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Numerical analysis of a parabolic hemivariational inequality for semipermeable media

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A B S T R A C T

In this paper, we consider the numerical solution of a model problem in the form of a parabolic hemivariational inequality that arises in applications of semipermeable media. The model problem is first studied as a particular case of an abstract parabolic hemivariational inequality. A general fully discrete numerical method is introduced for the abstract parabolic hemivariational inequality, where the time derivative of the unknown solution is approximated by the backward divided difference. A Céa's type inequality is shown as a preparation for error estimation. Then the general result is specialized for the numerical solution of the model problem and an optimal order error estimate with the use of linear finite elements is derived. Finally numerical examples are presented to show the performance of the numerical solutions and the emphasis is to illustrate numerical convergence orders that match the theoretically predicted optimal first order convergence of the linear element solutions with respect to the finite element mesh-size and the time step-size.

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1. Introduction

In recent years, a family of nonlinear problems, known as hemivariational inequalities, has attracted much attention in research community. The notion of hemivariational inequalities was introduced in early 1980s by Panagiotopoulos [1], responding to the need in modeling and studying engineering problems involving non-smooth, non-monotone and/or multi-valued physical laws and conditions. Mathematical theory of hemivariational inequalities can be found in several research monographs (e.g., [2–7]) and many journal articles. Since no closed-form solution formula can be expected for a hemivariational inequality arising in applications, numerical methods are needed to solve hemivariational inequalities. An early comprehensive reference on the numerical solution of hemivariational inequalities is [8] where convergence of finite element solutions and solution algorithms is discussed. In [9], an optimal order error bound is derived for linear finite element solutions of a stationary variational–hemivariational inequality, and this is followed by a series of papers on optimal order error bounds for linear finite element solutions of various hemivariational inequalities or variational–hemivariational inequalities, e.g., [10–12] for stationary variational–hemivariational inequalities and [13–15] for some time-dependent variational–hemivariational inequalities, cf. the recent survey [16].

Hemivariational inequalities of parabolic type arise in problems in heat conduction, electrostatics, flows through porous media with semipermeability conditions (cf. [17]). Parabolic hemivariational inequalities are also studied in a

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number of other publications, e.g., [18–22], [7, Chapter 7]. In this paper, we present numerical analysis of a parabolic hemivariational inequality arising in applications in semipermeable media [23]. Note that in [24], numerical analysis is provided for a stationary analogue of the hemivariational inequality. For definiteness, we choose a model with a boundary semipermeability term. The study of the parabolic hemivariational inequality will be carried out through that of an abstract general parabolic variational–hemivariational inequality and follow [22] for the solution existence and uniqueness. The parabolic variational–hemivariational inequality reduces to a parabolic hemivariational inequality when certain term is dropped from the formulation.

In the study of hemivariational inequalities, we will need notions of the generalized directional derivative and the generalized subdifferential in the sense of Clarke, introduced in [25]. We will only use real Banach spaces in this paper. For a Banach space X, we denote its norm by $\|\cdot\|_X$, its topological dual by X^*, and the duality pairing of X and X^* by $\langle\cdot,\cdot\rangle$. Let $\varphi\colon X\to\mathbb{R}$ be a locally Lipschitz functional. Its Clarke generalized directional derivative at a point $x\in X$ in a direction $z\in X$ is defined by

$$\varphi^0(x;z)=\limsup_{y\to x,\,\lambda\downarrow 0}\frac{\varphi(y+\lambda z)-\varphi(y)}{\lambda}.$$

The Clarke subdifferential of φ at x is a subset of X^* given by

$$\partial\varphi(x)=\left\{\zeta\in X^*\mid \varphi^0(x;z)\ge\langle\zeta,z\rangle\ \ \forall\, z\in X\right\}.$$

Discussions of the subdifferential in the sense of Clarke can be found in the books [3,6,26]. We will also make use of the notion of the subdifferential in convex analysis for a convex functional. The reader is referred to [27] for details on the convex subdifferential.

Now we introduce the parabolic hemivariational inequality for applications in media with boundary semipermeability (cf. [23]). Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with $d\le 3$ for applications. Since the boundary $\partial\Omega$ is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu}$ exists a.e. on $\partial\Omega$. Decompose the boundary $\partial\Omega$ into two parts Γ_D and Γ_N. Let $[0,T]$ be the time interval with a given $T>0$. The pointwise formulation of the model problem is

$$\dot u-\Delta u=f\quad\text{in }\Omega\times(0,T), \tag{1.1}$$

$$u=0\quad\text{on }\Gamma_D\times(0,T), \tag{1.2}$$

$$-\frac{\partial u}{\partial\nu}\in\partial j(u)+\partial\phi_I(u)\quad\text{on }\Gamma_N\times(0,T), \tag{1.3}$$

$$u|_{t=0}=u_0\quad\text{in }\Omega. \tag{1.4}$$

Here $\dot u$ denotes the time derivative of u, j is a locally Lipschitz continuous function and ∂j denotes its generalized subdifferential, $I\subset\mathbb{R}$ is an interval, ϕ_I is the indicator function of I and $\partial\phi_I$ denotes its convex subdifferential. In the context of an application with boundary temperature, the unknown function u represents the temperature and the interval I gives the range of the temperature on the boundary Γ_N.

The rest of the paper is organized as follows. In Section 2, we introduce an abstract parabolic variational–hemivariational inequality and review a result for its solution existence and uniqueness; this abstract parabolic variational–hemivariational inequality contains the model problem (1.1)–(1.4) as a special case. In Section 3, we present a fully discrete scheme for the abstract parabolic variational–hemivariational inequality where the time derivative is approximated by the backward divided difference and the spatial variable is approximated by a Galerkin method. We derive a Céa's type inequality that is the starting point for convergence order error estimation. In Section 4, we specialize the results from the previous section for the numerical solution of the model problem and proceed further to derive an optimal order error estimate when the linear finite element method is applied for the spatial discretization. Note that in deriving error estimates for the numerical solution of the parabolic variational–hemivariational inequality, we need to assume certain solution regularity. It is possible to prove the convergence of the numerical solution to the solution of the parabolic variational–hemivariational inequality under the minimal solution regularity available from the solution existence result, by extending the arguments presented in [28] on the convergence of the numerical solution for an elliptic variational–hemivariational inequality. In Section 5, we report computer simulation results of the fully discrete scheme in solving the model problem, with an emphasis on evidence of numerical convergence orders of the numerical solutions that match the theoretical predictions from Section 4.

2. An abstract parabolic variational–hemivariational inequality

The model problem (1.1)–(1.4) will be studied as a particular case of an abstract parabolic variational–hemivariational inequality. To formulate the abstract problem, we introduce two spaces V and H such that V is a strictly convex, reflexive and separable Banach space, H is a separable Hilbert space, and $V\subset H\subset V^*$ is an evolution triple with both embeddings continuous, dense and compact. Let $K\subset V$ be a non-empty, closed and convex set in V. The norms in V and H are denoted by $\|\cdot\|_V$ and $\|\cdot\|_H$, and the norm in $\mathbb{R}^d$ is denoted by $|\cdot|$. We use $\langle\cdot,\cdot\rangle$ for the duality pairing between V^* and V, and use $(\cdot,\cdot)$ for the inner product in H; they are related by the identity

$$\langle v^*,v\rangle=(v^*,v)\quad\forall\, v^*\in H,\ v\in V. \tag{2.1}$$

In addition, we need another reflexive Banach space U, and denote by $\langle\cdot,\cdot\rangle_{U^*\times U}$ for the duality pairing between U^* and U, and by $\|\cdot\|_U$ the norm in U. Let $[0,T]$ be the time interval of interest for the study of the inequality problem, with a given $T>0$. We further define Bochner spaces $\mathcal{V}=L^2(0,T;V)$, $\mathcal{H}=L^2(0,T;H)$, $\mathcal{U}=L^2(0,T;U)$, $\mathcal{U}^*=L^2(0,T;U^*)$, and $\mathcal{W}=\{v\in\mathcal{V}\mid \dot v\in\mathcal{V}^*\}$.

Consider the following parabolic variational–hemivariational inequality [22].

Problem 2.1. Find $u\in\mathcal{W}$ such that $u(t)\in K$ for a.e. $t\in(0,T)$,

$$\begin{aligned}&\int_0^T\left[\langle\dot u(t)+Au(t)+\gamma^*\xi(t),v(t)-u(t)\rangle+\Phi(v(t))-\Phi(u(t))\right]dt\\&\qquad\ge\int_0^T\langle f(t),v(t)-u(t)\rangle\,dt\quad\forall\,v\in\mathcal{V},\ v(t)\in K\end{aligned}\tag{2.2}$$

where

$$\xi\in\mathcal{U}^*,\quad \xi(t)\in\partial J(\gamma u(t))\quad\text{a.e. }t\in(0,T),\tag{2.3}$$

and

$$u(0)=u_0.\tag{2.4}$$

We impose the following assumptions.

$\underline{H(A)}$ The operator $A\colon V\to V^*$ is Lipschitz continuous: for some constant $L_A>0$,

$$\|A(u)-A(v)\|_{V^*}\le L_A\|u-v\|_V\quad\forall\,u,v\in V,\tag{2.5}$$

and satisfies the inequality

$$\langle A(u)-A(v),u-v\rangle\ge m_1\|u-v\|_V^2-m_2\|u-v\|_H^2\quad\forall\,u,v\in V,\tag{2.6}$$

with constants $m_1>0$ and $m_2\ge 0$.

