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Minimal-energy Splines: I. Plane Curves with Angle Constraints

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This is the first in a series of papers on minimal-energy splines. The paper is devoted to plane minimal-energy splines with angle constraints. We first consider minimal-energy spline segments, then general minimal-energy spline curves. We formulate problems for minimal-energy spline segments and curves, prove the existence of solutions, justify the Lagrange multiplier rules, and obtain some nice properties (e.g., the infinite smoothness). Finally, we report our computational experience on minimal-energy splines.

1. Introduction

According to Bernoulli–Euler theory [8], the differential equation of a bent elastica describing the resistance of the elastica to bending can be obtained by minimizing the strain energy of the elastica—which is proportional to the integral of the square of the curvature taken along the elastica. We shall call a curve a minimal-energy spline, if the curve considered to be a bent elastica minimizes the strain energy. When the deformation of an elastica is small, one may drop the high-order term for the curvature, and obtain the celebrated cubic spline.

We emphasize the role played by the length in defining a minimal-energy spline. We quote the following statement from Birkhoff and de Boor [1] (κ denoting the curvature): curiously, an absolute minimum to $\int \kappa^2 \, ds$ does not exist except in the trivial case of a straight line; this is because one can construct large loops joining given endpoints with given endslopes, of length $2\pi r$ and curvature $\kappa = O(1/r)$, for arbitrarily large r—hence with $\int \kappa^2 \, ds$ less than any preassigned positive number.

The minimal-energy spline has been considered by others under different names. In Lee and Forsythe [7], many interesting relations are obtained for non-linear spline curves through a formal calculus of variations. Malcolm [9] implements an algorithm for computing non-linear spline functions. Jerome [6], and Fisher and Jerome [3]

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provide the mathematical basis for non-linear spline curves. In particular, in [6], the existence of a non-linear spline curve is proved, which has never been done before. In [3], the Lagrange multiplier relations and C^∞ regularity for a non-linear spline curve are proved. In Golomb and Jerome [5], using the Lagrange multiplier relations, extremal interpolations are defined and studied in much detail.

Our minimal-energy splines in this paper are different from those in papers listed above in two respects. We have angle constraints at end points, and a prescribed length for the spline. These constraints make the constraint set of a minimal-energy spline problem more rigid than those considered before. Hence, we use new tools to get theoretical results. We focus our attention on constrained minimization problems for the study of minimal-energy splines. This will allow us to generalize naturally the results in this paper to other kinds of minimal-energy splines with different constraint settings, in particular, the boundary conditions considered in the papers listed above will be special cases in our forthcoming papers. Another advantage is that we are able to obtain efficient algorithms to compute minimal-energy splines numerically.

Our plan: in this series of papers, we try to supply minimal-energy splines with various constraints with a profound mathematical basis, find nice properties of the splines, and report our computational experience on splines. In the first paper of the series, we study the plane minimal-energy splines with angle constraints thoroughly. In the forthcoming papers, we shall study plane minimal-energy splines with other kinds of boundary conditions and space minimal-energy spline curves.

This paper is organized as follows. In section 2, we study plane minimal-energy spline segments with angle constraints. In section 3, we study general plane minimal-energy splines with angle constraints. In section 4, we state some results on the computation of minimal-energy splines with angle constraints, and provide the graphs of some minimal-energy splines.

2. Minimal-energy spline segments

2.1. *Formulation of the problem and the existence of a solution*

First, we form the problem for a minimal-energy spline segment.

Given two points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ on the plane, $\alpha_1, \alpha_2 \in \mathbb{R}$, and a positive number l, we are interested in finding a curve that is of length l, connects P_1 and P_2, has direction angles α_1 at P_1, α_2 at P_2, such that the curve minimizes the energy:

$$\int_0^l \tfrac{1}{2}\kappa(s)^2\, ds, \tag{2.1}$$

where $\kappa(s)$ is the curvature of the curve.

In the following, we use f' as the derivative of a function $f(s)$ with respect to the arc-length parameter s. Since on an arc-length parametrized curve $\{(x(s), y(s)) \mid 0 \leqslant s \leqslant l\}$,

$$x'(s)^2 + y'(s)^2 = 1, \tag{2.2}$$

there is a function $\theta(s)$ such that on the curve:

$$x'(s) = \cos\theta(s), \tag{2.3}$$

$$y'(s) = \sin\theta(s). \tag{2.4}$$

Obviously, $\theta(s)$ may be interpreted as the tangent direction angle of the curve at the point $(x(s), y(s))$. We obtain $(x(s), y(s))$ through:

$$x(s) = x_1 + \int_0^s \cos\theta(s)\,ds, \tag{2.5}$$

$$y(s) = y_1 + \int_0^s \sin\theta(s)\,ds, \tag{2.6}$$

and the curvature through:

$$\kappa(s) = \theta'(s). \tag{2.7}$$

Thus, we give the following definition.

