littlewood maximal function is the following:

**Theorem 2.2**

\[ \|M(v)(x)\|_{L^p(\Omega)} \leq C\|v\|_{L^p(\Omega)} \quad \text{for any } 1 < p \leq +\infty. \]

\[ |\{x \in \Omega : Mv(x) \geq \lambda\}| \leq \frac{C}{\lambda} \|v\|_{L^1(\Omega)}. \]

The first inequality is call strong $p-p$ estimates and the second is call weak $1-1$ estimates. This theorem says that the measures of $\{|v(x)| > \lambda\}$ and $\{Mv(x) > \lambda\}$ decay roughly in the same way. However $Mu(x) \leq 1$ is much more stable and geometrical than $|u(x)| \leq 1$ if $u$ is merely an $L^p$ function. The reason is that $Mu$ is invariant with respect to scaling. Another aspect
of the maximal function is that \( \{M u \geq \lambda\} \) and \( \{|u| \geq \lambda\} \) have roughly the same measure.

Likewise we will replace (2.11) by
\[
\left( \mathcal{M} |D^2 u|^2 \right) (x) \leq 1.
\] (2.14)

If \( \mathcal{M} |D^2 u|^2 (x_0) \leq 1 \), one would see that \( D^2 u (x) \) is really \( \leq 1 \) at \( x_0 \) in all scales in the sense of \( L^2 \).

In fact we will show that
\[
\|B_1 \cap \{ \mathcal{M} |D^2 u|^2 > \lambda_0^2 \} \| \leq \varepsilon \left( \|B_1 \cap \{ \mathcal{M} |D^2 u|^2 > 1 \} \| + \|B_1 \cap \{ \mathcal{M} (f^2) > \delta_0^2 \} \| \right)
\]
(2.15)
where \( \delta_0 \) can be taken as small as possible since it is about the data.

Now comes the question, how can we prove (2.15)?

Let’s first examine a limiting case that \( \{ \mathcal{M} |D^2 u|^2 > 1 \} \) has big measure, for example \( |B_1 \cap \{ \mathcal{M} |D^2 u|^2 > 1 \} | = |B_1| \), i.e., \( \{ |\mathcal{M} |D^2 u|^2 | \leq 1 \} \) has measure 0 or is a point, say.

This further reduces the question to the following.

Under what condition on the data \( f \), and \( \mathcal{M} |D^2 u|^2 \leq 1 \) at some point, can we conclude that \( \{ \mathcal{M} |D^2 u|^2 > \lambda_0^2 \} \) has small measure?

Let us again look at a limiting case of this question in which \( f = 0 \). However the answer is clear in this case since now \( u \) is harmonic. The condition that \( \mathcal{M} |D^2 u|^2 \leq 1 \) at some point, provides a bound, say \( N_0 \) on \( D^2 u \) and hence \( \{ \mathcal{M} |D^2 u|^2 > N_0^2 \} = \emptyset \) which has measure 0.

If \( f \) is small, then \( \{ \mathcal{M} |D^2 u|^2 > N_0^2 + 1 \} \) will have small measure by the standard energy estimates. Namely that
\[
(N_0^2 + 1) |\{ x : \mathcal{M} |D^2 u| > N_0^2 + 1 \} | \leq C \int |D^2 u|^2 \leq C \int |f|^2 \ll 1,
\]
which is exactly what we want.

Let us go back to the general case of (2.15).

First of all one should understand Lemma 2.7 by its scaling invariant form: which says that if the density of the set \( \{ \mathcal{M} |D^2 u|^2 > \lambda_0 \} \) in a ball \( B \):
\[
\frac{|\{ \mathcal{M} |D^2 u|^2 > \lambda_0 \} \cap B|}{|B|} \geq \varepsilon,
\]
then \( B \subset \{ \mathcal{M} |D^2 u|^2 > 1 \} \).

One immediately observes that if one can cover the set by balls (or by cubes as in the Calderón-Zygmund decomposition) so that the density
\[
\frac{|\{ \mathcal{M} |D^2 u|^2 > \lambda_0 \} \cap B|}{|B|} = \varepsilon
\]
and at the same time these balls are disjoint, then clearly $B$ is $\frac{1}{\epsilon}$ times as big as $\{ M|D^2 u|^2 > \lambda_0 \} \cap B$ in this situation. We conclude that (2.15) is true by taking sum over these balls.

Of course the above expectation, that is the set is covered by disjoint balls with the exact density in each of these balls, is not true. However, it is almost true. There are two ways of arranging these coverings. One is called Calderón Zygmund decomposition and the other is called Vitali covering lemma. The Calderón Zygmund decomposition arranges cubes in an almost optimal way so that in each cube the density is almost $\epsilon$ and the cubes are disjoint. In Vitali covering lemma, one can arrange the balls so they are disjoint and the balls cover a major portion of the set.

**Lemma 2.3 (Vitali)** Let $C$ be a class of balls in $\mathbb{R}^n$ with bounded radius. Then there is a finite or countable sequence $B_i \in C$ of disjoint balls such that

$$\bigcup_{B \in C} B \subset \bigcup_i 5B_i,$$

where $5B_i$ is the ball with the same center as $B_i$ and radius five times big.

We will use the following in this paper.

**Theorem 2.4 (Modified Vitali)** Let $0 < \epsilon < 1$ and let $C \subset D \subset B_1$ be two measurable sets with $|C| < \epsilon |B_1|$ and satisfying the following property: for every $x \in B_1$ with $|C \cap B_r(x)| \geq \epsilon |B_r|$, $B_r(x) \cap B_1 \subset D$. Then $|D| \geq \frac{1}{20^n \epsilon} |C|$.

**Proof.** Since $|C| < \epsilon |B_1|$, we see that for almost every $x \in C$, there is an $r_x < 2$ so that $|C \cap B_{r_x}(x)| = \epsilon |B_{r_x}|$ and $|C \cap B_r(x)| < \epsilon |B_r|$ for all $2 > r > r_x$. By Vitali’ covering lemma, there are $x_1, x_2, \ldots$, so that $B_{r_{x_1}}(x_1), B_{r_{x_2}}(x_2), \ldots$ are disjoint and $\bigcup_k B_{5r_{x_k}}(x_k) \cap B_1 \supset C$.