$\underline{H(J)}$ The functional $J\colon U\to\mathbb{R}$ is locally Lipschitz continuous, for constants $c\ge 0$ and $\alpha_J\ge 0$,

$$\|\xi\|_{U^*}\le c\,(1+\|u\|_U)\quad\forall\,u\in U,\ \xi\in\partial J(u)\tag{2.7}$$

and

$$\langle\xi-\eta,u-v\rangle_{U^*\times U}\ge-\alpha_J\|u-v\|_U^2\quad\forall\,u,v\in U,\ \xi\in\partial J(u),\ \eta\in\partial J(v).\tag{2.8}$$

$\underline{H(\Phi)}$ The functional $\Phi\colon V\to\mathbb{R}$ is convex and continuous.

$\underline{H(\gamma)}$ The operator $\gamma\in\mathcal{L}(V,U)$ is compact, the associated Nemytskii operator

$$\bar\gamma\colon M^{2,2}(0,T;V,V^*)\to\mathcal{U}$$

defined by $(\bar\gamma v)(t)=\gamma(v(t))$ is also compact, and for any $\delta>0$, there is a constant $c=c(\delta)$ such that

$$\|\gamma v\|_U\le\delta\,\|v\|_V+c\,\|v\|_H\quad\forall\,v\in V.\tag{2.9}$$

$\underline{H_0}$ $f\in H^1(0,T;V^*)$, $u_0\in V$, and there exist $\xi_0\in\partial J(\gamma u_0)$ and $\eta_0\in\partial\Phi(u_0)$ such that $Au_0+\gamma^*\xi_0+\eta_0-f(0)\in H$.

We comment that (2.8) is equivalent to the inequality

$$J^0(u;v-u)+J^0(v;u-v)\le\alpha_J\|u-v\|_U^2\quad\forall\,u,v\in U,$$

known as the relaxed monotonicity condition in the literature. In $\underline{H(\gamma)}$, the space $M^{2,2}(0,T;V,V^*)$ is used, which is a Banach space defined by

$$M^{2,2}(0,T;V,V^*)=L^2(0,T;V)\cap BV^2(0,T;V^*)$$

with the norm $\|\cdot\|_{L^2(0,T;V)}+\|\cdot\|_{BV^2(0,T;V^*)}$. Here,

$$BV^2(0,T;V^*)=\left\{v\colon[0,T]\to V^*\mid\|v\|_{BV^2(0;T;V^*)}<\infty\right\}$$

and $\|v\|_{BV^2(0;T;V^*)}$ is defined as follows. Let $\pi:0=t_0<t_1<\cdots<t_N=T$ be a partition of the interval $[0,T]$ into N sub-intervals, and let Π be the family of all such partitions. Then,

$$\|v\|_{BV^2(0,T;V^*)}^2=\sup_{\pi\in\Pi}\sum_{i=0}^{N-1}\|v(t_{i+1})-v(t_i)\|_{V^*}^2.$$

The next result regarding Problem 2.1 holds [22, Theorem 6.1].

Theorem 2.2. *Under the assumptions $\underline{H(A)}$, $\underline{H(J)}$, $\underline{H(\Phi)}$, $\underline{H(\gamma)}$ and $\underline{H_0}$, Problem 2.1 has a unique solution.*

Remark 2.3. Let us make some remarks on the assumptions since in the literature, different assumptions are found to ensure solution existence and uniqueness on Problem 2.1 and this may cause confusion.

On $\underline{H(A)}$: In [22], the operator A is assumed to be pseudomonotone and coercive, and satisfy a linear growth condition and (2.6). By [29, Proposition 27.6 (a)], a Lipschitz continuous, monotone operator is pseudomonotone. By [29, Proposition 27.6 (f)], the summation of a pseudomonotone operator and a strongly continuous operator is pseudomonotone. We define an operator $B: V \to V^*$ by

$$\langle Bu, v\rangle = -m_2(u, v)_H, \quad u, v \in V.$$

This operator is strongly continuous due to the compact embedding of V to H. By the conditions (2.5)–(2.6), the operator $A - B: V \to V^*$ is Lipschitz continuous and monotone, and is hence pseudomonotone. Thus, the operator $A = (A - B) + B$ is pseudomonotone, being the sum of a pseudomonotone operator and a strongly continuous operator. The linear growth condition of A is a consequence of (2.5), and the coercivity follows from (2.6). That is why in this paper, we assume the more easily checked Lipschitz continuity condition (2.5) and do not assume the pseudomonotonicity and the linear growth condition.

On $\underline{H(\Phi)}$: In [22], the functional $\Phi: V \to \mathbb{R} \cup \{+\infty\}$ is assumed to be proper, convex and lower semicontinuous. In this paper, with K a non-empty, closed and convex set in V, we consider the particular form $\tilde{\Phi} + I_K$ for the functional Φ in [22], where I_K is the indicator functional of K, whereas $\tilde{\Phi}: V \to \mathbb{R}$ is convex and lower semicontinuous. By [27], a l.s.c. convex functional on a Banach space is continuous. So in this paper we assume $\tilde{\Phi}: V \to \mathbb{R}$ is convex and continuous from the outset (and use the symbol Φ for $\tilde{\Phi}$).

In [7, Chapter 7], a smallness condition of the form $\alpha_J \|\gamma\|^2 < m_1$ is required. Following [22], the smallness assumption can be dropped at the expense of assuming $\underline{H(\gamma)}$.

For the model problem and problems in other applications, typically V is $H^1(\Omega)$ or its closed subspace, $H = L^2(\Omega)$, $U = L^2(\Gamma)$ for Γ a subset of $\partial\Omega$ with a positive boundary measure and γ is a trace operator, or $U = L^2(\Omega_0)$ for $\Omega_0 = \Omega$ or a measurable subset of Ω and γ is an embedding operator, or we have vector versions of these spaces. For such settings, $\underline{H(\gamma)}$ is valid, cf. [21, pp. 460–461].

By a standard localization argument, we can derive the following localized version of Problem 2.1.

Problem 2.4. Find $u \in \mathcal{W}$ such that for a.e. $t \in (0, T)$, $u(t) \in K$,

$$\begin{aligned}&\langle \dot{u}(t) + Au(t) + \gamma^*\xi(t), v - u(t)\rangle + \Phi(v) - \Phi(u(t)) \\ &\quad \ge \langle f(t), v - u(t)\rangle \quad \forall\, v \in K,\end{aligned} \tag{2.10}$$

where

$$\xi \in \mathcal{U}^*, \quad \xi(t) \in \partial J(\gamma u(t)) \quad \text{a.e. } t \in (0, T), \tag{2.11}$$

and

$$u(0) = u_0. \tag{2.12}$$

Introduce a companion problem.

Problem 2.5. Find $u \in \mathcal{W}$ such that for a.e. $t \in (0, T)$, $u(t) \in K$,

$$\begin{aligned}&\langle \dot{u}(t) + Au(t), v - u(t)\rangle + J^0(\gamma u(t); \gamma(v - u(t))) + \Phi(v) - \Phi(u(t)) \\ &\quad \ge \langle f(t), v - u(t)\rangle \quad \forall\, v \in K,\end{aligned} \tag{2.13}$$

and

$$u(0) = u_0. \tag{2.14}$$

Let us show that Problems 2.4 and 2.5 are equivalent.

Theorem 2.6. *Under the assumptions stated in Theorem 2.2, Problem 2.5 has a unique solution. Moreover, Problems 2.4 and 2.5 are equivalent in the sense that $u \in \mathcal{W}$ is the solution of Problem 2.4 if and only if $u \in \mathcal{W}$ is the solution of Problem 2.5.*

Proof. Let $u \in \mathcal{W}$ be the solution of Problem 2.4, guaranteed by Theorem 2.2. Since

$$\langle \gamma^*\xi(t), v - u(t)\rangle = \langle \xi(t), \gamma(v - u(t))\rangle_{U^*\times U} \le J^0(\gamma u(t); \gamma(v - u(t))),$$

we derive (2.13) from (2.10). Thus, $u \in \mathcal{W}$ is a solution of Problem 2.5.

