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On the Cauchy problem of 3-D energy-critical Schrödinger equations with subcritical perturbations

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Abstract

We investigate the global well-posedness, scattering and blow up phenomena when the 3-D quintic nonlinear Schrödinger equation, which is energy-critical, is perturbed by a subcritical nonlinearity $\lambda_1|u|^p u$. We find when the quintic term is defocussing, then the solution is always global no matter what the sign of λ_1 is. Scattering will occur either when the perturbation is defocussing and $\frac{4}{3} < p < 4$ or when the mass of the solution is small enough and $\frac{4}{3} \leqslant p < 4$. When the quintic term is focusing, we show the blow up for certain solutions.

1. Introduction

We study the initial value problem for the 3-D energy-critical problem of nonlinear Schrödinger equation with subcritical perturbations

$$\begin{cases} iu_t + \Delta u = \lambda_1|u|^p u + \lambda_2|u|^4 u, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $u(t,x)$ is a complex-valued function in space–time $\mathbb{R}_t \times \mathbb{R}_x^3$, the initial data $u_0 \in H_x^1$, λ_1, λ_2 are nonzero real constants and $0 < p < 4$.

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This equation has Hamiltonian,

$$E\big(u(t)\big)=\int\left[\frac{1}{2}\big|\nabla u(t,x)\big|^2+\frac{\lambda_1}{p+2}\big|u(t,x)\big|^{p+2}+\frac{\lambda_2}{6}\big|u(t,x)\big|^6\right]dx. \tag{1.2}$$

As (1.2) is preserved by the flow corresponding to (1.1), we shall refer to it as energy and often write $E(u)$ for $E(u(t))$.

A second conserved quantity we will rely on is the mass $M(u)(t)=\|u(t)\|_{L_x^2(\mathbb{R}^n)}$. As the mass is conserved, we will often write $M(u)$ for $M(u)(t)$.

One of the motivations for considering this problem is the failure of the equation to be scale invariant. Removing the subcritical term $\lambda_1|u|^{p_1}u$, one recovers the energy-critical nonlinear Schrödinger equation

$$\begin{cases}(i\partial_t+\Delta)v=\lambda_2|v|^4v,\\ v(0)=v_0,\end{cases} \tag{1.3}$$

which is invariant under the scaling $v\mapsto v^\lambda$, where

$$v^\lambda(t,x):=\lambda^{-\frac{1}{2}}v\left(\frac{t}{\lambda^2},\frac{x}{\lambda}\right). \tag{1.4}$$

More precisely, the scaling $v\mapsto v^\lambda$ maps a solution of (1.3) to another solution of it, and v and v^λ have the same energy.

The energy-critical nonlinear Schrödinger equation has a long history. In the focusing case ($\lambda_2<0$), an argument of Glassey [9], shows that certain Schwartz solution will blow up in finite time; for instance, this will occur whenever the potential energy exceeds the kinetic energy. In the defocussing case ($\lambda_2>0$), it is known that if the initial data v_0 has finite energy, then the equation is locally well-posed (see, for instance, [4,5]). That is, there exists a unique local-in-time solution that lies in $C_t^0\dot H_x^1\cap L_{t,x}^{10}$ and the map from the initial data to the solution is Lipschitz continuous. If, in addition, the energy is small, it is known that the solution exists globally in time and scattering occurs; that is, there exist solutions $u_\pm$ of the free Schrödinger equation $(i\partial_t+\Delta)u_\pm=0$ such that $\|u(t)-u_\pm(t)\|_{\dot H_x^1}\to 0$ as $t\to\pm\infty$. However, for initial data with large energy, the local well-posedness arguments do not extend to give global well-posedness.

Global well-posedness in $\dot H_x^1(\mathbb{R}^3)$ for the energy-critical NLS in the case of large finite-energy, radially-symmetric initial data was first obtained by Bourgain [2,3] and subsequently by Grillakis [11]. Tao [17], settled the problem for arbitrary dimensions (with an improvement in the final bound due to a simplification of the argument), but again only for radially-symmetric data. A major breakthrough in the field was made by Colliander, Keel, Staffilani, Takaoka, and Tao in [7], where they obtained global well-posedness and scattering for the energy-critical NLS in dimension $n=3$ with arbitrary initial data.

The method they used relies heavily on the scale invariance of the equation in (1.3), therefore, adding a subcritical perturbation to the equation which destroys the scale invariance, is of particular interest.

Motivated by this problem, we consider here the problem (1.1). We are interested in global well-posedness, the scattering result and the finite time blow up of (1.1). More precisely, we seek to answer the following questions: under what conditions of λ_1, λ_2 and p will the solution be globally well-posed, or has scattering, or blow up in finite time for certain solution?

We first restrict our attention to the case when the quintic term is defocussing, i.e., $\lambda_2 > 0$. We find that the solution is always global well-posed whether the subcritical term is defocussing or focusing. The scattering theory is available when $\lambda_1 > 0$ and $\frac{4}{3} < p < 4$ or when the mass is small enough and $\frac{4}{3} \leqslant p < 4$. When the quintic term is focusing, we show finite time blow up for certain Schwartz solution. We also include the results of scattering in Σ space when both terms are defocussing, where

$$\Sigma = \left\{ f \in H_x^1,\ xf \in L^2 \right\}. \tag{1.5}$$

We show if initial data $u_0 \in \Sigma$, we can lower the value p down to $1 < p < 4$.

The approach we use to prove the global well-posedness and scattering in H_x^1 space is "perturbative." However, we should notice that, although the perturbation approach is classical and has a long history, we are enlightened here mainly by the work [2,7].

More precisely, in order to get global well-posedness, we need to show the "good local well-posedness" which means the time interval on which we have a well-posed solution depends only on the H_x^1 norm of the initial data, rather than the profile of the initial data, as the classical local theory can tell (see the local well-posed theorem, Proposition 2.1). This good local well-posedness combining with the global kinetic energy control yields global well-posedness. Since (1.1) is time translation invariant, we need only to show the well-posedness on the time interval $[0, T]$, for some small $T = T(\|u_0\|_{H_x^1})$.

On the interval $[0, T]$, we try to approximate (1.1) by (1.3) with the same initial data and achieve this by solving the difference equation with 0-data. By choosing T small enough but depending only on $\|u_0\|_{H_x^1}$, we can prove the difference problem is solvable and the solution stays small on $[0, T]$, therefore close the proof of good local well-posedness.

In order to prove scattering, one usually needs a priori information about the decay of the solution. An example of such a priori estimate on which our analysis relies is the interaction Morawetz inequality which has appeared in [7,8], etc. (For more research on Morawetz inequality, one can see [13–15].) Indeed, Morawetz estimate is useful when both nonlinearities are defocussing since in this case, we have the global space–time control on the global solution of (1.1),

$$\|u\|_{L^4(\mathbb{R};L^4)} \lesssim \|u\|_{L^\infty(\mathbb{R};H_x^1)} \lesssim C\left(\|u_0\|_{H_x^1}\right).$$

However, the Morawetz control is not immediately useful for our case. Indeed, in order to get scattering result, we require the global solution obey the stronger decay,

$$\sup_{(q,r)\text{-admissible}} \|u\|_{L^q(\mathbb{R};H^{1,r})} \lesssim C\left(\|u_0\|_{H_x^1}\right). \tag{1.6}$$

To prove (1.6), we chop time $\mathbb{R}$ into finite time intervals such that on each subinterval, Morawetz norm is small, then we try to compare (1.1) and (1.3) on each subinterval. The main point here is that if on the time slab I, the Morawetz norm of the solution of (1.1) is small, the subcritical term will be small and therefore the solutions of the two equations will stay close on I. The final space–time bound (1.6) follows by summing estimates on each subinterval together.

The scattering can occur even if the subcritical term is focusing but the mass is small since when the mass is small, the subcritical term will also be small (in certain norms defined later), therefore we can compare two systems (1.1) and (1.3) with the same initial data globally in time and got global space–time estimate for the solution u of (1.1) from which the scattering results follow.

Now, we collect our results into the following theorems.

Theorem 1.1. *Let $u_0 \in H_x^1(\mathbb{R}^3)$ and $\lambda_2 > 0$, then there exists a unique global solution $u(t,x)$ to* (1.1) *such that*

$$\|u\|_{L^q(I;H^{1,r})} \leqslant C\big(\|u_0\|_{H_x^1}, |I|\big) \tag{1.7}$$

for any I compact and (q,r) admissible.

