NAME (PRINT): _________________________________

I pledge to NOT disclose the content of this exam to anyone (SIGN BELOW):

______________________________________________

MATH 115: Introduction to Real Analysis
Midterm II, Fall 2013

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Rules of the exam

- You have 50 minutes to complete this exam.
- Show your work! – any answer without an explanation will get you zero points.
- Please read the questions carefully; some ask for more than one thing.
- When applicable, BOX the answer.
- Do not forget to write your name.

Good luck!
PROBLEM 1: (25 points) Define each of the terms listed below:

1. Convergence in measure for a sequence of functions \( f_n : E \to \mathbb{R} \)

2. Uniform integrable sequence of functions

3. Measurable function

4. Simple function

5. Almost everywhere pointwise convergence for a sequence of functions \( f_n : E \to \mathbb{R} \)
PROBLEM 2: (20 points) State and prove Fatou’s Lemma.
PROBLEM 3: (20 points) Assume that $E$ has finite measure and $f, f_n : E \to \mathbb{R}$ are measurable functions for every $n \geq 1$.

1. If $\{f_n\} \to f$ pointwise a.e. on $E$ then $\{f_n\} \to f$ in measure on $E$.

2. Conversely, if $\{f_n\} \to f$ in measure on $E$ then there exists a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on $E$ to $f$.

Illustrate by a counterexample that the full converse of part 1. above does not hold.
PROBLEM 4: (20 points) Let $f, g : E \to \mathbb{R}$ be measurable functions. Show that their sum $f + g$ and their product $fg$ are also measurable.
PROBLEM 5: (15 points) Let $f : [0, 1] \to [0, \infty)$ be a measurable function. Suppose that there exists $M > 0$ such that $\int_{1 - \frac{1}{n}}^{1} f(x) \, d\mu(x) \leq M$ for all $n \geq 1$. Then show that $f$ is Lebesgue integrable on $[0, 1]$. 
BEAUTIFUL PROBLEM : (10 points) Solve one of the following problems at your choice

1. If $f : [0, 2] \rightarrow \mathbb{R}$ is an increasing function then show that $\int_0^2 (x - 1)f(x)\,dx \geq 0$.

2. If $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function then show that

$$\int_0^1 \left( \int_0^1 |f(x) + f(y)|\,dx \right)\,dy \geq \int_0^1 |f(x)|\,dx.$$