Now we prove the solution uniqueness of Problem 2.5. Let $u_1, u_2 \in \mathcal{W}$ be two solutions of Problem 2.5. Then for a.e. $t \in (0, T)$, $u_1(t), u_2(t) \in K$ and for any $v \in K$,

$$\langle \dot{u}_1(t) + Au_1(t), v - u_1(t)\rangle + J^0(\gamma u_1(t); \gamma(v - u_1(t))) + \Phi(v) - \Phi(u_1(t)) \geq \langle f(t), v - u_1(t)\rangle, \tag{2.15}$$

$$\langle u_2'(t) + Au_2(t), v - u_2(t)\rangle + J^0(\gamma u_2(t); \gamma(v - u_2(t))) + \Phi(v) - \Phi(u_2(t)) \geq \langle f(t), v - u_2(t)\rangle. \tag{2.16}$$

We take $v = u_2(t)$ in (2.15), $v = u_1(t)$ in (2.16), and add the two inequalities to obtain

$$\begin{aligned}&\langle \dot{u}_1(t) - \dot{u}_2(t), u_1(t) - u_2(t)\rangle + \langle Au_1(t) - Au_2(t), u_1(t) - u_2(t)\rangle \\ &\quad \leq J^0(\gamma u_1(t); \gamma(u_2(t) - u_1(t))) + J^0(\gamma u_2(t); \gamma(u_1(t) - u_2(t))).\end{aligned}$$

Then,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_1(t) - u_2(t)\|_H^2 + m_1\|u_1(t) - u_2(t)\|_V^2 - m_2\|u_1(t) - u_2(t)\|_H^2 \\ &\quad \leq \alpha_J\|\gamma(u_1(t) - u_2(t))\|_U^2 \\ &\quad \leq \frac{m_1}{2}\|u_1(t) - u_2(t)\|_V^2 + c\,\|u_1(t) - u_2(t)\|_H^2.\end{aligned}$$

So

$$\frac{d}{dt}\|u_1(t) - u_2(t)\|_H^2 + \|u_1(t) - u_2(t)\|_V^2 \leq c\,\|u_1(t) - u_2(t)\|_H^2.$$

Since $u_1(0) - u_2(0) = 0$, by the Gronwall inequality, we derive from the above inequality that $\|u_1(t) - u_2(t)\|_H^2 = 0$, i.e., $u_1 = u_2$ and a solution of Problem 2.5 is unique. ■

3. Numerical approximation of the abstract parabolic variational–hemivariational inequality

For the numerical solution of Problem 2.5, we let $V^h \subset V$ be a finite element space approximating V, and let $K^h = V^h \cap K$ and assume it is non-empty. The space V^h is constructed over a finite element partition of $\overline{\Omega}$ from a regular family of finite element meshes. The parameter h represents the mesh-size of the finite element partition. For the discretization of the time derivative, we use a finite difference. Given a positive integer N, let $k = T/N$ be the time step-size. The time nodal points are $t_n = kn$, $n = 0, \ldots, N$. For a continuous function g defined on the interval $[0, T]$ we write $g_n = g(t_n)$, $n = 0, \ldots, N$.

In this section, we will assume the regularities

$$u \in C([0, T]; V), \quad u \in H^2(0, T; V^*), \quad f \in C([0, T]; V^*). \tag{3.1}$$

Note that $u \in H^2(0, T; V^*)$ implies $\dot{u} \in C([0, T]; V^*)$. An immediate consequence of the regularity assumptions is that (2.13) holds for $t \in (0, T)$. Moreover, the first part of (3.1) implies that $u_0 \in K$.

Let $u_0^h \in K^h$ be an appropriate approximation of u_0. The discrete scheme for Problem 2.5 is the following.

Problem 3.1. Find $u^{hk} = \{u_n^{hk}\}_{n=0}^N \subset K^h$ such that for $1 \leq n \leq N$,

$$\begin{aligned}&\left(\frac{u_n^{hk} - u_{n-1}^{hk}}{k}, v^h - u_n^{hk}\right) + \langle Au_n^{hk}, v^h - u_n^{hk}\rangle + J^0(\gamma u_n^{hk}; \gamma(v^h - u_n^{hk})) + \Phi(v^h) - \Phi(u_n^{hk}) \\ &\quad \geq \langle f_n, v^h - u_n^{hk}\rangle \quad \forall\, v^h \in K^h,\end{aligned} \tag{3.2}$$

and

$$u_0^{hk} = u_0^h. \tag{3.3}$$

The inequality (3.2) can be rewritten as

$$\begin{aligned}&(u_n^{hk}, v^h - u_n^{hk}) + k\langle Au_n^{hk}, v^h - u_n^{hk}\rangle + kJ^0(\gamma u_n^{hk}; \gamma(v^h - u_n^{hk})) + k\,\Phi(v^h) - k\,\Phi(u_n^{hk}) \\ &\quad \geq k\langle f_n, v^h - u_n^{hk}\rangle + (u_{n-1}^{hk}, v^h - u_n^{hk}) \quad \forall\, v^h \in K^h.\end{aligned} \tag{3.4}$$

Existence and uniqueness of a solution $u_n^{hk} \in K^h$ to (3.4), for $k > 0$ sufficiently small, follow from a corresponding result for elliptic variational–hemivariational inequality [16].

The rest of the section focuses on error analysis for the numerical solution. Denote

$$e_n = u_n - u_n^{hk}, \quad 0 \leq n \leq N$$

for the numerical solution errors. Let $v_n^h \in K^h$ be arbitrary but fixed. Then by (2.6),

$$\begin{aligned}m_1\|e_n\|_V^2 - m_2\|e_n\|_H^2 &\leq \langle Au_n - Au_n^{hk}, u_n - u_n^{hk}\rangle \\ &= \langle Au_n - Au_n^{hk}, u_n - v_n^h\rangle + \langle Au_n, v_n^h - u_n\rangle \\ &\quad + \langle Au_n, u_n - u_n^{hk}\rangle + \langle Au_n^{hk}, u_n^{hk} - v_n^h\rangle.\end{aligned} \tag{3.5}$$

From (2.13) at $t = t_n$ with $v = u_n^{hk}$,

$$\langle Au_n, u_n - u_n^{hk}\rangle \le \langle \dot{u}_n, u_n^{hk} - u_n\rangle + J^0(\gamma u_n; \gamma(u_n^{hk} - u_n)) + \Phi(u_n^{hk}) - \Phi(u_n) - \langle f_n, u_n^{hk} - u_n\rangle. \tag{3.6}$$

From (3.2) with $v^h = v_n^h \in K^h$,

$$\langle Au_n^{hk}, u_n^{hk} - v_n^h\rangle \le \langle \frac{u_n^{hk} - u_{n-1}^{hk}}{k}, v_n^h - u_n^{hk}\rangle + J^0(\gamma u_n^{hk}; \gamma(v_n^h - u_n^{hk})) + \Phi(v_n^h) - \Phi(u_n^{hk}) - \langle f_n, v_n^h - u_n^{hk}\rangle. \tag{3.7}$$

Use (3.6) and (3.7) in (3.5),

$$\begin{aligned} m_1\|e_n\|_V^2 - m_2\|e_n\|_H^2 \le{}& \langle Au_n - Au_n^{hk}, u_n - v_n^h\rangle + \langle Au_n, v_n^h - u_n\rangle \\ &+ \langle \dot{u}_n, u_n^{hk} - u_n\rangle + \langle \frac{u_n^{hk} - u_{n-1}^{hk}}{k}, v_n^h - u_n^{hk}\rangle \\ &+ J^0(\gamma u_n; \gamma(u_n^{hk} - u_n)) + J^0(\gamma u_n^{hk}; \gamma(v_n^h - u_n^{hk})) \\ &+ \Phi(v_n^h) - \Phi(u_n) - \langle f_n, v_n^h - u_n\rangle. \end{aligned} \tag{3.8}$$

Define a residual type quantity on K,

$$R_n(v) = \langle \dot{u}_n + Au_n, v - u_n\rangle + J^0(\gamma u_n; \gamma(v - u_n)) + \Phi(v) - \Phi(u_n) - \langle f_n, v - u_n\rangle. \tag{3.9}$$

Then we can rewrite (3.8) as

$$\begin{aligned} m_1\|e_n\|_V^2 - m_2\|e_n\|_H^2 \le{}& \langle Au_n - Au_n^{hk}, u_n - v_n^h\rangle + \langle \dot{u}_n - \frac{u_n^{hk} - u_{n-1}^{hk}}{k}, u_n^{hk} - v_n^h\rangle \\ &+ R_n(v_n^h) + \Delta_n(J^0), \end{aligned} \tag{3.10}$$

where

$$\Delta_n(J^0) = J^0(\gamma u_n; \gamma(u_n^{hk} - u_n)) + J^0(\gamma u_n^{hk}; \gamma(v_n^h - u_n^{hk})) - J^0(\gamma u_n; \gamma(v_n^h - u_n)). \tag{3.11}$$

Denote

$$E_n = \dot{u}_n - \frac{u_n - u_{n-1}}{k}. \tag{3.12}$$

Note that

$$\|E_n\|_{V^*} \le \|\ddot{u}\|_{L^1(t_{n-1},t_n;V^*)}$$

and thus,

$$\|E_n\|_{V^*}^2 \le k\,\|\ddot{u}\|_{L^2(t_{n-1},t_n;V^*)}^2. \tag{3.13}$$

Write

$$\begin{aligned} \langle \dot{u}_n - \frac{u_n^{hk} - u_{n-1}^{hk}}{k}, u_n^{hk} - v_n^h\rangle &= \langle E_n + \frac{e_n - e_{n-1}}{k}, (-e_n) + (u_n - v_n^h)\rangle \\ &= -\left(\langle \frac{e_n - e_{n-1}}{k}, e_n\rangle + \langle E_n, e_n\rangle\right) \\ &\quad + \langle \frac{e_n - e_{n-1}}{k}, u_n - v_n^h\rangle + \langle E_n, u_n - v_n^h\rangle. \end{aligned}$$