Theorem 1.2. *Let $u_0 \in H_x^1$, and u the unique global solution to* (1.1), *then there exists unique $u_\pm \in H_x^1$ such that*

$$\big\|u(t) - e^{it\Delta}u_\pm\big\|_{H_x^1} \to 0, \quad \text{as } t \to \pm\infty,$$

in each of the two cases:

(i) $\lambda_1 > 0$, $\lambda_2 \geqslant 0$, $\frac{4}{3} < p < 4$,
(ii) $\lambda_2 > 0$, $\frac{4}{3} \leqslant p < 4$ *and* $M \leqslant c(\|\nabla u_0\|_2)$,

where $c(\|\nabla u_0\|_2)$ is a small constant depending only on $\|\nabla u_0\|_2$.
If $u_0 \in \Sigma$, $\lambda_1 > 0$, $\lambda_2 > 0$ and $1 < p < 4$, then there exists unique $u_\pm \in \Sigma$ such that

$$\big\|u(t) - e^{it\Delta}u_\pm\big\|_{\Sigma} \to 0, \quad \text{as } t \to \pm\infty.$$

Remark. In the case $\lambda_2 = 0$, $\frac{4}{3} < p < 4$, Theorem 1.2 gives an alternative proof of scattering for energy subcritical problem that had appeared in [1,10].

Theorem 1.3. *Let $\lambda_2 < 0$, $u_0 \in \Sigma$ and $u(x,t)$ the classical solution of* (1.1). *Denote $d_0 := \operatorname{Im}\int r\bar{u}_0(u_0)_r\,dx > 0$. Then in each of the following cases*:

(i) $\lambda_1 > 0$, $0 \leqslant p < 4$, $E < 0$;
(ii) $\lambda_1 < 0$, $\frac{4}{3} < p < \infty$, $E < 0$;
(iii) $\lambda_1 < 0$, $0 \leqslant p \leqslant \frac{4}{3}$, *and* $E + C(\lambda_1, \lambda_2, p)M^2 < 0$,

there exists $0 < T_* \leqslant C\frac{\|xu_0\|_2^2}{d_0}$ such that

$$\lim_{t \to T_*} \big\|\nabla u(t)\big\|_2 = \infty.$$

Remark 1.4. It looks weird when we compare conditions (i) and (iii), since we need additional condition $E + C(\lambda_1, \lambda_2, p)M^2 < 0$ to get blow-up when the first nonlinearity is focusing which is easier to lead to blow-up. However, we should note that the condition $E(t) < 0$ is easier to be satisfied when both $\lambda_i < 0$. In other words, even the kinetic energy of u is small, which means there is no blow-up for $\|\nabla u(t)\|_2$, $E(t)$ can still be negative just by requiring mass large enough. So in order to get blow-up of the kinetic energy, it is necessary to add a size restriction on mass, as shown in the last point.

Now, let us say a couple of words about the higher-dimensional case. Sometime after this work is done, we are informed that E. Rychman and M. Visan settled the problem of global well-posedness and scattering for 4-D energy-critical NLS [16] and was finally solved by M. Visan for all higher dimensions [19,20]. Their methods again rely heavily on the scale invariance of n-dimensional energy-critical NLS

$$iu_t + \Delta u = \lambda_2 |u|^{\frac{4}{n-2}} u, \tag{1.8}$$

so it is reasonable to try to extend the idea of this paper to deal with the energy-critical problem with subcritical perturbations in high dimensions. Indeed, in dimensions 4, 5, 6, since the critical nonlinearity $\lambda_2 |u|^{\frac{4}{n-2}} u$ is Lipschitz continuous in certain space with one derivation and scale like $L^\infty \dot{H}^1_x$, we can successfully extend the approach in this paper to dimensions $n = 4, 5, 6$. However, such kind of strategy cannot work in higher dimensions due to the low order of the nonlinearity, we need to consult the very recent work of Tao and Visan [18] and need more careful analysis. We will discuss them elsewhere [21].

1.1. Notation

We will often use the notation $X \lesssim Y$ whenever there exists some constant C so that $X \leqslant CY$. Similarly, we will use $X \sim Y$ if $X \lesssim Y \lesssim X$. The derivative operator ∇ refers to the space variable only.

We use $L^r_x(\mathbb{R}^3)$ to denote the Banach space of functions $f : \mathbb{R}^3 \to \mathbb{C}$ whose norm

$$\|f\|_r = \left(\int_{\mathbb{R}^3} \left| f(x) \right|^r dx \right)^{\frac{1}{r}}$$

is finite, with the usual modifications when $r = \infty$. For any non-negative integer k, we denote by $H^{k,r}(\mathbb{R}^3)$ the Sobolev space defined as the closure of smooth compactly supported functions in the norm

$$\|f\|_{H^{k,r}} = \sum_{|\alpha| \leqslant k} \left\| \frac{\partial^\alpha}{\partial x^\alpha} f \right\|_r,$$

when $r = 2$, we denote it by H^k.

For a time slab I, we use $L^q(I; L^r)$ to denote the space–time norm

$$\|u\|_{L^q(I;L^r)} = \left(\int_I \left(\int_{\mathbb{R}^3} \left| u(t,x) \right|^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}}$$

with the usual modifications when q or r is infinite.

Let $U(t) = e^{it\Delta}$ be the free Schrödinger propagator, and this propagator preserves H^k norms and obeys the dispersive inequality

$$\left\| U(t) f \right\|_{L^\infty_x} \lesssim |t|^{-\frac{3}{2}} \|f\|_{L^1_x}$$

for all time $t \neq 0$.

We also recall Duhamel's formula

$$u(t) = U(t-t_0)u(t_0) - i\int_{t_0}^{t} U(t-s)(iu_t + \Delta u)(s)\,ds. \tag{1.9}$$

We say a pair (q,r) admissible if $\frac{2}{q}+\frac{3}{r}=\frac{3}{2}$ and $2 \leqslant r \leqslant 6$. Let us now also record the following standard Strichartz estimates that we will invoke throughout the paper (for a proof see, for example, [12]).

Lemma 1.5. *Let I be a compact time interval, and let $u : I \times \mathbb{R}^n \to \mathbb{C}$ be a unique solution to the forced Schrödinger equation*

$$iu_t + \Delta u = \sum_{m=1}^{M} F_m$$

for some functions $F_1, \ldots, F_M$, then we have

$$\|u\|_{L^q(I;\dot{H}^{k,r})} \lesssim \|u(t_0)\|_{\dot{H}^k} + C\sum_{m=1}^{M} \|\nabla^k F_m\|_{L^{q_m'}(I;L^{r_m'})}$$

for any time $t_0 \in I$ and any admissible exponents $(q_1,r_1),\ldots,(q_m,r_m)$. As usual, p' denotes the dual exponent to p, that is $\frac{1}{p}+\frac{1}{p'}=1$.

Now, let us recall the interaction Morawetz estimate for the solution u of (1.1).

Lemma 1.6. [7,8] *Let I be a compact interval, λ_1 and λ_2 non-negative constants and u the solution to* (1.1) *on the time–space slab $I \times \mathbb{R}^3$, then*

$$\|u\|_{L^4(I;L^4)} \lesssim \|u\|_{L^\infty(I;H_x^1)}.$$

It is useful to define several spaces and give estimates of the nonlinearities in terms of these spaces. Given $0 < p < 4$, define

$$\rho = \frac{3(p+2)}{p+3}, \qquad \gamma = \frac{4(p+2)}{p}, \qquad \rho^* = 3p+6,$$

and ρ', γ' the dual number of ρ, γ, then it is easy to verify that (γ,ρ) is an admissible pair and obeys

$$\frac{1}{\gamma'} = 1 - \frac{p}{4} + \frac{p+1}{\gamma}, \tag{1.10}$$

$$\frac{1}{\rho'} = \frac{p}{\rho^*} + \frac{1}{\rho}, \tag{1.11}$$

$$\frac{1}{\rho^*} = \frac{1}{\rho} - \frac{1}{3}. \tag{1.12}$$

For a time slab $I \subset \mathbb{R}$, we define

$$\dot{X}_I^0 = L^{\frac{10}{3}}\big(I; L^{\frac{10}{3}}\big) \cap L^{10}\big(I; L^{\frac{30}{13}}\big) \cap L^{\gamma}\big(I; L^{\rho}\big),$$
$$\dot{X}_I^1 = \big\{f; \nabla f \in \dot{X}_I^0\big\}, \qquad X_I^1 = \dot{X}_I^0 \cap \dot{X}_I^1,$$

then we have:

Lemma 1.7. *Let I be a time slab with finite length, then for $i = 0, 1$, we have*

$$\big\|\nabla^i\big(|u|^4 u\big)\big\|_{L^{\frac{10}{7}}(I; L^{\frac{10}{7}})} \lesssim \|u\|_{\dot{X}_I^1}^4 \|u\|_{\dot{X}_I^i},$$
$$\big\|\nabla^i\big(|u|^p u\big)\big\|_{L^{\gamma'}(I; L^{\rho'})} \lesssim |I|^{1-\frac{p}{4}} \|u\|_{\dot{X}_I^1}^p \|u\|_{\dot{X}_I^i}.$$