Definition 2.1. *A plane curve* $\{(x^*(s), y^*(s)) \mid 0 \leqslant s \leqslant l\}$ *is said to be a minimal-energy spline segment of length* l, *passing through* $P_1 = (x_1, y_1)$ *with direction angle* α_1, *and* $P_2 = (x_2, y_2)$ *with direction angle* α_2, *if*

$$x^*(s) = x_1 + \int_0^s \cos\theta^*(s)\,ds, \tag{2.8}$$

$$y^*(s) = y_1 + \int_0^s \sin\theta^*(s)\,ds, \tag{2.9}$$

where $\theta^* \in H(P_1, P_2; \alpha_1, \alpha_2; l)$ *is such that*:

$$E(\theta^*) = \inf\{E(\theta) \mid \theta \in H(P_1, P_2; \alpha_1, \alpha_2; l)\} \tag{2.10}$$

with the energy:

$$E(\theta) = \int_0^l \tfrac{1}{2}\theta'(s)^2\,ds \tag{2.11}$$

and the constraint set:

$$H(P_1, P_2; \alpha_1, \alpha_2; l) = \left\{\theta \in H^1(0, l) \mid \theta(0) = \alpha_1, \theta(l) = \alpha_2, \int_0^l \cos\theta(s)\,ds = x_2 - x_1, \int_0^l \sin\theta(s)\,ds = y_2 - y_1\right\}. \tag{2.12}$$

Remark. In [5], the representation (2.8, 2.9) is called a normal representation.

Let us state some conditions on the input data to guarantee the non-emptiness of the constraint set.

Lemma 2.2. $H(P_1, P_2; \alpha_1, \alpha_2; l) \neq \varnothing$ *iff either of the following holds*:

$$l^2 > (x_2 - x_1)^2 + (y_2 - y_1)^2, \tag{2.13}$$

$$l^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,\ \alpha_1 = \alpha_2 \text{ and } \sin\alpha_1 = (y_2 - y_1)/l, \cos\alpha_1 = (x_2 - x_1)/l. \tag{2.14}$$

Proof. Assume $H(P_1, P_2; \alpha_1, \alpha_2; l) \neq \varnothing$.

Since the distance between P_1 and P_2 is $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$, we have the inequality:

$$l^2 \geqslant (x_2 - x_1)^2 + (y_2 - y_1)^2. \tag{2.15}$$

If $l^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, then $H(P_1, P_2; \alpha_1, \alpha_2; l)$ contains exactly one element $\theta = \alpha_1$, thus we must have:

$$\alpha_1 = \alpha_2 \quad \text{and} \quad \sin\alpha_1 = \frac{y_2 - y_1}{l}, \quad \cos\alpha_1 = \frac{x_2 - x_1}{l}. \tag{2.16}$$

Therefore either (2.13) or (2.14) holds.

Conversely, if (2.14) holds, then $\theta = \alpha_1$ is an element of $H(P_1, P_2; \alpha_1, \alpha_2; l)$. Now assume (2.13), let us show $H(P_1, P_2; \alpha_1, \alpha_2; l) \neq \varnothing$. We take a curve $\{(\bar{x}(s), \bar{y}(s)), 0 \leqslant s \leqslant \bar{l})\}$, with

$$\bar{x}(s) = x_1 + \int_0^s \cos\bar{\theta}(s)\,ds,$$

$$\bar{y}(s) = y_1 + \int_0^s \sin\bar{\theta}(s)\,ds,$$

$$\bar{x}(\bar{l}) = x_2, \qquad \bar{y}(\bar{l}) = y_2,$$

$\bar{\theta}$ is of class $C^1, \bar{\theta}(0) = \alpha_1, \bar{\theta}(\bar{l}) = \alpha_2,$

$\bar{l} < l.$

Let $s_0 \in (0, \bar{l})$. We define a function $\theta(s)$ on $[0, l]$ as follows:

$$\theta(s) = \begin{cases} \bar{\theta}(s) & 0 \leqslant s \leqslant s_0, \\ \bar{\theta}(s_0) + 4\pi(s - s_0)/(l - \bar{l}), & s_0 \leqslant s \leqslant s_0 + (l - \bar{l})/2, \\ \bar{\theta}(s_0) + 4\pi - 4\pi(s - s_0)/(l - \bar{l}), & s_0 + (l - \bar{l})/2 \leqslant s \leqslant s_0 + l - \bar{l}, \\ \bar{\theta}(s - l + \bar{l}), & s_0 + l - \bar{l} \leqslant s \leqslant l. \end{cases}$$

Then $\theta \in H^1(0, l), \theta(0) = \bar{\theta}(0) = \alpha_1, \theta(l) = \bar{\theta}(\bar{l}) = \alpha_2$.