Since

$$\left(\frac{e_n - e_{n-1}}{k}, e_n\right) = \frac{1}{2\,k}\left(\|e_n\|_H^2 - \|e_{n-1}\|_H^2 + \|e_n - e_{n-1}\|_H^2\right) \ge \frac{1}{2\,k}\left(\|e_n\|_H^2 - \|e_{n-1}\|_H^2\right),$$

we have

$$\begin{aligned} \langle \dot{u}_n - \frac{u_n^{hk} - u_{n-1}^{hk}}{k}, u_n^{hk} - v_n^h\rangle \le{}& -\frac{1}{2\,k}\left(\|e_n\|_H^2 - \|e_{n-1}\|_H^2\right) - \langle E_n, e_n\rangle \\ &+ \langle \frac{e_n - e_{n-1}}{k}, u_n - v_n^h\rangle + \langle E_n, u_n - v_n^h\rangle. \end{aligned} \tag{3.14}$$

For the J^0 terms in (3.11), note that

$$J^0(\gamma u_n^{hk}; \gamma(v_n^h - u_n^{hk})) \le J^0(\gamma u_n^{hk}; \gamma(u_n - u_n^{hk})) + J^0(\gamma u_n^{hk}; \gamma(v_n^h - u_n)),$$

and by (2.8),

$$J^0(\gamma u_n;\gamma(u_n^{hk}-u_n))+J^0(\gamma u_n^{hk};\gamma(u_n-u_n^{hk}))\le \alpha_J\|\gamma e_n\|_U^2.$$

Let $\delta>0$ be a small number to be chosen. Applying (2.9),

$$J^0(\gamma u_n;\gamma(u_n^{hk}-u_n))+J^0(\gamma u_n^{hk};\gamma(u_n-u_n^{hk}))\le \delta\,\|e_n\|_V^2+c_\delta\|e_n\|_H^2.$$

Use the condition (2.7),

$$\begin{aligned}-J^0(\gamma u_n;\gamma(v_n^h-u_n))&\le c\,(1+\|\gamma u_n\|_U)\,\|\gamma(v_n^h-u_n)\|_U,\\ J^0(\gamma u_n^{hk};\gamma(v_n^h-u_n))&\le c\left(1+\|\gamma u_n^{hk}\|_U\right)\|\gamma(v_n^h-u_n)\|_U\\ &\le c\,(1+\|\gamma u_n\|_U+\|\gamma e_n\|_U)\,\|\gamma(v_n^h-u_n)\|_U.\end{aligned}$$

Thus,

$$\Delta_n(J^0)\le \delta\,\|e_n\|_V^2+c_\delta\|e_n\|_H^2+c\,(1+\|\gamma u_n\|_U+\|\gamma e_n\|_U)\,\|\gamma(u_n-v_n^h)\|_U. \tag{3.15}$$

From

$$\langle Au_n-Au_n^{hk},u_n-v_n^h\rangle\le\|Au_n-Au_n^{hk}\|_{V^*}\|u_n-v_n^h\|_V$$

and the Lipschitz continuity of A, we have

$$\langle Au_n-Au_n^{hk},u_n-v_n^h\rangle\le L_A\|e_n\|_V\|u_n-v_n^h\|_V.$$

Then we deduce from (3.10) that

$$\begin{aligned}&m_1\|e_n\|_V^2-m_2\|e_n\|_H^2+\frac{1}{2\,k}\left(\|e_n\|_H^2-\|e_{n-1}\|_H^2\right)\\ &\quad\le L_A\|e_n\|_V\|u_n-v_n^h\|_V-\langle E_n,e_n\rangle+\langle\frac{e_n-e_{n-1}}{k},u_n-v_n^h\rangle+\langle E_n,u_n-v_n^h\rangle+R_n(v_n^h)\\ &\qquad+\delta\,\|e_n\|_V^2+c_\delta\|e_n\|_H^2+c\,(1+\|\gamma u_n\|_U+\|\gamma e_n\|_U)\,\|\gamma(u_n-v_n^h)\|_U.\end{aligned} \tag{3.16}$$

Using the modified Cauchy inequality

$$a\,b\le\delta\,a^2+c\,b^2\quad\forall\,a,b\in\mathbb{R},\ \ c=\frac{1}{4\,\delta},$$

we have

$$\begin{aligned}L_A\|e_n\|_V\|u_n-v_n^h\|_V&\le\delta\,\|e_n\|_V^2+c\,\|u_n-v_n^h\|_V^2,\\ -\langle E_n,e_n\rangle&\le\delta\,\|e_n\|_V^2+c\,\|E_n\|_{V^*}^2.\end{aligned}$$

Moreover,

$$c\,\|\gamma e_n\|_U\|\gamma(u_n-v_n^h)\|_U\le c\,\|e_n\|_V\|u_n-v_n^h\|_V\le\delta\,\|e_n\|_V^2+c\,\|u_n-v_n^h\|_V^2,$$

and

$$\langle E_n,u_n-v_n^h\rangle\le\|E_n\|_{V^*}\|u_n-v_n^h\|_V\le\frac12\,\|E_n\|_{V^*}^2+\frac12\,\|u_n-v_n^h\|_V^2.$$

Thus, from (3.16) and $\|u_n\|_V\le\|u\|_{C([0,T];V)}$,

$$\begin{aligned}&(m_1-4\,\delta)\,\|e_n\|_V^2-(m_2+c_\delta)\,\|e_n\|_H^2+\frac{1}{2\,k}\left(\|e_n\|_H^2-|e_{n-1}|^2\right)\\ &\quad\le c\left[\|u_n-v_n^h\|_V^2+\|E_n\|_{V^*}^2+\|\gamma(u_n-v_n^h)\|_U\right]+R_n(v_n^h)+\langle\frac{e_n-e_{n-1}}{k},u_n-v_n^h\rangle,\end{aligned}$$

where the constant c depends on $\|u\|_{C([0,T];V)}$. Choose $\delta=m_1/8$ to obtain

$$\begin{aligned}&\frac{m_1}{2}\,\|e_n\|_V^2-(m_2+c_\delta)\,\|e_n\|_H^2+\frac{1}{2\,k}\left(\|e_n\|_H^2-\|e_{n-1}\|_H^2\right)\\ &\quad\le c\left[\|u_n-v_n^h\|_V^2+\|E_n\|_{V^*}^2+\|\gamma(u_n-v_n^h)\|_U\right]+R_n(v_n^h)+\langle\frac{e_n-e_{n-1}}{k},u_n-v_n^h\rangle,\end{aligned}$$

or

$$\begin{aligned}\|e_n\|_H^2-\|e_{n-1}\|_H^2+k\,\|e_n\|_V^2&\le c\,k\,\|e_n\|_H^2+c\,k\left[\|u_n-v_n^h\|_V^2+\|E_n\|_{V^*}^2+\|\gamma(u_n-v_n^h)\|_U\right]\\ &\quad+c\,k\,|R_n(v_n^h)|+c\,\langle e_n-e_{n-1},u_n-v_n^h\rangle.\end{aligned}$$

Replace n by i in the previous inequality and sum over i from 1 to n:

$$\begin{aligned}\|e_n\|_H^2 - \|e_0\|_H^2 + k\sum_{i=1}^n \|e_i\|_V^2 \\ &\le c\,k\sum_{i=1}^n \|e_i\|_H^2 + c\,k\sum_{i=1}^n \left[\|u_i - v_i^h\|_V^2 + \|E_i\|_{V^*}^2 + \|\gamma(u_i - v_i^h)\|_U\right] \\ &\quad + c\,k\sum_{i=1}^n |R_i(v_i^h)| + c\sum_{i=1}^n \langle e_i - e_{i-1}, u_i - v_i^h\rangle. \end{aligned} \tag{3.17}$$

Note that

$$\begin{aligned} c\sum_{i=1}^n \langle e_i - e_{i-1}, u_i - v_i^h\rangle \\ &= c\,\langle e_n, u_n - v_n^h\rangle + c\sum_{i=1}^{n-1}\langle e_i, (u_i - v_i^h) - (u_{i+1} - v_{i+1}^h)\rangle - c\,\langle e_0, u_1 - v_1^h\rangle \\ &\le \frac{1}{2}\,\|e_n\|_H^2 + c\,\|u_n - v_n^h\|_H^2 + \frac{1}{2}\,k\sum_{i=1}^{n-1}\|e_i\|_V^2 \\ &\quad + c\,k^{-1}\sum_{i=1}^{n-1}\|(u_i - v_i^h) - (u_{i+1} - v_{i+1}^h)\|_{V^*}^2 + c\,\|e_0\|_H^2 + c\,\|u_1 - v_1^h\|_H^2. \end{aligned}$$

Then from (3.17),

$$\begin{aligned}\|e_n\|_H^2 + k\sum_{i=1}^n \|e_i\|_V^2 &\le c\,k\sum_{i=1}^n \left[\|e_i\|_H^2 + \|u_i - v_i^h\|_V^2 + \|\gamma(u_i - v_i^h)\|_U + \left|R_i(v_i^h)\right| + \|E_i\|_{V^*}^2\right] \\ &\quad + c\left(\|e_0\|_H^2 + \|u_1 - v_1^h\|_H^2 + \|u_n - v_n^h\|_H^2\right) \\ &\quad + c\,k^{-1}\sum_{i=1}^{n-1}\|(u_i - v_i^h) - (u_{i+1} - v_{i+1}^h)\|_{V^*}^2. \end{aligned} \tag{3.18}$$