Proof. The first estimate is a direct application of the Hölder inequality. In view of (1.10) and (1.11), we have by Hölder and Sobolev embedding,

$$\begin{aligned}\big\|\nabla^i |u|^p u\big\|_{L^{\gamma'}(I; L^{\rho'})} &\lesssim |I|^{1-\frac{p}{4}} \|u\|_{L^{\gamma}(I; L^{\rho^*})} \big\|\nabla^i u\big\|_{L^{\gamma}(I; L^{\rho})} \\ &\lesssim |I|^{1-\frac{p}{4}} \|u\|_{L^{\gamma}(I; \dot{H}^{1,\rho})}^p \big\|\nabla^i u\big\|_{L^{\gamma}(I; L^{\rho})} \lesssim |I|^{1-\frac{p}{4}} \|u\|_{\dot{X}_I^1}^p \|u\|_{\dot{X}_I^i}. \qquad \square\end{aligned}$$

In the proof of scattering part, we need to control the subcritical term by using the Morawetz control, and this can be achieved through interpolation. Let I be a time slab, when $\frac{4}{3} < p \leqslant \frac{8}{5}$, we define

$$\dot{Y}_I^0 = L^{10}\big(I; L^{\frac{30}{13}}\big) \cap L^{\frac{10}{3}}\big(I; L^{\frac{10}{3}}\big) \cap L^2\big(I; L^6\big), \qquad \dot{Y}_I^1 = \big\{f; \nabla f \in \dot{Y}_I^0\big\}, \qquad Y_I^1 = \dot{Y}_I^0 \cap \dot{Y}_I^1;$$

when $\frac{8}{5} < p < 4$, we define

$$\dot{Y}_I^0 = L^{10}\big(I; L^{\frac{30}{13}}\big) \cap L^{\frac{10}{3}}\big(I; L^{\frac{10}{3}}\big),$$

with the same modifications to $\dot{Y}_I^1$ and Y_I^1. Then we have:

Lemma 1.8. *Let $i = 0, 1$, then in the case when $\frac{8}{5} < p < 4$, we have*

$$\big\|\nabla^i\big(|u|^p u\big)\big\|_{L^{\frac{10}{7}}(I; L^{\frac{10}{7}})} \lesssim \|u\|_{L^4(I; L^4)}^{\frac{8-2p}{3}} \|u\|_{\dot{Y}_I^1}^{\frac{5p-8}{3}} \big\|\nabla^i u\big\|_{\dot{Y}_I^0} \lesssim \|u\|_{L^4(I; L^4)}^{\frac{8-2p}{3}} \|u\|_{Y_I^1}^{\frac{5p-5}{3}}. \tag{1.13}$$

In the case when $\frac{4}{3} < p \leqslant \frac{8}{5}$, we can find an admissible pair (q_0, r_0) such that

$$\big\|\nabla^i\big(|u|^p u\big)\big\|_{L^{q_0'}(I; L^{r_0'})} \lesssim M^{8-5p} \|u\|_{L^4(I; L^4)}^{6p-8} \big\|\nabla^i u\big\|_{\dot{Y}_I^0}. \tag{1.14}$$

In either cases, we always have

$$\big\|\nabla^i\big(|u|^4 u\big)\big\|_{L^{\frac{10}{7}}(I; L^{\frac{10}{7}})} \lesssim \|u\|_{\dot{Y}_I^1}^4 \big\|\nabla^i u\big\|_{\dot{Y}_I^0}.$$

Proof. In the case $\frac{8}{5} < p < 4$, applying interpolation gives that

$$\|u\|_{L^{\frac{5p}{2}}(I;L^{\frac{5p}{2}})} \lesssim \|u\|_{L^4(I;L^4)}^{\frac{8-2p}{3p}} \|u\|_{L^{10}(I;L^{10})}^{\frac{5p-8}{3p}},$$

therefore, we have by Hölder inequality

$$\begin{aligned}\big\|\nabla^i\big(|u|^p u\big)\big\|_{L^{\frac{10}{7}}(I;L^{\frac{10}{7}})} &\lesssim \|u\|^p_{L^{\frac{5p}{2}}(I;L^{\frac{5p}{2}})} \big\|\nabla^i u\big\|_{L^{\frac{10}{3}}(I;L^{\frac{10}{3}})} \\ &\lesssim \|u\|_{L^4(I;L^4)}^{\frac{8-2p}{3}} \|u\|_{L^{10}(I;L^{10})}^{\frac{5p-8}{3}} \|\nabla u\|_{L^{\frac{10}{3}}(I;L^{\frac{10}{3}})}\end{aligned}$$

which, by our notation and Sobolev embedding, can be controlled by (1.13).

In the case when $\frac{4}{3} < p \leqslant \frac{8}{5}$, let $q = (\frac{3}{2} - \frac{2}{p})^{-1}$, $r = (\frac{2}{p} - 1)^{-1}$, $q_0 = (\frac{5}{2} - \frac{3}{2}p)^{-1}$, $r_0 = (p - \frac{7}{6})^{-1}$, it is easy to verify that (q_0, r_0) is admissible and

$$\frac{1}{q_0'} = \frac{p}{q} + \frac{1}{2}; \qquad \frac{1}{r_0'} = \frac{p}{r} + \frac{1}{6}.$$

Using Hölder's inequality and interpolation gives that

$$\begin{aligned}\big\|\nabla^i\big(|u|^p u\big)\big\|_{L^{q_0'}(I;L^{r_0'})} &\lesssim \|u\|^p_{L^q(I;L^r)} \big\|\nabla^i u\big\|_{L^2(I;L^6)} \\ &\lesssim \|u\|^{8-5p}_{L^\infty(I;L^2)} \|u\|^{6p-8}_{L^4(I;L^4)} \big\|\nabla^i u\big\|_{L^2(I;L^6)} \\ &\lesssim M^{8-5p} \|u\|^{6p-8}_{L^4(I;L^4)} \big\|\nabla^i u\big\|_{\dot{Y}_I^0}.\end{aligned}$$

The last inequality in Lemma 1.8 is a direct consequence of Hölder's inequality. □

We will also use $\dot{Z}_I^0$ to denote $L^{\frac{10}{3}}(I; L^{\frac{10}{3}}) \cap L^{10}(I; L^{\frac{30}{13}})$ with the same modifications to $\dot{Z}_I^1$ and Z_I^1. (We see the definition of $\dot{Z}_I^0$ agrees with $\dot{Y}_I^0$ in some cases, however, this will not make confusion since we use them in different settings.)

2. Proof of global well-posedness

As we said before, the proof will be splitted to two parts: global kinetic energy control and "good local well-posedness." More precisely, we will show that there exists a small constant $T = T(\|u_0\|_{H_x^1})$ such that (1.1) is well-posed on $[0, T]$, since the equation in (1.1) is time translation invariant, this combining with the global kinetic energy control gives immediately the global well-posedness. To begin with, we list the well-known local theory of (1.1).

2.1. Local theory

Proposition 2.1. [4,5] *Let $u_0 \in H_x^1$, λ_1 and λ_2 be nonzero real constants, and $0 < p < 4$. Then, there exists a unique strong H_x^1-solution u to* (1.1) *defined on a maximal time interval $(-T_{\min}, T_{\max})$. Moreover, $u \in L^q(I; H^{1,r})$ for any (q, r) admissible and every compact time interval $I \subset (-T_{\min}, T_{\max})$ and the following properties hold*:

- *If $T_{\max} < \infty$, then*

$$\text{either} \quad \lim_{t \to T_{\max}} \|u(t)\|_{H_x^1} = \infty \quad \text{or} \quad \sup_{(q,r)\text{-admissible}} \|u\|_{L^q((0,T_{\max});H^{1,r})} = \infty.$$

Similarly, if $T_{\min} < \infty$, then

$$\text{either} \quad \lim_{t \to -T_{\min}} \|u(t)\|_{H_x^1} = \infty \quad \text{or} \quad \sup_{(q,r)\text{-admissible}} \|u\|_{L^q(((-T_{\min},0);H^{1,r}))} = \infty.$$

- *The solution u depends continuously on the initial data u_0 in the following sense: the functions $T_{\min}$ and $T_{\max}$ are lower semicontinuous from H_x^1 to $(0,\infty]$. Moreover, if $u_0^{(m)} \to u_0$ in H_x^1 and if $u^{(m)}$ is the maximal solution to (1.1) with initial data $u_0^{(m)}$, then $u^{(m)} \to u$ in $L_t^q H_x^1([-S,T] \times \mathbb{R}^n)$ for every $q < \infty$ and every interval $[-S,T] \subset (-T_{\min}, T_{\max})$.*

2.2. Kinetic energy control

As $E = E(t) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u(t,x)|^2 + \frac{\lambda_1}{p+2}|u(t,x)|^{p+2} + \frac{\lambda_2}{6}|u(t,x)|^6\,dx$, we immediately see that for $\lambda_1 > 0$, $\lambda_2 > 0$, $\|\nabla u(t)\|_2^2 \lesssim E$, uniformly in t.

Whenever $\lambda_1 < 0$ and $\lambda_2 > 0$, we remark the inequality

$$-\frac{|\lambda_1|}{p+2}\big|u(t,x)\big|^{p+2} + \frac{|\lambda_2|}{6}\big|u(t,x)\big|^6 \geqslant -C(\lambda_1,\lambda_2)\big|u(t,x)\big|^2$$

holds true for $0 < p < 4$, integrating over $\mathbb{R}^3$ and using the energy conservation (1.2) gives finally

$$\big\|\nabla u(t)\big\|_2^2 \lesssim E + M^2$$

uniformly in t.