Note that

$$\int_{s_0}^{s_0 + (l - \bar{l})/2} \cos\theta(s)\,ds = \int_{s_0 + (l - \bar{l})/2}^{s_0 + l - \bar{l}} \cos\theta(s)\,ds = 0.$$

We have

$$\begin{aligned} \int_0^l \cos\theta(s)\,ds &= \int_0^{s_0} \cos\theta(s)\,ds + \int_{s_0}^{s_0 + (l - \bar{l})/2} \cos\theta(s)\,ds \\ &\quad + \int_{s_0 + (l - \bar{l})/2}^{s_0 + l - \bar{l}} \cos\theta(s)\,ds + \int_{s_0 + l - \bar{l}}^{l} \cos\theta(s)\,ds \\ &= \int_0^{s_0} \cos\bar{\theta}(s)\,ds + \int_{s_0 + l - \bar{l}}^{l} \cos\bar{\theta}(s - l + \bar{l})\,ds \\ &= \int_0^{s_0} \cos\bar{\theta}(s)\,ds + \int_{s_0}^{\bar{l}} \cos\bar{\theta}(s)\,ds \\ &= \int_0^{\bar{l}} \cos\bar{\theta}(s)\,ds \\ &= \bar{x}(\bar{l}) - x_1 \\ &= x_2 - x_1. \end{aligned}$$

Similarly,

$$\int_0^l \sin\theta(s)\,\mathrm{d}s = y_2 - y_1.$$

Hence $\theta \in H(P_1, P_2; \alpha_1, \alpha_2; l)$, that is $H(P_1, P_2; \alpha_1, \alpha_2; l) \neq \emptyset$. □

From now on, we assume (2.13) or (2.14), thus the constraint set is non-empty. We are going to give an existence result of a minimal-energy spline segment.

Theorem 2.3. *Under the assumptions* (2.13) *or* (2.14), *there is a* $\theta^* \in H(P_1, P_2; \alpha_1, \alpha_2; l)$, *such that*

$$E(\theta^*) = \inf\{E(\theta) \mid \theta \in H(P_1, P_2; \alpha_1, \alpha_2; l)\}. \tag{2.17}$$

Proof. Denote $E = \inf\{E(\theta) \mid \theta \in H(P_1, P_2; \alpha_1, \alpha_2; l)\}$. Then there is a sequence $\{\theta_n\} \subset H(P_1, P_2; \alpha_1, \alpha_2; l)$, such that $E(\theta_n) \to E$. Thus, $\{\theta'_n\}$ is a bounded sequence in $L^2(0, l)$. Since

$$\theta_n(s) = \alpha_1 + \int_0^s \theta'_n(s)\,\mathrm{d}s,$$

we have

$$\|\theta_n\|_{L^2(0,l)} \leqslant C(|\alpha_1| + \|\theta'_n\|_{L^2(0,l)}),$$

that is $\{\theta_n\}$ is a bounded sequence in $L^2(0, l)$. Therefore, $\{\theta_n\}$ is a bounded sequence in $H^1(0, l)$. So we can find a subsequence $\{\theta_{n_k}\} \subset \{\theta_n\}$ and $\theta^* \in H^1(0, l)$, such that:

$$\theta_{n_k} \to \theta^* \text{ weakly in } H^1(0, l), \tag{2.18}$$

$$\theta_{n_k} \to \theta^* \text{ in } C([0, l]). \tag{2.19}$$

We shall prove

$$\theta^* \in H(P_1, P_2; \alpha_1, \alpha_2; l) \tag{2.20}$$

and

$$E(\theta^*) = E. \tag{2.21}$$

First we prove (2.20).
Since

$$\theta_{n_k}(0) = \alpha_1, \qquad \theta_{n_k}(l) = \alpha_2,$$

$$\int_0^l \cos\theta_{n_k}(s)\,\mathrm{d}s = x_2 - x_1, \qquad \int_0^l \sin\theta_{n_k}(s)\,\mathrm{d}s = y_2 - y_1,$$

