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A new class of fractional differential hemivariational inequalities with application to an incompressible Navier–Stokes system coupled with a fractional diffusion equation

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Abstract. This paper is devoted to the study of a new and complicated dynamical system, called a fractional differential hemivariational inequality, which consists of a quasilinear evolution equation involving the fractional Caputo derivative operator and a coupled generalized parabolic hemivariational inequality. Under certain general assumptions, existence and regularity of a mild solution to the dynamical system are established by employing a surjectivity result for weakly–weakly upper semicontinuous multivalued mappings, and a feedback iterative technique together with a temporally semi-discrete approach through the backward Euler difference scheme with quasi-uniform time-steps. To illustrate the applicability of the abstract results, we consider a nonstationary and incompressible Navier–Stokes system supplemented by a fractional reaction–diffusion equation, which is studied as a fractional hemivariational inequality.

Keywords: fractional differential hemivariational inequality, Clarke subgradient, C_0-semigroup, existence, Navier–Stokes system.

§ 1. Introduction

In 2008, Pang–Stewart [1] introduced and systematically studied a new kind of coupled dynamical systems on finite-dimensional spaces, called differential variational inequalities (DVIs, for short), which are formulated as a combination of (partial) differential equations and time-dependent variational inequalities. It was shown that DVIs can serve as a powerful and useful mathematical tool to model and solve a variety of problems in engineering areas, such as dynamic vehicle

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routing problems, electrical circuits with ideal diodes, Coulomb frictional problems for bodies in contact, economical dynamics, dynamic traffic networks, and so on. Since then, more and more researchers have been attracted to investigate both theoretical and numerical aspects of DVIs as well as their applications in economical dynamical systems and contact problems in mechanics. Let us mention a few representative results in this area. Liu–Motreanu–Zeng [2]–[4] and Liu–Migórski–Zeng [5] used the theory of semigroups, the Minty trick, the Filippov implicit function lemma and fixed point theorems for condensing multivalued operators to examine the existence of solutions and compactness of the solution set to a class of mixed differential variational inequalities in Banach spaces. By using Hamiltonian and Bellman's principle of optimality, Chen–Wang [6] formulated a dynamic Nash equilibrium problem of multiple players with shared constraints and dynamic decision processes (NEPSC, for short) as a differential variational inequality, and developed a regularized smoothing method to find a solution of dynamic NEPSC. Ke–Loi–Obukhovskii [7] introduced a new kind of fractional differential variational inequalities, and proved the existence of a delay solution to the inequality problem via a fixed point argument. Gwinner [8] used the monotonicity method of Browder and Minty, combined with the technique of Mosco convergence, to deliver a stability result for a class of differential variational inequalities. Employing topological methods from the theory of multivalued maps and some versions of the method of guiding functions, Liu–Loi–Obukhovskii [9] proved an existence theorem for periodic solutions of a global bifurcation problem described by a differential variational inequality in finite-dimensional spaces. For other results on DVIs the reader is referred to [10]–[21] and the references therein.

On the other hand, hemivariational inequalities were introduced by Panagiotopoulos [22], [23] in early 1980s to study engineering problems involving nonsmooth, nonmonotone and possibly multivalued constitutive relations and boundary conditions for deformable bodies. Since multivalued and nonmonotone constitutive laws appear often in applications, recently, Liu–Zeng–Motreanu [24] introduced the new notion of a differential hemivariational inequality (DHVI, for short). DHVI is a valuable and efficient mathematical modeling tool to explore the nonsmooth contact problems in mechanics, semipermeability problems, abnormal diffusion phenomena, etc. For example, Migórski–Zeng [25] used the idea of DHVIs to examine a dynamic adhesive viscoelastic contact problem with friction and nonlinear Kelvin–Voigt viscoelastic constitutive law. More recently, Zeng–Liu–Migórski [26] combined the Rothe method with the surjectivity of multivalued pseudomonotone operators and properties of the Clarke generalized subgradient to establish the existence of solutions to a class of fractional differential hemivariational inequalities in Banach spaces, and applied their abstract results to study a frictional quasistatic contact problem for viscoelastic materials with adhesion. For more details on these topics, we refer to [27]–[35] and the references therein.

Let $\alpha \in (0,1)$ be fixed, H a Hilbert space with the inner product $(\cdot\,,\cdot)_H$, and E and V Banach spaces such that (V,H,V^*) becomes an evolution triple of spaces. Given $z_0 \in E$ and $w_0 \in V$, we formulate the following fractional differential hemivariational inequality of parabolic–parabolic type.

Problem 1.1. Find functions $w\colon [0,T]\to V$ and $z\colon [0,T]\to E$ such that

$$ {}_0^C D_t^\alpha z(t) = Sz(t) + g(t,z(t),w(t)) \quad \text{for a.e. } t\in[0,T], \tag{1.1}$$

$$ w\in \mathrm{SOL}(V,F,J,G)(z), \tag{1.2}$$

$$ z(0)=z_0, \tag{1.3}$$

where ${}_0^C D_t^\alpha$ stands for the fractional derivative operator in the sense of Caputo (see Definition 2.1) and $\mathrm{SOL}(V,F,J,G)(z)$ is the solution set of the generalized parabolic hemivariational inequality: given a function $z\colon [0,T]\to E$, find $w\colon [0,T]\to V$ such that

$$ (w'(t),v)_H + \langle F(z(t),w(t)),v\rangle + J^0(z(t),\gamma w(t);\gamma v) \geqslant \langle G(t,z(t)),v\rangle \tag{1.4}$$

for all $v\in V$ and a.e. $t\in[0,T]$, and

$$ w(0)=w_0. \tag{1.5}$$

Unlike the existing literature on differential hemivariational inequalities, in the present paper, the subsystem (1.4) is of parabolic type rather than elliptic.

The goal of the present work is twofold. The first aim is to establish the existence of a mild solution to Problem 1.1, based on a surjectivity result for weakly–weakly upper semicontinuous multivalued mappings, and a feedback iterative technique together with a temporally semi-discrete approach based on the backward Euler difference scheme with quasi-uniform time-steps. The second aim of the paper is to study, through our theoretical findings, a nonstationary and incompressible Navier–Stokes system described by a fractional reaction-diffusion equation.

The rest of the paper is organized as follows. In Section 2 we survey preliminary material needed later in the paper. In Section 3 we deliver an existence theorem of mild solutions to Problem 1.1 by using a surjectivity theorem and a feedback iterative approximation method. To illustrate the applicability of our results, in Section 4 we investigate a nonstationary Navier–Stokes equation with nonmonotone and multivalued frictional boundary condition driven by a fractional reaction–diffusion equation.

§ 2. Mathematical background

In this section we review preliminary materials needed later in the paper. More details can be found in [36]–[42].

Throughout the paper, the symbols "$\rightharpoonup$" and "$\to$" stand for the weak and the strong convergences in various spaces, respectively. Let X be a Banach space with its dual space X^*. A single-valued mapping $A\colon X\to X^*$ is said to be *weakly continuous* if $Au_n\rightharpoonup Au$ in X^* as $n\to\infty$ whenever the sequence $\{u_n\}$ is such that $u_n\rightharpoonup u$ in X as $n\to\infty$ for some $u\in X$.

Definition 2.1. Let E be a Banach space, $\alpha\in(0,1)$ and $0<T<+\infty$.

(i) The Riemann–Liouville fractional integral of order α of $f\colon[0,T]\to E$ is defined by

$$ {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds \quad \text{for a.e. } t\in[0,T],$$

where Γ is the Gamma function, $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}\,dt$.

(ii) The Caputo fractional derivative of order α of $f\colon [0,T] \to E$ is defined by

$$ {}_0^C D_t^\alpha f(t) = {}_0 I_t^{1-\alpha} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds \quad \text{for a.e. } t \in [0,T]. $$

Let E be a reflexive Banach space and $S\colon D(S) \subset E \to E$ be the infinitesimal generator of a C_0-semigroup $\{\mathscr{S}(t)\}_{t\geqslant 0}$ in E. Given two constants $0 < T < +\infty$ and $\alpha \in (0,1)$, a function $f\colon [0,T] \times E \to E$ and an element $x_0 \in E$, we say that a function $x \in C([0,T];E)$ is a *mild solution* to the following fractional Cauchy problem (cf. e.g. [43])

$$ \begin{cases} {}_0^C D_t^\alpha x(t) = Sx(t) + f(t,x(t)) & \text{for a.e. } t \in [0,T], \\ x(0) = x_0, \end{cases} $$

if

$$ x(t) = \mathscr{E}(t)x_0 + \int_0^t (t-s)^{\alpha-1} \mathscr{F}(t-s) f(s,x(s))\, ds \quad \text{for all } t \in [0,T], $$

where $\mathscr{E}(t)$ and $\mathscr{F}(t)$ are defined by

$$ \mathscr{E}(t) = \int_0^\infty \xi_\alpha(\theta) \mathscr{S}(t^\alpha \theta)\, d\theta, \qquad \mathscr{F}(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) \mathscr{S}(t^\alpha\theta)\, d\theta, \tag{2.1} $$

with

$$ \begin{aligned} \xi_\alpha(\theta) &= \frac{1}{\alpha} \theta^{-1-1/\alpha} \omega_\alpha(\theta^{-1/\alpha}), \\ \omega_\alpha(\theta) &= \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-\alpha n - 1} \frac{\Gamma(\alpha n + 1)}{n!} \sin(n\pi\alpha), \qquad \theta \in (0,\infty). \end{aligned} $$

It can be shown that $\xi_\alpha(\theta) \geqslant 0$ for $\theta \in (0,+\infty)$ with $\int_0^\infty \xi_\alpha(\theta)\, d\theta = 1$ ([44] or [45], p. 4467). This means that ξ_α is a probability density function on $(0,\infty)$. Besides ([43], Remark 2.8),

$$ \int_0^\infty \theta^\beta \xi_\alpha(\theta)\, d\theta = \int_0^\infty \theta^{-\alpha\beta} \omega_\alpha(\theta)\, d\theta = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} \quad \text{for } \beta \in [0,1]. $$

For functions $\mathscr{E}$ and $\mathscr{F}$, we have the following result ([43], Lemma 2.9).

Lemma 2.2. *Let $S\colon D(S) \subset E \to E$ be the infinitesimal generator of a C_0-semigroup $\{\mathscr{S}(t)\}_{t\geqslant 0}$ in a Banach space E such that $\|\mathscr{S}(t)\| \leqslant M_S$ for all $t \geqslant 0$, where $M_S > 0$ is a constant. Then the functions $\mathscr{E}$ and $\mathscr{F}$ defined in* (2.1) *have the following properties.*

(i) *For any $t \geqslant 0$, $\mathscr{E}(t)$ and $\mathscr{F}(t)$ are both linear such that $\|\mathscr{E}(t)\| \leqslant M_S$ and $\|\mathscr{F}(t)\| \leqslant M_S/\Gamma(\alpha)$.*

(ii) *$\{\mathscr{E}(t)\}_{t\geqslant 0}$ and $\{\mathscr{F}(t)\}_{t\geqslant 0}$ are both strongly continuous.*

(iii) *For each $t > 0$, $\mathscr{E}(t)$ and $\mathscr{F}(t)$ are also compact provided that $\mathscr{S}(t)$ is compact.*

Next, we recall a convergence theorem ([41], Theorem 3.13), which provides a powerful tool in the study of evolutionary inclusion problems.

Theorem 2.3. *Let $0 < T < \infty$ and $F \colon X \to 2^Y$ be an upper semicontinuous multivalued mapping from a Hausdorff locally convex space X to the closed convex subsets of a Banach space Y endowed with the weak topology. Let $\{x_n\}$ and $\{y_n\}$ be sequences of functions such that*

(i) *$x_n \colon [0,T] \to X$ and $y_n \colon [0,T] \to Y$ are measurable functions for all $n \in \mathbb{N}$;*

(ii) *for a.e. $t \in (0,T)$ and for every neighborhood $\mathcal{N}(0)$ of 0 in $X \times Y$, there exists an $n_0 \in \mathbb{N}$ such that $(x_n(t), y_n(t)) \in \mathrm{Gr}(F) + \mathcal{N}(0)$ for all $n \geqslant n_0$;*

(iii) *there is a function $x \colon [0,T] \to X$ such that $x_n(t) \to x(t)$ for a.e. $t \in (0,T)$;*

(iv) *$y_n \in L^1(0,T;Y)$ and $y_n \to y$ weakly in $L^1(0,T;Y)$, for some $y \in L^1(0,T;Y)$.*

Then $(x(t), y(t)) \in \mathrm{Gr}(F)$, i.e., $y(t) \in F(x(t))$ for a.e. $t \in (0,T)$.

Let X be a Banach space. A function $J \colon X \to \mathbb{R}$ is said to be *locally Lipschitz* at $u \in X$ if there exist a neighborhood $N(u)$ of u in X and a constant $L_u > 0$, depending on the neighborhood $N(u)$, such that

$$|J(w) - J(v)| \leqslant L_u \|w - v\|_X \quad \text{for all } w, v \in N(u).$$

Definition 2.4 ([37]). Given a locally Lipschitz function $J \colon X \to \mathbb{R}$, we denote by $J^0(u;v)$ the generalized directional derivative of J at the point $u \in X$ in the direction $v \in X$ defined by

$$J^0(u;v) = \limsup_{\lambda \to 0^+,\, w \to u} \frac{J(w + \lambda v) - J(w)}{\lambda}.$$

The generalized gradient of $J \colon X \to \mathbb{R}$ at $u \in X$ is given by

$$\partial J(u) = \{\xi \in X^* \mid J^0(u;v) \geqslant \langle \xi, v \rangle \text{ for all } v \in X\}.$$

Basic properties of the generalized directional derivative and the generalized subgradient are collected in the result below, see e.g. [41], Proposition 3.23.

Proposition 2.5. *Assume that the function $J \colon X \to \mathbb{R}$ is locally Lipschitz. Then the following assertions hold.*

(i) *For every $x \in X$, the function $X \ni v \mapsto J^0(x;v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $J^0(x;\lambda v) = \lambda J^0(x;v)$ for all $\lambda \geqslant 0$, $v \in X$ and $J^0(x; v_1 + v_2) \leqslant J^0(x;v_1) + J^0(x;v_2)$ for all $v_1, v_2 \in X$.*

(ii) *For every $v \in X$ we have $J^0(x;v) = \max\{\langle \xi, v \rangle \mid \xi \in \partial J(x)\}$.*

(iii) *The function $X \times X \ni (u,v) \mapsto J^0(u;v) \in \mathbb{R}$ is upper semicontinuous.*

The next result is proved in [25], Lemma 7.

Lemma 2.6. *Let E and X be Banach spaces. Let $J \colon E \times X \to \mathbb{R}$ be such that*

(i) *for each $z \in E$, the function $u \mapsto J(z,u)$ is locally Lipschitz;*

(ii) *for each $v \in X$, the function $E \times X \ni (y,u) \mapsto J^0(y,u;v) \in \mathbb{R}$ is upper semicontinuous.*

Then the generalized gradient $E \times X \ni (y,u) \mapsto \partial J(y,u) \subset X^*$ is upper semicontinuous from $E \times X$ endowed with the norm topology to the subsets of X^* endowed with the weak topology.

Furthermore, we state a compactness result for an embedding between the Bochner–Lebesgue spaces. Its proof can be found in [46], Proposition 2.8. Let $0 < T < \infty$ and $I = [0,T]$. Let us denote by χ a finite partition of the interval I such that $[0,T] = \bigcup_{i=1}^{n} \overline{\eta}_i$, where $\{\eta_i\}_{1\leqslant i\leqslant n}$ is a family of disjoint subintervals $\eta_i = (l_i, r_i)$. Let $\mathscr{H}$ be the family of all such partitions, X be a Banach space and $1 \leqslant q < \infty$. We introduce the function space $\mathrm{BV}^q(0,T;X)$ defined by

$$\mathrm{BV}^q(0,T;X) = \biggl\{ v\colon [0,T] \to X \biggm| \sup_{\chi\in\mathscr{H}} \sum_{\eta_i\in\chi} \|v(r_i) - v(l_i)\|_X^q < \infty \biggr\}.$$

The seminorm on the space $\mathrm{BV}^q(0,T;X)$ is defined by

$$\|v\|_{\mathrm{BV}^q(0,T;X)}^q = \sup_{\chi\in\mathscr{H}} \sum_{\eta_i\in\chi} \|v(r_i) - v(l_i)\|_X^q \quad \text{for all } v \in \mathrm{BV}^q(0,T;X).$$

Suppose that $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$, and X, Z are Banach spaces such that the embedding of X to Z is continuous. Put $M^{p,q}(0,T;X,Z) = L^p(0,T;X) \cap \mathrm{BV}^q(0,T;Z)$. Then the space $M^{p,q}(0,T;X,Z)$ endowed with the norm

$$\|v\|_{M^{p,q}(0,T;X,Z)} = \|v\|_{L^p(0,T;X)} + \|v\|_{\mathrm{BV}^q(0,T;Z)} \quad \text{for all } v \in M^{p,q}(0,T;X,Z)$$

is a Banach space.

Proposition 2.7. *Let $1 \leqslant p,q < \infty$, and let $X \subset Y \subset Z$ be Banach spaces such that X is reflexive, the embedding $X \subset Y$ is compact, and the embedding $Y \subset Z$ is continuous. Then any bounded set in the space $M^{p,q}(0,T;X,Z)$ is relatively compact in $L^p(0,T;Y)$.*

We end the section by recalling a surjectivity result for weakly–weakly upper semicontinuous multivalued mappings (see [47], Theorem 8). This result will play an important role for solvability analysis of the discrete approximate problem, see Problem 3.3 in Section 3. Given a normed space X with dual X^*, a multivalued mapping $F\colon X \to 2^{X^*}$ is said to be *coercive* if

$$\lim_{\|u\|_X\to\infty} \frac{1}{\|u\|_X} \inf_{u^*\in F(u)} \langle u^*, u\rangle = \infty.$$

Theorem 2.8. *Let X be a reflexive Banach space with its dual X^* and $F\colon X \to 2^{X^*}$ be a multivalued mapping with nonempty, bounded, closed, and convex values. If F is coercive and upper semicontinuous from X endowed with the weak topology to the subsets of X^* endowed with the weak topology, then F is surjective.*

§ 3. Fractional differential hemivariational inequalities of parabolic–parabolic type

This section is devoted to exploring an abstract dynamical system called a fractional differential hemivariational inequality of parabolic–parabolic type, which consists of a quasilinear evolution equation involving the fractional Caputo derivative operator coupled with a generalized parabolic hemivariational inequality. Our goal is to establish the existence of a mild solution to the dynamical system. The proof method is based on a surjectivity result for weakly–weakly upper semicontinuous multivalued mappings, and a feedback iterative technique together with a temporally semi-discrete approach based on the backward Euler difference with quasi-uniform time-steps. It should be mentioned that the main results established in this section provide not only the existence of a mild solution, but also an iterative algorithm to obtain an approximation sequence for a mild solution.