2.3. Good local well-posedness

Assume $\lambda_2 = 1$. Let T be a small constant to be specified later, and v the unique solution of (1.3) with initial data u_0, then by [7], (1.3) is globally well-posed and the global solution u satisfies the estimate

$$\|v\|_{L^q(\mathbb{R};\dot{H}^{1,r})} \leqslant C\big(\|u_0\|_{\dot{H}_x^1}\big), \qquad \|v\|_{L^q(\mathbb{R};L^r)} \leqslant C\big(\|u_0\|_{\dot{H}_x^1}\big)\|u_0\|_2 \tag{2.1}$$

for any (q,r) admissible. So we need only to solve the 0-data initial value problem for the difference equation of w,

$$\begin{cases} iw_t + \Delta w = \lambda_1|v+w|^p(v+w) + |v+w|^4(v+w) - |v|^4 v, \\ w(0) = 0 \end{cases} \tag{2.2}$$

on the time interval $[0,T]$. In order to solve (2.2), we need to subdivide $[0,T]$ into finite subintervals such that on each subinterval, the influence of v to problem (2.2) is small.

Let η be a small constant, in view of (2.1), we can divide $\mathbb{R}$ into subintervals $I_0, \dots, I_{J-1}$ such that on each I_j,

$$\|v\|_{X^1_{I_j}} \sim \eta, \quad 0 \leqslant j \leqslant J-1,$$

here, $J \leqslant C(\|u_0\|_{H^1_x}, \eta)$. Of course, we are only concerned about the subintervals that have nonempty intersection with $[0, T]$, so without loss of generality and renaming the intervals if necessary, we can write

$$[0, T] = \bigcup_{j=0}^{J'-1} I_j, \quad I_j = [t_j, t_{j+1}],$$

with $J' \leqslant J$ and on each I_j, $\|v\|_{X^1_{I_j}} \lesssim \eta$. Now we aim to solve (2.2) on each I_j by inductive arguments. More precisely, we show that for each $0 \leqslant j \leqslant J'-1$, (2.2) has a unique solution w on I_j such that

$$\|w\|_{X^1_{I_j}} + \|w\|_{L^\infty(I_j; H^1_x)} \leqslant (2C)^j T^{1-\frac{p}{4}}. \tag{2.3}$$

Assume (2.2) has been solved on I_{j-1} and the solution w satisfies the bound (2.3) for $j-1$, let us consider the problem on I_j.

Define the solution map

$$\Gamma u(t) = U(t-t_j)w(t_j) - i\int_{t_j}^{t} U(t-s)\big[\lambda_1 |v+w|^p (v+w) + |v+w|^4 (v+w) - |v|^4 v\big](s)\, ds,$$

then we will show that Γ maps the compact set

$$\mathcal{B} = \big\{w\colon \|w\|_{L^\infty(I_j; H^1_x)} + \|w\|_{X^1_{I_j}} \leqslant (2C)^j T^{1-\frac{p}{4}}\big\}$$

into itself and is contractive under the weak topology $\dot{X}^0_{I_j}$. Indeed, by using Lemma 1.7, Strichartz estimate and Hölder's inequality, we have

$$\begin{aligned}
&\|\Gamma w\|_{L^\infty(I_j; H^1_x)} + \|\Gamma w\|_{X^1_{I_j}} \\
&\quad \leqslant C\big\|w(t_j)\big\|_{H^1} + C\big\||v+w|^4(v+w) - |v|^4 v\big\|_{L^{\frac{10}{7}}(I_j; H^{1,\frac{10}{7}})} \\
&\qquad + C\big\||v+w|^p(v+w)\big\|_{L^{\gamma'}(I_j; H^{1,\rho'})} \\
&\quad \leqslant C\big\|w(t_j)\big\|_{H^1} + C\sum_{i=0}^{4} \|v\|^i_{X^1_{I_j}} \|w\|^{5-i}_{X^1_{I_j}} + CT^{1-\frac{p}{4}} \|v+w\|^{p+1}_{X^1_{I_j}} \\
&\quad \leqslant C\big\|w(t_j)\big\|_{\dot{H}^1} + C\sum_{i=0}^{4} \eta^i \|w\|^{5-i}_{X^1_{I_j}} + CT^{1-\frac{p}{4}} \|w\|^{p+1}_{X^1_{I_j}} + CT^{1-\frac{p}{4}} \eta^{p+1}.
\end{aligned}$$

Plugging the inductive assumption $\|w(t_j)\|_{H_x^1} \leqslant (2C)^{j-1}T^{1-\frac{p}{4}}$, we see that for $w \in \mathcal{B}$,

$$\begin{aligned}
&\|\Gamma w\|_{L^\infty(I_j;H_x^1)} + \|\Gamma w\|_{X^1_{I_j}} \\
&\quad \leqslant C(2C)^{j-1}T^{1-\frac{p}{4}} &(2.4)\\
&\qquad + C(2C)^j T^{1-\frac{p}{4}}\eta^4 + CT^{1-\frac{p}{4}}\eta^{p+1} &(2.5)\\
&\qquad + C\sum_{i=0}^{3}\big[(2C)^j T^{1-\frac{p}{4}}\big]^{5-i}\eta^i + C\big((2C)^j T^{1-\frac{p}{4}}\big)^{p+1}T^{1-\frac{p}{4}}. &(2.6)
\end{aligned}$$

It is easy to observe that $(2.4) = \frac{1}{2}(2C)^j T^{1-\frac{p}{4}}$. (2.5) is a linear term with respect to the quantity $T^{1-\frac{p}{4}}$, we can choose η small depending only on the Strichartz constant appearing in (2.5) such that

$$(2.5) \leqslant \frac{1}{8}(2C)^j T^{1-\frac{p}{4}}.$$

Fix this η, note (2.6) is a higher order term with respect to the quantity $T^{1-\frac{p}{4}}$, we can choose T small enough such that

$$(2.6) \leqslant \frac{1}{8}(2C)^j T^{1-\frac{p}{4}}.$$

Of course T will depend on j, however, since $j \leqslant J' - 1 \leqslant C(\|u_0\|_{H_x^1})$, we can choose T to be a small constant depending only on $\|u_0\|_{H_x^1}$ and η, therefore is uniform in the process of induction. By the same token, we can also show that for $w_1, w_2 \in \mathcal{B}$,

$$\|\Gamma w_1 - \Gamma w_2\|_{\dot X^0_{I_j}} \leqslant \frac{1}{2}\|w_1 - w_2\|_{\dot X^0_{I_j}}.$$

Thus, a direct application of the fixed point theorem gives a unique solution w of (2.2) on I_j which satisfies the bound (2.3). Therefore, we get a unique solution of (2.2) on $[0,T]$ such that

$$\|w\|_{X^1_{[0,T]}} \leqslant \sum_{j=0}^{J'-1}\|w\|_{X^1_{I_j}} \leqslant \sum_{j=0}^{J'-1}(2C)^j T^{1-\frac{p}{4}} \leqslant C(2C)^J T^{1-\frac{p}{4}} \leqslant C.$$

Since on $[0,T]$, $u = v + w$, we get a unique solution of (1.1) on $[0,T]$ such that

$$\|u\|_{X^1_{[0,T]}} \leqslant \|w\|_{X^1_{[0,T]}} + \|v\|_{X^1_{[0,T]}} \leqslant C\big(\|u_0\|_{H_x^1}\big).$$

As we mentioned before, this "good local well-posedness" combining with the global kinetic energy control gives finally the global well-posedness. However, since the solution is connected one interval by another, it does not possess global space–time bound. In the following, we will discuss several cases where the global solution have the enough decay to imply scattering.

3. Scattering in H_x^1 space

We first show the global space–time bound for the global solution, then construct the asymptotic state in the last subsection.

3.1. Global space–time bound for $\frac{4}{3} < p < 4$ and $\lambda_1, \lambda_2 > 0$

Since in this case, both nonlinearities are defocussing, without loss of generality, we assume $\lambda_1 = \lambda_2 = 1$. Therefore, by Theorem 1.1 and Lemma 1.6, (1.1) has a unique global solution u obeying the global space–time control

$$\|u\|_{L^4(\mathbb{R};L^4)} < C\big(\|u_0\|_{H_x^1}\big). \tag{3.1}$$

Let ε be a small constant to be specified later, in view of (3.1), we can divide $\mathbb{R}$ into finitely many subintervals $J_0, J_1, \dots, J_K$ with $K \leqslant C(\|u_0\|_{H_x^1}, \varepsilon)$ such that on each J_k,

$$\|u\|_{L^4(J_k, L^4)} \sim \varepsilon.$$

Now we need to give the estimate of u on each J_k. Fix one of the subinterval J_{k_0} and denote it by $[a,b]$. The idea here is to estimate u on $[a,b]$ via the energy-critical problem.