From the choice of $B_{r_{x_k}}$, we have

$$|C \cap B_{5r_{x_k}}(x_k)| < \epsilon |B_{5r_{x_k}}(x_k)| = 5^n \epsilon |B_{r_{x_k}}(x_k)| = 5^n |C \cap B_{r_{x_k}}(x_k)|.$$

We also notice that

$$|B_{r_{x_k}}(x_k)| \leq 4^n |B_{r_{x_k}}(x_k) \cap B_1|$$

since $x_k \in B_1$ and $r_{x_k} \leq 2$. 
Putting everything together,

\[ |C| = | \bigcup_k B_{5r(x_k)}(x_k) \cap C| \]
\[ \leq \sum_k |B_{5r(x_k)}(x_k) \cap C| \]
\[ \leq 5^n \sum_k \varepsilon |B_{r(x_k)}(x_k)| \]
\[ \leq 20^n \sum_k \varepsilon |B_{r(x_k)}(x_k) \cap B_1| \]
\[ = 20^n \varepsilon | \bigcup B_{r(x_k)}(x_k) \cap B_1| \]
\[ \leq 20^n \varepsilon |D|. \]

This finishes the proof.

The reader can compare the above theorem with the following so called Calderón–Zygmund Decomposition.

**Lemma 2.5 (Calderón–Zygmund Decomposition)** Let \( C \subset D \subset Q_1 \) be measurable such that

1. \( |C| < \varepsilon \),
2. if \( |Q \cap C| \geq \varepsilon |Q| \) for a cube then \( 3Q \subset D \).

Then \( |C| \leq \varepsilon |D| \).

Notice that this lemma is from a scaling invariant micro-condition to \( |A| \leq \varepsilon |B| \), a global estimate.

The details are provided in the next section.

We remark that one can prove Theorem 2.2 using either Theorem 2.4 or Lemma 2.5.

### 2.3 \( W^{2,p} \) Estimates

Now we prove Theorem 2.1. We only need to prove it for \( p > 2 \) since the statement for \( p < 2 \) follows from the standard duality argument.

The starting point of the estimates is the following classical estimates. See [9], page 317.

**Lemma 2.6** If

\[ \begin{align*}
\Delta u &= f & \text{in } B_1, \\
u &= 0 & \text{on } \partial B_1,
\end{align*} \]

then

\[ \int_{B_1} |D^2 u|^2 \leq C \int_{B_1} |f|^2. \]
Lemma 2.7 There is a constant $N_1$ so that for any $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ and if $u$ is a solution of (2.1) in a domain $\Omega \supset B_4$, with
\[
\left\{ \mathcal{M} \left( |f| \right)^2 \leq \delta^2 \right\} \cap \left\{ \mathcal{M} \left| \mathbf{D}^2 u \right|^2 \leq 1 \right\} \cap B_1 \neq \emptyset
\]
then
\[
\left| \left\{ \mathcal{M} \left| \mathbf{D}^2 u \right|^2 > N_1^2 \right\} \cap B_1 \right| < \varepsilon |B_1|.
\]

Proof. From condition (2.16), we see that there is a point $x_0 \in B_1$ so that
\[
\int_{B_r(x_0)} \left| \mathbf{D}^2 u \right|^2 \leq 1 \quad \text{and} \quad \int_{B_r(x_0)} |f|^2 \leq \delta^2,
\]
for all $B_r(x_0) \subset \Omega$ and consequently we have
\[
\int_{B_4} \left| \mathbf{D}^2 u \right|^2 \leq 2^n \quad \text{and} \quad \int_{B_4} |f|^2 \leq 2^n \delta^2.
\]
Then
\[
\int_{B_4} \left| \nabla u - \nabla u_{B_4} \right|^2 \leq C_1.
\]
Let $v$ be the solution of the following equation
\[
\left\{ \begin{array}{l}
\triangle v = 0 \\
v = u - \left( \nabla u \right)_{B_4} \cdot \mathbf{x} - \bar{u}_{B_4} \quad \text{on} \quad \partial B_4.
\end{array} \right.
\]
Then by the minimality of harmonic function with respect to energy in $B_4$,
\[
\int_{B_4} \left| \nabla v \right|^2 \leq \int_{B_4} \left| \nabla u - \nabla u_{B_4} \right|^2 \leq C_1.
\]
Now we can use the local $C^{1,1}$ estimates that there is a constant $N_0$ so that
\[
\left\| \mathbf{D}^2 v \right\|_{L^\infty(B_1)}^2 \leq N_0^2.
\]
At the same time we have,
\[
\int_{B_3} \left| \mathbf{D}^2 (u - v) \right|^2 \leq C \int_{B_4} f^2 \leq C \delta^2.
\]
From the weak $1 - 1$ estimate,
\[
\lambda \left| \{ x \in B_3 : \mathcal{M}_{B_3} \left| \mathbf{D}^2 (u - v) \right|^2 (x) > \lambda \} \right| \leq C \int_{B_3} \left| \mathbf{D}^2 (u - v) \right|^2 \leq C \int_{B_4} f^2 \leq C \delta^2.
\]
Consequently,
\[ \{ x \in B_1 : \mathcal{M}_{B_3} |D^2(u - v)|^2(x) > N_0^2 \} \leq C\delta^2. \]

Now we claim that
\[ \{ x \in B_1 : \mathcal{M} |D^2 u|^2 > N_1^2 \} \subset \{ x \in B_1 : \mathcal{M}_{B_3} |D^2(u - v)|^2 > N_0^2 \}, \]
where \( N_1^2 = \max(4N_0^2, 2^n) \).

Actually if \( y \in B_3 \), then
\[ |D^2 u(y)|^2 = |D^2 u(y)|^2 - 2|D^2 v(y)|^2 + 2|D^2 v(y)|^2 \leq 2|D^2 u(y) - D^2 v(y)|^2 + 2N_0^2. \]

Let \( x \) be a point in \( \{ x \in B_1 : \mathcal{M}_{B_3} |D^2(u - v)|^2(x) \leq N_0^2 \} \).

If \( r \leq 2 \) we have \( B_r(x) \subset B_3 \) and
\[ \sup_{r \leq 2} \int_{B_r(x)} |D^2 u|^2 \leq 2\mathcal{M}_{B_3}(|D^2(u - v)|^2)(x) + 2N_0^2 \leq 4N_1^2. \]

Now for \( r > 2 \), since \( x_0 \in B_r(x) \subset B_{2r}(x_0) \), we have
\[ \int_{B_r(x)} |D^2 u|^2 \leq \frac{1}{|B_r|} \int_{B_{2r}(x_0)} |D^2 u|^2 \leq 2^n, \]
where we have used (2.18). This says that \( \mathcal{M} (|D^2 u|^2)(x) \leq N_1^2. \)

This establishes the claim.

Finally, we have
\[ |\{ x \in B_1 : \mathcal{M}(|D^2 u|^2) > N_1^2 \}| \leq |\{ x \in B_1 : \mathcal{M}_{B_3}(|D^2(u - v)|^2 > N_0^2 \}| \leq \frac{C \delta^2}{N_0^2} \int f^2 \leq \frac{C \delta^2}{N_0^2} < C \delta^2 = \varepsilon |B_1|, \]
by taking \( \delta \) satisfying the last identity above. This completes the proof.