Applying Gronwall's inequality, we conclude that if $k > 0$ is small enough, then

$$\begin{aligned}\max_n \|e_n\|_H^2 + k\sum_{n=1}^N \|e_n\|_V^2 &\le c\,k\sum_{n=1}^N \left[\|u_n - v_n^h\|_V^2 + \|\gamma(u_n - v_n^h)\|_U + \left|R_n(v_n^h)\right| + \|E_n\|_{V^*}^2\right] \\ &\quad + c\left(\|e_0\|_H^2 + \max_n \|u_n - v_n^h\|_H^2\right) \\ &\quad + c\,k^{-1}\sum_{n=1}^{N-1}\|(u_n - v_n^h) - (u_{n+1} - v_{n+1}^h)\|_{V^*}^2. \end{aligned}$$

By (3.13),

$$\sum_{n=1}^N \|E_n\|_{V^*}^2 \le k\,\|\ddot{u}\|_{L^2(0,T;V^*)}^2.$$

Therefore, for all $v_n^h \in K^h$, $1 \le n \le N$,

$$\begin{aligned}\max_n \|u_n - u_n^{hk}\|_H^2 + k\sum_{n=1}^N \|u_n - u_n^{hk}\|_V^2 &\le c\,k\sum_{n=1}^N \left[\|u_n - v_n^h\|_V^2 + \|\gamma(u_n - v_n^h)\|_U + \left|R_n(v_n^h)\right|\right] \\ &\quad + c\left(k^2 + \|u_0 - u_0^h\|_H^2 + \max_n \|u_n - v_n^h\|_H^2\right) \\ &\quad + c\,k^{-1}\sum_{n=1}^{N-1}\|(u_n - v_n^h) - (u_{n+1} - v_{n+1}^h)\|_{V^*}^2. \end{aligned} \tag{3.19}$$

This is a Céa's type inequality and it is the starting point for convergence order error estimation, as is illustrated in Section 4.

4. Numerical analysis of the model problem

Our main goal in this section is to derive error estimates for numerical solutions of the model problem (1.1)–(1.4). For the source function f and the initial value u_0, we assume

$$f \in C([0,T]; L^2(\Omega)), \quad u_0 \in H^2(\Omega). \tag{4.1}$$

In addition, we assume $j: \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous,

$$|\partial j(r)| \le c\,(1+|r|) \quad \forall\, r \in \mathbb{R} \tag{4.2}$$

for some constant c, and there exists a constant α such that

$$j^0(r_1; r_2 - r_1) + j^0(r_2; r_1 - r_2) \le \alpha\, |r_1 - r_2|^2 \quad \forall\, r_1, r_2 \in \mathbb{R}. \tag{4.3}$$

Let

$$\begin{aligned} V &= \left\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D \right\}, \\ K &= \{ v \in V \mid v \in I \text{ on } \Gamma_N \}, \\ H &= L^2(\Omega), \\ U &= L^2(\Gamma_N). \end{aligned}$$

and let $\gamma: V \to U$ be the trace operator. The weak formulation of the problem (1.1)–(1.4) is to find $u: \Omega \times (0,T) \to \mathbb{R}$ such that

$$u(0) = u_0 \quad \text{in } \Omega, \tag{4.4}$$

and for $t > 0$,

$$\begin{aligned} u(t) \in K, \quad \langle \dot u(t), v - u(t)\rangle + \int_\Omega \nabla u(t)\cdot\nabla(v - u(t))\,dx + \int_{\Gamma_N} j^0(u(t); v - u(t))\,ds \\ \ge \int_\Omega f(t)(v - u(t))\,dx \quad \forall\, v \in K, \end{aligned} \tag{4.5}$$

where if $\dot u(t)$ is in $L^2(\Omega)$, then

$$\langle \dot u(t), v\rangle = \int_\Omega \dot u(t)\, v\, dx.$$

Moreover, $u(t)$ and $f(t)$ stand for $u(\boldsymbol{x}, t)$ and $f(\boldsymbol{x}, t)$, respectively.

Define a functional

$$J(v) = \int_{\Gamma_N} j(v)\, ds, \quad v \in V.$$

Then $J: V \to \mathbb{R}$ is locally Lipschitz continuous, and from (4.2) and (4.3), we have their counterparts (2.7) and (2.8) with possibly different constant (cf. [6]). Introduce an auxiliary problem of finding $w: \Omega \times (0,T) \to \mathbb{R}$ such that

$$w(0) = u_0 \quad \text{in } \Omega, \tag{4.6}$$

and for $t > 0$,

$$\begin{aligned} w(t) \in K, \quad \langle \dot w(t), v - w(t)\rangle + \int_\Omega \nabla w(t)\cdot\nabla(v - w(t))\,dx + J^0(w(t); v - w(t)) \\ \ge \int_\Omega f(t)(v - w(t))\,dx \quad \forall\, v \in K, \end{aligned} \tag{4.7}$$

Applying Theorem 2.6 and recalling the last paragraph of Remark 2.3, we can conclude that under the stated assumptions on f and j, the problem defined by (4.6) and (4.7) admits a unique solution $w \in \mathcal{W}$. Since [6]

$$J^0(u; v) \le \int_{\Gamma_N} j^0(u; v)\, ds \quad \forall\, u, v \in V, \tag{4.8}$$

the solution w satisfies (4.4) and (4.5), i.e., $w \in \mathcal{W}$ is a solution of the problem defined by (4.4) and (4.5). The solution uniqueness can be verified in a standard way. Thus, the problem defined by (4.4) and (4.5) has a unique solution $u \in \mathcal{W}$.

Given a positive integer N, let $k = T/N$ be the time-step, and let $t_n = nk$, $0 \le n \le N$, be the nodal points of the time interval $[0,T]$. For simplicity, we assume Ω is a polyhedral/polygonal domain, and express Γ_N as the union of flat components $\Gamma_{N,i}$ for $1 \le i \le i_N$:

$$\Gamma_N = \cup_{i=1}^{i_N} \Gamma_{N,i}.$$

Let $\{\mathcal{T}^h\}_{h>0}$ be a regular family of finite element partitions of $\overline{\Omega}$ such that each partition $\mathcal{T}^h$ is compatible with the splitting of the boundary into Γ_D and Γ_N, i.e., if the intersection of a face (in 3D) or a side (in 2D) of an element in $\mathcal{T}^h$ with Γ_D or Γ_N has a positive relative measure, then the face or the side lies entirely on a flat component of Γ_D or Γ_N. Here, $h > 0$ denotes the mesh-size parameter. Let $V^h \subset V$ be the linear finite element space corresponding to the partition $\mathcal{T}^h$ and let $K^h = V^h \cap K$. Then the numerical method is to find $u^{hk} = \{u_n^{hk}\}_{n=0}^N \subset K^h$ such that

$$u_0^{hk} = u_0^h, \tag{4.9}$$

and for $1 \le n \le N$,

$$\int_\Omega \frac{u_n^{hk} - u_{n-1}^{hk}}{k}\,(v^h - u_n^{hk})\,dx + \int_\Omega \nabla u_n^{hk}\cdot\nabla(v^h - u_n^{hk})\,dx + \int_{\Gamma_N} j^0(u_n^{hk}; v^h - u_n^{hk})\,ds$$
$$\ge \int_\Omega f_n(v^h - u_n^{hk})\,dx \quad \forall\, v^h \in K^h, \tag{4.10}$$

where f_n stands for the function $f(\boldsymbol{x}, t_n)$.

For error estimation of the numerical solution, we assume the solution regularity

$$u \in C([0,T]; H^2(\Omega)) \cap H^1(0,T; H^1(\Omega)) \cap H^2(0,T; L^2(\Omega)), \tag{4.11}$$

$$u|_{\Gamma_{N,i}} \in C([0,T]; H^2(\Gamma_{N,i})), \quad 1 \le i \le i_N. \tag{4.12}$$

With the same trick used in proving the solution existence for the problem defined by (4.4) and (4.5), based on applications of the inequality (4.8), it can be seen that the Céa's type inequality (3.19) holds for the numerical solution defined by (4.9) and (4.10), where in the definition of the residual type quantity R_n, the expression J^0 is replaced by $\int_{\Gamma_N} j^0$. As a preparation in applying (3.19) for error estimation, we first derive some relations for the solution u of the problem (4.5), under the regularity assumption (4.11). Let v in (4.5) to be of the form $u(t) \pm v$ with $v \in C_0^\infty(\Omega)$ be arbitrary to obtain

$$\int_\Omega (\dot u(t)\,v + \nabla u(t)\cdot\nabla v)\,dx = \int_\Omega f(t)\,v\,dx \quad \forall\, v \in C_0^\infty(\Omega).$$

By an integration by parts,

$$\int_\Omega (\dot u(t)\,v - \Delta u(t)\,v)\,dx = \int_\Omega f(t)\,v\,dx \quad \forall\, v \in C_0^\infty(\Omega).$$

Therefore,

$$\dot u(t) - \Delta u(t) = f(t) \quad \text{a.e. in } \Omega,\ t \in (0,T). \tag{4.13}$$

Consider the residual term, for $v \in V$,

$$R_n(v) = \int_\Omega [\dot u_n(v - u_n) + \nabla u_n\cdot\nabla(v - u_n) - f_n(v - u_n)]\,dx + \int_{\Gamma_N} j^0(u_n; v - u_n)\,ds.$$