Let v be the solution of

$$\begin{cases} iv_t + \Delta v = |v|^4 v, \\ v(a) = u(a), \end{cases} \tag{3.2}$$

then by [7], (3.2) is globally well-posed and the global solution v satisfies $\|v\|_{L^q(\mathbb{R};H^{1,r})} \leqslant C(\|u(a)\|_{H_x^1}) \leqslant C(\|u_0\|_{H_x^1})$ for any (q,r) admissible. Let η be a small constant, we divide $\mathbb{R}$ into subintervals $I_0, I_1, \dots, I_{J-1}$ such that

$$\|v\|_{Y^1_{I_j}} \sim \eta, \quad 0 < j \leqslant J-1.$$

Note the total number of the subintervals $J \leqslant C(\|u_0\|_{H_x^1}, \eta)$. Again, we are only concerned with the subintervals which have nonempty intersection with $[a,b]$, so changing names if necessary, we can assume that

$$[a,b] = \bigcup_{j=0}^{J'} I_j, \quad I_j = [t_j, t_{j+1}],\ J' \leqslant J-1,$$

and $\|v\|_{Y^1_{I_j}} \lesssim \eta$.

Let w be the solution of the difference between u and v on $[a,b]$, w satisfies,

$$\begin{cases} iw_t + \Delta w = |u|^p u + |v+w|^4(v+w) - |v|^4 v, \\ w(a) = 0, \end{cases} \tag{3.3}$$

then our aim is to estimate w on each I_j, $0 \leqslant j \leqslant J'$. Indeed, by inductive arguments, we will show that

$$\|w\|_{L^\infty(I_j;H_x^1)}, \|w\|_{Y^1_{I_j}} \leqslant (2C)^j \varepsilon^{6p-8}, \tag{3.4}$$

when $\frac{4}{3} < p \leqslant \frac{8}{5}$ and

$$\|w\|_{L^\infty(I_j;H_x^1)}, \|w\|_{Y^1_{I_j}} \leqslant (2C)^j \varepsilon^{\frac{8-2p}{3}}$$

when $\frac{8}{5} < p < 4$ for each $0 \leqslant j \leqslant J'$. Since the two cases are similar, we only consider the case when $\frac{4}{3} < p \leqslant \frac{8}{5}$.

Assume (3.4) holds for $j-1$, let us consider the estimate of w on I_j, by Duhamel (1.9), w satisfies the equation

$$w(t) = U(t-t_j)w(t_j) - i\int_{t_j}^{t} U(t-s)\big(|u|^p u + |v+w|^4(v+w) - |v|^4 v\big)(s)\,ds,$$

applying Strichartz estimate and Lemma 1.8, we have

$$\begin{aligned}
\|w\|_{Y_{I_j}} &\leqslant C\big\|w(t_j)\big\|_{H^1} + C\big\||u|^p u\big\|_{L^{q_0'}(I_j;H^{1,r_0'})} + C\big\||v+w|^4(v+w) - |v|^4 v\big\|_{L^{\frac{10}{7}}(I_j;H^{1,\frac{10}{7}})} \\
&\leqslant C\big\|w(t_j)\big\|_{H^1} + C\|u\|^{8-5p}_{L^\infty(I_j;L^2)}\|u\|^{6p-8}_{L^4(I_j;L^4)}\|u\|_{L^2(I_j;H^{1,6})} + C\sum_{i=0}^{4}\|v\|^i_{Y^1_{I_j}}\|w\|^{5-i}_{Y^1_{I_j}} \\
&\leqslant C\big\|w(t_j)\big\|_{H^1} + CM^{8-5p}\varepsilon^{6p-8}\big(\|v\|_{Y^1_{I_j}} + \|w\|_{Y^1_{I_j}}\big) + C\sum_{i=0}^{4}\eta^i\|w\|^{5-i}_{Y^1_{I_j}} \\
&\leqslant C\big\|w(t_j)\big\|_{H^1} + C\varepsilon^{6p-8}\eta + C\big(\varepsilon^{6p-8} + \eta^4\big)\|w\|_{Y^1_{I_j}} + C\sum_{i=0}^{3}\eta^i\|w\|^{5-i}_{Y^1_{I_j}}.
\end{aligned}$$

Plugging the inductive assumption

$$\big\|w(t_j)\big\|_{H_x^1} \leqslant (2C)^{j-1}\varepsilon^{6p-8},$$

we have

$$\|w\|_{Y^1_{I_j}} \leqslant C(2C)^{j-1}\varepsilon^{6p-8} + C\varepsilon^{6p-8}\eta + C\big(\varepsilon^{6p-8} + \eta^4\big)\|w\|_{Y^1_{I_j}} + C\sum_{i=0}^{3}\eta^i\|w\|^{5-i}_{Y^1_{I_j}}. \tag{3.5}$$

It is easy to verify that $(3.5) \leqslant \frac{2}{3}(2C)^j\varepsilon^{6p-8}$, therefore, by choosing ε and η small enough and using the standard continuity argument gives that

$$\|w\|_{Y^1_{I_j}} \leqslant (2C)^j\varepsilon^{6p-8}.$$

Applying Strichartz again gives the estimate

$$\|w\|_{L^\infty(I_j;H_x^1)} \leqslant (2C)^j \varepsilon^{6p-8}.$$

Collecting all the estimates on each I_j together gives

$$\|w\|_{Y^1_{[a,b]}} \leqslant \sum_{j=0}^{J'} \|w\|_{Y^1_{I_j}} \leqslant \sum_{j=0}^{J'} (2C)^j \varepsilon^{6p-8} \leqslant (2C)^{J'} \varepsilon^{6p-8} \lesssim 1,$$

and therefore the estimates of u on $[a,b]$,

$$\|u\|_{Y^1_{[a,b]}} \leqslant \|v\|_{Y^1_{[a,b]}} + \|w\|_{Y^1_{[a,b]}} \leqslant C\big(\|u_0\|_{H_x^1}\big).$$

Since $[a,b]$ is chosen arbitrarily, we have

$$\|u\|_{Y^1_{\mathbb{R}}} \leqslant \sum_{k=0}^{K} \|u\|_{Y^1_{J_k}} \leqslant C\big(\|u_0\|_{H_x^1}\big)K \lesssim C\big(\|u_0\|_{H_x^1}\big).$$

Applying Strichartz yields immediately that

$$\|u\|_{L^q(\mathbb{R};H^{1,r})} \leqslant C\big(\|u_0\|_{H_x^1}\big) \quad \text{for all } (q,r) \text{ admissible}.$$

3.2. Global space–time bound for $\frac{4}{3} < p < 4$, $\lambda_1 > 0$, and $\lambda_2 = 0$

Assume $\lambda_1 = 1$, and u the unique global solution of the subcritical problem

$$iu_t + \Delta u = |u|^p u$$

with initial data u_0. Then by Lemma 1.6, u satisfies

$$\|u\|_{L^4(\mathbb{R};L^4)} \leqslant C\big(\|u_0\|_{H_x^1}\big).$$

Let η be a small constant, we can divide $\mathbb{R}$ into subintervals $I_0, I_1, \ldots, I_{J-1}$ such that

$$\|u\|_{L^4(I_j;L^4)} \sim \eta.$$

Note $J = J(\|u_0\|_{H_x^1}, \eta)$. We aim to give the space–time bound of u for each j.

Fix one of the subintervals $I_j = [t_j, t_{j+1}]$, using Strichartz and Lemma 1.8, we have

$$\begin{aligned}\|u\|_{Y^1_{I_j}} &\lesssim \big\|u(t_j)\big\|_{H_x^1} + \big\||u|^p u\big\|_{L^{\frac{10}{7}}(I_j;L^{\frac{10}{7}})} \lesssim C\big(\|u_0\|_{H_x^1}\big) + \|u\|^{\frac{8-2p}{3}}_{L^4(I_j;L^4)} \|u\|^{\frac{5p-5}{3}}_{Y^1_{I_j}} \\ &\lesssim C\big(\|u_0\|_{H_x^1}\big) + \eta^{\frac{8-2p}{3}} \|u\|^{\frac{5p-5}{3}}_{Y^1_{I_j}}\end{aligned}$$

for $\frac{8}{5} < p < 4$ and

$$\|u\|_{Y^1_{I_j}} \lesssim C\big(\|u_0\|_{H^1_x}\big) + CM^{8-5p}\eta^{6p-8}\|u\|_{Y^1_{I_j}}$$

for $\frac{4}{3} < p \leqslant \frac{8}{5}$.