An immediate consequence of the above lemma is the following corollary.

**Corollary 2.8** Assume \( u \) is a solution in a domain \( \Omega \) and a ball \( B \) so that \( 4B \subset \Omega \). If \( | \{ x : \mathcal{M} \left( |D^2 u|^2 \right) > N_1^2 \} \cap B | \geq \varepsilon |B| \), then
\[ B \subset \left\{ x : \mathcal{M} \left( |D^2 u|^2 \right)(x) > 1 \right\} \cup \{ |f| > \delta^2 \}. \]
CHAPTER 2. \( L^p \) ESTIMATES

The moral of Corollary 2.8 is that the set \( \{ x : \mathcal{M} \left( |D^2 u|^2 \right) > 1 \} \) is much bigger than the set \( \{ x : \mathcal{M} \left( |D^2 u|^2 \right) > N_1^2 \} \) modulo \( \{ \mathcal{M}(f^2) > \delta^2 \} \) if

\[
\left| \left\{ x : \mathcal{M} \left( |D^2 u|^2 \right) > N_1^2 \right\} \cap B \right| = \varepsilon |B|
\]

As said in the previous section, we will cover a good portion of the set \( \{ x : \mathcal{M} \left( |D^2 u|^2 \right) > N_1^2 \} \) by disjoint balls so that in each of the balls the density of the set is \( \varepsilon \). As an application of Corollary 2.8, we will show the decay of the measure of the set \( \{ x : \mathcal{M} \left( |D^2 u|^2 \right) > N_1^2 \} \).

The covering is a careful choice of balls as in Vitali covering lemma.

**Corollary 2.9** Assume that \( u \) is a solution in a domain \( \Omega \supset B_4 \), with the condition that \( \left| \left\{ x \in B_1 : \mathcal{M} \left( |D^2 u|^2 \right) > N_1^2 \right\} \right| \leq \varepsilon |B_1| \). Then for \( \varepsilon_1 = 20^n \varepsilon \),

1. \( |B_1| \cap \{ \mathcal{M} \left( |D^2 u|^2 \right) > N_1^2 \} \leq \varepsilon_1 \left( |B_1| \cap \{ \mathcal{M} \left( |D^2 u|^2 \right) > 1 \} \right) \leq \varepsilon_1 \left( |B_1| \cap \{ \mathcal{M} \left( |f|^2 \right) > \delta^2 \} \right) \).
2. \( |B_1| \cap \{ \mathcal{M} \left( |D^2 u|^2 \right) > N_1^2 \} \leq \varepsilon_1 \left( |B_1| \cap \{ x \in B_1 : \mathcal{M} \left( |D^2 u|^2 \right) > \lambda^2 \} \right) \leq \varepsilon_1 \left( |B_1| \cap \{ \mathcal{M} \left( |f|^2 \right) > \delta^2 \lambda^2 \} \right) \).
3. \( |B_1| \cap \{ \mathcal{M} \left( |D^2 u|^2 \right) > (N_1^2)^k \} \leq \sum_{i=1}^k \varepsilon_1 \left( |B_1| \cap \{ \mathcal{M} \left( |f|^2 \right) > \delta^2 \left( N_1^2 \right)^{k-i} \} \right) \leq \varepsilon_1 \left( |B_1| \cap \{ \mathcal{M} \left( |D^2 f|^2 \right) > 1 \} \right) \).

**Proof.** (1) is a direct consequence of Corollary 2.8 and Theorem 2.4 on

\[
C = B_1 \cap \{ \mathcal{M} \left( |D^2 u|^2 \right) > N_1^2 \}, \quad D = B_1 \cap \{ \mathcal{M} \left( |D^2 u|^2 \right) > 1 \} \cup \{ \mathcal{M} \left( |f|^2 \right) > \delta^2 \}.
\]

(2) is obtained by applying (1) to the equation \( \triangle \left( \lambda^{-1} u \right) = \lambda^{-1} f \).

(3) is an iteration of (2) by \( \lambda = N_1, (N_1)^2, \ldots \).

**Theorem 2.10** If \( \triangle u = f \) in \( B_4 \) then

\[
\int_{B_1} |D^2 u|^p \leq C \int_{B_6} |f|^p + |u|^p.
\]

**Proof.** Without lose of generality, we may assume that \( \|f\|_p \) is small and the measure \( \{ x \in B_1 : \mathcal{M}|D^2 u|^2 > N_1^2 \} \leq \varepsilon |B_1| \) by multiplying the
2.4. $W^{1,p}$ ESTIMATES

function by a small constant. We will show that $\mathcal{M}(|D^2u|^2) \in L^\frac{p}{2}(B_1)$ from which it follows that $D^2u \in L^p(B_1)$. Since $f \in L^p$, we have that $\mathcal{M}(|f|^2) \in L^\frac{p}{2}$ with small norm. Suppose

$$\|f\|_{L^p(B_1)} = \delta.$$

Then

$$\sum_{i=1}^{+\infty} (N_1)^i \left| \left\{ \mathcal{M} \left( |f|^2 \right) > \delta^2 (N_1)^{2i} \right\} \right| \leq \frac{p^i}{\delta^p(N-1)} \|f\|_{L^p(B_1)}^p \leq C\|f\|_{L^p(B_2)}^p \leq C.$$

Hence

$$\int_{B_1} |D^2u|^p \leq \int_{B_1} \left( \mathcal{M} \left( |D^2u|^2 \right) \right)^\frac{p}{2} dx$$

$$= p \int_0^{+\infty} \lambda^{p-1} \left| \left\{ x \in B_1 : \mathcal{M} |D^2u|^2 \geq \lambda^2 \right\} \right| d\lambda$$

$$\leq p \left( |B_1| + \sum_{k=1}^{+\infty} (N_1)^{kp} \left| \left\{ x \in B_1 : \mathcal{M} |D^2u|^2 > (N_1)^{2k} \right\} \right| \right)$$

$$\leq p \left( |B_1| + \sum_{i=1}^{+\infty} N_1^{kp} \sum_{i=1}^k \varepsilon_1 \left| \left\{ x \in B_1 : \mathcal{M} |f|^2 \geq \delta^2 N_1^{2(k-i)} \right\} \right| \right.$$

$$+ \sum_{k=1}^{+\infty} N_1^{kp} \varepsilon_1 \left| \left\{ x : \mathcal{M} \left( |D^2u|^2 \right) \geq 1 \right\} \right|$$

$$\leq p \left( |B_1| + \sum_{i=1}^{+\infty} N_1^{ip} \varepsilon_1 \sum_{k \geq i} N_1^{(k-i)p} \left| B_1 \cap \left\{ \mathcal{M} |f|^2 \geq \delta^2 N_1^{2(k-i)} \right\} \right| \right.$$

$$+ \sum_{k=1}^{+\infty} N_1^{kp} \varepsilon_1 \left| B_1 \cap \left\{ \mathcal{M} |D^2u|^2 \geq 1 \right\} \right|$$

$$\leq C,$$

if we take $\varepsilon_1$ so that $N_1^p \varepsilon_1 < 1$ and the theorem follows.