The functional setting of the abstract dynamical system, Problem 1.1, is as follows. Let $V \hookrightarrow H \hookrightarrow V^*$ be an evolution triple of spaces (or Gelfand triple of spaces), i.e., $(V, \|\cdot\|_V)$ is a reflexive and separable Banach space with its dual space $(V^*, \|\cdot\|_{V^*})$ and $(H, \|\cdot\|_H)$ is a separable Hilbert space such that the embedding of V to H is continuous and V is dense in H. In addition, we need two reflexive Banach spaces $(E, \|\cdot\|_E)$, $(X, \|\cdot\|_X)$. Also, we suppose that the embedding of V to H is compact, and the operator $\gamma\colon V \to X$ is bounded, linear and compact. For $0 < T < +\infty$ fixed, we denote by $C([0,T];E)$ the space of continuous functions from $[0,T]$ into E, and introduce the following Bochner–Lebesgue spaces:

$$
\begin{aligned}
&\mathcal{V} := L^2(0,T;V), && \mathcal{H} := L^2(0,T;H), && \mathcal{V}^* := L^2(0,T;V^*),\\
&\mathcal{X} := L^2(0,T;X), && \mathcal{X}^* := L^2(0,T;X^*), && \mathcal{E} := L^2(0,T;E),\\
&\mathcal{E}^* := L^2(0,T;E^*), && \mathcal{W} := \{u \in \mathcal{V} \mid v' \in \mathcal{V}^*\},
\end{aligned}
$$

where $u' = \partial u/\partial t$ stands for the time derivative of u, which is understood in the sense of distributions. In what follows, we adopt the symbols $\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle_{X^* \times X}$, $\langle \cdot, \cdot \rangle_{\mathcal{V}^* \times \mathcal{V}}$, and $\mathcal{L}(V;X)$ for the duality pairing between V^* and V, that between X^* and X, that between $\mathcal{V}^*$ and $\mathcal{V}$, and the space of bounded linear operators from V to X, respectively. We set $L^p(0,T)_+ := \{u \in L^p(0,T) \mid u(t) \geqslant 0 \text{ for a.e. } t \in (0,T)\}$.

For convenience, in the rest of the paper, we denote by $C > 0$ a generic constant whose value may change from line to line.

The mild solutions of Problem 1.1 are understood as follows.

Definition 3.1. A pair of functions (z, w) with $z \in C([0,T];E)$ and $w \in \mathcal{W}$ is called a *mild solution of Problem* 1.1 if

$$
z(t) = \mathscr{E}(t) z_0 + \int_0^t (t-s)^{\alpha-1} \mathscr{F}(t-s)\, g(s, z(s), w(s))\, ds \quad \text{for all } t \in [0,T], \tag{3.1}
$$

$$
\begin{aligned}
&\int_0^T (w'(t), v(t))_H\, dt + \int_0^T J^0(z(t), \gamma w(t); \gamma v(t))\, dt + \int_0^T \langle F(z(t), w(t)), v(t) \rangle\, dt\\
&\qquad \geqslant \int_0^T \langle G(t, z(t)), v(t) \rangle\, dt \quad \text{for all } v \in \mathcal{V},
\end{aligned} \tag{3.2}
$$

$$
w(0) = w_0, \tag{3.3}
$$

where $\{\mathscr{E}(t)\}_{t \geqslant 0}$ and $\{\mathscr{F}(t)\}_{t \geqslant 0}$ are given in (2.1).

To explore the existence of a mild solution for Problem 1.1, we impose the following assumptions.

Assumption $H(S)$**.** $S\colon D(S) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $\{\mathscr{S}(t)\}_{t>0}$ in E such that $\mathscr{S}(t)$ is compact for each $t > 0$.

Assumption $H(F)$**.** The operator $F\colon E\times V\to V^*$ is such that

(i) for each $y\in E,\ u\mapsto F(y,u)$ is weakly continuous;

(ii) one of the following conditions holds

(ii)$_1$ there exists a constant $c_F>0$ such that

$$\|F(y,u)\|_{V^*}\leqslant c_F(1+\|u\|_V)\ \text{for all}\ u\in V\ \text{and}\ y\in E;\tag{3.4}$$

(ii)$_2$ there exists a constant $c_{\mathcal{F}}>0$ such that

$$\|\mathcal{F}(y,u)\|_{\mathcal{V}^*}\leqslant c_{\mathcal{F}}\|u\|_{\mathcal{V}}(1+\|u\|_{L^\infty(0,T;H)})\tag{3.5}$$

for all $u\in\mathcal{V}\cap L^\infty(0,T;H)$ and all $y\in\mathcal{E}$, where $\mathcal{F}\colon\mathcal{E}\times\mathcal{V}\to\mathcal{V}^*$ is the Nemytskii operator corresponding to F given by $\mathcal{F}(y,u)(t)=F(y(t),u(t))$ for all $t\in[0,T],\ u\in\mathcal{V}$ and $y\in\mathcal{E}$;

(iii) there are constants $m_F>0,\ l_F,d_F\geqslant 0$ and $e_F\in\mathbb{R}$ such that

$$\langle F(y,u),u\rangle\geqslant m_F\|u\|_V^2-l_F\|u\|_H^2-d_F\|u\|_V+e_F\quad\text{for all}\ u\in V\ \text{and}\ y\in E;$$

(iv) for any sequences $\{u_n\}\subset\mathcal{V}\cap L^\infty(0,T;H)$ and $\{y_n\}\subset\mathcal{E}$ such that $\{u_n\}$ is bounded in $L^\infty(0,T;H)$, $y_n\to y$ in $\mathcal{E}$ for some $y\in\mathcal{E}$, and $u_n\rightharpoonup u$ in $\mathcal{V}$ and $\mathcal{H}$ for some $u\in\mathcal{V}$, we have $\mathcal{F}(y_n,u_n)\rightharpoonup\mathcal{F}(y,u)$ in $\mathcal{V}^*$.

Assumption $H(J)$**.** $J\colon E\times X\to\mathbb{R}$ has the following properties:

(i) for each $y\in E$, the function $X\ni w\mapsto J(y,w)\in\mathbb{R}$ is locally Lipschitz;

(ii) for each $y\in E$, $w\in X$ and $\xi\in\partial J(y,w)$, we have $\|\xi\|_{X^*}\leqslant c_J+d_J\|w\|_X$ with $c_J\geqslant 0$ and $d_J>0$;

(iii) for each $v\in X$, the function $E\times X\ni(y,u)\mapsto J^0(y,u;v)\in\mathbb{R}$ is upper semicontinuous;

(iv) there is a $\theta\in[1,2)$ such that $J^0(y,w;-w)\leqslant L_J+M_J\|w\|_X^\theta$ for all $y\in E$ and $w\in X$ with $L_J\geqslant 0$ and $M_J>0$.

Assumption $H(\gamma)$**.** $\gamma\colon V\to X$ is bounded, linear, and compact such that its Nemytskii operator $\overline{\gamma}\colon\mathcal{V}\to\mathcal{X}$, defined by $(\overline{\gamma}w)(t)=\gamma(w(t))$ for all $t\in[0,T]$ and $w\in\mathcal{V}$, is compact from $M^{2,2}(0,T;V,V^*)$ to $\mathcal{X}$.

Assumption $H(G)$**.** $G\colon[0,T]\times E\to V^*$ is such that

(i) for each $z\in E$, the function $t\mapsto G(t,z)$ is measurable on $[0,T]$;

(ii) for every $t\in[0,T]$, the function $z\mapsto G(t,z)\in V^*$ is continuous;

(iii) there exists a function $\rho_G\in L^2(0,T)_+$ such that $\|G(t,z)\|_{V^*}\leqslant\rho_G(t)$ for all $(t,z)\in[0,T]\times E$.

Assumption $H(g)$**.** $g\colon[0,T]\times E\times H\to E$ is such that

(i) for each $(z,w)\in E\times H,\ t\mapsto g(t,z,w)$ is measurable on $[0,T]$;

(ii) for all $z\in E$ and a.e. $t\in[0,T]$, the function $H\ni w\mapsto g(t,z,w)$ is continuous;

(iii) there exists a function $\rho_g\in L^{1/\beta}(0,T)_+$ with $0<\beta<\alpha$ satisfying

$$M_S\|\rho_g\|_{L^{1/\beta}(0,T)}T^{(1+\zeta)(1-\beta)}<\Gamma(\alpha)(1+\zeta)^{1-\beta}$$

with $\zeta=(\alpha-1)/(1-\beta)$ and $\|g(t,z_1,w)-g(t,z_2,w)\|_E\leqslant\rho_g(t)\|z_1-z_2\|_E$, for all $z_1,z_2\in E,\ w\in H$ and a.e. $t\in[0,T]$, and for each bounded set D in H, we can find a function $\psi_D\in L^{1/\beta}(0,T)_+$ such that $\|g(t,0,w)\|_E\leqslant\psi_D(t)$ for all $w\in D$ and a.e. $t\in[0,T]$.

Remark 3.2. An operator $N\colon V \to V^*$ is said to be of *generalized Navier–Stokes type* (see e.g. [48]) if $N(v) = Av + B[v]$ for all $v \in V$, where

(i) $A \in L(V;V^*)$ is a symmetric operator satisfying
$\langle Av, v\rangle \geqslant \alpha_A\|v\|_V^2 - \beta_A\|v\|_H^2$ for all $v \in V$ with some constants $\alpha_A > 0$ and $\beta_A \geqslant 0$;

(ii) $B\colon V \times V \to V^*$, $B[v] := B(v,v)$ is a bilinear continuous function such that
$\langle B(u,v), v\rangle = 0$ for all $u, v \in V$, and $B[\,\cdot\,]\colon V \to V^*$ is weakly continuous.

Let $\mu\colon \mathbb{R}_+ \to (0,+\infty)$ be a continuous function such that $c_\mu \leqslant \mu(s) \leqslant d_\mu$ for all $s \in \mathbb{R}_+$ with two constants $d_\mu \geqslant c_\mu > 0$. It can be shown that the operator $F\colon E \times V \to V^*$ defined by $F(y,u) = \mu(e_\mu\|y\|_E)Au + B[u]$ satisfies the hypothesis $H(F)$ with $d_F = e_F = 0$ and $e_\mu > 0$, where $A(\,\cdot\,) + B[\,\cdot\,]$ is a generalized Navier–Stokes type operator; see Lemma 4.9 below.

Since $\{\mathscr{S}(t)\}_{t>0}$ is a C_0-semigroup, in what follows, let $M_S > 0$ be such that $\|\mathscr{S}(t)\| \leqslant M_S$ for all $t \geqslant 0$. We are now in a position to invoke an iterative approximation approach together with a temporally semi-discrete technique based on the backward Euler difference with a variable time step length for Problem 1.1. For $n \in \mathbb{N}$, we use the symbol $\mathcal{T}_n$ to denote a partition of the time interval $[0,T]$:

$$\mathcal{T}_n = \{0 = t_n^0 < t_n^1 < \cdots < t_n^n = T\}.$$

We write $\tau_n^k = t_n^k - t_n^{k-1}$, $1 \leqslant k \leqslant n$, for the lengths of the time subintervals. Note that the quantity $\max_{1\leqslant k\leqslant n} \tau_n^k$ represents the mesh-size of $\mathcal{T}_n$. We will assume the sequence of time grids $\{\mathcal{T}_n\}$ to be quasi-uniform.

Assumption $H(\mathcal{T})$**.** The sequence of time grids $\{\mathcal{T}_n\}$ satisfies two conditions:

(i) $\max_{1\leqslant k\leqslant n} \tau_n^k \to 0$ as $n \to \infty$;

(ii) there exists a constant $c_{\mathcal{T}} > 0$ such that $\max_{1\leqslant k\leqslant n} \tau_n^k \leqslant c_{\mathcal{T}} \min_{1\leqslant k\leqslant n} \tau_n^k$ for all $n \in \mathbb{N}$.

Given a sequence of time grids $\{\mathcal{T}_n\}$, we propose a feedback iterative approximated system corresponding to Problem 1.1 as follows.

Problem 3.3. Find sequences $\{w_n^k\}_{k=0}^n \subset V$, $\{\xi_n^k\}_{k=1}^n \subset X^*$, and a function $z_n \in C([0,T];E)$ such that $w_n^0 = w_0$, $z_n(0) = z_0$,

$$\frac{w_n^k - w_n^{k-1}}{\tau_n^k} + F(z_n(t_n^{k-1}), w_n^k) + \gamma^*\xi_n^k = G_n^k \quad \text{for } k = 1,\ldots,n \tag{3.6}$$

with $\xi_n^k \in \partial J(z_n(t_n^{k-1}), \gamma w_n^k)$, and

$$z_n(t) = \mathscr{E}(t)z_0 + \int_0^t (t-s)^{\alpha-1}\mathscr{F}(t-s)g(s, z_n(s), w_n(s))\,ds \quad \text{for all } t \in [0, t_n^k], \tag{3.7}$$

for $k = 1,\ldots,n$, where the element G_n^k and function $t \mapsto w_n(t)$ are defined by

$$\begin{aligned} G_n^k &= \begin{cases} G(0, z_0) & \text{if } k = 1, \\ \dfrac{1}{\tau_n^{k-1}} \displaystyle\int_{t_n^{k-2}}^{t_n^{k-1}} G(s, z_n(s))\,ds & \text{if } k \geqslant 2, \end{cases} \\ w_n(t) &= w_n^k + \frac{t - t_n^k}{\tau_n^k}(w_n^k - w_n^{k-1}) \end{aligned} \tag{3.8}$$

for all $t \in (t_n^{k-1}, t_n^k]$ and $k = 1,\ldots,n$, respectively.

The following lemma provides a basic result on the solvability of Problem 3.3.

Lemma 3.4. *Let $n \in \mathbb{N}$ be fixed. Assume that $H(S)$, $H(F)$, $H(J)$, $H(G)$, $H(g)$, $H(\gamma)$ and $H(\mathcal{T})$ are fulfilled. Then there exists an $n_0 \in \mathbb{N}$ such that Problem* 3.3 *has at least one solution for all $n \geqslant n_0$.*

Proof. Let $n_0 \in \mathbb{N}$ be a fixed integer such that $n_0 > c_{\mathcal{T}} l_F T$, and let $n \geqslant n_0$. Since $n \min_{1\leqslant k\leqslant n} \tau_n^k \leqslant T$, we have $\max_{1\leqslant k\leqslant n} \tau_n^k \leqslant c_{\mathcal{T}} T/n$ by the quasi-uniformity of $\mathcal{T}_n$. Hence, for $1 \leqslant k \leqslant n$,

$$\frac{1}{\tau_n^k} \geqslant \frac{n}{c_{\mathcal{T}} T} \geqslant \frac{n_0}{c_{\mathcal{T}} T} > l_F.$$

For every fixed $n \geqslant n_0$, we prove the result by induction on k.

First, for $k = 1$, $G_n^1 = G(0, z_0)$. Let us introduce a set-valued mapping $\mathcal{G}_1\colon V \to 2^{V^*}$ defined by

$$\mathcal{G}_1(w) = \frac{i^* i}{\tau_n^1} w + F(z_n(t_0), w) + \gamma^* \partial J(z_n(t_0), \gamma w) \quad \text{for all } w \in V, \tag{3.9}$$

where $i\colon V \to H$ denotes the embedding operator from V to H with its dual operator $i^*\colon H \to V^*$. Observe that if we are able to show that $\mathcal{G}_1$ is onto V^*, then there exist $w \in V$ and $\xi_n^1 \in \partial J(z_n(t_0), \gamma w)$ satisfying (3.6) with $k = 1$. Hence, we will apply the surjectivity result of Theorem 2.8 to $\mathcal{G}_1$. This requires verification of all the assumptions stated in Theorem 2.8, i.e., $\mathcal{G}_1$ is a coercive and weakly–weakly upper semicontinuous multivalued mapping with nonempty, bounded, closed, and convex values in V^*. By the hypotheses $H(J)$ and [41], Proposition 3.23, (iv), we know that for each $w \in V$ fixed, the set $\gamma^* \partial J(z_0, \gamma w)$ is nonempty, bounded, closed, and convex in V^*. Moreover, Lemma 2.6 shows that the multivalued mapping $X \ni x \mapsto \partial J(z_0, x) \subset X^*$ is strongly–weakly upper semicontinuous. The latter combined with the compactness of γ and [49], Theorem 1.2.8, shows that $w \mapsto \gamma^* \partial J(z_0, \gamma w)$ is weakly–weakly upper semicontinuous. In addition, using the fact that $i^* i/\tau_n^1$ is a bounded and linear operator, we conclude from $H(F)$ (i) that the mapping $\mathcal{G}_1$ is also weakly–weakly upper semicontinuous.