Choosing η sufficiently small depending only on $\|u_0\|_{H^1_x}$, we have

$$\|u\|_{Y^1_{I_j}} \leqslant C\big(\|u_0\|_{H^1_x}\big).$$

Since I_j is taken arbitrarily, we have

$$\|u\|_{Y^1_{\mathbb{R}}} \leqslant JC\big(\|u_0\|_{H^1_x}\big) \leqslant C\big(\|u_0\|_{H^1_x}\big).$$

Applying Strichartz estimate again yields that

$$\|u\|_{L^q(\mathbb{R};H^{1,r})} \leqslant C\big(\|u_0\|_{H^1_x}\big) \quad \text{for any } (q,r) \text{ admissible.}$$

3.3. *Global space–time bound for $\frac{4}{3} \leqslant p < 4$ and mass is small*

The idea here is that we can approximate (1.1) by (1.3) globally in time when the mass is small enough. By time reversal symmetry, we need only to consider the positive time direction.

Let η be a small constant, and v the global solution of (1.3) with the same initial data u_0. Then in view of the global space–time bound of v, we can divide $\mathbb{R}^+$ into finitely many subintervals $I_0, \ldots, I_{J-1}$, such that

$$\|v\|_{\dot{Z}^1_{I_j}} \sim \eta, \quad 0 \leqslant j \leqslant J-1,$$

here, $J = J(\|\nabla u_0\|_2)$.

By (1.9) and Strichartz estimate, we have

$$\|v\|_{\dot{Z}^0_{I_j}} \leqslant \big\|v(t_j)\big\|_2 + C\big\||v|^4 v\big\|_{L^{\frac{10}{7}}(I_j;L^{\frac{10}{7}})} \leqslant \big\|v(t_j)\big\|_2 + C\|v\|^4_{\dot{Z}^1_{I_j}}\|v\|_{\dot{Z}^0_{I_j}} \leqslant M + C\eta^4\|v\|_{\dot{Z}^0_{I_j}}.$$

Taking η small such that $C\eta^4 < \frac{1}{2}$, we have

$$\|v\|_{\dot{Z}^0_{I_j}} \leqslant 2M, \tag{3.6}$$

therefore, we can make M small enough such that

$$\|v\|_{Z^1_{I_j}} \leqslant \|v\|_{\dot{Z}^0_{I_j}} + \|v\|_{\dot{Z}^1_{I_j}} \lesssim \eta. \tag{3.7}$$

Let w be the difference equation on $\mathbb{R}^+$ between v and u, w satisfies

$$\begin{cases} iw_t + \Delta w = \lambda_1|v+w|^p(v+w) + |v+w|^4(v+w) - |v|^4 v, \\ w(0) = 0. \end{cases} \tag{3.8}$$

Then we aim to solve w on $\mathbb{R}^+$, or equivalently, solve w on each I_j, $0 \leqslant j \leqslant J-1$. Again, we achieve this via inductive arguments, we will show that for each j, there exists unique solution w on I_j such that

$$\|w\|_{L^\infty(I_j;H^1_x)} + \|w\|_{Z^1_{I_j}} \leqslant (2C)^j M^{2-\frac{p}{2}}. \tag{3.9}$$

Assume (3.8) has been solved on I_{j-1} and the solution w on I_{j-1} satisfies (3.9), let us consider (3.8) on I_j.

Define the solution map

$$\Gamma w(t) = U(t-t_j)w(t_j) - i\int_{t_j}^{t} U(t-s)\big(\lambda_1|v+w|^p(v+w) + |v+w|^4(v+w) - |v|^4 v\big)(s)\,ds,$$

then it suffices to show that Γ maps

$$\mathcal{B} = \big\{w\colon\ \|w\|_{L^\infty(I_j;H^1_x)} + \|w\|_{Z^1_{I_j}} \leqslant (2C)^j M^{2-\frac{p}{2}}\big\}$$

into itself and is contractive under the weak topology $\dot{Z}^0_{I_j}$.

By Hölder and interpolation, we have that

$$\begin{aligned}
&\big\||v+w|^p(v+w)\big\|_{L^{\frac{10}{7}}(I_j;H^{1,\frac{10}{7}})} \\
&\quad\leqslant \|v+w\|^p_{L^{\frac{5}{2}p}(I_j;L^{\frac{5}{2}p})} \|v+w\|_{L^{\frac{10}{3}}(I_j;H^{1,\frac{10}{3}})} \\
&\quad\leqslant C\|v\|^p_{L^{\frac{5}{2}p}(I_j;L^{\frac{5}{2}p})} \|v\|_{L^{\frac{10}{3}}(I_j;H^{1,\frac{10}{3}})} + C\|v\|^p_{L^{\frac{5}{2}p}(I_j;L^{\frac{5}{2}p})} \|w\|_{L^{\frac{10}{3}}(I_j;H^{1,\frac{10}{3}})} \\
&\qquad + C\|w\|^p_{L^{\frac{5}{2}p}(I_j;L^{\frac{5}{2}p})} \|v\|_{L^{\frac{10}{3}}(I_j;H^{1,\frac{10}{3}})} + C\|w\|^p_{L^{\frac{5}{2}p}(I_j;L^{\frac{5}{2}p})} \|w\|_{L^{\frac{10}{3}}(I_j;H^{1,\frac{10}{3}})}.
\end{aligned}$$

Noting by interpolation,

$$\|f\|_{L^{\frac{5p}{2}}(I_j;L^{\frac{5p}{2}})} \leqslant \|f\|^{\frac{4-p}{2p}}_{\dot{Z}^0_{I_j}} \|f\|^{\frac{3p-4}{2p}}_{\dot{Z}^1_{I_j}},$$

plugging inductive assumptions (3.6) and (3.7), we have

$$\big\||v+w|^p(v+w)\big\|_{L^{\frac{10}{7}}(I_j;H^{1,\frac{10}{7}})} \leqslant M^{2-\frac{p}{2}}\eta^{\frac{3p}{2}-1} + M^{2-\frac{p}{2}}\eta^{\frac{3p}{2}-2}\|w\|_{Z^1_{I_j}} + \eta\|w\|^p_{Z^1_{I_j}} + \|w\|^{p+1}_{Z^1_{I_j}}.$$

Applying Hölder's inequality gives directly

$$\big\||v+w|^4(v+w) - |v|^4 v\big\|_{L^{\frac{10}{7}}(I_j;H^{1,\frac{10}{7}})} \leqslant C\sum_{i=0}^{4}\eta^i\|w\|^{5-i}_{Z^1_{I_j}}.$$

Plugging these nonlinear estimates and applying Strichartz's estimates gives that

$$
\begin{aligned}
&\|\Gamma w\|_{L^\infty(I_j;H_x^1)} + \|\Gamma w\|_{Z^1_{I_j}} \\
&\quad \leqslant C\big\|w(t_j)\big\|_{H_x^1} + CM^{2-\frac{p}{2}}\eta^{\frac{3p}{2}-1} && (3.10)\\
&\qquad + \big(CM^{2-\frac{p}{2}}\eta^{\frac{3p}{2}-2} + C\eta^4\big)\|w\|_{Z^1_{I_j}} && (3.11)\\
&\qquad + C\eta\|w\|^p_{Z^1_{I_j}} + \|w\|^{p+1}_{Z^1_{I_j}} + C\sum_{i=0}^{3}\eta^i\|w\|^{5-i}_{Z^1_{I_j}}. && (3.12)
\end{aligned}
$$

By inductive assumption, $\|w(t_j)\|_{H_x^1} \leqslant (2C)^{j-1}M^{2-\frac{p}{2}}$, letting η small, we see that

$$(3.10) \leqslant \frac{2}{3}(2C)^j M^{2-\frac{p}{2}};$$

(3.11) is a linear term w.r.t. $\|w\|_{Z^1_{I_j}}$, therefore we can choose η, M small such that

$$(3.11) \leqslant \frac{1}{16}(2C)^j M^{2-\frac{p}{2}};$$

(3.12) is a high order term w.r.t. $\|w\|_{Z^1_{I_j}}$, therefore by choosing M small enough, we get

$$(3.12) \leqslant \frac{1}{16}(2C)^j M^{2-\frac{p}{2}}.$$

Of course, M will depend on j. However, since $j \leqslant J \leqslant C(\|\nabla u_0\|_2)$, we can choose M is a small constant depending only on $\|\nabla u_0\|_2$ therefore uniform in the process of induction. By the same token, one can easily get

$$\|\Gamma w_1 - \Gamma w_2\|_{\dot Z^0_{I_j}} \leqslant \frac{3}{4}\|w_1 - w_2\|_{\dot Z^0_{I_j}}$$

for $w_1, w_2 \in \mathcal{B}$. Applying fixed point theorem gives a unique solution in $\mathcal{B}$, and closing the induction as well.

Therefore, (3.8) has a unique solution w on $\mathbb{R}^+$ such that

$$\|w\|_{Z^1_{\mathbb{R}^+}} \leqslant \sum_{j=0}^{J-1}\|w\|_{Z^1_{I_j}} \leqslant \sum_{j=0}^{J-1}(2C)^j M^{2-\frac{p}{2}} \leqslant (2C)^J M^{2-\frac{p}{2}} \leqslant C.$$

This in turn gives unique solution of (1.1) on $\mathbb{R}^+$ such that

$$\|u\|_{Z^1_{\mathbb{R}^+}} \leqslant \|w\|_{Z^1_{\mathbb{R}^+}} + \|v\|_{Z^1_{\mathbb{R}^+}} \leqslant C\big(\|u_0\|_{H_x^1}\big).$$

Applying Strichartz again yields that

$$\|u\|_{L^q(\mathbb{R}^+;H^{1,r})} \leqslant C\big(\|u_0\|_{H_x^1}\big)$$

for any (q, r) admissible.