We remark that our methods can be adapted to prove the same result by using the Caldron Zygmund decomposition.

The advantage of Vitali covering lemma is that it holds on any manifolds whereas the Caldron Zygmund decomposition requires cubes which give a clean cut in Euclidean spaces which are rare to find on manifolds.

2.4 $W^{1,p}$ estimates

One can easily adapt the methods in the preceding section to obtain $W^{1,p}$ estimates of the following type:
CHAPTER 2. $L^p$ ESTIMATES

Theorem 2.11 If

$$\Delta u = \text{div } \mathbf{f} = \sum_{i=1}^{n} \partial_i f_i \quad \text{in } B_1$$

then

$$\int_{B_1^2} |\nabla u|^p \leq C_p \int_{B_1} |f|^p + |u|^p \quad \text{for } 1 < p < +\infty.$$ 

Theorem 2.11 is proved by the following elementary energy estimates lemma and the steps as in the previous section.

Lemma 2.12 If

$$\begin{cases}
\Delta u = \text{div } \mathbf{f} & \text{in } B_1, \\
u = 0 & \text{on } \partial B_1,
\end{cases}$$

then

$$\int_{B_1} |\nabla u|^2 \leq \int_{B_1} |f|^2.$$ 

For a proof of this elementary lemma, see [9], page 297.
Chapter 3

Harmonic Maps

3.1 Introduction and Statement of Results

In a recent work we have made some progress in understanding the regularity theory of biharmonic maps. Since the techniques that we use are based on a simplified treatment of the regularity theory for harmonic map, we present here the argument for regularity of harmonic maps as an introduction to our work for the more complicated situation of biharmonic maps. To orient the reader, we briefly recall the basic references to the subject. For harmonic maps of surfaces, Morrey ([21]), Schoen ([25]), and Helein ([13], [14]) provided the classic regularity results. In higher dimensions, the corresponding regularity results are due to Hildebrandt-Kaul-Widman ([16]), Schoen-Uhlenbeck ([24]), Evans ([10]) and Bethuel([1]).

First we will present an elementary proof of Helein’s [13] regularity theory for weakly harmonic maps from compact surface to spheres. Our proof is more elementary because it does not rely on the structure theory of the Hardy spaces ([8]). We will first derive the continuity for harmonic maps of surfaces, we will then indicate how our method can be adapted to study stationary harmonic maps when the dimension of the manifold of the domain is greater than two, this again simplifies an earlier result of Evans [10]. At the end of the paper we will also give an easy proof of the $C^{1,\alpha}$ regularity of harmonic maps once the solution is continuous.

We would like to remark that, regularity results of Helein [13] and Evans [10] have also been extended to arbitrary target manifolds (in [14] and [1], cf also the excellent book of Helein [15] for a complete treatment of the theory). At this moment, it is not clear how to extend our method to treat the case of general target manifolds, we hope to extend this type of argument to cover
3.2 Weakly Harmonic Maps

Suppose $M^m$, $N^n$ are Riemannian manifolds we assume $N^n$ to be isometrically embedded in Euclidean space $R^k$ and let $u: M \rightarrow N$ be a smooth map. Denote its differential by $du: TM \rightarrow TN$. In local coordinates $\{x^i\}$ on $M$ and ambient Euclidean coordinates $\{u^\alpha\}$, the Riemannian metric $g$ on $M$ is represented by $g = g_{ij} dx^i dx^j$; and we denote $du$ by the matrix $(\frac{\partial u^\alpha}{\partial x^i})$.

**Definition 4** The energy density of $u$ at $x \in M$ is defined by

$$e(u) = \frac{1}{2} |du|^2 = \frac{1}{2} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j} g^{ij}$$

where $g^{ij}$ is the inverse matrix of $g_{ij}$, i.e. $g^{ij} g_{jk} = \delta_i^k$.

**Definition 5** The energy of $u$ is defined by

$$E(u) = \int e(u) dVol$$

where

$$dVol = \sqrt{\det(g_{ij})} dx^1 \ldots dx^m.$$ 

**Definition 6** A map $u: M \rightarrow N$ a.e. and $du \in L^2(M, R^k)$ is harmonic if it is a critical point of the energy in the sense of calculus of variations, i.e., if for each smooth deformation of $u_t$ such that $u_0 = u$, we have

$$\frac{d}{dt} E(u_t) \bigg|_{t=0} = 0.$$ 

**Theorem 3.1 (Helein [13], [14])** Any harmonic map from a surface is Hölder continuous.

In the following we shall give a simple proof of above theorem in the case the target manifold is the standard sphere.

Let us recall the following standard estimates from the linear theory. Denote by $B_1$ the unit ball in $R^n$. Denote by $\int_{B_1} f$ the average integral of $f$ over $B_1$. 

the general case.
3.2. WEAKLY HARMONIC MAPS

Lemma 3.2 Suppose $u$ is a scalar weak solution of

$$
\begin{align*}
\text{div}(A(x)du) &= \text{div}(F) = \sum_{i=1}^{m} \frac{\partial F_i}{\partial x^i} \text{ on } B \\
u &= 0 \text{ on } \partial B_1
\end{align*}
$$

with $\lambda I \leq A(x) \leq \Lambda I$ and that $A(x)$ is H"older continuous in $B_1$, then for any $1 < q < \infty$ there is a constant $C$ depending only on $q$, the dimension $m$, two elliptic constants $\lambda, \Lambda$ and $\|A\|_{C^\alpha(B_1)}$ such that

$$
\|du\|_{L^q(B_1)} \leq C\|F\|_{L^q(B_1)}.
$$

Theorem 3.3 Any harmonic map from a two dimensional disk to sphere $S^n$ is H"older continuous.