Next, we verify that $\mathcal{G}_1$ is coercive. Let $w \in V$ and $w^* \in \mathcal{G}_1(w)$ be arbitrary. Then there exists a $\xi \in \partial J(z_0, \gamma w)$ such that

$$w^* = \frac{i^* i}{\tau_n^1} w + F(z_0, w) + \gamma^* \xi.$$

Thanks to $H(F)$ (iii),

$$\langle w^*, w \rangle \geqslant \frac{\|w\|_H^2}{\tau_n^1} + m_F \|w\|_V^2 - l_F \|w\|_H^2 - d_F \|w\|_V + e_F - \langle \xi, \gamma w \rangle_{X^* \times X}. \tag{3.10}$$

By the Young inequality and $H(J)$ (iv), it follows that

$$\begin{aligned} \langle \xi, \gamma w \rangle_{X^* \times X} &\geqslant -J^0(z_0, \gamma w; -\gamma w) \geqslant -L_J - M_J \|\gamma w\|_X^\theta \\ &\geqslant -L_J - M_J \|\gamma\|^\theta \|w\|_V^\theta \geqslant -L_J - \frac{(M_J \|\gamma\|^\theta)^q}{q\varepsilon^q} - \frac{m_F}{2} \|w\|_V^2, \end{aligned} \tag{3.11}$$

where $\varepsilon = (m_F/\theta)^{\theta/2}$ and $q = 2/(2-\theta)$. Inserting (3.11) into (3.10) yields

$$\langle w^*, w\rangle \geqslant \left(\frac{1}{\tau_n^1} - l_F\right)\|w\|_H^2 + \frac{m_F}{2}\|w\|_V^2 - d_F\|w\|_V - L_J - \frac{(M_J\|\gamma\|^\theta)^q}{q\varepsilon^q}. \quad (3.12)$$

Since $n \geqslant n_0$, $1/\tau_n^1 > l_F$ and $\mathcal{G}_1$ is coercive.

Hence all the hypotheses of Theorem 2.8 are verified. Applying this result, we infer that there exist elements $w_n^1 \in V$ and $\xi_n^1 \in X^*$ with $\xi_n^1 \in \partial J(z_n(t_0), \gamma w_n^1)$ such that (3.6) holds for $k = 1$. Moreover, the function w_n given by (3.8) is well defined on $[0, t_n^1]$ and belongs to $C([0, t_n^1]; V) \subset L^2(0, t_n^1; V)$.

Consider the integral equation (3.7) for $k = 1$. Following the arguments in the proof of [43], Theorem 3.1, we know that there exists a unique function $z_n \in C([0, t_n^1]; E)$ such that (3.7) holds for $k = 1$.

Now, we assume that there exist $z_n \in C([0, t_n^{k-1}]; E)$, $w_n^0, w_n^1, \ldots, w_n^{k-1} \in V$ and $\xi_n^1, \xi_n^2, \ldots, \xi_n^{k-1} \in X^*$ such that (3.6) and (3.7) hold with $k-1$. The integral $\int_{t_n^{k-2}}^{t_n^{k-1}} G(s, z_n(s))\, ds$ is well defined. As in the case $k = 1$, we can verify that the multivalued mapping $\mathcal{G}_k$ given by

$$\mathcal{G}_k(w) = \frac{i^*i}{\tau_n^k} w + F(z_n(t_n^{k-1}), w) + \gamma^*\, \partial J(z_n(t_n^{k-1}), \gamma w) \quad \text{for all } w \in V$$

is surjective. Hence, there exist elements $w_n^k \in V$ and $\xi_n^k \in X^*$ with $\xi_n^k \in \partial J(z_n(t_n^{k-1}),\ \gamma w_n^k)$ such that (3.6) is satisfied. We see that $w_n \in C([0, t_n^k]; E) \subset L^2(0, t_n^k; V)$. Applying [43], Theorem 3.1, again, we have the existence of a unique function $z_n \in C([0, t_n^k]; E)$ satisfying (3.7). This completes the proof of the lemma.

Next, we provide *a priori* bounds for the solutions to the feedback iterative approximate system, Problem 3.3.

Lemma 3.5. *Let $n_1 \in \mathbb{N}$ be such that $n_1 \geqslant 2c_3c_{\mathcal{T}}T$, where*

$$c_3 = \max\left\{\frac{8d_F^2}{m_F} + |e_F| + L_J + \frac{(M_J\|\gamma\|^\theta)^q}{q\varepsilon^q}, l_F\right\}, \qquad \varepsilon = \left(\frac{m_F}{\theta}\right)^{\theta/2}, \quad q = \frac{2}{2-\theta}. \quad (3.13)$$

Under the hypotheses of Lemma 3.4, there exists a constant $C > 0$ such that the following bounds hold for each $n \geqslant \max\{n_0, n_1\}$:

$$\max_{1\leqslant k\leqslant n} \|w_n^k\|_H \leqslant C, \quad (3.14)$$

$$\sum_{k=1}^n \|w_n^k - w_n^{k-1}\|_H^2 \leqslant C, \quad (3.15)$$

$$\sum_{k=1}^n \tau_n^k \|w_n^k\|_V^2 \leqslant C. \quad (3.16)$$

Proof. Let us suppose that $\xi_n^k \in X^*$ with $\xi_n^k \in \partial J(z_n(t_n^{k-1}), \gamma w_n^k)$ and (3.6) holds. We act on (3.6) by the test function w_n^k to get

$$\left(\frac{w_n^k - w_n^{k-1}}{\tau_n^k}, w_n^k\right)_H + \langle F(z_n(t_n^{k-1}), w_n^k) - G_n^k, w_n^k\rangle + \langle \xi_n^k, \gamma(w_n^k)\rangle_{X^*\times X} = 0.$$

Taking into account the hypothesis $H(F)$ (iii) and the equality $(w, w-v)_H = (\|w\|_H^2 - \|v\|_H^2 + \|w-v\|_H^2)/2$ for $w, v \in H$, we have

$$\frac{1}{2\tau_n^k}(\|w_n^k\|_H^2 - \|w_n^{k-1}\|_H^2 + \|w_n^k - w_n^{k-1}\|_H^2) + m_F\|w_n^k\|_V^2 - l_F\|w_n^k\|_H^2 \\ - d_F\|w_n^k\|_V + e_F + \langle \xi_n^k, \gamma(w_n^k)\rangle_{X^*\times X} \leqslant \|G_n^k\|_{V^*}\|w_n^k\|_V.$$

It follows from this inequality along with the growth condition $H(J)$ (iv) and (3.11) with $w = w_n^k$ that

$$\begin{aligned}\frac{\|G_n^k\|_{V^*}^2}{m_F} &+ \frac{m_F}{4}\|w_n^k\|_V^2 \geqslant \|G_n^k\|_{V^*}\|w_n^k\|_V \\ &\geqslant \frac{1}{2\tau_n^k}(\|w_n^k\|_H^2 - \|w_n^{k-1}\|_H^2 + \|w_n^k - w_n^{k-1}\|_H^2) + m_F\|w_n^k\|_V^2 - l_F\|w_n^k\|_H^2 \\ &\quad - \frac{8d_F^2}{m_F} - \frac{m_F}{8}\|w_n^k\|_V^2 + e_F - J^0\big(z_n(t_n^{k-1}), \gamma w_n^k; -\gamma w_n^k\big) \\ &\geqslant \frac{1}{2\tau_n^k}(\|w_n^k\|_H^2 - \|w_n^{k-1}\|_H^2 + \|w_n^k - w_n^{k-1}\|_H^2) + \frac{3m_F}{8}\|w_n^k\|_V^2 \\ &\quad - l_F\|w_n^k\|_H^2 - \frac{8d_F^2}{m_F} + e_F - L_J - \frac{(M_J\|\gamma\|^\theta)^q}{q\varepsilon^q}.\end{aligned}$$

Hence,

$$2\tau_n^k\,\frac{\|G_n^k\|_{V^*}^2}{m_F} + c_3\tau_n^k(1 + \|w_n^k\|_H^2) \\ \geqslant \|w_n^k\|_H^2 - \|w_n^{k-1}\|_H^2 + \|w_n^k - w_n^{k-1}\|_H^2 + \tau_n^k\,\frac{m_F}{4}\|w_n^k\|_V^2,$$

where $c_3 > 0$ is defined in (3.13). We sum up these inequalities over k from 1 to i with $1 \leqslant i \leqslant n$ to obtain

$$\|w_n^i\|_H^2 - \|w_n^0\|_H^2 + \sum_{k=1}^{i}\|w_n^k - w_n^{k-1}\|_H^2 + \frac{m_F}{4}\sum_{k=1}^{i}\tau_n^k\|w_n^k\|_V^2 \\ \leqslant \frac{2}{m_F}\sum_{k=1}^{i}\tau_n^k\|G_n^k\|_{V^*}^2 + c_3\sum_{k=1}^{i}\tau_n^k + c_3\sum_{k=1}^{i}\tau_n^k\|w_n^k\|_H^2$$

and

$$(1 - c_3\tau_n^i)\|w_n^i\|_H^2 + \sum_{k=1}^{i}\|w_n^k - w_n^{k-1}\|_H^2 + \frac{m_F}{4}\sum_{k=1}^{i}\tau_n^k\|w_n^k\|_V^2 \\ \leqslant \frac{2}{m_F}\sum_{k=1}^{i}\tau_n^k\|G_n^k\|_{V^*}^2 + c_3\sum_{k=1}^{i}\tau_n^k + c_3\sum_{k=1}^{i-1}\tau_n^k\|w_n^k\|_H^2 + \|w_n^0\|_H^2.$$

By making use of the condition $H(G)$ (iii), the form of G_n and the Hölder inequality,

$$\tau_n^k \|G_n^k\|_{V^*}^2 = \frac{\tau_n^k}{(\tau_n^{k-1})^2} \left(\int_{t_n^{k-2}}^{t_n^{k-1}} \|G(s, z_n(s))\|_{V^*} \, ds \right)^2 \leqslant \frac{\tau_n^k}{(\tau_n^{k-1})^2} \left(\int_{t_n^{k-2}}^{t_n^{k-1}} \rho_G(s) \, ds \right)^2 \leqslant \frac{\tau_n^k}{\tau_n^{k-1}} \int_{t_n^{k-2}}^{t_n^{k-1}} \rho_G(s)^2 \, ds.$$

Then recall $H(\mathcal{T})$ (ii) to obtain

$$\tau_n^k \|G_n^k\|_{V^*}^2 \leqslant c_{\mathcal{T}} \int_{t_n^{k-2}}^{t_n^{k-1}} \rho_G(s)^2 \, ds. \tag{3.17}$$

Thus,

$$\frac{2}{m_F} \sum_{k=1}^{i} \tau_n^k \|G_n^k\|_{V^*}^2 \leqslant \frac{2c_{\mathcal{T}}}{m_F} \int_0^T \rho_G(s)^2 \, ds + \frac{2c_{\mathcal{T}} T \|G(0, z_0)\|_{V^*}^2}{m_F} := m_G > 0.$$

Note that $\sum_{k=1}^{i} \tau_n^k \leqslant T$. Therefore, we have

$$(1 - c_3 \tau_n^i) \|w_n^i\|_H^2 + \sum_{k=1}^{i} \|w_n^k - w_n^{k-1}\|_H^2 + \frac{m_F}{4} \sum_{k=1}^{i} \tau_n^k \|w_n^k\|_V^2 \leqslant m_G + c_3 T + c_3 \sum_{k=1}^{i-1} \tau_n^k \|w_n^k\|_H^2 + \|w_0\|_H^2. \tag{3.18}$$

Since $n \geqslant n_1$, we have $1 - c_3 \tau_n^i \geqslant 1/2$. By the discrete version of the Gronwall inequality ([50], Lemma 7.25), the inequality (3.14) holds. The estimates (3.15) and (3.16) are consequences of (3.14) and (3.18). This completes the proof.

For every fixed $n \in \mathbb{N}$, we introduce piecewise constant interpolating functions $\overline{w}_n$, ξ_n and G_n by

$$\overline{w}_n(t) = \begin{cases} w_n^k, & t \in (t_{k-1}, t_k], \\ w_0, & t = 0, \end{cases}$$

$$\xi_n(t) = \xi_n^k \text{ for } t \in (t_{k-1}, t_k] \quad \text{and} \quad G_n(t) = G_n^k \text{ for } t \in (t_{k-1}, t_k].$$

The following lemma provides *a priori* bounds for these functions.

Lemma 3.6. *Keep the assumptions of Lemma* 3.4. *Then the following bounds hold*:

$$\|w_n\|_{C([0,T];H)} \leqslant C, \tag{3.19}$$

$$\|\overline{w}_n\|_{L^\infty(0,T;H)} \leqslant C, \tag{3.20}$$

$$\|\overline{w}_n\|_{\mathcal{V}} \leqslant C, \tag{3.21}$$

$$\|w_n\|_{\mathcal{V}} \leqslant C, \tag{3.22}$$

$$\|\xi_n\|_{\mathcal{X}^*} \leqslant C, \tag{3.23}$$

$$\|w_n'\|_{\mathcal{V}^*} \leqslant C, \tag{3.24}$$

$$\|\overline{w}_n\|_{M^{2,2}(0,T;V,V^*)} \leqslant C, \tag{3.25}$$

where $C > 0$ is independent of n.

Proof. Obviously, for each $n \in \mathbb{N}$, w_n belongs to $C([0,T];H)$. For any $t \in [0,T]$, there is a k between 1 and n such that $t \in (t_n^{k-1}, t_n^k]$. Using the boundedness of the sequence $\{\|w_n^k\|_H\}_{k=1}^n$ (see (3.14)), we get

$$\begin{aligned}\|w_n(t)\|_H &\leqslant \|w_n^k\|_H + \frac{(t_n^k - t)}{\tau_n^k}\|w_n^k - w_n^{k-1}\|_H \\ &\leqslant \|w_n^k\|_H + \|w_n^k - w_n^{k-1}\|_H \leqslant 2\|w_n^k\|_H + \|w_n^{k-1}\|_H \leqslant C\end{aligned}$$

for all $t \in (t_n^{k-1}, t_n^k]$, $k = 1, \dots, n$. This leads to the bound (3.19). Moreover, the bound (3.20) is a consequence of (3.14). By the equality

$$\|\overline{w}_n\|_{\mathcal{V}}^2 = \int_0^T \|\overline{w}_n(s)\|_V^2\, ds = \sum_{k=1}^n \int_{t_n^{k-1}}^{t_n^k} \|\overline{w}_n(s)\|_V^2\, ds = \sum_{k=1}^n \tau_n^k \|w_n^k\|_V^2$$

and (3.16), we obtain (3.21). Concerning the function w_n, we have

$$\begin{aligned}\|w_n\|_{\mathcal{V}}^2 &= \int_0^T \|w_n(s)\|_V^2\, ds = \sum_{k=1}^n \int_{t_n^{k-1}}^{t_n^k} \|w_n(s)\|_V^2\, ds \\ &= \sum_{k=1}^n \int_{t_n^{k-1}}^{t_n^k} \left\| w_n^k + \frac{s - t_n^k}{\tau_n^k}(w_n^k - w_n^{k-1})\right\|_V^2 ds \\ &\leqslant 2\sum_{k=1}^n \int_{t_n^{k-1}}^{t_n^k} \left(\|w_n^k\|_V^2 + \left(\frac{t_n^k - s}{\tau_n^k}\right)^2 \|w_n^k - w_n^{k-1}\|_V^2 \right) ds \\ &\leqslant 2\sum_{k=1}^n \tau_n^k \big(\|w_n^k\|_V^2 + 2(\|w_n^k\|_V^2 + \|w_n^{k-1}\|_V^2)\big) \leqslant 10\sum_{k=1}^n \tau_n^k \|w_n^k\|_V^2 + 10\|w_0\|_V^2.\end{aligned}$$

So, (3.16) implies the bound (3.22). In addition, we deduce from $H(J)$ (ii), that

$$\begin{aligned}\|\xi_n\|_{\mathcal{X}^*}^2 &= \int_0^T \|\xi_n(s)\|_{X^*}^2\, ds = \sum_{k=1}^n \int_{t_n^{k-1}}^{t_n^k} \|\xi_n(s)\|_{X^*}^2\, ds = \sum_{k=1}^n \tau_n^k \|\xi_n^k\|_{X^*}^2 \\ &\leqslant \sum_{k=1}^n \tau_n^k (L_J + M_J\|\gamma w_n^k\|_X)^2 \leqslant 2L_J^2 \sum_{k=1}^n \tau_n^k + 2M_J^2\|\gamma\|^2 \sum_{k=1}^n \tau_n^k \|w_n^k\|_V^2.\end{aligned}$$

The latter combined with (3.16) guarantees the validity of (3.23). In addition, we observe that (3.6) can be formulated as follows:

$$w_n'(t) + F(z_n(\delta_n(t)), \overline{w}_n(t)) + \gamma^*\xi_n(t) = G_n(t) \tag{3.26}$$

for a.e. $t \in [0,T]$, where the function $\delta_n \colon [0,T] \to [0,T]$ is defined by

$$\delta_n(t) = t_n^{k-1} \quad \text{for } t \in [t_n^{k-1}, t_n^k) \quad \text{and} \quad k = 1, 2, \dots, n. \tag{3.27}$$

Given any $v \in \mathcal{V}$, we multiply the equality (3.26) by $v(t)$ and integrate the resulting equality over $[0,T]$ to get

$$\langle G_n, v\rangle_{\mathcal{V}^*\times\mathcal{V}} - \langle F(z_n(\delta_n), \overline{w}_n), v\rangle_{\mathcal{V}^*\times\mathcal{V}} - \langle \xi_n, \overline{\gamma}v\rangle_{\mathcal{X}^*\times\mathcal{X}} = (w_n', v)_{\mathcal{H}} = \langle w_n', v\rangle_{\mathcal{V}^*\times\mathcal{V}},$$

where $\overline{\gamma}\colon \mathcal{V} \to \mathcal{X}$ is the Nemytskii operator of γ defined by $(\overline{\gamma}w)(t) = \gamma w(t)$ for all $t \in [0, T]$. This results in

$$\|w_n'\|_{\mathcal{V}^*} \leqslant \|G_n\|_{\mathcal{V}^*} + \|F(z_n(\delta_n), \overline{w}_n)\|_{\mathcal{V}^*} + \|\gamma\| \|\xi_n\|_{\mathcal{X}^*}. \tag{3.28}$$

The bound (3.17) turns out to be

$$\begin{aligned}\|G_n\|_{\mathcal{V}^*}^2 &= \sum_{k=1}^{n} \int_{t_n^{k-1}}^{t_n^k} \|G_n(s)\|_{V^*}^2 \, ds = \sum_{k=1}^{n} \tau_n^k \|G_n^k\|_{V^*}^2 \\ &\leqslant c_{\mathcal{T}} \|\rho_G\|_{L^2(0,T)} + T\|G(0, z_0)\|_{V^*}^2. \end{aligned}\tag{3.29}$$

If $H(F)\,(\mathrm{ii})_1$ holds, i.e., (3.4) is satisfied, then we see from (3.21) that

$$\begin{aligned}\|F(z_n(\delta_n), \overline{w}_n)\|_{\mathcal{V}^*}^2 &= \int_0^T \|F(z_n(\delta_n(t)), \overline{w}_n(t))\|_{V^*}^2 \, dt \\ &\leqslant \int_0^T c_F^2 (1 + \|\overline{w}_n(t)\|_V)^2 \, dt \leqslant 2c_F^2 T + 2c_F^2 \|\overline{w}_n\|_{\mathcal{V}}^2. \end{aligned}$$

The latter together with (3.23), (3.28) and (3.29) shows the validity of (3.24). Further, when $H(F)\,(\mathrm{ii})_2$ is satisfied, namely, (3.5) holds, we use (3.20), (3.21), (3.28) and (3.29) to conclude that the estimate (3.24) holds.