Perform an integration by parts and apply the boundary condition $v - u_n = 0$ on Γ_D,

$$R_n(v) = \int_\Omega (\dot u_n - \Delta u_n - f_n)(v - u_n)\,dx + \int_{\Gamma_N}\left[\frac{\partial u_n}{\partial \nu}(v - u_n) + j^0(u_n; v - u_n)\right] ds.$$

Making use of (4.13), we obtain

$$R_n(v) = \int_{\Gamma_N}\left[\frac{\partial u_n}{\partial \nu}(v - u_n) + j^0(u_n; v - u_n)\right] ds.$$

Since $u \in C([0,T]; H^2(\Omega))$, we have $\partial u/\partial\nu|_{\Gamma_N} \in C([0,T]; L^2(\Gamma_N))$ and

$$\left|\int_{\Gamma_N}\frac{\partial u_n}{\partial\nu}(v - u_n)\,ds\right| \le c\left\|\frac{\partial u}{\partial\nu}\right\|_{C([0,T];L^2(\Gamma_N))}\|v - u_n\|_{L^2(\Gamma_N)}.$$

Also,

$$\left|\int_{\Gamma_N} j^0(u_n; v - u_n)\,ds\right| \le \int_{\Gamma_N} c\,(1 + |u_n(\boldsymbol{x})|)\,|v(\boldsymbol{x}) - u_n(\boldsymbol{x})|\,ds$$
$$\le c\left(1 + \|u_n\|_{C([0,T];L^2(\Gamma_N))}\right)\|v - u_n\|_{L^2(\Gamma_N)}.$$

Thus,

$$\left|R_n(v_n^h)\right| \le c(u)\,\|v_n^h - u_n\|_{L^2(\Gamma_N)}. \tag{4.14}$$

Therefore, from (3.19), we have that for all $v_n^h \in K^h$, $1 \le n \le N$,

$$\begin{aligned}\max_n \|u_n - u_n^{hk}\|^2_{L^2(\Omega)} + k \sum_{n=1}^N \|u_n - u_n^{hk}\|^2_{H^1(\Omega)} &\le c\, k \sum_{n=1}^N \left[\|u_n - v_n^h\|^2_{H^1(\Omega)} + \|u_n - v_n^h\|_{L^2(\Gamma_N)} \right] \\ &\quad + c \left[k^2 + \|u_0 - u_0^h\|^2_{L^2(\Omega)} + \max_n \|u_n - v_n^h\|^2_{L^2(\Omega)} \right] \\ &\quad + c\, k^{-1} \sum_{n=1}^{N-1} \|(u_n - v_n^h) - (u_{n+1} - v_{n+1}^h)\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.15}$$

Now let us choose $v_n^h = \Pi^h u_n \in V^h$, the finite element interpolant of u_n. Then by the finite element interpolation error estimates [30–32] and recalling the solution regularity (4.11)–(4.12), we have

$$\|u_n - \Pi^h u_n\|^2_{H^1(\Omega)} \le c\, h^2 \|u_n\|^2_{H^2(\Omega)},$$
$$\|u_n - \Pi^h u_n\|^2_{L^2(\Omega)} \le c\, h^4 \|u_n\|^2_{H^2(\Omega)},$$
$$\|u_n - \Pi^h u_n\|_{L^2(\Gamma_N)} \le c\, h^2 \left(\sum_{i=1}^{i_N} \|u_n\|^2_{H^2(\Gamma_{N,i})} \right)^{1/2}.$$

Moreover,

$$(u_n - \Pi^h u_n) - (u_{n+1} - \Pi^h u_{n+1}) = \left(I - \Pi^h\right)(u_n - u_{n+1}),$$
$$\|(u_n - \Pi^h u_n) - (u_{n+1} - \Pi^h u_{n+1})\|^2_{L^2(\Omega)} \le c\, h^2 \|u_n - u_{n+1}\|^2_{H^1(\Omega)}.$$

Now

$$u_n - u_{n+1} = -\int_{t_n}^{t_{n+1}} \dot u(t)\, dt,$$
$$\|u_n - u_{n+1}\|^2_{H^1(\Omega)} \le k \int_{t_n}^{t_{n+1}} \|\dot u(t)\|^2_{H^1(\Omega)} dt.$$

Then,

$$k^{-1} \sum_{n=1}^{N-1} \|(u_n - \Pi^h u_n) - (u_{n+1} - \Pi^h u_{n+1})\|^2_{L^2(\Omega)} \le c\, h^2 \|\dot u\|^2_{L^2(0,T;H^1(\Omega))}.$$

Therefore, from (4.15),

$$\max_n \|u_n - u_n^{hk}\|^2_{L^2(\Omega)} + k \sum_{n=1}^N \|u_n - u_n^{hk}\|^2_{H^1(\Omega)} \le c \left(k^2 + h^2\right). \tag{4.16}$$

5. Numerical examples

In this section, we report computer simulation results on two numerical examples. In the first numerical example, there is no constraint on the value of the unknown solution, i.e., the inequality problem is posed over an entire space. In the second numerical example, there is constraint on the range of the values of the unknown solution. The focus of the numerical examples is to provide numerical evidence of the first order convergence in both h and k.

In the numerical examples, we take $\Omega = (0, 1) \times (0, 1)$, $\Gamma_N = \{0\} \times (0, 1)$ and $\Gamma_D = \partial\Omega \backslash \Gamma_N$. We use uniform triangulations on the domain $\overline{\Omega}$ such that the unit interval [0, 1] on each of the four sides of Ω is split into $1/h$ equal sub-intervals, cf. Fig. 1.

The corresponding finite element space V^h is constructed from the standard continuous piecewise linear functions. Let $u_{ref} = u^{h_0,k_0}$ be a reference solution in computing numerical solution errors; choose, e.g., $h_0 = 1/512$ and $k_0 = 1/1024$. Then the numerical convergence orders in h are computed from the errors $\|u_{ref} - u^{h,k_0}\|_{H^1(\Omega)}$ for $h = 1/8, 1/16, \ldots, 1/128$, whereas the numerical convergence orders in k are computed from the errors $\|u_{ref} - u^{h_0,k}\|_{H^1(\Omega)}$ for $k = 1/16, 1/32, \ldots, 1/256$.

5.1. Numerical example without constraint

We consider the following problem

$$\dot u - \Delta u = f \quad \text{in } \Omega \times (0, T] \tag{5.1}$$

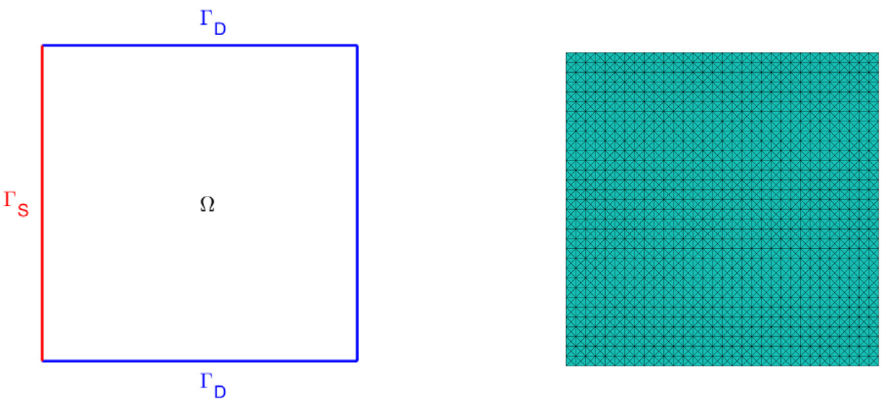

Fig. 1. Domain Ω and its uniform triangulation with $h = 1/32$.

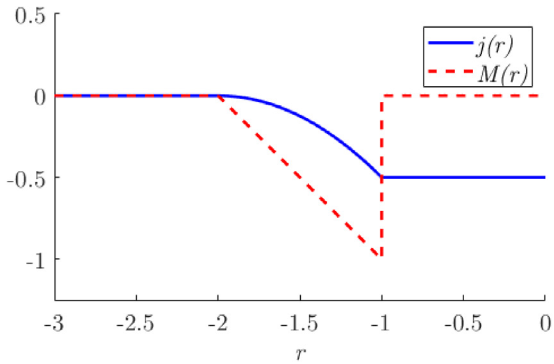

Fig. 2. M and j for the numerical example without constraint.

$$u = 0 \quad \text{on } \Gamma_D \times [0, T] \tag{5.2}$$

$$-\frac{\partial u}{\partial \nu} \in M(u) \quad \text{on } \Gamma_N \times [0, T] \tag{5.3}$$

$$u = 0 \quad \text{in } \Omega, \tag{5.4}$$

where $T = 0.5$, $f(\boldsymbol{x}, t) = -30(1 - te^{-t})$ and the multivalued function $M : \mathbb{R} \to 2^{\mathbb{R}}$ is given by

$$M(r) = \begin{cases} 0 & \text{for } r \in (-\infty, -2] \\ -2 - r & \text{for } r \in (-2, -1) \\ [-1, 0] & \text{for } r = -1 \\ 0 & \text{for } r \in (-1, +\infty). \end{cases}$$

Note that $M(r) = \partial j(r)$ and (see Fig. 2)

$$j(r) = \begin{cases} 0 & \text{for } r \le -2, \\ -\frac{1}{2} r^2 - 2r - 2 & \text{for } -2 < r < -1, \\ -\frac{1}{2} & \text{for } r \ge -1. \end{cases}$$

An iterative scheme with Lagrangian multiplier (see [24]) is used to find the solutions at each time step. We report the reference solution at the moments $t = \frac{3}{64}, \frac{6}{64}, \frac{9}{64}, \frac{12}{64}, \frac{16}{64}, \frac{20}{64}, \frac{23}{64}, \frac{26}{64}, \frac{29}{64}, \frac{32}{64}$ in Figs. 3–4, and the corresponding Lagrangian multipliers at these moments in Fig. 5.

