I pledge to NOT disclose the content of this exam to anyone (SIGN BELOW):
Rules of the exam

• You have 55 minutes to complete this exam.
• Show your work! – any answer without an explanation will get you zero points.
• Please read the questions carefully; some ask for more than one thing.
• When applicable, BOX the answer.
• Do not forget to write your name.

Good luck!

Joke 1: A Mathematician, a Biologist and a Physicist are sitting in a street cafe watching people going in and coming out of the house on the other side of the street.

First they see two people going into the house. Time passes. After a while they notice three persons coming out of the house.

The Physicist: "The measurement wasn’t accurate."

The Biologist’s conclusion: "They have reproduced!"

The Mathematician: "If now exactly 1 person enters the house then it will be empty again."

Joke 2: A physicist, an engineer and a mathematician were all in a hotel sleeping when a fire broke out in their respective rooms.

The physicist woke up, saw the fire, ran over to his desk, pulled out his CRC, and began working out all sorts of fluid dynamics equations. After a couple minutes, he threw down his pencil, got a graduated cylinder out of his suitcase, and measured out a precise amount of water. He threw it on the fire, extinguishing it, with not a drop wasted, and went back to sleep.

The engineer woke up, saw the fire, ran into the bathroom, turned on the faucets full-blast, flooding out the entire apartment, which put out the fire, and went back to sleep.

The mathematician woke up, saw the fire, ran over to his desk, began working through theorems, lemmas, hypotheses, you-name-it, and after a few minutes, put down his pencil triumphantly and exclaimed, "I have *proven* that I *can* put the fire out!" He then went back to sleep.

Joke 3: Einstein dies and goes to heaven only to be informed that his room is not yet ready. "I hope you will not mind waiting in a dormitory. We are very sorry, but it’s the best we can do and you will have to share the room with others,” he is told by the doorman (say his name is Pete). Einstein says that this is no problem at all and that there is no need to make such a great fuss. So Pete leads him to the dorm. They enter and Albert is introduced to all of the present inhabitants. "See, here is your first room mate. He has an IQ of 180!” "Why that’s wonderful!” Says Albert. "We can discuss mathematics!” "And here is your second room mate. His IQ is 150!” "Why that’s wonderful!” Says Albert. "We can discuss physics!” "And here is your third room mate. His IQ is 100!” "That Wonderful! We can discuss the latest plays at the theater!” Just then another man moves out to capture Albert’s hand and shake it. "I’m your last room mate and I’m sorry, but my IQ is only 80.” Albert smiles back at him and says, "So, where do you think interest rates are headed?”
PROBLEM 1: (25 points) Define each of the terms listed below:

1. Convergence in measure for a sequence of functions $f_n : E \to \mathbb{R}$
2. Uniform integrable sequence of functions
3. The Lebesgue integral of a bounded measurable function
4. Simple function
5. Uniform convergence for a sequence of functions $f_n : E \to \mathbb{R}$
PROBLEM 2: (20 points) State and prove Chebychev’s Inequality.
PROBLEM 3: (20 points) Let \( \{f_n\} \) be a sequence of integrable functions on \( E \) for which \( f_n \to f \) a.e. on \( E \) and \( f \) is integrable over \( E \). Show that \( \int_E |f - f_n| \, d\mu \to 0 \) if and only if \( \lim_{n \to \infty} \int_E |f_n| = \int_E |f| \, d\mu \).
PROBLEM 4: (20 points) Assume that $E$ has finite measure and $f, f_n : E \to \mathbb{R}$ are measurable functions for every $n \geq 1$.

1. If $\{f_n\} \to f$ pointwise a.e. on $E$ then $\{f_n\} \to f$ in measure on $E$.

2. Conversely, if $\{f_n\} \to f$ in measure on $E$ then there exists a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on $E$ to $f$.

Illustrate by a counterexample that the full converse of part 1. above does not hold.
PROBLEM 5: (15 points). Solve, at your choice, one of the following problems:

a) Let $\{f_n\}$ be a sequence of nonnegative measurable functions that converges to $f$ pointwise on $E$. Let $M \geq 0$ be such that $\int_E f_n d\mu \leq M$ for all $n$. Show that $\int_E f d\mu \leq M$. Verify that this property is equivalent to Fatou's Lemma.

b) If $f : [-1,1] \to \mathbb{R}$ is a continuous function then show that

$$2 \int_{[-1,1]} f^2(t) dt \geq \left( \int_{[-1,1]} f(t) dt \right)^2 + 3 \left( \int_{[-1,1]} t f(t) dt \right)^2.$$
BEAUTIFUL PROBLEM : (10 points) Solve one of the following problems at your choice

1. Evaluate the following integral $\int_0^2 \frac{\ln(x+1)}{x^2+1}dx$.

2. If $f : [0,1] \to \mathbb{R}$ is a continuous function then show that

$$\int_0^1 \left( \int_0^1 |f(x)+f(y)| \, dy \right) \, dx \geq \int_0^1 |f(x)| \, dx.$$