In view of (3.21), to prove the boundedness of $\overline{w}_n$ in $M^{2,2}(0, T; V, V^*)$, it remains to examine the boundedness of $\{\overline{w}_n\}$ in $\mathrm{BV}^2(0, T; V^*)$. For the $\mathrm{BV}^2(0, T; V^*)$-seminorm of $\overline{w}_n$, write

$$\|\overline{w}_n\|_{\mathrm{BV}^2(0,T;V^*)}^2 = \sum_{k=1}^{M_n} \left\| w_n^{m_n^k - 1} - w_n^{m_n^{k-1} - 1} \right\|_{V^*}^2,$$

where the equality is attained at a partition whose vertices are located at the grid intervals indexed by $m_n^0, m_n^1, \ldots, m_n^{M_n - 1}, m_n^{M_n}$ with $m_n^0 = 1$ and $m_n^{M_n} = n$. Hence, we have

$$\begin{aligned}\|\overline{w}_n\|_{\mathrm{BV}^2(0,T;V^*)}^2 &= \sum_{k=1}^{M_n} \left\| w_n^{m_n^k - 1} - w_n^{m_n^{k-1} - 1} \right\|_{V^*}^2 \\ &\leqslant \sum_{k=1}^{M_n} (m_n^k - m_n^{k-1}) \sum_{j = m_n^{k-1}+1}^{m_n^k} \|w_n^{j-1} - w_n^{j-2}\|_{V^*}^2 \\ &\leqslant \sum_{k=1}^{M_n} (m_n^k - m_n^{k-1}) \max_{1 \leqslant l \leqslant n} \tau_n^l \sum_{j = m_n^{k-1}+1}^{m_n^k} \tau_n^j \left\| \frac{w_n^{j-1} - w_n^{j-2}}{\tau_n^j} \right\|_{V^*}^2 \\ &\leqslant n \max_{1 \leqslant l \leqslant n} \tau_n^l \sum_{k=2}^{n} \tau_n^k \left\| \frac{w_n^{k-1} - w_n^{k-2}}{\tau_n^k} \right\|_{V^*}^2 \leqslant c_{\mathcal{T}} T \|w_n'\|_{\mathcal{V}^*}^2. \end{aligned}$$

Therefore, from (3.24), we derive the bound (3.25). This completes the proof of the lemma.

We end the section with an examination of critical convergence theorem for the functions $\{w_n\}$, $\{\overline{w}_n\}$, $\{\xi_n\}$ and $\{z_n\}$, and to reveal that the limit point of $\{(z_n, w_n)\}$ is also a mild solution of Problem 1.1.

Theorem 3.7. *Assume* $H(S)$, $H(F)$, $H(J)$, $H(G)$, $H(g)$, $H(\gamma)$ *and* $H(\mathcal{T})$. *Then, there exist* $(w, \xi, z) \in \mathcal{W} \times \mathcal{X}^* \times C([0,T]; E)$, *and a subsequence of indices such that for this subsequence (still denoted by* n*) we have*

$$\overline{w}_n \rightharpoonup w \quad \text{in } \mathcal{V} \quad \text{and } \overline{w}_n \to w \ \text{ weakly-}* \text{ in } L^\infty(0,T;H), \tag{3.30}$$

$$w_n \rightharpoonup w \quad \text{in } \mathcal{V}, \tag{3.31}$$

$$w_n' \rightharpoonup w' \quad \text{in } \mathcal{V}^*, \tag{3.32}$$

$$w_n \rightharpoonup w \quad \text{in } \mathcal{W}, \tag{3.33}$$

$$w_n \to w \quad \text{in } \mathcal{H}, \tag{3.34}$$

$$\xi_n \rightharpoonup \xi \quad \text{in } \mathcal{X}^*, \tag{3.35}$$

$$z_n \to z \quad \text{in } C([0,T]; E). \tag{3.36}$$

Moreover, $(z, w) \in C([0,T]; E) \times \mathcal{W}$ *is a solution to the problem* (3.1)–(3.3).

Proof. By (3.20) and (3.21), we are able to find an element $w \in \mathcal{V} \cap L^\infty(0,T;H)$ and a subsequence of $\{\overline{w}_n\}$, still denoted by $\{\overline{w}_n\}$, such that (3.30) holds. Moreover, the boundedness of $\{w_n\}$ in $\mathcal{V}$ enables us to assume that $w_n \rightharpoonup \widetilde{w}$ in $\mathcal{V}$ for some $\widetilde{w} \in \mathcal{V}$. We claim that $w = \widetilde{w}$. Indeed, note that

$$\begin{aligned} \|\overline{w}_n - w_n\|^2_{\mathcal{V}^*} &= \int_0^T \|\overline{w}_n(t) - w_n(t)\|^2_{V^*}\, dt = \sum_{k=1}^n \int_{t_n^{k-1}}^{t_n^k} \|\overline{w}_n(t) - w_n(t)\|^2_{V^*}\, dt \\ &= \sum_{k=1}^n \int_{t_n^{k-1}}^{t_n^k} (t_n^k - t)^2 \left\| \frac{w_n^k - w_n^{k-1}}{\tau_n^k} \right\|^2_{V^*} dt \\ &= \sum_{k=1}^n \frac{(\tau_n^k)^3}{3} \left\| \frac{w_n^k - w_n^{k-1}}{\tau_n^k} \right\|^2_{V^*} \leqslant \frac{(\tau_n^{\max})^2}{3} \|w_n'\|^2_{\mathcal{V}^*}. \end{aligned}$$

We use (3.24) to see that $\overline{w}_n - w_n \to 0$ in $\mathcal{V}^*$. On the other hand, the facts (3.30) and $w_n \rightharpoonup \widetilde{w}$ in $\mathcal{V}$ imply that $\overline{w}_n - w_n \rightharpoonup w - \widetilde{w}$ in $\mathcal{V}$. The continuity of the embedding of $\mathcal{V}$ to $\mathcal{V}^*$ shows that $\overline{w}_n - w_n \rightharpoonup w - \widetilde{w}$ in $\mathcal{V}^*$. Combining this with the convergence $\overline{w}_n - w_n \to 0$ in $\mathcal{V}^*$ proves the claim. So $w = \widetilde{w}$, and consequently (3.31) is satisfied.

Thanks to (3.24), by passing to a subsequence if necessary, we find a function $w^* \in \mathcal{V}^*$ such that $w_n' \rightharpoonup w^*$ in $\mathcal{V}^*$. Using [42], Proposition 23.19, (3.31), and the continuity of the embedding of V to H, we have $w^* = w'$, which means that the convergence (3.32) holds. Obviously, the convergence relations (3.31) and (3.32) guarantee the result (3.33). Since the embedding of V to H is compact, so is the embedding of $\mathcal{W}$ to $\mathcal{H}$. Hence, (3.34) follows directly from (3.33). In addition, by (3.23), we may suppose that there exist $\xi \in \mathcal{X}^*$ and a subsequence of $\{\xi_n\}$, still denoted by $\{\xi_n\}$, such that the convergence (3.35) holds.

Let us prove the convergence (3.36). Since the embedding of $\mathcal{W}$ to $C([0,T]; H)$ is continuous, $w \in C([0,T]; H)$. Arguing as in the proof of [43], Theorem 3.1, we deduce from the hypothesis $H(g)$ (iii) that there exists a unique mild solution $z \in C([0,T]; E)$ to the following fractional evolution equation:

$$\begin{cases} {}_0^C D_t^\alpha z(t) = Sz(t) + g(t, z(t), w(t)) & \text{for a.e. } t \in [0,T], \\ z(0) = z_0, \end{cases}$$

which means that z satisfies the integral equation

$$z(t) = \mathscr{E}(t)z_0 + \int_0^t (t-s)^{\alpha-1}\mathscr{F}(t-s)g(s,z(s),w(s))\,ds \quad \text{for all } t\in[0,T]. \tag{3.37}$$

Since both $\{\mathscr{E}(t)\}$ and $\{\mathscr{F}(t)\}$ are bounded, we see from $H(g)$ (iii) that

$$\begin{aligned}\|z_n(t)-z(t)\|_E &\leqslant \frac{M_S}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|g(s,z_n(s),w_n(s))-g(s,z(s),w(s))\|_E\,ds\\ &\leqslant \frac{M_S}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|g(s,z_n(s),w_n(s))-g(s,z(s),w_n(s))\|_E\,ds\\ &\quad + \frac{M_S}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|g(s,z(s),w_n(s))-g(s,z(s),w(s))\|_E\,ds\\ &\leqslant \frac{M_S}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\rho_g(s)\,\|z_n(s)-z(s)\|_E\,ds + r_n(t),\end{aligned}$$

where the function $r_n\colon [0,T]\to\mathbb{R}_+$ is defined by

$$r_n(t) = \frac{M_S}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|g(s,z(s),w_n(s))-g(s,z(s),w(s))\|_E\,ds \quad \text{for all } t\in[0,T].$$

It follows from the Hölder inequality that

$$\begin{aligned}\int_0^t (t-s)^{\alpha-1}\rho_g(s)\,ds &\leqslant \left(\int_0^t \rho_g(s)^{1/\beta}\,ds\right)^{\beta}\left(\int_0^t (t-s)^{\zeta}\,ds\right)^{1-\beta}\\ &\leqslant \frac{\|\rho_g\|_{L^{1/\beta}(0,T)}T^{(1+\zeta)(1-\beta)}}{(1+\zeta)^{1-\beta}}\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}r_n(t) &\leqslant \frac{M_S}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\zeta}\,ds\right)^{1-\beta}\left(\int_0^t \|g(s,z(s),w_n(s))-g(s,z(s),w(s))\|_E^{1/\beta}\,ds\right)^{\beta}\\ &\leqslant \frac{M_S}{\Gamma(\alpha)}\left(\frac{T^{1+\zeta}}{1+\zeta}\right)^{1-\beta}\left(\int_0^T \|g(s,z(s),w_n(s))-g(s,z(s),w(s))\|_E^{1/\beta}\,ds\right)^{\beta}\end{aligned} \tag{3.39}$$

for all $t\in[0,T]$, where $\zeta := (\alpha-1)/(1-\beta) > -1$ since $0<\beta<\alpha$. Hence, we have

$$\begin{aligned}&\left(1-\frac{M_S\|\rho_g\|_{L^{1/\beta}(0,T)}T^{(1+\zeta)(1-\beta)}}{\Gamma(\alpha)(1+\zeta)^{1-\beta}}\right)\sup_{s\in[0,T]}\|z_n(s)-z(s)\|_E\\ &\qquad\leqslant \frac{M_S}{\Gamma(\alpha)}\left(\frac{T^{1+\zeta}}{1+\zeta}\right)^{1-\beta}\left(\int_0^T \|g(s,z(s),w_n(s))-g(s,z(s),w(s))\|_E^{1/\beta}\,d\tau\right)^{\beta}\end{aligned} \tag{3.40}$$

for all $t\in[0,T]$. We see from (3.19) that

$$D = \left(\bigcup_{t\in[0,T]}\{w_n(t)\}\right)\cup\left(\bigcup_{t\in[0,T]}\{w(t)\}\right) \tag{3.41}$$

is a bounded set in H. Note that

$$\begin{aligned}\|g(s,z(s),w_n(s))-g(s,z(s),w(s))\|_E &\leqslant \|g(s,z(s),w_n(s))-g(s,0,w_n(s))\|_E\\ &\quad+\|g(s,0,w_n(s))\|_E+\|g(s,0,w(s))-g(s,z(s),w(s))\|_E+\|g(s,0,w(s))\|_E\\ &\leqslant 2\rho_g(t)\|z(s)\|_E+2\psi_D(s)\end{aligned}$$

for all $s\in[0,T]$. So the function $s\mapsto\|g(s,z(s),w_n(s))-g(s,z(s),w(s))\|_E$ belongs to $L^{1/\beta}(0,T)$ (see the hypothesis $H(g)$ (iii)). Since $w_n\to w$ in $\mathcal{H}$, by passing to a subsequence if necessary, we have $w_n(t)\to w(t)$ in H for a.e. $t\in(0,T)$ as $n\to\infty$. Then apply the Lebesgue dominated convergence theorem to obtain

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t\|g(\tau,z(\tau),w_n(\tau))-g(\tau,z(\tau),w(\tau))\|_E^{1/\beta}\,d\tau\\ &\quad=\int_0^t\lim_{n\to\infty}\|g(\tau,z(\tau),w_n(\tau))-g(\tau,z(\tau),w(\tau))\|_E^{1/\beta}\,d\tau=0 \qquad (3.42)\end{aligned}$$

for all $t\in[0,T]$. Passing to the limit as $n\to\infty$ in (3.40) and using (3.42), we obtain the convergence (3.36).

For any $t\in[0,T]$, we have $|\delta_n(t)-t|\leqslant\max_{1\leqslant k\leqslant n}\tau_n^k$. Thus,

$$\sup_{t\in[0,T]}|\delta_n(t)-t|\to 0\quad\text{as } n\to\infty, \qquad (3.43)$$

where we have applied the condition $H(\mathcal{T})$(i). Therefore, for each $t\in[0,T]$,

$$\begin{aligned}\|z_n(\delta_n(t))-z(t)\|_E &\leqslant \bigg\|\int_0^{\delta_n(t)}(\delta_n(t)-s)^{\alpha-1}\mathscr{F}(\delta_n(t)-s)g(s,z_n(s),w_n(s))\,ds\\ &\qquad-\int_0^t(t-s)^{\alpha-1}\mathscr{F}(t-s)g(s,z(s),w(s))\,ds\bigg\|_E\\ &\qquad+\|\mathscr{E}(t)z_0-\mathscr{E}(\delta_n(t))z_0\|_E\leqslant L_1+L_2+L_3+L_4+L_5+L_6, \qquad (3.44)\end{aligned}$$

where $L_1,\dots,L_6$ are defined by

$$\begin{aligned}L_1&:=\|\mathscr{E}(t)z_0-\mathscr{E}(\delta_n(t))z_0\|_E,\\ L_2&:=\int_0^{\delta_n(t)-\varepsilon}(\delta_n(t)-s)^{\alpha-1}\|[\mathscr{F}(\delta_n(t)-s)-\mathscr{F}(t-s)]g(s,z_n(s),w_n(s))\|_E\,ds,\\ L_3&:=\int_{\delta_n(t)-\varepsilon}^{\delta_n(t)}(\delta_n(t)-s)^{\alpha-1}\|[\mathscr{F}(\delta_n(t)-s)-\mathscr{F}(t-s)]g(s,z_n(s),w_n(s))\|_E\,ds,\\ L_4&:=\int_0^{\delta_n(t)}|(\delta_n(t)-s)^{\alpha-1}-(t-s)^{\alpha-1}|\,\|\mathscr{F}(t-s)g(s,z_n(s),w_n(s))\|_E\,ds,\\ L_5&:=\int_0^{\delta_n(t)}(t-s)^{\alpha-1}\|\mathscr{F}(t-s)[g(s,z_n(s),w_n(s))-g(s,z(s),w(s))]\|_E\,ds,\\ L_6&:=\int_{\delta_n(t)}^t(t-s)^{\alpha-1}\|\mathscr{F}(t-s)g(s,z(s),w(s))\|_E\,ds,\end{aligned}$$

respectively, and $0 < \varepsilon < \delta_n(t)$. Since $\mathscr{E}$ is continuous, we have

$$L_1 = \|\mathscr{E}(t)z_0 - \mathscr{E}(\delta_n(t))z_0\|_E \to 0 \quad \text{as } n \to \infty. \tag{3.45}$$

It follows from the condition $H(g)$ (iii) that

$$\begin{aligned}\|g(s, z_n(s), w_n(s))\|_E &\leqslant \|g(s, z_n(s), w_n(s)) - g(s, 0, w_n(s))\|_E + \|g(s, 0, w_n(s))\|_E \\ &\leqslant \rho_g(s)\|z_n(s)\|_E + \psi_D(s),\end{aligned}$$

where $D \subset H$ is given in (3.41). This together with (3.36) yields that the sequence $\{\|g(\,\cdot\,, z_n(\,\cdot\,), w_n(\,\cdot\,))\|_E\}$ is bounded in $L^{1/\beta}(0,T)$. Let $m_3 > 0$ be such that $\|g(\,\cdot\,, z_n(\,\cdot\,), w_n(\,\cdot\,))\|_{L^{1/\beta}(0,T)} \leqslant m_3$ for all $n \in \mathbb{N}$. We apply the Hölder inequality again to deduce that

$$\begin{aligned}&\int_0^{\delta_n(t)-\varepsilon} (\delta_n(t) - s)^{\alpha-1}\|g(s, z_n(s), w_n(s))\|_E\, ds \\ &\qquad\leqslant \left(\frac{\delta_n(t)^{\zeta+1} - \varepsilon^{\zeta+1}}{1+\zeta}\right)^{1-\beta} \|g(\,\cdot\,, z_n(\,\cdot\,), w_n(\,\cdot\,))\|_{L^{1/\beta}(0,T)} \\ &\qquad\leqslant \left(\frac{\delta_n(t)^{\zeta+1} - \varepsilon^{\zeta+1}}{1+\zeta}\right)^{1-\beta} m_3 \leqslant \left(\frac{T^{\zeta+1} - \varepsilon^{\zeta+1}}{1+\zeta}\right)^{1-\beta} m_3.\end{aligned}$$