Proof. We first fix some $1 < q < 2$ and denote $p = \frac{2q}{2-q}$. We want to show that if $E_1(u) = \frac{1}{2} \int_{B_1} |\nabla u|^2$ is small enough, then for some fixed $s < 1/2$ we have

$$
(\int_{B_1} |u - A_{1}|^p)^{\frac{1}{p}} \leq \frac{1}{2} (\int_{B_1} |u - A_0|^p)^{\frac{1}{p}},
$$

for some constant vectors $A_0, A_1$ satisfying

$$
|A_0 - A_1|^p \leq C \int_{B_1} |u - A_0|^p).
$$

Then a rescaling yields,

$$
(\int_{B_{sk}} |u - A_k|^p)^{\frac{1}{p}} \leq \frac{1}{2} (\int_{B_{sk-1}} |u - A_{k-1}|^p)^{\frac{1}{p}},
$$

with

$$
|A_k - A_{k-1}|^p \leq C \int_{B_{sk-1}} |u - A_{k-1}|^p.
$$

It then follows from (1.3) and (1.4) that

$$
|A_k - A_{k-1}| \leq C_p(\frac{1}{2})^k (\int_{B_1} |u - A_0|^p)^{1/p} \leq C_p(\frac{1}{2})^k E_1[u]
$$

for some constant $C_p$ depending only on $C$ and $p$. Thus the sequence $A_k$ converges exponentially to a vector $A$, with

$$
|A_k - A| \leq C_p(\frac{1}{2})^k E_1[u].
$$

We have also from (1.3)

$$
(\int_{B_{sk}} |u - A_k|^p)^{\frac{1}{p}} \leq (\frac{1}{2})^k (\int_{B_1} |u - A_0|^p)^{\frac{1}{p}}.
$$
The Hölder continuity of \( u \) follows from the decay estimates (3.6) and (3.7).

Let us recall the equation of harmonic map to spheres \( u = (u^1, \ldots, u^n) \) as
\[
-\Delta u^\alpha = u^\alpha |du|^2.
\]
(3.8)

An important observation of Helein is to rewrite this equation when the target space is \( S^n \):
\[
-\Delta u^\alpha = u^\alpha |du|^2 = \sum_{\beta=1}^{n} \sum_{k=1}^{m} (u^\alpha u^\beta_k - u^\beta u^\alpha_k) u^\beta_k.
\]
(3.9)

Let us assume \( \int_{B_r} |\nabla u|^2 \leq \epsilon \) for some \( \epsilon \) small. Let \( \frac{1}{2} \leq r \leq 1 \) be such that \( \int_{B_r} |u - A_0|^p \leq 8 \int_{B_1} |u - A_0|^p \) for some constant vector \( A_0 \); it turns out the proof below works for any constant vector \( A_0 \). Let \( h \) be the harmonic function such that \( u = h \) on \( \partial B_r \). We have
\[
|dh(x)|^p \leq C \int_{B_r} |h - A_0|^p \leq C(\int_{B_1} |u - A_0|^p)
\]
(3.10)

for \( |x| \leq \frac{r}{4} \). From (3.9) we have,
\[
-\Delta (u^\alpha - h^\alpha) = \sum_{\beta=1}^{n} \sum_{k=1}^{m} \partial_k [(u^\alpha u^\beta_k - u^\beta u^\alpha_k)(u^\beta - A^\beta_0)].
\]
(3.11)

denote \( E = \frac{1}{2} \int_{B_1} |Du|^2 \) as before, then for any fixed \( 1 < q < 2, p = \frac{2q}{2-q} \) we have from Lemma 3.2 and the Hölder inequality that
\[
\int_{B_r} |d(u - h)|^q \leq C \int_{B_r} \sum_{\alpha,\beta,k} [(u^\alpha u^\beta_k - u^\beta u^\alpha_k)(u^\beta - A^\beta_0)]^q
\]
\[
\leq C \sum_{\alpha,\beta,k} (\int_{B_r} [(u^\alpha u^\beta_k - u^\beta u^\alpha_k)]^2 (\int_{B_r} |u^\beta - A_0|^p)^{2/q}
\]
\[
\leq CE_{r}^{q/2} (\int_{B_1} |u - A_0|^p)^{q/2}.
\]
(3.12)

Now for any \( s < \frac{1}{4} < r \) we have, via Sobolev inequality,
\[
\frac{1}{s^2} \int_{B_s} |u - h(0)|^p dx \leq \frac{2^p-1}{s^2} \int_{B_s} |u - h|^p dx + \frac{2^p-1}{s^2} \int_{B_s} |h - h(0)|^p dx
\]
\[
\leq \frac{2^p-1}{s^2} (\int_{B_s} |u - h|^p) + \frac{2^p-1}{s^2} \int_{B_s} |h - h(0)|^p dx
\]
\[
\leq \frac{2^p-1}{s^2} (\int_{B_1} |d(u - h)|^q dx)^{p/q} + 2^{p-1} C s^p \sup_{B_{\frac{r}{4}}} |dh|^p
\]
\[
\leq C \frac{E_{r}^{q/2}}{s^2} (\int_{B_1} |u - A_0|^p) + C s^p (\int_{B_1} |u - A_0|^p).
\]
(3.13)

(3.14)
Now, taking $s$ small such that the second term is less than
\[
\frac{1}{2p+2} \int_{B_1} |u - A_0|^p
\]
and then taking $E = E_1$ small such that the first term is less than
\[
\frac{1}{2p+2} \int_{B_1} |u - A_0|^p
\]
so we have by letting $A_1 = h(0)$
\[
\int_{B_r} |u - A_1|^p dx \leq \frac{1}{2p} \int_{B_1} |u - A_0|^p.
\]
This proves (3.1). We then observe that for $A_1 = h(0) = \gamma_{\partial B_r} h = \gamma_{\partial B_r} u,$
\[
|A_1 - A_0| = |h(0) - A_0| = |\gamma_{\partial B_r}(u - A_0)|
\leq (\gamma_{\partial B_r} |u - A_0|^p)^{\frac{1}{p}}
\leq C(\gamma_{B_1} |u - A_0|^p)^{\frac{1}{p}}.
\]
Thus (3.2) is satisfied for any constant vector $A_0$. From (3.1) and (3.2), we conclude that $u$ is Hölder continuous by the arguments in (3.3) to (3.7).

**Remark** The main point in the above argument is that the right hand side of (3.11) is the divergence of a quadratic oscillation of $u$ while the left hand side of (3.11) is the divergence of the linear oscillation of $u$. The indices in $L^p$ norms in (3.13) are from different sources. The index $p$ on the left is from the Sobolev embedding and the index $p$ on the right hand side is from the Hölder inequality (3.12). These two match only in dimension 2. In our later work on biharmonic maps we again observe a similar matching of indices in dimension 4.

We will use BMO semi-norm to study the case when these indices do not match in Section 3.3, where we will have $\frac{mq}{m-q}$ on the left hand side and $\frac{2q}{2-q}$ on the right of the inequality (3.13).