3.4. Construction of the asymptotic state

In any of above cases, we have shown that the unique global solution $u(t,x)$ satisfies the global space–time bound

$$\|u\|_{L^q(\mathbb{R};H^{1,r})} \leqslant C\big(\|u_0\|_{H^1_x}\big)$$

for any admissible pair (q,r).

Define

$$u_+(t) = u_0 - i\int_0^t U(-s)\big(\lambda_1|u|^p u + \lambda_2|u|^4 u\big)(s)\,ds,$$

then we show as $t\to\infty$, $u_+(t)$ is a Cauchy sequence in H^1_x. Indeed, by Strichartz's estimate,

$$\begin{aligned}\big\|u_+(t_1) - u_+(t_2)\big\|_{H^1_x} &\lesssim \big\|\lambda_1|u|^p u + \lambda_2|u|^4 u\big\|_{L^{\frac{10}{7}}([t_1,t_2];H^{1,\frac{10}{7}})}\\ &\lesssim \|u\|^{p+1}_{Z^1_{[t_1,t_2]}} + \|u\|^5_{Z^1_{[t_1,t_2]}}\end{aligned}$$

which tends to 0 as $t_1, t_2 \to \infty$. Therefore, we see that

$$u_+ := u_0 - i\int_0^\infty U(-s)\big(\lambda_1|u|^p u + \lambda_2|u|^4 u\big)(s)\,ds \tag{3.13}$$

is an absolutely convergent integral in H^1_x. Using the same strategy again gives that

$$\big\|U(-t)u(t) - u_+\big\|_{H^1_x} \leqslant \|u\|^{p+1}_{Z^1_{[t,\infty)}} + \|u\|^5_{Z^1_{[t,\infty)}} \to 0,$$

as $t\to\infty$.

Similar arguments can be used to construct the asymptotic state in the negative direction

$$u_- = u_0 + i\int_{-\infty}^0 \big(\lambda_1|u|^p u + \lambda_2|u|^4 u\big)(s)\,ds.$$

4. Scattering in Σ for $\lambda_1 > 0$, $\lambda_2 > 0$ and $1 < p < 4$

We first note that the space Σ is defined in (1.5) and, since both nonlinearities are defocussing, we assume $\lambda_1 = \lambda_2 = 1$.

Let $H(t) = x + 2it\nabla$ be the Galilean operator, it has been shown in [6] that

$$H(t) = U(t)xU(-t) = 2ite^{\frac{i|x|^2}{4t}}\nabla_x\big(e^{-\frac{i|x|^2}{4t}}\cdot\big),\quad \forall t > 0,$$

and for $F(z)=G(|z|^2)z$,

$$H(t)F(u)=\frac{\partial F(u)}{\partial z}H(t)u+\frac{\partial F(u)}{\partial \bar z}\overline{H(t)u}.$$

From these properties, one sees that for $t\in I$, $I=[a,b)$,

$$H(t)u(t)=U(t-a)H(a)u(a)-i\int_a^t U(t-s)H(s)\big(|u|^p u(s)+|u|^4u(s)\big)\,ds,$$

hence by Strichartz's estimate,

$$\begin{aligned}\|Hu\|_{L^{10/3}(I;L^{10/3})}\leqslant{}&\big\|H(a)u(a)\big\|_2+C\|u\|_{L^{10/3}(I;L^{10/3})}^{\frac{4-p}{2}}\|u\|_{L^{10}(I;L^{10})}^{\frac{3p-4}{2}}\|Hu\|_{L^{10/3}(I;L^{10/3})}\\&+C\|u\|_{L^{10}(I;L^{10})}^4\|Hu\|_{L^{10/3}(I;L^{10/3})}.\end{aligned}\tag{4.1}$$

From this estimate, we see that $\sup_{(q,r)\text{-admissible}}\|Hu\|_{L^q(I;L^r)}$ is bounded as long as $\|u\|_{L^q(I;H^{1,r})}$ is bounded for $(q,r)=(\frac{10}{3},\frac{10}{3})$, $(10,\frac{30}{13})$. (One may cut I into subintervals such that u is small on each subinterval, and get estimate of Hu via (4.1), then sum all the estimates on each subintervals together.) In particular, we have

$$\|Hu\|_{L^q(I;L^r)}\leqslant C\big(\|u_0\|_\Sigma,|I|\big).$$

Next, we show the time decay of solution caused by the spatial decay of u_0.

Let $h(t)=\|H(t)u(t)\|_2^2+8t^2(\frac{1}{p+2}\|u(t)\|_{p+2}^{p+2}+\frac16\|u(t)\|_6^6)$, then it follows that

$$h'(t)=t\left(\frac{4(4-3p)}{p+2}\big\|u(t)\big\|_{p+2}^{p+2}-\frac{16}{3}\big\|u(t)\big\|_6^6\right):=t\theta(t).\tag{4.2}$$

If $\frac43\leqslant p<4$, by integrating (4.2) from 0 to t, we get

$$\begin{aligned}&\big\|H(t)u(t)\big\|_2^2+\frac{8t^2}{p+2}\big\|u(t)\big\|_{p+2}^{p+2}+\frac{4t^2}{3}\big\|u(t)\big\|_6^6\\&\quad=\|xu_0\|_2^2+\frac{4}{p+2}(4-3p)\int_0^t s\int_{\mathbf{R}^3}\big|u(s,x)\big|^{p+2}\,dx\,ds-\frac{16}{3}\int_0^t s\int_{\mathbf{R}^3}\big|u(s,x)\big|^6\,dx\,ds,\end{aligned}$$

which implies that

$$\big\|u(t)\big\|_{p+2}^{p+2}+\big\|u(t)\big\|_6^6\leqslant C\frac{\|xu_0\|_2^2}{t^2}.\tag{4.3}$$

If $0 < p < \frac{4}{3}$, by integrating (4.2) from 1 to t, we get

$$h(t) \leqslant h(1) + \int_1^t s\theta(s)\,ds, \quad \text{for } t \geqslant 1,$$

in particular,

$$\frac{8t^2}{p+2}\left\|u(t)\right\|_{p+2}^{p+2} \leqslant h(1) + \frac{4(4-3p)}{p+2}\int_1^t s\left\|u(s)\right\|_{p+2}^{p+2}\,ds.$$

Using Gronwall's inequality yields:

$$t^2\left\|u(t)\right\|_{p+2}^{p+2} \leqslant C + \frac{4-3p}{2}\int_1^t \frac{1}{s}\left\|u(s)\right\|_{p+2}^{p+2}\,ds, \qquad \left\|u(t)\right\|_{p+2}^{p+2} \leqslant Ct^{-\frac{3p}{2}}. \tag{4.4}$$

Inserting this inequality into (4.2), we get

$$h(t) \leqslant C\left(1 + t^{2-\frac{3p}{2}}\right).$$

Therefore, we have

$$\left\|u(t)\right\|_6^6 \leqslant Ct^{-\frac{3p}{2}}, \quad t \geqslant 1. \tag{4.5}$$

Now, we show that the decay estimates imply small Strichartz norm near infinity. We begin by estimating the nonlinear terms.

In the case $1 < p < \frac{4}{3}$, denote δ the constant such that $(\delta, p+2)$ is admissible. Using (4.4), (4.5) and noticing that $p > 1$ means that $2p > \delta - 2$, we have

$$\begin{aligned}\left\||u|^p u\right\|_{L^{\delta'}([T,\infty);H^{1,\frac{p+2}{p+1}})} &\leqslant \|u\|^p_{L^{(1-2/\delta)^{-1}p}([T,\infty);L^{p+2})}\|Du\|_{L^\delta([T,\infty);L^{p+2})} \\ &\leqslant CT^{1-\frac{2+2p}{\delta}}\|Du\|_{L^\delta([T,\infty);L^{p+2})},\end{aligned} \tag{4.6}$$

$$\begin{aligned}\left\||u|^4 u\right\|_{L^2([T,\infty);H^{1,\frac{6}{5}})} &\leqslant C\|u\|^4_{L^\infty([T,\infty);L^6)}\|Du\|_{L^2([T,\infty);L^6)} \\ &\leqslant CT^{-p}\|Du\|_{L^2([T,\infty);L^6)}.\end{aligned} \tag{4.7}$$

In the case $\frac{4}{3} \leqslant p < 4$, using (4.3) and direct computation, one gets that

$$\left\||u|^p u\right\|_{L^{\delta'}([T,\infty);H^{1,\frac{p+2}{p+1}})} \leqslant CT^{\frac{4-5p}{2(p+2)}}\|Du\|_{L^\delta([T,\infty);L^{p+2})}, \tag{4.8}$$

$$\left\||u|^4 u\right\|_{L^2([T,\infty);H^{1,\frac{6}{5}})} \leqslant CT^{-4}\|Du\|_{L^2([T,\infty);L^6)}. \tag{4.9}$$