Recall that $\mathscr{S}(t)$ is compact for all $t > 0$. Then we know from Lemma 2.2 that $\mathcal{F}(t)$ is continuous in the uniform operator topology, and so

$$\begin{aligned}L_2 &\leqslant \sup_{s\in[0,\delta_n(t)-\varepsilon]} \|\mathscr{F}(\delta_n(t) - s) - \mathscr{F}(t - s)\| \\ &\qquad\times \int_0^{\delta_n(t)-\varepsilon} (\delta_n(t) - s)^{\alpha-1}\|g(s, z_n(s), w_n(s))\|_E\, ds \\ &\leqslant \sup_{s\in[0,\delta_n(t)-\varepsilon]} \|\mathscr{F}(\delta_n(t) - s) - \mathscr{F}(t - s)\| \\ &\qquad\times \left(\frac{T^{\zeta+1} - \varepsilon^{\zeta+1}}{1+\zeta}\right)^{1-\beta} m_3 \to 0 \quad \text{as } n \to \infty \text{ and } \varepsilon \to 0.\end{aligned} \tag{3.46}$$

We put $M_{\mathscr{F}} = M_S/\Gamma(\alpha)$. Then

$$\begin{aligned}L_3 &= \int_{\delta_n(t)-\varepsilon}^{\delta_n(t)} (\delta_n(t) - s)^{\alpha-1}\|[\mathscr{F}(\delta_n(t) - s) - \mathscr{F}(t - s)]g(s, z_n(s), w_n(s))\|_E\, ds \\ &\leqslant 2M_{\mathscr{F}}\left(\frac{\varepsilon^{\zeta+1}}{\zeta+1}\right)^{1-\beta} m_3 \to 0 \quad \text{as } n \to \infty \text{ and } \varepsilon \to 0,\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}L_4 &= \int_0^{\delta_n(t)} [(\delta_n(t) - s)^{\alpha-1} - (t - s)^{\alpha-1}]\|\mathscr{F}(t - s)g(s, z_n(s), w_n(s))\|_E\, ds \\ &\leqslant M_{\mathscr{F}} \int_0^{\delta_n(t)} [(\delta_n(t) - s)^{\alpha-1} - (t - s)^{\alpha-1}][\rho_g(s)\|z_n(s)\|_E + \psi_D(s)]\, ds \\ &\leqslant M_{\mathscr{F}} \int_0^{\delta_n(t)} [(\delta_n(t) - s)^{\alpha-1} - (t - s)^{\alpha-1}][\rho_g(s)m_4 + \psi_D(s)]\, ds,\end{aligned}$$

where the first equality is obtained by using the fact that $\delta_n(t) \leqslant t$ for all $t \in [0, T]$, and $m_4 > 0$ is independent of n (see (3.36)). We put

$$p_n(s) = \begin{cases} (\delta_n(t) - s)^{\alpha-1} & \text{if } s \in [0, \delta_n(t)], \\ 0 & \text{otherwise,} \end{cases}$$

$$q_n(s) = \begin{cases} (t - s)^{\alpha-1} & \text{if } s \in [0, \delta_n(t)], \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} L_4 &\leqslant M_{\mathscr{F}} \int_0^{\delta_n(t)} [(\delta_n(t) - s)^{\alpha-1} - (t-s)^{\alpha-1}][\rho_g(s) m_4 + \psi_D(s)]\, ds \\ &= M_{\mathscr{F}} \int_0^t [p_n(s) - q_n(s)][\rho_g(s) m_4 + \psi_D(s)]\, ds. \end{aligned}$$

It follows from the convergence (3.43) that $q_n(s) - p_n(s) \to 0$ as $n \to \infty$ for a.e. $s \in [0, t]$. So applying the Lebesgue dominated convergence theorem, we find

$$\begin{aligned} \lim_{n\to\infty} L_4 &\leqslant \lim_{n\to\infty} M_{\mathscr{F}} \int_0^t [p_n(s) - q_n(s)][\rho_g(s) m_4 + \psi_D(s)]\, ds \\ &= M_{\mathscr{F}} \int_0^t \lim_{n\to\infty} [p_n(s) - q_n(s)][\rho_g(s) m_4 + \psi_D(s)]\, ds = 0. \end{aligned} \tag{3.48}$$

For the term L_5, we obtain

$$\begin{aligned} L_5 &\leqslant M_{\mathscr{F}} \int_0^{\delta_n(t)} (t-s)^{\alpha-1} \|[g(s, z_n(s), w_n(s)) - g(s, z(s), w(s))]\|_E \, ds \\ &\leqslant \frac{M_{\mathscr{F}} (t^{\zeta+1} - (t - \delta_n(t))^{\zeta+1})^{1-\beta}}{(\zeta+1)^{1-\beta}} \\ &\quad \times \left(\int_0^{\delta_n(t)} \|[g(s, z_n(s), w_n(s)) - g(s, z(s), w(s))]\|_E^{1/\beta} \, ds \right)^\beta \\ &\leqslant \frac{M_{\mathscr{F}} (t^{\zeta+1} - (t - \delta_n(t))^{\zeta+1})^{1-\beta}}{(\zeta+1)^{1-\beta}} \\ &\quad \times \left(\int_0^{T} \|[g(s, z_n(s), w_n(s)) - g(s, z(s), w(s))]\|_E^{1/\beta} \, ds \right)^\beta . \end{aligned}$$

The convergence relations (3.34), (3.36) and the Lebesgue dominated convergence theorem guarantee that

$$\begin{aligned} \lim_{n\to\infty} L_5 &\leqslant \lim_{n\to\infty} \frac{M_{\mathscr{F}} (t^{\zeta+1} - (t - \delta_n(t))^{\zeta+1})^{1-\beta}}{(\zeta+1)^{1-\beta}} \\ &\quad \times \left(\int_0^{T} \|[g(s, z_n(s), w_n(s)) - g(s, z(s), w(s))]\|_E^{1/\beta} \, ds \right)^\beta = 0. \end{aligned} \tag{3.49}$$

Also,

$$L_6 \leqslant \left(\frac{(t-\delta_n(t))^{\zeta+1}}{\zeta+1} \right)^{1-\beta} M_{\mathscr{F}} \|g(\,\cdot\,, z(\,\cdot\,), w(\,\cdot\,))\|_{L^{1/\beta}(0,T)} \to 0 \quad \text{as } n \to \infty. \tag{3.50}$$

Taking (3.44)–(3.50) into account, we have

$$\|z_n(\delta_n(t)) - z(t)\|_E \to 0 \quad \text{as } n \to \infty \tag{3.51}$$

for all $t \in [0,T]$. It remains to prove that $(z,w) \in C([0,T];E) \times \mathcal{W}$ is a mild solution of Problem 1.1. First, we consider the term with G_n. Note that

$$\begin{aligned} &\left\| \frac{1}{\tau_n^{k-1}} \int_{t_n^{k-2}}^{t_n^{k-1}} G(s, z_n(s))\, ds - \frac{1}{\tau_n^{k-1}} \int_{t_n^{k-2}}^{t_n^{k-1}} G(s, z(s))\, ds \right\|_{V^*} \\ &\qquad \leqslant \frac{1}{\tau_n^{k-1}} \int_{t_n^{k-2}}^{t_n^{k-1}} \|G(s, z_n(s)) - G(s, z(s))\|_{V^*}\, ds \\ &\qquad \leqslant \sup_{s \in [0,T]} \|G(s, z_n(s)) - G(s, z(s))\|_{V^*}, \end{aligned}$$

and we apply the hypothesis $H(G)$, the convergence (3.36) and [51], Lemma 3.3, to deduce

$$G_n \to G(\,\cdot\,, z(\,\cdot\,)) \quad \text{in } \mathcal{V}^*. \tag{3.52}$$

Next, we shall show that

$$\mathcal{F}(z_n(\delta_n), \overline{w}_n) \rightharpoonup \mathcal{F}(z,w) \quad \text{in } \mathcal{V}^*. \tag{3.53}$$

Since $\{\overline{w}_n\} \subset \mathcal{V} \cap L^\infty(0,T;H)$ and $\{z_n(\delta_n)\} \subset \mathcal{E}$ are such that $\{\overline{w}_n\}$ is bounded in $L^\infty(0,T;H)$, $z_n(\delta_n) \to z$ in $\mathcal{E}$ (see (3.51)) and $\overline{w}_n \rightharpoonup w$ in $\mathcal{V}$ and $\mathcal{H}$, we can apply the condition $H(F)$ (iv) to obtain the convergence (3.53). It follows from the convergence (3.32) that

$$(w_n', v)_{\mathcal{H}} = \langle w_n', v\rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle w', v\rangle_{\mathcal{V}^* \times \mathcal{V}} = (w', v)_{\mathcal{H}} \tag{3.54}$$

for all $v \in \mathcal{V}$. Therefore, combining (3.35) and (3.52)–(3.54), we obtain

$$\begin{aligned} &\int_0^T (w'(t), v(t))_H\, dt + \int_0^T \langle F(z(t), w(t)), v(t)\rangle\, dt + \int_0^T \langle \xi(t), v(t)\rangle_{X^* \times X}\, dt \\ &\qquad = \int_0^T \langle G(t, z(t)), v(t)\rangle\, dt, \end{aligned} \tag{3.55}$$

$$w(0) = w_0, \tag{3.56}$$

for all $v \in \mathcal{V}$ and

$$z(t) = \mathscr{E}(t) z_0 + \int_0^t (t-s)^{\alpha-1} \mathscr{F}(t-s) g(s, z(s), w(s))\, ds \quad \text{for all } t \in [0,T]. \tag{3.57}$$

Evidently, if we can verify that $\xi(t) \in \partial J(z(t), \gamma w(t))$ for a.e. $t \in [0,T]$, then $(z,w) \in C([0,T];E) \times \mathcal{W}$ is a solution to the problem (3.1)–(3.3). According to $H(\gamma)$, Proposition 2.7, and (3.25), the sequence $\{\overline{\gamma}\,\overline{w}_n\}$ is relatively compact in $\mathcal{X}$. Moreover, it follows from the convergence (3.30) that

$$\overline{\gamma}\,\overline{w}_n \to \overline{\gamma}\,\overline{w} \quad \text{in } \mathcal{X} \text{ as } n \to \infty. \tag{3.58}$$

Hence, at least along a subsequence, we have $\gamma\overline{w}_n(t) \to \gamma w(t)$ in X as $n \to \infty$, for a.e. $t \in [0,T]$. Since $z_n \to z$ in $C([0,T];E)$, one has $z_n(t) \to z(t)$ in E for all $t \in [0,T]$. Using Theorem 2.3, Lemma 2.6 and (3.58), we conclude that $\xi(t) \in \partial J(z(t), \gamma w(t))$ for a.e. $t \in [0,T]$. Finally, it follows from the definition of the Clarke subgradient that

$$\int_0^T \langle \xi(t), v(t)\rangle_{X^*\times X}\, dt \leqslant \int_0^T J^0(y(t), \gamma w(t); \gamma v(t))\, dt.$$

The latter combined with (3.55)–(3.57) shows that $(z,w) \in C([0,T];E) \times \mathcal{W}$ is a solution to the problem (3.1)–(3.3). This completes the proof.

§ 4. Incompressible Navier–Stokes equations coupled with fractional diffusion equations

To illustrate the applicability of the theoretical results established in Section 3, we will study a nonstationary incompressible Navier–Stokes system involving a fractional diffusion equation. The system will be formulated as a fractional hemivariational inequality.

We consider an incompressible fluid flow in a bounded, open, and connected domain Ω in $\mathbb{R}^d$, $d = 2,3$, with a C^1 boundary $\Gamma = \partial\Omega$ expressed as $\Gamma = \overline{\Gamma}_C \cup \overline{\Gamma}_D$ for some disjoint relatively open sets Γ_C, Γ_D such that $\operatorname{meas}(\Gamma_D) > 0$. Given any $T \in (0,\infty)$, we put $\mathcal{Q} = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, $\Sigma_D = \Gamma_D \times (0,T)$, and $\Sigma_C = \Gamma_C \times (0,T)$. Let $\mathbb{S}^d$ be the set of real symmetric matrices of dimension d. The symbols "$\cdot$" and "$:$" stand for the inner products in $\mathbb{R}^d$ and $\mathbb{S}^d$ which are given by $\boldsymbol{\xi}\cdot\boldsymbol{\eta} = \xi_i\eta_i$ and $\boldsymbol{\tau} : \boldsymbol{\sigma} = \tau_{ij}\sigma_{ij}$, respectively. The corresponding norms are denoted by $\|\cdot\|_{\mathbb{R}^d}$ and $\|\cdot\|_{\mathbb{S}^d}$. The basic notation needed in this section is provided in Table 1. In the definitions of divergence operators, the index that follows a comma represents the partial derivative with respect to the corresponding component of the space variable $\boldsymbol{x}$.

It is well known that the nonstationary flow of a viscous incompressible fluid is modeled by the following Navier–Stokes equations with divergence-free condition (or solenoidal condition)

$$\begin{cases} \dfrac{\partial \boldsymbol{u}(\boldsymbol{x},t)}{\partial t} - \mu\Delta\boldsymbol{u}(\boldsymbol{x},t) & \\ \quad +(\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla)\boldsymbol{u}(\boldsymbol{x},t) + \nabla\pi(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x},t) & \text{in } \mathcal{Q}, \\ \nabla\cdot\boldsymbol{u}(t) = 0 & \text{in } \mathcal{Q}, \end{cases}$$

where μ represents the kinematic viscosity coefficient. In many studies of the Navier–Stokes equations, μ is taken to be a constant. In this paper, we allow μ to depend on the concentration of some chemical solution or chemical substances. Denoting the concentration field by $y = y(t,\boldsymbol{x})$, we consider a kinematic viscosity

Table 1. Nomenclature

Symbol	Description
$\boldsymbol{\nu} = (\nu_i)$	unit outward normal vector on Γ
$\boldsymbol{x} \in \overline{\Omega} = \Omega \cup \Gamma$	position vector
indices i, j, k, l	they range between 1 and d; the summation convention over repeated indices is used
$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$	velocity field of the fluid
$\pi = \pi(\boldsymbol{x}, t)$	pressure of the fluid
$y = y(\boldsymbol{x}, t)$	concentration of a chemical solution or a chemical substance
$\boldsymbol{f}(y) = \boldsymbol{f}(\boldsymbol{x}, t, y)$	external force field
Div	divergence operator for tensor-valued functions, $\mathrm{Div}\, \boldsymbol{S} = (S_{ij,j})$
div	divergence operator for vector-valued functions, $\mathrm{div}\, \boldsymbol{u} = (u_{i,i})$
$\boldsymbol{\sigma}$	total stress tensor of the fluid
$u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$	normal component of the velocity field $\boldsymbol{u}$ on Γ
$\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$	tangential component of the velocity field $\boldsymbol{u}$ on Γ
$S_\nu = (\boldsymbol{S}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$	normal component of the extra stress tensor field $\boldsymbol{S}$ on Γ
$\boldsymbol{S}_\tau = \boldsymbol{S}\boldsymbol{\nu} - S_\nu \boldsymbol{\nu}$	tangential component of the extra stress tensor field $\boldsymbol{S}$ on Γ
$\boldsymbol{D}(\boldsymbol{u})$	symmetric part of the velocity gradient of the fluid $\boldsymbol{D}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top)/2$
$\nabla \pi$	$\nabla \pi = (\partial \pi / \partial x_i)_{i=1,\ldots,d}$
$\nabla \cdot \boldsymbol{u}$	$\nabla \cdot \boldsymbol{u} = \mathrm{div}\, \boldsymbol{u} = \sum_{i=1}^d \partial u_i / \partial x_i$
$(\boldsymbol{v} \cdot \nabla)\boldsymbol{u}$	$(\boldsymbol{v} \cdot \nabla)\boldsymbol{u} = \left(\sum_{j=1}^d v_j \partial u_i / \partial x_j\right)_{i=1,\ldots,d}$

coefficient of the form $\mu(\|y(t)\|_{L^2(\Omega)}/|\Omega|)$. The constraint $\nabla \cdot \boldsymbol{u}(t) = 0$ reflects the fact that the viscous fluid is incompressible. The expression $(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t)$ represents the convective term. The functions π and $\boldsymbol{f}$ are the pressure field and the density of external volume forces, respectively. In most references on the Navier–Stokes equations, $\boldsymbol{f}$ is a given function of $\boldsymbol{x}$ and t only. We consider the case when the density function $\boldsymbol{f}$ depends on the concentration field $y = y(t, \boldsymbol{x})$. In other words, we will study the following Navier–Stokes equations for a viscous incompressible fluid:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}(\boldsymbol{x},t)}{\partial t} - \mu\left(\dfrac{\|y(t)\|_{L^2(\Omega)}}{|\Omega|}\right)\Delta \boldsymbol{u}(\boldsymbol{x},t) & \\ \quad +(\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla)\boldsymbol{u}(\boldsymbol{x},t) + \nabla\pi(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x},t,y(t)) & \text{in } \mathcal{Q}, \\ \nabla\cdot\boldsymbol{u}(t) = 0 & \text{in } \mathcal{Q}. \end{cases} \tag{4.1}$$

Next, we turn our attention to boundary conditions for the system (4.1). On the part Γ_D of the boundary, we assume that the fluid is adhered to the wall. In mathematical terms, this refers to the homogeneous Dirichlet condition for the velocity field $\boldsymbol{u}$ on Γ_D,

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Sigma_D. \tag{4.2}$$

On the part Γ_C of the boundary, we consider the no-flux condition, i.e., the normal component of the velocity field $\boldsymbol{u}$ vanishes on Γ_C:

$$u_\nu(t) = 0 \quad \text{on } \Sigma_C. \tag{4.3}$$

The total stress tensor for the incompressible fluid is given by $\boldsymbol{\sigma} = -\pi \boldsymbol{I} + \boldsymbol{S}$ in $\mathcal{Q}$, where $\boldsymbol{I} \in \mathbb{S}^d$ is the identity matrix and the vector-valued function $\boldsymbol{S}$: $\Omega \times (0,T) \times \mathbb{S}^d \to \mathbb{S}^d$ denotes the viscous (or extra) response of the total stress tensor $\boldsymbol{\sigma}$. We consider a general constitutive law by allowing the viscous stress tensor $\boldsymbol{S}$ to depend on the concentration field $y(t)$,

$$\boldsymbol{S} = 2\mu\left(\frac{\|y(t)\|_{L^2(\Omega)}}{|\Omega|}\right)\boldsymbol{D}(\boldsymbol{u}) \quad \text{in } \mathcal{Q}.$$

Regarding the tangential direction on Γ_C, we focus on the general multivalued and non-monotone boundary condition of the form

$$-\boldsymbol{S}_\tau(t) \in \partial j_\tau(\boldsymbol{x}, y(t), \boldsymbol{u}_\tau(t)) \quad \text{on } \Sigma_C, \tag{4.4}$$

where ∂j_τ stands for the generalized gradient in the sense of Clarke of a given function j_τ which is locally Lipschitz with respect to the last variable. From the mechanical viewpoint, the boundary condition (4.4) can be regarded as a generalized multivalued and non-monotone frictional law, and the energy potential function j_τ is considered to depend on the concentration field $y(t,\boldsymbol{x})$. This friction law for the fluid is reasonable since in the process of industrial production, the concentration of a chemical solution or a chemical substance will directly affect the friction of the fluid on (a part of) the boundary. The initial velocity of the fluid is prescribed by $\boldsymbol{u}_0$, i.e.,

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{4.5}$$

In the model under consideration, a chemical reaction-diffusion effect is involved in the incompressible fluid flow. On one hand, the underlying stochastic process for subdiffusion and superdiffusion is usually given by continuous-time random walk and Lévy process, respectively, and the corresponding macroscopic model for the probability density function of the particle appearing at certain time instance t and location $\boldsymbol{x}$ is given by a diffusion model with a fractional-order derivative in time or in space. On the other hand, we also consider the case when the velocity of the motion of the fluid will result in a rate change of the reaction–diffusion process. Thus, we use a fractional reaction-diffusion equation to describe the evolution of the concentration field (see [52], Sections 4 and 5, and [53], [54]):

$${}_0^C D_0^\alpha y(t) - k\Delta y = h(\boldsymbol{x}, t, y(t), \boldsymbol{u}(t)) \quad \text{in } \mathcal{Q}, \tag{4.6}$$

where $\alpha \in (0,1)$ is given and $k > 0$ denotes the diffusion coefficient; for convenience, we take $k = 1$. The initial concentration of a chemical substance is given by

$$y(0) = y_0 \quad \text{in } \Omega. \tag{4.7}$$

The boundary condition for the concentration field y is

$$\frac{\partial y(t)}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \Sigma. \tag{4.8}$$

Summarizing the relations (4.1)–(4.8), we have the following initial-boundary value problem for the coupled system of the nonstationary Navier–Stokes equations and a fractional reaction–diffusion equation.