### 3.3 Stationary Harmonic Maps

In this section we modify the argument in section 3.2 to give an alternative proof of Evans’ theorem that the singular set of a stationary harmonic map from an $m$-manifold to a sphere has $m - 2$ Hausdorff measure zero.
**Definition 7** (Stationary Harmonic Maps) Let \( u \) be a harmonic map from a manifold \( M \) (possible with boundary) to another compact manifold \( N \). We say that \( u \) is stationary if

\[
\frac{d}{dt} E(u(\varphi(t))) = 0 \text{ at } t = 0,
\]

where \( \varphi(t) : M \to M \) is a smooth one parameter family of diffeomorphism such that \( \varphi(0) = \text{identity} \).

**Definition 8** A function \( f \) defined on a smooth domain \( \Omega \subset \mathbb{R}^m \) is in BMO, i.e. function of bounded mean oscillation, if

\[
||u||_{BMO(\Omega)} = \sup_B \int_B |u - u_B| dx < +\infty, \tag{3.15}
\]

where \( u_B = \frac{1}{|B|} \int_B u \), the supremum is taken over all balls with \( B \subset \Omega \).

We will use the following classical result of John and Nirenberg([18], cf also Chapter IV of Stein [?]).

**Theorem 3.4** For any \( 1 < p < \infty \) there exists an \( C_p \) (which depends only on \( p \) and dimension \( m \)) such that if \( u \in BMO(\Omega) \), then

\[
\frac{1}{C_p} ||u||_{BMO(\Omega)} \leq \sup_B (\int_B |u - u_B|^p dx)^{\frac{1}{p}} \leq C_p ||u||_{BMO(\Omega)}
\]

where the supremum is taken over all balls with \( B \subset \Omega \).

The following monotonicity formula is proved by Schoen-Uhlenbeck ([24]) in the case of minimizing harmonic maps, and by P. Price ([23]) for stationary harmonic maps.

**Theorem 3.5** A stationary harmonic map from the \( m \)-dimensional Euclidean disk satisfies the following monotonicity formula: The scaling invariant energy

\[
E(r) = r^2 \int_{B_r} |du|^2 = \int_{B_1} |du(rx)|^2.
\]

is monotonically increasing in \( r \) for all concentric balls \( B_r = B(x, r) \subset B_1 \).

The following regularity result is due to Evans ([10]):

**Theorem 3.6** A stationary harmonic map from the \( m \)-dimensional Euclidean disk to sphere \( S^n \) is Hölder continuous except a set of \( m-2 \) dimensional Hausdorff measure zero.
Our proof below is patterned after the two dimensional argument. In higher dimensions the exponents resulting from the two inequalities (3.12) and (3.13) do not match so we show instead that the BMO norm of the map decays. In fact we have to show the decay of the map in every scale. The monotonicity formula makes the control in every scale possible.

We will show that when \( E_1(u) \) is small enough, we can choose \( s \) small, so that

\[
||u||_{BMO(B_s)} \leq \frac{1}{2}||u||_{BMO(B_1)},
\]

(3.16)

Then an iteration of (3.16) yields,

\[
||u||_{BMO(B_{s^k})} \leq \frac{1}{2^k}||u||_{BMO(B_1)}.
\]

(3.17)

We start with the equation of a harmonic map \( u = (u^1, \ldots, u^n) \) with target \( S^n \) as in (3.8). We also assume \( \int_{B_1}|du|^2 \leq \epsilon \) for some \( \epsilon \) small here \( B_1 \) is the unit ball in \( \mathbb{R}^m \). The monotonicity formula asserts \( E_r = r^{2-m} \int_{B_r(0)} |du|^2 \leq 2\epsilon \) for all \( 0 < r < 1 \). A little reflection will show that for any \( B_r(x) \subset B_{\frac{1}{2}}(0) \) we also have for some constant \( C_m \):

\[
r^2 \int_{B_r(x)} |du|^2 \leq C_m \epsilon.
\]

(3.18)

Now fix \( r \) and \( x_0 \) with \( B_r(x_0) \subset B_{\frac{1}{2}}(0) \); we abbreviate \( B_r \) for \( B_r(x_0) \); and choose \( r/2 \leq r_1 \leq r \) so that

\[
\int_{\partial B_{r_1}} |u - A_0| \leq 8 \int_{B_{r_1}} |u - A_0|
\]

(3.19)

here \( A_0 = \frac{1}{|B_{r_1}|} u \).

Let \( h \) be the harmonic function such that \( u = h \) on \( \partial B_{r_1} \). We have from (3.19):

\[
|dh(x)|^p \leq C_p r_1^{-p} \int_{B_{r_1}} |u - A_0|^p
\]

for \( |x| \leq \frac{r_1}{4} \). Then from equation (3.11) we have with \( E_r = r^2 \int_{B_r}|du|^2 \), that exactly as (3.12) that

\[
\int_{B_{r_1}} |d(u - h)|^q \leq CE_{r_1}^{q/2} \left( \int_{B_1}|u - A_0|^{2q} \right)^{\frac{2-q}{q}} r_1^{-q}.
\]

(3.20)
CHAPTER 3. HARMONIC MAPS

Now taking $p = \frac{mq}{m-q}$, for any $s < r_1/4$, we have, via Sobolev inequality and the John-Nirenberg’s inequality as in Theorem 3.4.

\[
\frac{1}{s^m} \int_{B_s} |u - h(x_0)|^p dx \leq \frac{2^{p-1}}{s^m} \int_{B_{r_1}} |u - h|^p dx + \frac{2^{p-1}}{s^m} \int_{B_s} |h - h(x_0)|^p dx \\
\leq C \frac{2^{p-1}}{s^m} \left( \int_{B_{r_1}} |d(u - h)|^q dx \right)^{\frac{p}{q}} + \frac{2^{p-1}}{s^m} \int_{B_s} |h - h(x_0)|^p dx \\
\leq C \frac{2^{p-1}}{s^m} \left( \int_{B_{r_1}} |d(u - h)|^q dx \right)^{\frac{p}{q}} + C 2^{p-1} s^p \sup_{B_{r_1}/4} |dh|^p \\
\leq C \frac{2^{p-1}}{s^m} \left( \int_{B_{r_1}} |u - A_0|^{2\frac{q}{r-1}} dx \right)^{\frac{p}{2} \left(\frac{2-q}{q}\right)} + C \frac{s^p}{r_1} \left( \int_{B_{r_1}} |u - A_0|^p dx \right) \\
\leq C \frac{2^{p-1}}{s^m} E_{r_1}^{p/2} (\|u\|_{BMO(B_r)})^p + C \frac{s^p}{r_1} (\|u\|_{BMO(B_r)})^p. \quad (3.21)
\]