Taking T sufficiently large such that

$$CT^{\gamma} < \frac{1}{16},$$

where γ is the maximal constant among $1 - \frac{2+2p}{\delta}$, $-p$, $\frac{4-5p}{2(p+2)}$, we have, by Strichartz estimates and the nonlinear estimates that on $[T, \infty)$,

$$\left\|\nabla^i u\right\|_{L^2 L^6 \cap L^\delta L^{p+2}} \leqslant \left\|u(T)\right\|_{\dot{H}^i} + \frac{1}{2}\left\|\nabla^i u\right\|_{L^2 L^6 \cap L^\delta L^{p+2}},$$
$$\left\|\nabla^i u\right\|_{L^2 L^6 \cap L^\delta L^{p+2}} \leqslant 2\left\|u(T)\right\|_{\dot{H}^i} \leqslant C\|u_0\|_{H^i},$$

where $i = 0, 1$, therefore, we have

$$\begin{aligned}
&\left\|\nabla^i u\right\|_{L^2(\mathbf{R}^+; L^6) \cap L^\delta(\mathbf{R}^+; L^{p+2})} \\
&\quad \leqslant \left\|\nabla^i u\right\|_{L^2([0,T); L^6) \cap L^\delta([0,T); L^{p+2})} + \left\|\nabla^i u\right\|_{L^2([T,\infty); L^6) \cap L^\delta([T,\infty); L^{p+2})} \\
&\quad \leqslant C\big(\|u_0\|_{\Sigma}\big),
\end{aligned}$$

hence

$$\|u\|_{L^q(\mathbf{R}^+; H^{1,r})} \leqslant C\big(\|u_0\|_{\Sigma}\big), \quad (q, r) \text{ admissible}, \tag{4.10}$$

by Strichartz estimate. This, in turn, gives that

$$\|Hu\|_{L^q(\mathbf{R}^+; L^r)} \leqslant C\big(\|u_0\|_{\Sigma}\big), \quad (q, r) \text{ admissible}. \tag{4.11}$$

Now, we show the scattering also holds in Σ.

Let u_+ be the same with (3.13) and $B = \{\nabla_x, x, I\}$, then we have

$$\begin{aligned}
\left\|U(-t)u(t) - u_+\right\|_{\Sigma} &= \left\| B \int_t^\infty U(-\tau)\big(|u|^p u(\tau) + |u|^4 u(\tau)\big)\, d\tau \right\|_2 \\
&= \left\| \int_t^\infty U(t)BU(-\tau)\big(|u|^p u(\tau) + |u|^4 u(\tau)\big)\, d\tau \right\|_2,
\end{aligned}$$

by noting $U(t)$ is unitary on L^2. Using

$$U(t)BU(-\tau) = U(t-\tau)U(\tau)BU(-\tau) = U(t-\tau)\big\{\nabla_x, H(\tau), I\big\},$$

we have

$$\left\|U(-t)u(t) - u_+\right\|_{\Sigma} \leqslant \left\||u|^p u + \|u|^4 u\right\|_{L^{\frac{10}{7}}(\mathbf{R}^+; H^{1,\frac{10}{7}})} + \left\|H(\cdot)\big(|u|^p u + |u|^4 u\big)\right\|_{L^{\frac{10}{7}}(\mathbf{R}^+; L^{\frac{10}{7}})},$$

which by (4.6)–(4.9) and (4.10), (4.11) gives that

$$\left\| U(-t)u(t) - u_+ \right\|_\Sigma \to 0, \quad \text{as } t \to \infty.$$

5. Blow-up

Recalling by the assumption,

$$\int \frac{1}{2}|\nabla u|^2 + \frac{\lambda_1}{p+2}|u|^{p+2} + \frac{\lambda_2}{6}|u|^6\,dx = E < 0. \tag{5.1}$$

On the other hand, let

$$y(t) = \operatorname{Im}\int r\bar{u}u_r\,dx,$$

then by straightforward calculation, one has

$$\dot{y}(t) = -2\int |\nabla u|^2\,dx - 2\lambda_2\int |u|^6\,dx - \frac{3p\lambda_1}{p+2}\int |u|^{p+2}\,dx, \qquad \frac{d}{dt}\int |xu|^2\,dx = -4y(t).$$

Now, we bound $\dot{y}(t)$ from below. In the case $0 \leqslant p < 4$ and $\lambda_1 > 0$, inserting (5.1), we have

$$\begin{aligned}\dot{y}(t) &= -2\int |\nabla u|^2\,dx + 6\int |\nabla u|^2\,dx + \frac{12\lambda_1}{p+2}\int |u|^{p+2}\,dx - 6E - \frac{3p\lambda_1}{p+2}\int |u|^{p+2}\,dx \\ &= 4\int |\nabla u|^2\,dx - 6E + \frac{3\lambda_1(4-p)}{p+2}\int |u|^{p+2}\,dx \geqslant 4\int |\nabla u|^2\,dx.\end{aligned}$$

In the case $\lambda_1 < 0,\ 0 \leqslant p < 4$, and $E + C(\lambda_1, \lambda_2, p)M^2 < 0$, we have

$$\begin{aligned}\dot{y}(t) &\geqslant -2\int |\nabla u|^2\,dx - \lambda_2\int |u|^6\,dx - \lambda_2\int |u|^6\,dx \\ &\geqslant -2\int |\nabla u|^2\,dx - \lambda_2\int |u|^6\,dx + 3\int |\nabla u|^2\,dx + \frac{6\lambda_1}{p+2}\int |u|^{p+2}\,dx - 3E \\ &\geqslant \int |\nabla u|^2\,dx - \lambda_2\int |u|^6\,dx + \frac{6\lambda_1}{p+2}\int |u|^{p+2}\,dx - 3E.\end{aligned}$$

Since by Young's equality and interpolation

$$\|u\|_{p+2}^{p+2} \leqslant M^{2-\frac{p}{2}}\|u\|_6^{\frac{3p}{2}} \leqslant M^{2-\frac{p}{2}}\left(\varepsilon^{-\frac{p}{4-p}} + \varepsilon\|u\|_6^6\right).$$

Letting ε be such that

$$C|\lambda_1|M^{2-\frac{p}{2}}\varepsilon = |\lambda_2|,$$

we have

$$\frac{6|\lambda_1|}{p+2}\int |u|^{p+2}\,dx \leqslant |\lambda_2|\int |u|^6\,dx + C(\lambda_1,\lambda_2,p)M^2,$$

therefore

$$\dot{y}(t) \geqslant \int |\nabla u|^2 - 3\big(E + C(\lambda_1,\lambda_2,p)M^2\big) > \int |\nabla u|^2.$$

In the case $\lambda_1 < 0$, $\frac{4}{3} < p < 4$ and $E < 0$, we have

$$\begin{aligned}\dot{y}(t) \geqslant{}& -2\int |\nabla u|^2\,dx - 2\lambda_2\int |u|^6\,dx + \frac{3p}{2}\int |\nabla u|^2\,dx\\&+\frac{p\lambda_2}{2}\int |u|^6\,dx - \frac{3p}{2}E > \left(\frac{3p}{2}-2\right)\int |\nabla u|^2\,dx.\end{aligned}$$

In the case $\lambda_1 < 0$, $p > 4$ and $E < 0$, we have

$$\dot{y}(t) \geqslant 4\int |\nabla u|^2\,dx.$$

In the above, we get the estimate

$$\dot{y}(t) \geqslant c\int |\nabla u|^2\,dx$$

for some positive constant c. Now we follow the argument in [9] to deduce the blow-up of kinetic energy.

By the assumption,

$$y(0) = \operatorname{Im}\int r\bar{u}_0 u_{0r}\,dx > 0, \qquad \dot{y}(t) > 0,$$

we have that

$$y(t) > 0, \quad \forall t \geqslant 0,$$

which, in turn, implies that

$$\big\|xu(t)\big\|_2 < \|xu_0\|_2, \quad \forall t > 0,$$

hence by using the Cauchy inequality $y(t) \leqslant \|xu(t)\|_2\|\nabla u(t)\|_2$, we have

$$\big\|\nabla u(t)\big\|_2 \geqslant \frac{y(t)}{\|xu_0\|_2},$$

therefore we deduce a one order ODE for $y(t)$

$$\begin{cases} \dot{y}(t) > c\frac{y^2(t)}{\|xu_0\|_2^2}, \\ y(0) > 0, \end{cases}$$

from which, we conclude that $\exists T^* \leqslant C\frac{\|xu_0\|_2^2}{d_0}$ such that

$$\lim_{t\to T^*} \left\|\nabla u(t)\right\|_2 \geqslant \lim_{t\to T^*} \frac{y(t)}{\|xu_0\|_2} = \infty.$$

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