Problem 4.1. Find a velocity field $\boldsymbol{u}\colon \Omega \times [0,T] \to \mathbb{R}^d$, a pressure field $\pi\colon \Omega \times [0,T] \to \mathbb{R}$, and a concentration field $y\colon \Omega \times [0,T] \to \mathbb{R}$ such that

$$\frac{\partial \boldsymbol{u}(t)}{\partial t} - \mu\left(\frac{\|y(t)\|_{L^2(\Omega)}}{|\Omega|}\right)\Delta \boldsymbol{u}(t) + (\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t) + \nabla\pi(t) = \boldsymbol{f}(\boldsymbol{x},t,y(t)) \quad \text{in } \mathcal{Q}, \tag{4.9}$$

$$\nabla\cdot\boldsymbol{u}(t) = 0 \quad \text{in } \mathcal{Q}, \tag{4.10}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Sigma_D, \tag{4.11}$$

$$u_\nu(t) = 0 \quad \text{on } \Sigma_C, \tag{4.12}$$

$$-\boldsymbol{S}_\tau(t) \in \partial j_\tau(\boldsymbol{x}, y(t), \boldsymbol{u}_\tau(t)) \quad \text{on } \Sigma_C, \tag{4.13}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega, \tag{4.14}$$

and

$${}_0^C D_t^\alpha y(t) - \Delta y = h(\boldsymbol{x},t,y(t),\boldsymbol{u}(t)) \quad \text{in } \mathcal{Q}, \tag{4.15}$$

$$\frac{\partial y(t)}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \Sigma, \tag{4.16}$$

$$y(0) = y_0 \quad \text{in } \Omega. \tag{4.17}$$

In the study of Problem 4.1, we impose the following conditions.

Condition $H(f)$**.** $\boldsymbol{f}\colon \Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}^d$ is such that

(i) for each $r\in\mathbb{R}$, the function $(\boldsymbol{x},t)\mapsto \boldsymbol{f}(\boldsymbol{x},t,r)$ belongs to $L^2(\Omega\times(0,T))$;

(ii) for a.e. $(\boldsymbol{x},t)\in\Omega\times(0,T)$, the function $\mathbb{R}\ni r\mapsto \boldsymbol{f}(\boldsymbol{x},t,r)\in\mathbb{R}^d$ is continuous;

(iii) there exists a function $c_f\in L^2(\Omega\times[0,T])_+$ such that $\|\boldsymbol{f}(\boldsymbol{x},t,r)\|_{\mathbb{R}^d}\leqslant c_f(\boldsymbol{x},t)$ for all $(\boldsymbol{x},t,r)\in\Omega\times[0,T]\times\mathbb{R}$.

Condition $H(j_\tau)$**.** $j_\tau\colon \Gamma_C\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ is such that

(i) $j_\tau(\,\cdot\,,r,\boldsymbol{\xi})$ is measurable on Γ_C for all $(r,\boldsymbol{\xi})\in\mathbb{R}\times\mathbb{R}^d$ and $j_\tau(\,\cdot\,,0,\boldsymbol{0})\in L^1(\Gamma_C)$;

(ii) $j_\tau(\boldsymbol{x},r,\cdot\,)$ is locally Lipschitz continuous on $\mathbb{R}^d$ for all $r\in\mathbb{R}$ and a.e. $\boldsymbol{x}\in\Gamma_C$;

(iii) there exists $c_{j_\tau}>0$ such that $\|\partial j_\tau(\boldsymbol{x},r,\boldsymbol{\xi})\|_{\mathbb{R}^d}\leqslant c_{j_\tau}(1+\|\boldsymbol{\xi}\|_{\mathbb{R}^d})$ for all $(r,\boldsymbol{\xi})\in\mathbb{R}\times\mathbb{R}^d$ and a.e. $\boldsymbol{x}\in\Gamma_C$;

(iv) either $j_\tau(\boldsymbol{x},r,\cdot\,)$ or $-j_\tau(\boldsymbol{x},r,\cdot\,)$ is regular for a.e. $\boldsymbol{x}\in\Gamma_C$ and all $r\in\mathbb{R}$;

(v) $(r,\boldsymbol{\xi})\mapsto j_\tau^0(\boldsymbol{x},r,\boldsymbol{\xi};\boldsymbol{\eta})$ is upper semicontinuous for all $\boldsymbol{\eta}\in\mathbb{R}^d$ and a.e. $\boldsymbol{x}\in\Gamma_C$, where j_τ^0 denotes the Clarke directional derivative of $\boldsymbol{\xi}\mapsto j_\tau(\boldsymbol{x},r,\boldsymbol{\xi})$ in direction $\boldsymbol{\eta}$;

(vi) there exist $\theta\in[1,2)$, $d_{j_\tau}>0$ and $e_{j_\tau}>0$ such that $j_\tau^0(\boldsymbol{x},r,\boldsymbol{\xi};-\boldsymbol{\xi})\leqslant d_{j_\tau}+e_{j_\tau}\|\boldsymbol{\xi}\|_{\mathbb{R}^d}^\theta$ for a.e. $\boldsymbol{x}\in\Gamma_C$, all $r\in\mathbb{R}$ and all $\boldsymbol{\xi}\in\mathbb{R}^d$.

Condition $H(h)$**.** $h\colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) for all $(r, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^d$, the function $(\boldsymbol{x}, t) \mapsto h(\boldsymbol{x}, t, r, \boldsymbol{\xi})$ is measurable on $\Omega \times [0,T]$;

(ii) there exists a function $\varphi \in L_+^{1/\beta}(0,T)$ with $\beta \in (0, \alpha)$ such that

$$\|\varphi\|_{L^{1/\beta}(0,T)} T^{(1+\zeta)(1-\beta)} < \Gamma(\alpha)(1+\zeta)^{1-\beta}$$

with $\zeta = (\alpha - 1)/(1-\beta)$, $\boldsymbol{\xi} \mapsto h(\boldsymbol{x}, t, r, \boldsymbol{\xi})$ is continuous on $\mathbb{R}^d$, and

$$|h(\boldsymbol{x}, t, r_1, \boldsymbol{\xi}) - h(\boldsymbol{x}, t, r_2, \boldsymbol{\xi})| \leqslant \varphi(t)|r_1 - r_2|$$

for all $r_1, r_2 \in \mathbb{R}$, $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $(\boldsymbol{x}, t) \in \Omega \times [0,T]$;

(iii) there exist $c_h \in L^2(\Omega)_+$ and $d_h \in L^\infty(0,T)_+$ such that $|h(\boldsymbol{x}, t, 0, \boldsymbol{\xi})| \leqslant c_h(\boldsymbol{x}) + d_h(t)\|\boldsymbol{\xi}\|_{\mathbb{R}^d}$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and a.e. $(\boldsymbol{x}, t) \in \Omega \times [0,T]$.

Condition $H(\mu)$**.** $\mu\colon \mathbb{R}_+ \to (0, +\infty)$ is continuous and $c_\mu \leqslant \mu(s) \leqslant d_\mu$ for all $s \in \mathbb{R}_+$ with constants $0 < c_\mu \leqslant d_\mu$.

To present a variational formulation of Problem 4.1, we need function spaces

$$V = \overline{U}^{H^1(\Omega;\mathbb{R}^d)}, \qquad H = \overline{K}^{L^2(\Omega;\mathbb{R}^d)},$$

where

$$\begin{aligned} U &= \{\boldsymbol{u} \in C^\infty(\overline{\Omega}; \mathbb{R}^d) \mid \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_D,\ u_\nu = 0 \text{ on } \Gamma_C\}, \\ K &= \{\boldsymbol{u} \in C^\infty(\overline{\Omega}; \mathbb{R}^d) \mid \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega,\ u_\nu = 0 \text{ on } \Gamma_C\}. \end{aligned}$$

It is well known that V with the standard norm $\|\cdot\|_{H^1(\Omega;\mathbb{R}^d)}$ is a separable and reflexive Banach space. The Korn inequality $\|\boldsymbol{u}\|_{H^1(\Omega;\mathbb{R}^d)} \leqslant c\|\boldsymbol{D}(\boldsymbol{u})\|_{L^2(\Omega;\mathbb{R}^d)}$ for all $\boldsymbol{u} \in V$ with some constant $c > 0$, shows that the norm $\|\cdot\|_V$ defined by $\|\boldsymbol{u}\|_V = \|\boldsymbol{D}(\boldsymbol{u})\|_{L^2(\Omega;\mathbb{R}^d)}$ for $\boldsymbol{u} \in V$, is equivalent to the norm $\|\boldsymbol{u}\|$. In the following, we adopt $\|\cdot\|_V$ as the norm in the space V, and the duality pairing between V^* and V is defined by $\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{V^*\times V} = \int_\Omega \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v})\, d\boldsymbol{x}$ for $\boldsymbol{u}, \boldsymbol{v} \in V$. Moreover, the space H, equipped with the norm $\|\boldsymbol{u}\|_H = \|\boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^d)}$ for $\boldsymbol{u} \in H$, is a separable Hilbert space. We have $V \subset H \simeq H^* \subset V^*$, and the embedding of V to H is dense, continuous, and compact. This means that (V, H, V^*) forms an evolution triple of spaces.

To derive the variational formulation of Problem 4.1, we assume that the problem has a sufficiently smooth solution $(\boldsymbol{u}, \pi, y)$. Let $\boldsymbol{v} \in V$, $t \in [0,T]$. A standard procedure based on (4.9)–(4.14) gives

$$\begin{aligned} &\int_\Omega \frac{\partial \boldsymbol{u}(t)}{\partial t} \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_\Omega \mu\biggl(\frac{\|y(t)\|_{L^2(\Omega)}}{|\Omega|}\biggr) \nabla \boldsymbol{u}(t) : \nabla \boldsymbol{v}\, d\boldsymbol{x} \\ &\qquad + \int_{\Gamma_C} j_\tau^0(\boldsymbol{x}, y(t), \boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau)\, d\Gamma + \int_\Omega \bigl((\boldsymbol{u}(t) \cdot \nabla)\boldsymbol{u}(t)\bigr) \cdot \boldsymbol{v}\, d\boldsymbol{x} \\ &\geqslant \int_\Omega \boldsymbol{f}(t, y(t)) \cdot \boldsymbol{v}\, d\boldsymbol{x} \quad \text{for all } \boldsymbol{v} \in V \text{ and a.e. } t \in [0,T]. \end{aligned} \tag{4.18}$$

Let $E = L^2(\Omega)$. We introduce an operator $S\colon D(S) \subset E \to E$ and a function $g\colon [0,T] \times E \times H \to E$ by

$$Su = \Delta u \quad \text{for all } u \in D(S), \tag{4.19}$$

$$g(t, y, \boldsymbol{v})(\boldsymbol{x}) = h(\boldsymbol{x}, t, y(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{x})) \quad \text{in } \Omega \tag{4.20}$$

for all $t \in [0,T]$, $y \in E$ and $\boldsymbol{v} \in H$, where the domain $D(S)$ of S is defined by

$$D(S) = \left\{ y \in H^2(\Omega) = W^{2,2}(\Omega) \;\middle|\; \frac{\partial y}{\partial \boldsymbol{\nu}} = 0 \text{ on } \Gamma \right\}.$$

With the above notation, the reaction-diffusion system (4.15)–(4.17) can be reformulated as

$$\begin{cases} {}_0^C D_t^\alpha y(t) = Sy(t) + g(t, y(t), \boldsymbol{u}(t)) \quad \text{for a.e. } t \in [0,T], \\ y(0) = y_0. \end{cases} \tag{4.21}$$

Remark 4.2. Since Ω is a C^1 domain, by [55], Theorem 4.2.2, the operator $S\colon D(S) \subset E \to E$ defined in (4.19) is the generator of a C_0-semigroup $\{\mathscr{S}(t)\}_{t \geqslant 0}$ of contractions on E, i.e. $\|\mathscr{S}(t)\| \leqslant 1$ for all $t \geqslant 0$, and for each $t > 0$ the operator $\mathscr{S}(t)$ is compact.

Next, let $Z = H^{1-\delta}(\Omega; \mathbb{R}^d)$ for any $\delta \in (0, 1/2)$. We need the following operators: the embedding operator $\gamma_1\colon V \to Z$; the trace operator $\gamma_2\colon Z \to H^{1/2-\delta}(\Gamma_C; \mathbb{R}^d)$; the embedding operator $\gamma_3\colon H^{1/2-\delta}(\Gamma_C; \mathbb{R}^d) \to L^2(\Gamma_C; \mathbb{R}^d)$; the trace operator $\gamma\colon V \to L^2(\Gamma_C; \mathbb{R}^d)$ defined by $\gamma = \gamma_3 \circ \gamma_2 \circ \gamma_1$.

Remark 4.3. The embedding operator $\gamma_1\colon V \to Z$ is compact, and both $\gamma_2\colon Z \to H^{1/2-\delta}(\Gamma_C; \mathbb{R}^d)$ and $\gamma_3\colon H^{1/2-\delta}(\Gamma_C; \mathbb{R}^d) \to L^2(\Gamma_C; \mathbb{R}^d)$ are continuous. Thus, the trace operator $\gamma\colon V \to L^2(\Gamma_C; \mathbb{R}^d) := X$ is also compact.

Combining (4.14), (4.17), (4.18), and (4.21), we arrive at the following variational formulation of Problem 4.1.

Problem 4.4. Find a velocity field $\boldsymbol{u}\colon \Omega \times [0,T] \to \mathbb{R}^d$ and a concentration field $y\colon \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \displaystyle\int_\Omega \frac{\partial \boldsymbol{u}(t)}{\partial t} \cdot \boldsymbol{v}\, d\boldsymbol{x} + \mu\Big(\frac{\|y(t)\|_{L^2(\Omega)}}{|\Omega|}\Big) \int_\Omega \nabla \boldsymbol{u}(t) : \nabla \boldsymbol{v}\, d\boldsymbol{x} \\ \qquad \displaystyle + \int_{\Gamma_C} j_\tau^0(\boldsymbol{x}, y(t), \boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau)\, d\Gamma + \int_\Omega \big((\boldsymbol{u}(t) \cdot \nabla)\boldsymbol{u}(t)\big) \cdot \boldsymbol{v}\, d\boldsymbol{x} \\ \quad \displaystyle \geqslant \int_\Omega \boldsymbol{f}(t, y(t)) \cdot \boldsymbol{v}\, d\boldsymbol{x} \quad \text{for all } \boldsymbol{v} \in V \text{ and for a.e. } t \in [0,T], \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \\ {}_0^C D_t^\alpha y(t) = Sy(t) + g(t, y(t), \boldsymbol{u}(t)) \quad \text{for a.e. } t \in [0,T], \\ y(0) = y_0. \end{cases} \tag{4.22}$$

Definition 4.5. A pair of functions $(y, \boldsymbol{u})$ with $y \in C([0,T];E)$ and $\boldsymbol{u} \in \mathcal{W}$ is called *a mild solution of Problem* 4.4 if

$$y(t) = \mathscr{E}(t)y_0 + \int_0^t (t-s)^{\alpha-1}\mathscr{F}(t-s)g(s,y(s),\boldsymbol{u}(s))\,ds \quad \text{for all } t \in [0,T],$$

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial \boldsymbol{u}(t)}{\partial t}\cdot \boldsymbol{v}(t)\,d\boldsymbol{x} + \mu\Big(\frac{\|y(t)\|_{L^2(\Omega)}}{|\Omega|}\Big)\int_\Omega \nabla \boldsymbol{u}(t) : \nabla \boldsymbol{v}(t)\,d\boldsymbol{x} \\
&\quad + \int_\Omega \big((\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t)\big)\cdot \boldsymbol{v}(t)\,d\boldsymbol{x}\,dt + \int_0^T\int_{\Gamma_C} j_\tau^0(\boldsymbol{x}, y(t), \boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau(t))\,d\Gamma\,dt \\
&\geqslant \int_0^T \int_\Omega \boldsymbol{f}(t,y(t))\cdot \boldsymbol{v}(t)\,d\boldsymbol{x}\,dt \quad \text{for all } \boldsymbol{v} \in \mathcal{V},
\end{aligned} \tag{4.23}$$

and $\boldsymbol{u}(0) = \boldsymbol{u}_0$.