Now, taking $s/r_1 \leq r_0$ small such that the second term is less than

\[
\frac{1}{2^{2p+1}} (\|u\|_{BMO(B_r)})^p
\]

and then taking $E$ small depending only on $r$ such that the first term is less than

\[
\frac{1}{2^{2p+1}} (\|u\|_{BMO(B_r)})^p
\]

so we have, by taking the supremum over balls $B \subset B_r(x_0)$, for all $s \leq s_0 = r_1 r_0$ and $r$ sufficiently small that

\[
\int_{B_s} |u - h(x_0)|^p dx \leq \frac{1}{2^{2p}} \|u\|_{BMO(B_r)}^P. \quad (3.22)
\]

Since

\[
\int_{B_s} |u| - \int_{B_s} u|^p \leq 2^p \int_{B_s} |u - h(x_0)|^p,
\]

we have

\[
\int_{B_s} |u - \int_{B_s} u|^p \leq \frac{1}{2^p} \|u\|_{BMO(B_r)}^P. \quad (3.23)
\]

We then vary $r$ and $x_0$ with $B_r = B_r(x_0) \subset B_{r_1}(0)$. As $B_r$ runs over all balls in $B_{r_1}(0)$, $B_s$ runs over all balls in $B_{s_0}(0)$. From this we conclude from (3.23) that (3.5) holds and this finishes the proof of the theorem.

3.4 $C^{1,\gamma}$ Regularity

We will prove the following regularity result in this section.
3.4. $C^{1,\gamma}$ REGULARITY

Theorem 3.7 If $u$ is a weakly harmonic map from $M^m$ to $N^n$ for $m \geq 2$ and $u$ is continuous in an open set in $M^m$, then $u$ is locally smooth there.

We use compactness argument to establish $C^{1,\gamma}$ regularity. Then the higher order regularity of the map follows from the harmonic equation and standard elliptic theory.

Suppose the $u$ is a weak solution of

$$\Delta u^\alpha = f^\alpha(x, \nabla u),$$

(3.24)

We would assume also that

$$|f^\alpha(x, P)| \leq A(1 + |P|^2),$$

(3.25)

for some constant $A$.

Our iteration scheme depends on a finer structure of the right hand side of (3.25) defined as

$$|f^\alpha(x, P)| \leq A(1 + \mu |P|^2),$$

(3.26)

for some constant $A$ sufficiently small and with some constant $\mu$ suitably small.

Remarks 1. We introduce (3.26) in order to trace the different decay rate of the constant term and the quadratic term in $f^\alpha$.

2. Once we know $u$ is continuous and satisfies a system of equations (3.24) and (3.25), then we can define

$$u_1(x) = \frac{u(rx) - u(0)}{c(r)},$$

(3.27)

where $c(r) = r + \sup_{B_1} |u(rx) - u(0)|$. Then $u_1$ satisfies equation of same type as that of (3.24) with

$$f_1^\alpha(x, P) = \frac{r^2}{c(r)} f^\alpha(rx, \frac{c(r)}{r} P).$$

Thus $f_1$ satisfies equation of type (3.26) with

$$|f_1(x, P)| \leq A_1(1 + \mu |P|^2)$$

with $A_1 = c(r)^{\frac{1}{2}} A$ and $\mu = c(r)^{\frac{3}{2}}$, both can be made arbitrarily small when $r$ is sufficiently small. Also, the $C^{1,\gamma}$ estimates of $u$ follows from that of $u_1$. Thus we may assume without loss of generality that the harmonic map in our proof of Theorem 3.7 below satisfies both (3.24) and (3.26). This will
be the only place in our proof where we will use the assumption that \( u \) is continuous.

3. We also remark that the \( C^{1,\gamma} \) regularity theory holds if we replace the \( \Delta u^\alpha \) by any elliptic systems, in particular it covers the case when \( \Delta \) is the Laplacian operator on a manifold.

**Theorem 3.8** Suppose \( w \) is a solution of (3.24) satisfying (3.26) in \( B_1 \) with \( \mu|w| \leq C_1 \), and \( AC_1 < 1 \) then

\[
\int_{B_{1/2}} |\nabla w|^2 \leq C \int_{B_1} (w^2 + 1).
\]

where \( C = C(C_1, A) \).

**Proof.** This is a Caccioppoli type inequality. Choose a smooth cut-off function \( \eta \) with \( \eta(x) = \eta(|x|) \) of compact support in \( B_1 \) and \( \eta = 1 \) on \( B_{1/2} \). Multiplying the equation (3.24) by \( \eta^2 w \) and integrate by parts, we have

\[
\int \eta^2 w (-\Delta w) = \int \eta^2 w f dx.
\]

As before, we have

\[
\int \eta^2 |\nabla w|^2 = \int \eta^2 w f - 2 \int \eta \nabla \eta \cdot w \nabla w \\
\leq \int A |w|^2 + \int AC_1 \eta^2 |\nabla w|^2 + \epsilon \int \eta^2 |\nabla w|^2 + \frac{1}{\epsilon} |\nabla \eta|^2 w^2 \\
\leq \int A n^2 (|w|^2 + 1) + \frac{1}{\epsilon} |\nabla \eta|^2 w^2 + (C_1 A + \epsilon) \int |\eta|^2 |\nabla w|^2.
\]

Now taking \( \epsilon \) small so that \( C_1 A + \epsilon < 1 \), we obtain the inequality immediately.

**Lemma 3.9** For any given \( \epsilon > 0 \), there is a \( \delta > 0 \) such that if \( A \leq \delta \) then for any solution \( w \) of (3.24) with (3.26) satisfying \( \mu|w| \leq C_1 \) and \( \int_{B_1} |w|^2 \leq 1 \), there exists some harmonic function \( h \) defined in \( B_{1/2} \) which approximates \( w \) in the sense that:

\[
\int_{B_{1/2}} |w - h|^2 \leq \epsilon^2.
\]

**Proof.** We prove the result by contradiction. Suppose there exist some \( \epsilon > 0 \), and sequences of \( w_n \) and \( f_n \) satisfying

\[
\begin{cases}
-\Delta w_n = f_n(x, \nabla w_n) \\
\int_{B_1} w_n^2 \leq 1, \\
|f_n| \leq \frac{1}{n} (1 + \mu|\nabla w_n|^2).
\end{cases}
\]


3.4. $C^{1,\gamma}$ REGULARITY

But for any harmonic function $v$ in $B_{\frac{1}{2}}$, we have

$$\int |w_n - v|^2 dx \geq \epsilon.$$  

We then have by (3.4) in the previous lemma,

$$\int_{B_{\frac{3}{4}}} |\nabla w_n|^2 dx \leq C.$$  

Hence $\{w_n\}$ has a convergent subsequence, which we still denoted as $w_n$, such that

$$w_n \rightharpoonup w \text{ weakly in } H^1 \text{ and } w_n \rightarrow w \text{ strongly in } L^2(B_{\frac{3}{4}}).$$

We will show that $w$ itself is harmonic in $B_{\frac{3}{4}}$, which is a contradiction. Actually, for any test function $\varphi \in C^\infty_0(B_{\frac{3}{4}})$,

$$\int \nabla \varphi \nabla w_n = \int \varphi f_n dx.$$  

Thus if we let $n$ go to infinity, we have

$$\int \nabla \varphi \nabla w = 0.$$  

Hence $w$ is actually harmonic. We have thus proved the assertion (3.9).