In order to show the numerical convergence order with respect to mesh size, we take $h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$ and fix time step $k = k_0 = \frac{1}{1024}$. The errors of these numerical solutions, i.e.

$$\left(k_0 \sum_{n=1}^{N} \| u_n^{h,k_0} - u_n^{h_0,k_0} \|_{H^1(\Omega)}^2 \right)^{1/2},$$

are reported in Table 1, from which the first order convergence in h is observed.

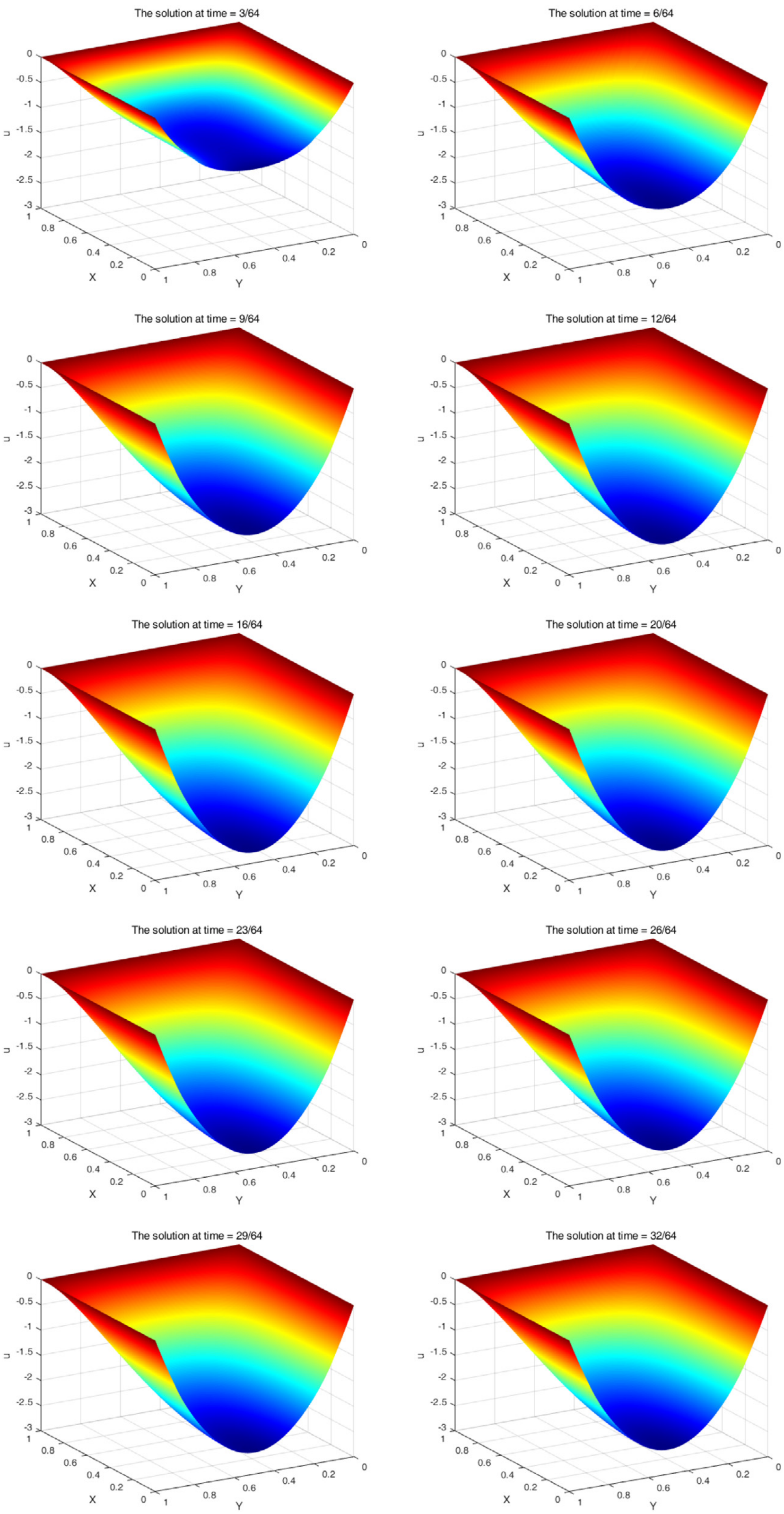

Fig. 3. Reference solution at the moments $t = \frac{3}{64}, \frac{6}{64}, \frac{9}{64}, \frac{12}{64}, \frac{16}{64}, \frac{20}{64}, \frac{23}{64}, \frac{26}{64}, \frac{29}{64}, \frac{32}{64}$.

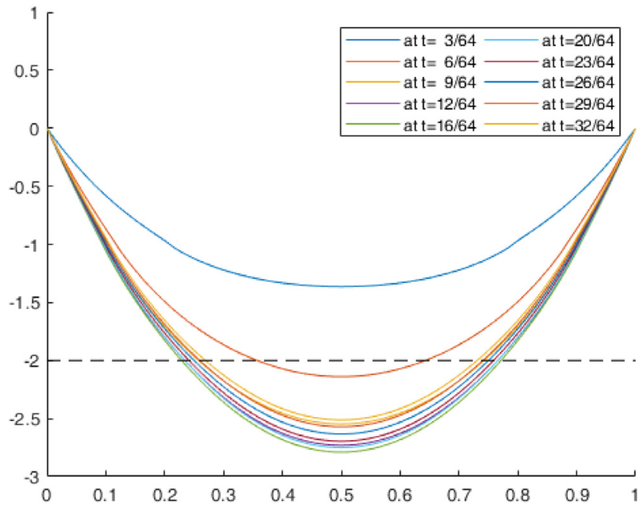

Fig. 4. Reference solution on Γ_N at the moments $t = \frac{3}{64}, \frac{6}{64}, \frac{9}{64}, \frac{12}{64}, \frac{16}{64}, \frac{20}{64}, \frac{23}{64}, \frac{26}{64}, \frac{29}{64}, \frac{32}{64}$.

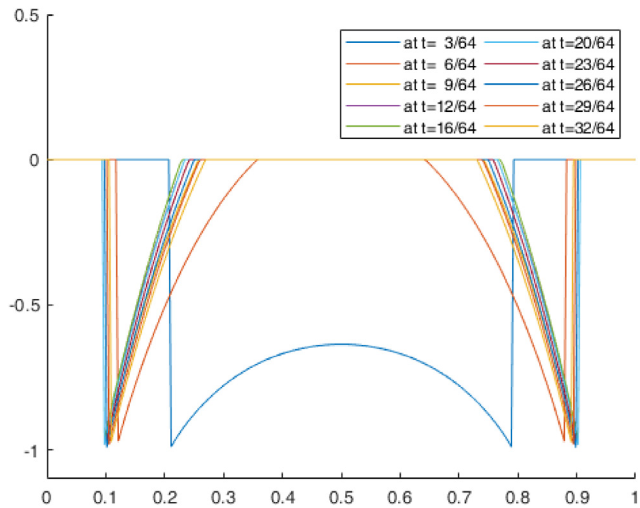

Fig. 5. Lagrangian multiplier on Γ_N at the moments $t = \frac{3}{64}, \frac{6}{64}, \frac{9}{64}, \frac{12}{64}, \frac{16}{64}, \frac{20}{64}, \frac{23}{64}, \frac{26}{64}, \frac{29}{64}, \frac{32}{64}$.

Table 1
Convergence order with respect to mesh size h.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
Error	1.6494e−01	8.3162e−02	4.2787e−02	2.1346e−02	1.0716e−02
conv. order	–	0.99	0.96	1.00	0.99

Table 2
Convergence order with respect to time step k.

k	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
Error	3.4185e−01	1.9290e−01	9.9814e−02	4.8450e−02	2.1772e−02
conv. order	–	0.83	0.95	1.04	1.15

In order to show the convergence order with respect to time step, we use the finest mesh with size $h = h_0 = \frac{1}{512}$ and take the time step $k = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$. The errors of these numerical solutions i.e.

$$\left(k \sum_{n=1}^{N} \| u_n^{h_0,k} - u_n^{h_0,k_0} \|_{H^1(\Omega)}^2 \right)^{1/2},$$

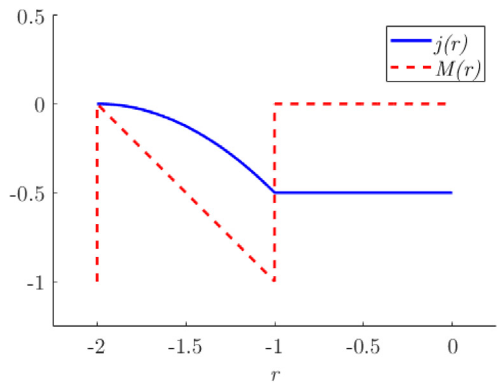

Fig. 6. M and j for the numerical example with constraint.