To prove solvability of Problem 4.4, we introduce an intermediate problem.

Problem 4.6. Find functions $\boldsymbol{u}\colon [0,T] \to V$ and $y\colon [0,T] \to E$ such that

$$\begin{cases}
\langle \boldsymbol{u}'(t), \boldsymbol{v}\rangle + \langle F(y(t), \boldsymbol{u}(t)), \boldsymbol{v}\rangle + J^0(y(t), \gamma\boldsymbol{u}(t); \gamma\boldsymbol{v}) \\
\qquad \geqslant \langle G(t, y(t)), \boldsymbol{v}\rangle \ \text{ for all } \boldsymbol{v} \in V \text{ and a.e. } t \in [0,T], \\
\boldsymbol{u}(0) = \boldsymbol{u}_0, \\
{}_0^C D_t^\alpha y(t) = Sy(t) + g(t, y(t), \boldsymbol{u}(t)) \ \text{ for a.e. } t \in [0,T], \\
y(0) = y_0,
\end{cases}$$

In Problem 4.6, the operators $G\colon [0,T]\times E \to V^*$, $F\colon E \times V \to V^*$ and the function $J\colon E \times X \to \mathbb{R}$ are given by

$$G(t,y) = \boldsymbol{f}(t,y) \quad \text{for } y \in E \text{ and } t \in [0,T], \tag{4.24}$$

$$\langle F(y,\boldsymbol{u}), \boldsymbol{v}\rangle = a(y,\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) \quad \text{for } \boldsymbol{u},\boldsymbol{v} \in V \text{ and } y \in E, \tag{4.25}$$

$$J(y,\boldsymbol{w}) = \int_{\Gamma_C} j_\tau(\boldsymbol{x}, y(\boldsymbol{x}), \boldsymbol{w}(\boldsymbol{x})_\tau)\,d\Gamma \quad \text{for } y \in E \text{ and } \boldsymbol{w} \in X. \tag{4.26}$$

Here, the forms $a\colon E\times V\times V \to \mathbb{R}$ and $b\colon V\times V\times V \to \mathbb{R}$ are defined by

$$\begin{aligned}
a(y,\boldsymbol{u},\boldsymbol{v}) &= \mu\Big(\frac{\|y\|_{L^2(\Omega)}}{|\Omega|}\Big)\int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v}\,d\boldsymbol{x}, \qquad \boldsymbol{u},\boldsymbol{v} \in V, \quad y \in E, \\
b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &= \int_\Omega \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\big)\cdot\boldsymbol{w}\,d\boldsymbol{x}, \qquad \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in V.
\end{aligned}$$

Based on Remark 4.2 and Definition 4.5, we now define the notion of a mild solution to Problem 4.6.

Definition 4.7. A pair of functions $(y, \boldsymbol{u})$ with $y \in C([0,T];E)$ and $\boldsymbol{u} \in \mathcal{W}$ is called a *mild solution to Problem* 4.6 if

$$y(t) = \mathscr{E}(t)y_0 + \int_0^t (t-s)^{\alpha-1}\mathscr{F}(t-s)g(s,y(s),\boldsymbol{u}(s))\,ds$$

for all $t \in [0, T]$,

$$\int_0^T \big(\langle \boldsymbol{u}'(t), \boldsymbol{v}(t)\rangle + \langle F(y(t), \boldsymbol{u}(t)), \boldsymbol{v}(t)\rangle\big)\, dt + \int_0^T J^0(y(t), \gamma \boldsymbol{u}(t); \boldsymbol{v}(t))\, dt \geqslant \int_0^T \langle G(t, y(t)), \boldsymbol{v}(t)\rangle\, dt \tag{4.27}$$

for all $\boldsymbol{v} \in \mathcal{V}$ and $\boldsymbol{u}(0) = \boldsymbol{u}_0$.

In the proof of existence of a mild solution to Problem 4.6, we need some properties of the functions F, G, J, and g.

Lemma 4.8. *Under the hypothesis $H(h)$, the function g defined in* (4.20) *satisfies the hypothesis $H(g)$.*

Proof. Hypotheses $H(h)$ (i) and (ii) imply directly that for each $(y, \boldsymbol{v}) \in E \times H$ the function $t \mapsto g(t, y, \boldsymbol{v})$ is measurable on $[0, T]$ and the function $\boldsymbol{v} \mapsto g(t, y, \boldsymbol{v})$ is continuous. Hence conditions $H(g)$ (i) and (ii) hold. Let $y_1, y_2 \in E$ and $\boldsymbol{v} \in V$ be arbitrary. Let D be a bounded subset of H. It follows from $H(h)$ (ii) and (iii) that

$$\begin{aligned} \|g(t, y_1, \boldsymbol{v}) - g(t, y_2, \boldsymbol{v})\|_E &= \left(\int_\Omega |h(\boldsymbol{x}, t, y_1(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{x})) - h(\boldsymbol{x}, t, y_2(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{x}))|^2\, d\boldsymbol{x}\right)^{1/2} \\ &\leqslant \left(\int_\Omega \varphi(t)^2 |y_1(\boldsymbol{x}) - y_2(\boldsymbol{x})|^2\, d\boldsymbol{x}\right)^{1/2} \leqslant \varphi(t)\|y_1 - y_2\|_E, \end{aligned}$$

and

$$\begin{aligned} \|g(t, 0, \boldsymbol{v})\|_E &\leqslant \left(\int_\Omega \big(c_h(\boldsymbol{x}) + d_h(t)\|\boldsymbol{v}(\boldsymbol{x})\|_{\mathbb{R}^d}\big)^2\, d\boldsymbol{x}\right)^{1/2} \\ &\leqslant \sqrt{2}\left(\int_\Omega \big(c_h(\boldsymbol{x})^2 + d_h(t)^2\|\boldsymbol{v}(\boldsymbol{x})\|^2_{\mathbb{R}^d}\big)\, d\boldsymbol{x}\right)^{1/2} \\ &\leqslant \sqrt{2}\,\big(\|c_h\|_{L^2(\Omega)} + d_h(t)\|\boldsymbol{v}\|_H\big) \leqslant \sqrt{2}\,\big(\|c_h\|_{L^2(\Omega)} + d_h(t)M_D\big), \end{aligned}$$

where $M_D > 0$ is such that $\|D\|_H \leqslant M_D$. Therefore, $H(g)$ (iii) holds with functions $\rho_g = \varphi$ and $\varphi_D(\cdot) = \sqrt{2}\,(\|c_h\|_{L^2(\Omega)} + d_h(\cdot)M_D)$. The lemma is proved.

Lemma 4.9. *The operator $F\colon E \times V \to V^*$ defined by* (4.25) *satisfies $H(F)$.*

Proof. The continuity of μ (see the hypothesis $H(\mu)$) implies that $y \mapsto a(y, \boldsymbol{u}, \boldsymbol{v})$ is continuous. Thanks to $H(\mu)$, we know that the function $\boldsymbol{u} \mapsto a(y, \boldsymbol{u}, \boldsymbol{v})$ is linear and bounded for all $(y, \boldsymbol{v}) \in E \times V$, and is thus weakly continuous. The functional

$$\boldsymbol{u} \mapsto b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\big) \cdot \boldsymbol{v}\, d\boldsymbol{x}$$

is weakly continuous too; see [56], Proposition 2.6. Hence, $H(F)$ (i) holds.

Let $\boldsymbol{u} \in V$. Using the Poincaré inequality and the relation

$$2\int_\Omega \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v})\, d\boldsymbol{x} = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}\, d\boldsymbol{x}, \tag{4.28}$$

we can find a constant $c_0 > 0$ such that

$$\begin{aligned}\|F(y,\boldsymbol{u})\|_{V^*} &= \sup_{\boldsymbol{v}\in V,\, \|\boldsymbol{v}\|_V=1} \langle F(y,\boldsymbol{u}),\boldsymbol{v}\rangle \leqslant \sup_{\boldsymbol{v}\in V,\, \|\boldsymbol{v}\|_V=1} \mu\Big(\frac{\|y\|_{L^2(\Omega)}}{|\Omega|}\Big)\|\boldsymbol{u}\|_V\|\boldsymbol{v}\|_V \\ &\quad + \sup_{\boldsymbol{v}\in V,\, \|\boldsymbol{v}\|_V=1} \|\boldsymbol{u}\|_{L^4(\Omega;\mathbb{R}^d)}\|\nabla\boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\|\boldsymbol{v}\|_{L^4(\Omega;\mathbb{R}^d)} \\ &\leqslant 2d_\mu\|\boldsymbol{u}\|_V + c_0\|\boldsymbol{u}\|_{L^4(\Omega;\mathbb{R}^d)}\|\nabla\boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^{d\times d})}. \qquad (4.29)\end{aligned}$$

In deriving (4.29), we used the Hölder inequality, the Sobolev embedding theorem and the hypothesis $H(\mu)$. From (4.29) together with the Korn and Cauchy–Schwarz inequalities, we find the existence of a constant $c_F > 0$ such that $\|F(y,\boldsymbol{u})\|_{V^*} \leqslant c_F\times(1+\|\boldsymbol{u}\|_V^2)$ for all $\boldsymbol{u}\in V$. Thus, for any $\boldsymbol{u}\in\mathcal{V}\cap L^\infty(0,T;H)$, it follows from (4.29) that

$$\begin{aligned}\|\mathcal{F}(z,\boldsymbol{u})\|_{\mathcal{V}^*} &\leqslant \Big(\int_0^T c_1\big(\|\boldsymbol{u}(t)\|_V + \|\boldsymbol{u}(t)\|_V\|\boldsymbol{u}(t)\|_H\big)^2\,dt\Big)^{1/2} \\ &\leqslant \sqrt{2c_1}\Big(\int_0^T \big(\|\boldsymbol{u}(t)\|_V^2 + \|\boldsymbol{u}(t)\|_V^2\|\boldsymbol{u}(t)\|_H^2\big)\,dt\Big)^{1/2} \\ &\leqslant \sqrt{2c_1}\,\|\boldsymbol{u}\|_{\mathcal{V}} + \sqrt{2c_1}\,\|\boldsymbol{u}\|_{\mathcal{V}}\|\boldsymbol{u}\|_{L^\infty(0,T;H)} = c_{\mathcal{F}}\|\boldsymbol{u}\|_{\mathcal{V}}(1+\|\boldsymbol{u}\|_{L^\infty(0,T;H)})\end{aligned}$$

for some $c_1 > 0$. Hence, $H(F)$ (ii)$_2$ is proved with $c_{\mathcal{F}} = \sqrt{2c_1}$.

Next, for any $\boldsymbol{u}\in U$, the Green formula yields that

$$\begin{aligned}b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}) &= \int_\Omega \sum_{i,j=1}^d u_i u_{j,i} u_j\,d\boldsymbol{x} \\ &= \int_\Omega \sum_{i,j=1}^d u_i\Big(\frac{u_j^2}{2}\Big)_{,i}\,d\boldsymbol{x} = -\frac{1}{2}\int_\Omega (\nabla\cdot\boldsymbol{u})\sum_{j=1}^d u_j^2\,d\boldsymbol{x} + \frac{1}{2}\int_\Gamma u_\nu \sum_{j=1}^d u_j^2\,d\Gamma = 0,\end{aligned}$$

where the last equality follows from the facts that $\nabla\cdot\boldsymbol{u} = 0$ in Ω, $\boldsymbol{u} = 0$ on Γ_D and $u_\nu = 0$ on Γ_C. Since the mapping $\boldsymbol{u}\mapsto\int_\Omega((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\boldsymbol{u}\,d\boldsymbol{x}$ is continuous and the embedding $U\subset V$ is dense, we have

$$b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}) = 0 \quad \text{for all } \boldsymbol{u}\in V.$$

This result combined with (4.28) and the hypothesis $H(\mu)$ implies $H(F)$ (iii) with $m_F = 2c_\mu$ and $l_F = d_F = e_F = 0$.

Let $\{\boldsymbol{u}_n\}\subset\mathcal{V}\cap L^\infty(0,T;H)$ and $\{z_n\}\subset\mathcal{E}$ be such that $\{\boldsymbol{u}_n\}$ is bounded in $L^\infty(0,T;H)$, $z_n\to z$ in $\mathcal{E}$ for some $z\in\mathcal{E}$, and $\boldsymbol{u}_n\rightharpoonup\boldsymbol{u}$ in $\mathcal{V}$ and $\mathcal{H}$ for some $\boldsymbol{u}\in\mathcal{V}$. Note that

$$\begin{aligned}&\Big|\int_0^T a(z_n(t),\boldsymbol{u}_n(t),\boldsymbol{v}(t))\,dt - \int_0^T a(z(t),\boldsymbol{u}(t),\boldsymbol{v}(t))\,dt\Big| \\ &\quad\leqslant \int_0^T \Big|\mu\Big(\frac{\|z_n(t)\|_{L^2(\Omega)}}{|\Omega|}\Big) - \mu\Big(\frac{\|z(t)\|_{L^2(\Omega)}}{|\Omega|}\Big)\Big|\int_\Omega \|\nabla\boldsymbol{u}_n(t)\|_{\mathbb{R}^{d\times d}}\|\nabla\boldsymbol{v}(t)\|_{\mathbb{R}^{d\times d}}\,d\boldsymbol{x}\,dt \\ &\qquad + \Big|\int_0^T \mu\Big(\frac{\|y\|_{L^2(\Omega)}}{|\Omega|}\Big)\int_\Omega (\nabla\boldsymbol{u}_n(t) - \nabla\boldsymbol{u}(t)) : \nabla\boldsymbol{v}(t)\,d\boldsymbol{x}\Big|.\end{aligned}$$

We use the continuity of μ and apply the Lebesgue dominated convergence theorem to find

$$\int_0^T a(z_n(t), \boldsymbol{u}_n(t), \boldsymbol{v}(t))\, dt \to \int_0^T a(z(t), \boldsymbol{u}(t), \boldsymbol{v}(t))\, dt,$$

because $V \ni \boldsymbol{w} \mapsto a(y, \boldsymbol{w}, \boldsymbol{v}) \in \mathbb{R}$ is a bounded and linear function for all $(y, \boldsymbol{v}) \in E \times V$. Moreover, as in Ahmed [57], we have

$$\int_0^T b(\boldsymbol{u}_n(t), \boldsymbol{u}_n(t), \boldsymbol{v}(t))\, dt \to \int_0^T b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v}(t))\, dt.$$

Therefore, we conclude that $\mathcal{F}(z_n, u_n) \rightharpoonup \mathcal{F}(y, u)$ in $\mathcal{V}^*$, which completes the proof.

Lemma 4.10. *The function $J \colon E \times X \to \mathbb{R}$ defined by* (4.26) *satisfies $H(J)$.*

Proof. Since $j_\tau(\,\cdot\,, r, \boldsymbol{v})$ is measurable on Γ_C for all $(r, \boldsymbol{v}) \in \mathbb{R} \times \mathbb{R}^d$ with $j_\tau(\,\cdot\,, 0, \mathbf{0}) \in L^1(\Gamma_C)$, and $j_\tau(\boldsymbol{x}, r, \cdot\,)$ is locally Lipschitz on $\mathbb{R}^d$ for all $r \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Gamma_C$, it follows from [41], Theorem 3.47 (i), that $H(J)$ (i) holds.

By the definition of J (see (4.26)) and [41], Theorem 3.47 (v), we know that for any $\boldsymbol{\xi} \in \partial J(y, \boldsymbol{w})$, there exists an $\boldsymbol{\eta} \in L^2(\Gamma_C; \mathbb{R}^d)$ such that

$$\langle \boldsymbol{\xi}, \boldsymbol{v} \rangle_{X^* \times X} = \int_{\Gamma_C} \boldsymbol{\eta}(\boldsymbol{x}) \cdot \boldsymbol{v}_\tau(\boldsymbol{x})\, d\Gamma \quad \text{for all } \boldsymbol{v} \in X.$$

The growth condition $H(j_\tau)$ (iii) ensures that

$$\begin{aligned} |\langle \boldsymbol{\xi}, \boldsymbol{v} \rangle_{X^* \times X}| &\leqslant \int_{\Gamma_C} c_{j_\tau} (1 + \|\boldsymbol{w}(\boldsymbol{x})_\tau\|_{\mathbb{R}^d}) \|\boldsymbol{v}(\boldsymbol{x})_\tau\|_{\mathbb{R}^d}\, d\Gamma \\ &\leqslant c_{j_\tau} \big(\sqrt{\operatorname{meas}(\Gamma_C)} + \|\boldsymbol{w}\|_X\big) \|\boldsymbol{v}\|_X. \end{aligned}$$

Thus, $H(J)$ (ii) is satisfied with $\alpha_J = c_{j_\tau} \sqrt{\operatorname{meas}(\Gamma_C)}$ and $\beta_J = c_{j_\tau}$.