**Corollary 3.10** For any $0 < \gamma < 1$ and $C_1$, there exist some $\epsilon > 0$ and $0 < \lambda < \frac{1}{2}$ such if $A \leq \epsilon$ and $w$ is a solution of (3.24) and (3.26) with $\mu|w| \leq C_1$ and $\int_{B_1}|w|^2 \leq 1$, then there is a linear function $l(x) = Bx + C$ such that

$$\int_{B_\lambda} |w - l|^2 \leq \lambda^{2(1+\gamma)}$$

and $|B| + |C| \leq C_0$ a universal constant.

**Proof.** Let $h$ be the harmonic function such that

$$\int_{B_{\frac{3}{4}}} |w - h|^2 \leq \epsilon^2,$$  

as in the statement of Lemma 3.9. By the triangle inequality, we have

$$\int_{B_{\frac{3}{4}}} |h|^2 \leq 2 \int_{B_1} (|w|^2 + 1) \leq C_0.$$
Since $h$ is harmonic, we have

$$|\nabla^2 h(x)|^2 \leq C \int_{B_{1/2}} |h|^2 \leq CC_0 \quad \text{for} \quad |x| \leq \frac{1}{4},$$

for some constant $C=C(m)$. If we now denote $l(x)$ be the first order Taylor polynomial of $h$ at 0, we have for $\lambda \leq \frac{1}{4}$,

$$\int_{B_\lambda} |w - l|^2 \ dx \leq 2 \int_{B_\lambda} |w - h|^2 \ dx + 2 \int_{B_\lambda} |h - l|^2 \ dx \leq C\lambda^{-m} \epsilon^2 + C\lambda^4.$$

by (3.4) and (3.4), where $C = C(m)$. Thus we can take $\lambda$ small so that the second term in (3.4) is less than

$$\frac{1}{2} \lambda^{2(1+\gamma)},$$

and then taking $\epsilon$ sufficiently small so that the first term in (3.4) is less than

$$\frac{1}{2} \lambda^{2(1+\gamma)},$$

we obtain (3.10) in the corollary.

We now prove Theorem 3.7.

Proof. (of Theorem 3.7) We first assert that by Remark(2) at the beginning of this section, we may assume that the continuous harmonic map we have also satisfy that $|u(x)| \leq 1$ for $x \in B_1$ and system (3.24) with conditions (3.26).

We will prove by induction the following statement (*):

There exist some constants $C_0$, $0 < \lambda < \frac{1}{2}$ and $\epsilon > 0$ such that for $|u| \leq 1$ and $u$ is a solution of (3.24) with (3.26) with $A \leq \epsilon$, there are linear functions $l_k(x) = B_k \cdot x + C_k$ such that

$$\int_{B_\lambda} |u - l_k|^2 \leq \lambda^{2(1+\gamma)k}$$

and the constants $B_k$ and $C_k$ satisfy

$$\lambda^{k}|B_k - B_{k+1}| + |C_k - C_{k+1}| \leq C_0 \lambda^{(\gamma+1)k},$$

with $C_0$ a universal constant.

Assuming statement (*), we can then argue as in the proof of Theorem 3.3 of section 3.2 that both $B_k$ and $C_k$ converge in an exponentially decay
rate to $B$ and $C$ respectively, and $u$ can be approximated by a linear function $l(x) = B \cdot x + C$ satisfying

\[ \int_{B_{\lambda k}} |u - l|^2 \leq 2\lambda^{2(1+\gamma)k} \]

for each $k$. Thus $u$ is in $C^{1,\gamma}$ by the usual Morrey estimates.

To prove the statement (*), we notice that the statement for $k = 0$ follows from our assumption on $u$ and the statement $k = 1$ follows from Corollary 3.10. Thus we need only to establish the inductive step.

We assume statement (*) for up to $k$. We first observe that from (3.29) that $|B_k| = |\nabla l_k| \leq \frac{C_0}{1-\lambda^2} \leq \frac{C_0}{1-2^{-\gamma}}$, similarly $|C_k| \leq \frac{C_0}{1-2^{-\gamma}}$.

Define $w(x) = \frac{(u - l_k)(\lambda^k x)}{\lambda^{(1+\gamma)k}}$. Then $w$ satisfies,

\[ \Delta w^\alpha = f_1^\alpha(x, \nabla w), \]

where

\[ f_1^\alpha(x, P) = \lambda^{(1-\gamma)k} f^\alpha(\lambda^k x, \lambda^k P + \nabla l_k). \]

By our assumption that $u$ satisfies (3.25), we have

\[ |f_1(x, P)| \leq A\lambda^{(1-\gamma)k}(1 + 2|\nabla l_k|^2) + 2\lambda^{(1+\gamma)k} A|P|^2. \]

We verify that $w$ satisfies the conditions of Corollary 3.26. To see this, we have $w$ satisfy (3.24) and (3.4), with

\[ 2|w|\lambda^{(1+\gamma)k} = 2(u - l_k)(\lambda^k x) \leq 2(1 + |\nabla l_k| + |C_k|) \leq 4(1 + \frac{C_0}{1-2^{-\gamma}}) \]

and

\[ \int_{B_1} |w|^2 = \lambda^{-2(1+\gamma)k} \int_{B_{\lambda k}} |u - l_k|^2 \leq 1. \]

Thus we may apply Corollary 3.10 to conclude that there exist some linear function $l(x) = Bx + C$ with $|B| + |C| \leq C_0$ some universal bound, and

\[ \int_{B_{\lambda k}} |w - l|^2 \leq \lambda^{2(1+\gamma)}. \tag{3.30} \]

We now define $l_{k+1} = l_k + \lambda^{(1+\gamma)k} l(\frac{x}{\lambda^k})$ and check that inequalities (3.28) and (3.29) in the statement (*) for $k + 1$ follow directly from (3.30) and the bounds on $l(x)$. We have thus finished the proof of the inductive step in statement (*) and hence the proof of Theorem 3.7.
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Address of the author:
Lihe Wang, Department of Mathematics, University of Iowa, Iowa city, Iowa 52240.

lwang@math.uiowa.edu

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