Table 3
Convergence order with respect to mesh size h.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
Error	1.7051e−01	8.5912e−02	4.2863e−02	2.1259e−02	1.0707e−02
conv. order	–	0.99	1.00	1.01	0.99

Table 4
Convergence order with respect to time step k.

k	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
Error	3.4076e−01	1.8692e−01	9.9067e−02	4.8670e−02	2.1540e−02
conv. order	–	0.87	0.92	1.03	1.18

are reported in Table 2. From this table, we can see the convergence with respect to the time-step k is of first order, which confirms the results obtained in our analysis.

5.2. Numerical example with constraint

The settings of this example are the same as those in Section 5.1, except that the multivalued function $M : \mathbb{R} \to 2^{\mathbb{R}}$ is given by (see Fig. 6)

$$M(r) = \begin{cases} \emptyset & \text{for } r \in (-\infty, -2), \\ (-\infty, 0] & \text{for } r = -2, \\ -2 - r & \text{for } r \in (-2, -1), \\ [-1, 0] & \text{for } r = -1, \\ 0 & \text{for } r \in (-1, +\infty). \end{cases} \tag{5.5}$$

Note that $M(r) = \partial j(r) + \partial \phi_I(r)$, where the locally Lipschitz continuous function j is the same as in the previous example, $I = [-2, \infty)$, and the indicator function is

$$\phi_I(r) = \begin{cases} 0 & \text{if } r \geq -2, \\ +\infty & \text{otherwise.} \end{cases}$$

A similar iterative scheme is used to obtain the numerical solutions as that used in Section 5.1. We report the reference solution at the moments $t = \frac{3}{64}, \frac{6}{64}, \frac{9}{64}, \frac{12}{64}, \frac{16}{64}, \frac{20}{64}, \frac{23}{64}, \frac{26}{64}, \frac{29}{64}, \frac{32}{64}$ in Figs. 7–8, and the corresponding Lagrangian multipliers in Fig. 9.

In order to show the convergence rate with respect to mesh size, we take $h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$ and fix time step $k = k_0 = \frac{1}{1024}$. The errors of these numerical solutions are reported in Table 3, from which the first order convergence in h is observed.

In order to show the convergence order with respect to the time step, we use the finest mesh with size $h = \frac{1}{512}$ and take $k = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$. The errors of these numerical solutions are reported in Table 4. From this table, we can see the convergence order with respect to the time-step is about 1.

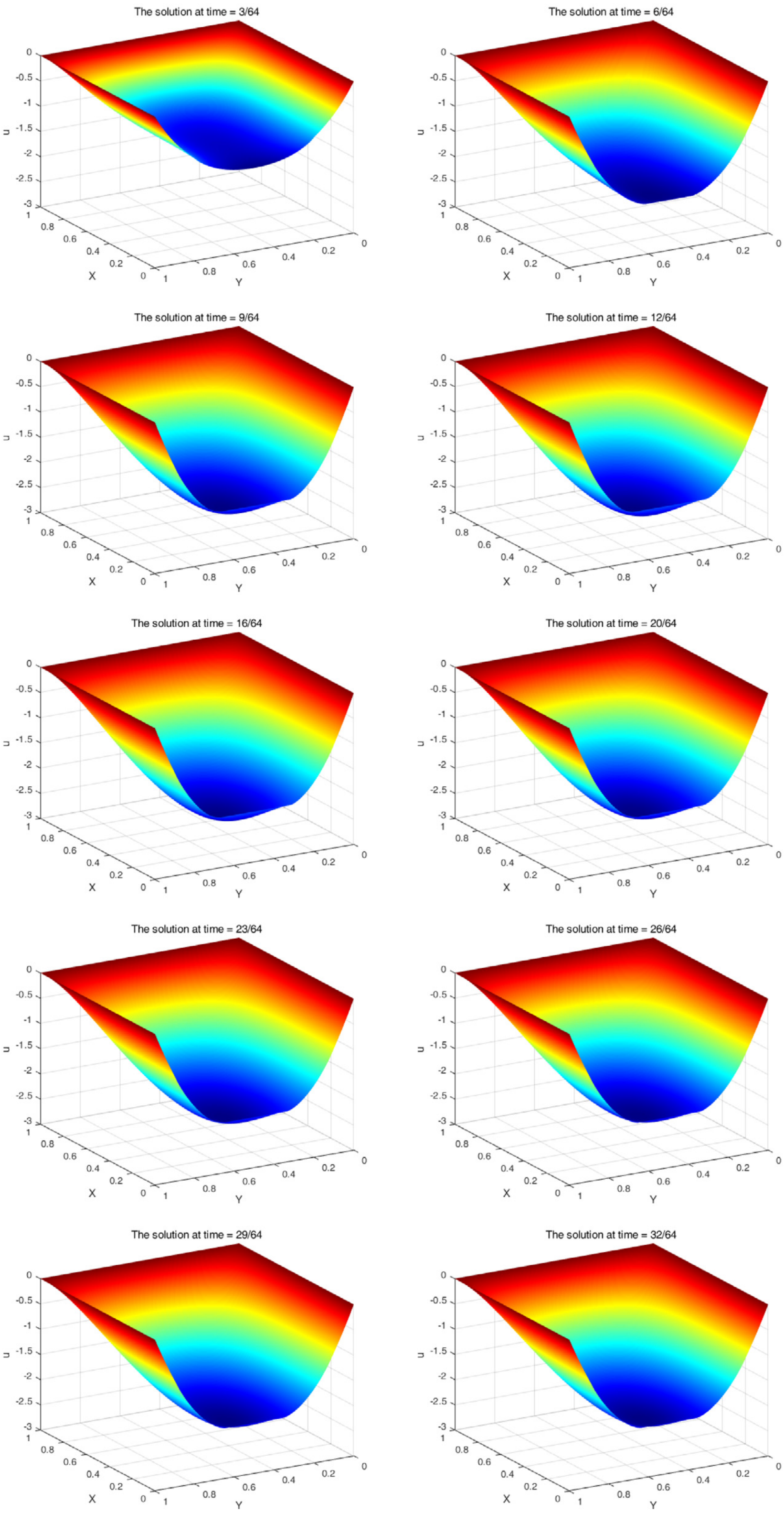

Fig. 7. Reference solution at the moments $t = \frac{3}{64}, \frac{6}{64}, \frac{9}{64}, \frac{12}{64}, \frac{16}{64}, \frac{20}{64}, \frac{23}{64}, \frac{26}{64}, \frac{29}{64}, \frac{32}{64}$.

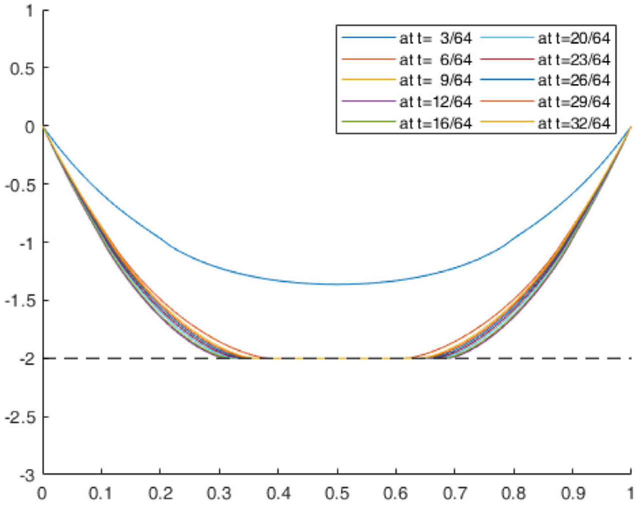

Fig. 8. Reference solution on Γ_N at the moments $t = \frac{3}{64}, \frac{6}{64}, \frac{9}{64}, \frac{12}{64}, \frac{16}{64}, \frac{20}{64}, \frac{23}{64}, \frac{26}{64}, \frac{29}{64}, \frac{32}{64}$.

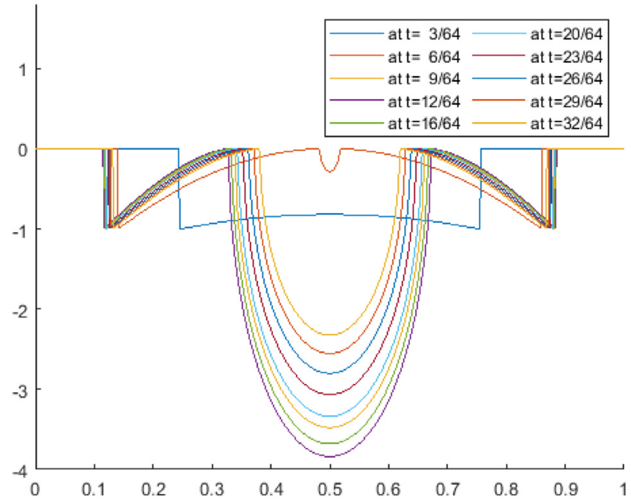

Fig. 9. Lagrangian multiplier on Γ_N at the moments $t = \frac{3}{64}, \frac{6}{64}, \frac{9}{64}, \frac{12}{64}, \frac{16}{64}, \frac{20}{64}, \frac{23}{64}, \frac{26}{64}, \frac{29}{64}, \frac{32}{64}$.

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