By $H(j_\tau)$ (iv), without loss of generality, we may assume that $j_\tau(\boldsymbol{x}, r, \cdot\,)$ is regular for all $r \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Gamma_C$, since the proof is similar in the case when $-j_\tau(\boldsymbol{x}, r, \cdot\,)$ is regular for all $r \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Gamma_C$. Invoking [41], Theorem 3.47 (viii), we deduce that $J(y, \cdot\,)$ is regular for all $y \in E$ and

$$\begin{aligned} J^0(y, \boldsymbol{w}; \boldsymbol{v}) &= \int_\Omega j_\tau^0(\boldsymbol{x}, y(\boldsymbol{x}), \boldsymbol{w}(\boldsymbol{x})_\tau; \boldsymbol{v}(\boldsymbol{x})_\tau)\, d\Gamma \\ &= \int_\Omega j_\tau'(\boldsymbol{x}, y(\boldsymbol{x}), \boldsymbol{w}(\boldsymbol{x})_\tau; \boldsymbol{v}(\boldsymbol{x})_\tau)\, d\Gamma = J'(y, \boldsymbol{w}; \boldsymbol{v}). \end{aligned} \tag{4.30}$$

Let $\{(y_n, \boldsymbol{w}_n)\} \subset E \times X$ be such that $y_n \to y$ in E and $\boldsymbol{w}_n \to \boldsymbol{w}$ in X as $n \to \infty$. We may assume, by passing to a subsequence if necessary, that $y_n(\boldsymbol{x}) \to y(\boldsymbol{x})$ and $\boldsymbol{w}_n(\boldsymbol{x}) \to \boldsymbol{w}(\boldsymbol{x})$ as $n \to \infty$ for a.e. $\boldsymbol{x} \in \Gamma_C$. The growth condition $H(j_\tau)$ (iii) guarantees that there exists a function $h_v \in L^1_+(\Gamma_C)$ such that $|j_\tau^0(\boldsymbol{x}, y_n(\boldsymbol{x}), \boldsymbol{w}_n(\boldsymbol{x})_\tau; \boldsymbol{v}(\boldsymbol{x})_\tau)| \leqslant h_v(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Gamma_C$. Hence, by the Fatou lemma (e.g. [41], Theorem 1.64), the hypothesis $H(j_\tau)$ (v), and (4.30), we have for all $\boldsymbol{v} \in X$,

$$\begin{aligned}\limsup_{n\to\infty} J^0(y_n, \boldsymbol{w}_n; \boldsymbol{v}) &= \limsup_{n\to\infty} \int_\Omega j_\tau^0(\boldsymbol{x}, y_n(\boldsymbol{x}), \boldsymbol{w}_n(\boldsymbol{x})_\tau; \boldsymbol{v}(\boldsymbol{x})_\tau)\, d\Gamma \\ &\leqslant \int_\Omega \limsup_{n\to\infty} j_\tau^0(\boldsymbol{x}, y_n(\boldsymbol{x}), \boldsymbol{w}_n(\boldsymbol{x})_\tau; \boldsymbol{v}(\boldsymbol{x})_\tau)\, d\Gamma \\ &\leqslant \int_\Omega j_\tau^0(\boldsymbol{x}, y(\boldsymbol{x}), \boldsymbol{w}(\boldsymbol{x})_\tau; \boldsymbol{v}(\boldsymbol{x})_\tau)\, d\Gamma = J^0(y, \boldsymbol{w}; \boldsymbol{v}).\end{aligned}$$

By the hypothesis $H(j_\tau)$ (vi) and [41], Theorem 3.47 (iv), we get

$$\begin{aligned}J^0(y, \boldsymbol{w}; -\boldsymbol{w}) &\leqslant \int_{\Gamma_C} j_\tau^0(\boldsymbol{x}, y(\boldsymbol{x}), \boldsymbol{w}(\boldsymbol{x})_\tau; -\boldsymbol{w}(\boldsymbol{x})_\tau)\, d\Gamma \\ &\leqslant \int_{\Gamma_C} (d_{j_\tau} + e_{j_\tau} \|\boldsymbol{w}(\boldsymbol{x})_\tau\|_{\mathbb{R}^d}^\theta)\, d\Gamma \leqslant c_J + d_J \|\boldsymbol{w}\|_X^\theta\end{aligned}$$

for all $y \in E$ and $\boldsymbol{w} \in X$ with some $c_J \geqslant 0$ and $d_J > 0$, where the last inequality is obtained by applying the Hölder inequality. So, $H(J)$ (iv) is verified.

Lemma 4.11. *The function $G\colon [0,T] \times E \to V^*$ defined by* (4.24) *satisfies the hypothesis $H(G)$.*

Proof. Condition $H(G)$ (i) is a consequence of the hypothesis $H(f)$ (i). The continuity of $r \mapsto \boldsymbol{f}(\boldsymbol{x}, t, r)$ implies that for all $t \in [0,T]$, the function $y \mapsto G(t,y)$ is continuous, i.e., $H(G)$ (ii) is valid. We use the condition $H(f)$ (iii) to find

$$\|G(t,y)\|_{V^*}^2 = \int_\Omega \|\boldsymbol{f}(\boldsymbol{x}, t, y(\boldsymbol{x}))\|_{\mathbb{R}^d}^2\, dx \leqslant \int_\Omega c_f(\boldsymbol{x}, t)^2\, dx = \|c_f(\,\cdot\,, t)\|_{L^2(\Omega)}^2$$

for a.e. $t \in [0,T]$ and all $y \in E$. Therefore, $H(G)$ (iii) is verified by taking $\rho_G := c_f\sqrt{|\Omega|}$.

Finally, we provide an existence result for a mild solution to Problem 4.1.

Theorem 4.12. *Assume $H(f)$, $H(j_\tau)$, $H(h)$, $H(\mu)$, $y_0 \in E$, and $u_0 \in V$. Then Problem* 4.1 *has at least one mild solution.*

Proof. We apply Theorem 3.7 to prove this result. Let us examine all the assumptions stated in Theorem 3.7. It follows from Lemmas 4.8–4.11 that functions g, F, J and G satisfy conditions $H(g)$, $H(F)$, $H(J)$, $H(G)$, respectively. Remark 4.2 indicates that S satisfies $H(S)$ with $M_S = 1$. It remains to show that $H(\gamma)$ holds. By Remark 4.3, we see that γ is linear, bounded and compact. Let $\{\boldsymbol{u}_n\}$ be a bounded sequence in $M^{2,2}(0,T;V,V^*)$. Thanks to Proposition 2.7, $\{\boldsymbol{u}_n\}$ is relatively compact in $L^2(0,T;H^{1-\delta}(\Omega;\mathbb{R}^d))$ due to the compactness of the embedding of $M^{2,2}(0,T;V,V^*)$ to $L^2(0,T;H^{1-\delta}(\Omega;\mathbb{R}^d))$, where $\delta \in (0,1/2)$ is given in Remark 4.3. Passing to a subsequence if necessary, we may assume that $\overline{\gamma}_1\boldsymbol{u}_n \to \overline{\gamma}_1\boldsymbol{u}$ in $L^2(0,T;H^{1/2+\delta}(\Omega;\mathbb{R}^d))$ for some $\boldsymbol{u} \in L^2(0,T;H^{1/2+\delta}(\Omega;\mathbb{R}^d))$, where $\overline{\gamma}_1\colon \mathcal{V} \to L^2(0,T;H^{1/2+\delta}(\Omega;\mathbb{R}^d))$ is the Nemytskii operator corresponding to γ_1. Combining this with the continuity of γ_2 and γ_3 (see Remark 4.3), we conclude that $\overline{\gamma}\boldsymbol{u}_n \to \overline{\gamma}\boldsymbol{u}$ in $\mathcal{X}$. Hence, $H(\gamma)$ holds.

Consequently, all conditions of Theorem 3.7 are verified. Applying that theorem, we know that there exists a pair of functions $(y, \boldsymbol{u}) \in C([0,T];E) \times \mathcal{W}$ which solves Problem 4.6. Moreover, the inequality

$$J^0(y, \boldsymbol{w}; \boldsymbol{v}) \leqslant \int_{\Gamma_C} j_\tau^0(\boldsymbol{x}, y(\boldsymbol{x}), \boldsymbol{w}(\boldsymbol{x})_\tau; \boldsymbol{v}(\boldsymbol{x})_\tau)\, d\Gamma \quad \text{for all } y \in E, \quad \boldsymbol{v}, \boldsymbol{w} \in X$$

implies that $(y, \boldsymbol{u}) \in C([0,T];E) \times \mathcal{W}$ is also a mild solution to Problem 4.4.

Bibliography

[1] J. S. Pang and D. E. Stewart, "Differential variational inequalities", *Math. Program.* **113**:2, Ser. A (2008), 345–424.

[2] Z. H. Liu, D. Motreanu, and S. D. Zeng, "Nonlinear evolutionary systems driven by mixed variational inequalities and its applications", *Nonlinear Anal. Real World Appl.* **42** (2018), 409–421.

[3] Z. H. Liu, D. Motreanu, and S. D. Zeng, "On the well-posedness of differential mixed quasi-variational-inequalities", *Topol. Methods Nonlinear Anal.* **51**:1 (2018), 135–150.

[4] Z. H. Liu, S. D. Zeng, and D. Motreanu, "Evolutionary problems driven by variational inequalities", *J. Differential Equations* **260**:9 (2016), 6787–6799.

[5] Z. H. Liu, S. Migórski, and S. D. Zeng, "Partial differential variational inequalities involving nonlocal boundary conditions in Banach spaces", *J. Differential Equations* **263**:7 (2017), 3989–4006.

[6] X. J. Chen and Z. Y. Wang, "Differential variational inequality approach to dynamic games with shared constraints", *Math. Program.* **146**:1-2, Ser. A (2014), 379–408.

[7] T. D. Ke, N. V. Loi, and V. Obukhovskii, "Decay solutions for a class of fractional differential variational inequalities", *Fract. Calc. Appl. Anal.* **18**:3 (2015), 531–553.

[8] J. Gwinner, "On a new class of differential variational inequalities and a stability result", *Math. Program.* **139**:1-2, Ser. B (2013), 205–221.

[9] Z. H. Liu, N. V. Loi, and V. Obukhovskii, "Existence and global bifurcation of periodic solutions to a class of differential variational inequalities", *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **23**:7 (2013), 1350125.

[10] B. Brogliato and A. Tanwani, "Dynamical systems coupled with monotone set-valued operators: formalisms, applications, well-posedness, and stability", *SIAM Rev.* **62**:1 (2020), 3–129.

[11] X. J. Chen and Z. Y. Wang, "Convergence of regularized time-stepping methods for differential variational inequalities", *SIAM J. Optim.* **23**:3 (2013), 1647–1671.

[12] L. S. Han, A. Tiwari, M. K. Camlibel, and J. S. Pang, "Convergence of time-stepping schemes for passive and extended linear complementarity systems", *SIAM J. Numer. Anal.* **47**:5 (2009), 3768–3796.

[13] L. S. Han and J. S. Pang, "Non-Zenoness of a class of differential quasi-variational inequalities", *Math. Program.* **121**:1, Ser. A (2010), 171–199.

[14] Y. R. Jiang and Z. C. Wei, "Weakly asymptotic stability for fractional delay differential mixed variational inequalities", *Appl. Math. Optim.* **84**:1 (2021), 273–297.

[15] Z. H. Liu and S. D. Zeng, "Differential variational inequalities in infinite Banach spaces", *Acta Math. Sci. Ser. B* (*Engl. Ed.*) **37**:1 (2017), 26–32.

[16] N. V. Loi, “On two-parameter global bifurcation of periodic solutions to a class of differential variational inequalities”, *Nonlinear Anal.* **122** (2015), 83–99.

[17] S. Migórski and S. D. Zeng, “A class of generalized evolutionary problems driven by variational inequalities and fractional operators”, *Set-Valued Var. Anal.* **27** (2019), 949–970.

[18] S. Migórski and S. D. Zeng, “Mixed variational inequalities driven by fractional evolution equations”, *Acta Math. Sci. Ser. B (Engl. Ed.)* **39**:2 (2019), 461–468.

[19] J. S. Pang and D. E. Stewart, “Solution dependence on initial conditions in differential variational inequalities”, *Math. Program.* **116**:1-2, Ser. B (2009), 429–460.

[20] J. L. Shen and J. S. Pang, “Linear complementarity systems: Zeno states”, *SIAM J. Control Optim.* **44**:3 (2005), 1040–1066.

[21] N. T. V. Anh and T. D. Ke, “Asymptotic behavior of solutions to a class of differential variational inequalities”, *Ann. Polon. Math.* **114**:2 (2015), 147–164.

[22] P. D. Panagiotopoulos, “Nonconvex energy functions. Hemivariational inequalities and substationarity principles”, *Acta Mech.* **48**:3-4 (1983), 111–130.

[23] P. D. Panagiotopoulos, *Hemivariational inequalities. Applications in mechanics and engineering*, Springer–Verlag, Berlin 1993.

[24] Z. H. Liu, S. D. Zeng, and D. Motreanu, “Partial differential hemivariational inequalities”, *Adv. Nonlinear Anal.* **7**:4 (2018), 571–586.

[25] S. Migórski and S. D. Zeng, “Hyperbolic hemivariational inequalities controlled by evolution equations with application to adhesive contact model”, *Nonlinear Anal. Real World Appl.* **43** (2018), 121–143.

[26] S. D. Zeng, Z. H. Liu, and S. Migórski, “A class of fractional differential hemivariational inequalities with application to contact problem”, *Z. Angew. Math. Phys.* **69**:2 (2018), 36.

[27] Y. R. Jiang, N. J. Huang, and Z. C. Wei, “Existence of a global attractor for fractional differential hemivariational inequalities”, *Discrete Contin. Dyn. Syst. Ser. B* **25**:4 (2020), 1193–1212.

[28] X. W. Li and Z. H. Liu, “Sensitivity analysis of optimal control problems described by differential hemivariational inequalities”, *SIAM J. Control Optim.* **56**:5 (2018), 3569–3597.

[29] X. W. Li, Z. H. Liu, and M. Sofonea, “Unique solvability and exponential stability of differential hemivariational inequalities”, *Appl. Anal.* **99**:14 (2020), 2489–2506.

[30] Z. H. Liu, D. Motreanu, and S. D. Zeng, “Nonlinear evolutionary systems driven by quasi-hemivariational inequalities”, *Math. Methods Appl. Sci.* **41**:3 (2018), 1214–1229.

[31] Z. H. Liu, D. Motreanu, and S. D. Zeng, “Generalized penalty and regularization method for differential variational-hemivariational inequalities”, *SIAM J. Optim.* **31**:2 (2021), 1158–1183.

[32] Z. H. Liu, V. T. Nguyen, J. C. Yao, and S. D. Zeng, “History-dependent differential variational-hemivariational inequalities with applications to contact mechanics”, *Evol. Equ. Control Theory* **9**:4 (2020), 1073–1087.

[33] S. Migórski and S. D. Zeng, “A class of differential hemivariational inequalities in Banach spaces”, *J. Global Optim.* **72**:4 (2018), 761–779.

[34] G. J. Tang, J. X. Cen, V. T. Nguyen, and S. D. Zeng, “Differential variational-hemivariational inequalities: existence, uniqueness, stability, and convergence”, *J. Fixed Point Theory Appl.* **22**:4 (2020), 83.

[35] S. D. Zeng, S. Migórski, and A. A. Khan, “Nonlinear quasi-hemivariational inequalities: existence and optimal control”, *SIAM J. Control Optim.* **59**:2 (2021), 1246–1274.

[36] V. Barbu, *Analysis and control of nonlinear infinite dimensional systems*, Math. Sci. Engrg., vol. 190, Academic Press, Inc., Boston, MA 1993.

[37] F. H. Clarke, *Optimization and nonsmooth analysis*, Canad. Math. Soc. Ser. Monogr. Adv. Texts, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York 1983; Russian transl., Nauka, Moscow 1988.

[38] Z. Denkowski, S. Migórski, and N. S. Papageorgiou, *An introduction to nonlinear analysis: theory*, Kluwer Acad. Publ., Boston, MA 2003.

[39] Z. Denkowski, S. Migórski, and N. S. Papageorgiou, *An introduction to nonlinear analysis: applications*, Kluwer Acad. Publ., Boston, MA 2003.

[40] L. Gasiński and N. S. Papageorgiou, *Nonlinear analysis*, Ser. Math. Anal. Appl., vol. 9, Chapman & Hall/CRC, Boca Raton, FL 2006.

[41] S. Migórski, A. Ochal, and M. Sofonea, *Nonlinear inclusions and hemivariational inequalities. Models and analysis of contact problems*, Adv. Mech. Math., vol. 26, Springer, New York 2013.

[42] E. Zeidler, *Nonlinear functional analysis and its applications*, vol. II/A, Springer-Verlag, New York 1990; vol. II/B.

[43] J. R. Wang and Y. Zhou, "A class of fractional evolution equations and optimal controls", *Nonlinear Anal. Real World Appl.* **12**:1 (2011), 262–272.

[44] F. Mainardi, P. Paradisi, and R. Gorenflo, "Probability distributions generated by fractional diffusion equations", *Econophysics: an emerging science*, Kluwer Acad. Publ., Dordrecht 2000, pp. 312–350.

[45] Y. Zhou and F. Jiao, "Nonlocal Cauchy problem for fractional evolution equations", *Nonlinear Anal. Real World Appl.* **11**:5 (2010), 4465–4475.

[46] P. Kalita, "Convergence of Rothe scheme for hemivariational inequalities of parabolic type", *Int. J. Numer. Anal. Model.* **10**:2 (2013), 445–465.

[47] B. Zeng and S. Migórski, "Evolutionary subgradient inclusions with nonlinear weakly continuous operators and applications", *Comput. Math. Appl.* **75**:1 (2018), 89–104.

[48] S. Migórski and A. Ochal, "Hemivariational inequalities for stationary Navier–Stokes equations", *J. Math. Anal. Appl.* **306**:1 (2005), 197–217.

[49] M. Kamenskii, V. Obukhovskii, and P. Zecca, *Condensing multivalued maps and semilinear differential inclusions in Banach spaces*, De Gruyter Ser. Nonlinear Anal. Appl., vol. 7, Walter de Gruyter & Co., Berlin 2001.

[50] W. Han and M. Sofonea, *Quasistatic contact problems in viscoelasticity and viscoplasticity*, AMS/IP Stud. Adv. Math., vol. 30, Amer. Math. Soc., Providence, RI; International Press, Somerville, MA 2002.

[51] C. Carstensen and J. Gwinner, "A theory of discretization for nonlinear evolution inequalities applied to parabolic Signorini problems", *Ann. Mat. Pura Appl.* (4) **177** (1999), 363–394.

[52] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach", *Phys. Rep.* **339**:1 (2000), 1–77.

[53] X. J. Li and C. J. Xu, "A space-time spectral method for the time fractional diffusion equation", *SIAM J. Numer. Anal.* **47**:3 (2009), 2108–2131.

[54] Y. N. Zhang, Z. Z. Sun, and H. L.Liao, "Finite difference methods for the time fractional diffusion equation on non-uniform meshes", *J. Comput. Phys.* **265** (2014), 195–210.

[55] I. I. Vrabie, *C_0-semigroups and applications*, North-Holland Math. Stud., vol. 191, North-Holland Publishing Co., Amsterdam 2003.

[56] J. Franců, "Weakly continuous operators. Applications to differential equations", *Appl. Math.* **39**:1 (1994), 45–56.

[57] N. U. Ahmed, "Optimal control of hydrodynamic flow with possible application to artificial heart", *Dynam. Systems Appl.* **1**:2 (1992), 103–119.

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