by (2.19), we have

$$\theta^*(0) = \lim_{k\to\infty} \theta_{n_k}(0) = \alpha_1, \tag{2.22}$$

$$\theta^*(l) = \lim_{k\to\infty} \theta_{n_k}(l) = \alpha_2, \tag{2.23}$$

$$\int_0^l \cos\theta^*(s)\,\mathrm{d}s = \lim_{k\to\infty} \int_0^l \cos\theta_{n_k}(s)\,\mathrm{d}s = x_2 - x_1, \tag{2.24}$$

$$\int_0^l \sin\theta^*(s)\,ds = \lim_{k\to\infty}\int_0^l \sin\theta_{n_k}(s)\,ds = y_2 - y_1. \tag{2.25}$$

Thus, $\theta^* \in H(P_1, P_2; \alpha_1, \alpha_2; l)$.

Now we prove (2.21). We write:

$$\int_0^l \tfrac{1}{2}\theta^{*\prime 2}\,ds = \int_0^l \tfrac{1}{2}\theta^{*\prime}(s)[\theta^{*\prime}(s) - \theta'_{n_k}(s)]\,ds + \int_0^l \tfrac{1}{2}\theta^{*\prime}(s)\theta'_{n_k}(s)\,ds. \tag{2.26}$$

By (2.18),

$$\int_0^l \tfrac{1}{2}\theta^{*\prime}(s)[\theta^{*\prime}(s) - \theta'_{n_k}(s)]\,ds \to 0 \text{ as } k\to\infty. \tag{2.27}$$

By Schwarz's inequality,

$$\left|\int_0^l \tfrac{1}{2}\theta^{*\prime}(s)\theta'_{n_k}(s)\,ds\right| \leqslant \sqrt{[E(\theta^*)]}\sqrt{[E(\theta_{n_k})]}. \tag{2.28}$$

Thus, let $k\to\infty$ in (2.26), we obtain

$$E(\theta^*) \leqslant \lim_{k\to\infty} E(\theta_{n_k}) = E. \tag{2.29}$$

On the other hand, since $\theta^* \in H(P_1, P_2; \alpha_1, \alpha_2; l)$, we must have

$$E(\theta^*) \geqslant E. \tag{2.30}$$

Therefore, $E(\theta^*) = E$. □

In general, a solution θ^* of (2.10) is not unique, hence a minimal-energy spline segment is not unique. For example, let $P_1 = (0, 0)$, $P_2 = (1, 0)$, $\alpha_1 = \alpha_2 = 0$, $l > 1$. If $\theta^*(s)$ is a solution of (2.10), then the function $-\theta^*(s)$ is a solution of (2.10) as well and $\theta^* \neq -\theta^*$. What kind of assumption on the input data will make a solution of (2.10) unique is an open question.

2.2. *Lagrange multipliers*

In this section, we justify the Lagrange multiplier rule for the constrained minimization problem (2.10).

Definition 2.4. *If X, Y are Banach spaces, a mapping $G: D(G) \subseteq X \to Y$ is called a submersion at $u_0 \in D(G)$ if:*

(i) *G is continuously Frechet-differentiable in a neighbourhood of u_0.*
(ii) *$G'(u_0): X \to Y$ is surjective, i.e. $R(G'(u_0)) = Y$.*
(iii) *The null space $N(G'(u_0))$ splits X, i.e. there exists a continuous projection operator of X on $N(G'(u_0))$.*

We shall use the following theorem of Ljusternik ([10]):

Theorem 2.5. *Let X, Y be two Banach spaces, $U \subseteq X$ an open set, $F: U \to R$, $G: U \to Y$. Consider the constrained-minimum problem:*

$$\text{Find } u_0 \in M = \{u \in U \mid G(u) = 0\}, \text{ such that } F(u_0) = \inf\{F(u) \mid u \in M\}. \tag{2.31}$$

Assume:

(i) *F is Frechet-differentiable at* u_0.

(ii) *G is a submersion at* u_0.

Then:

(1) *Necessary condition*: *if* u_0 *is a solution of the problem* (3.1), *then there exists a* $\Lambda \in Y^*$ *such that*:

$$F'(u_0)k - \Lambda(G'(u_0)k) = 0, \quad \forall k \in X.$$

(2) *Sufficient condition*: u_0 *is a local strict minimum point of* (3.1), *if*

(*a*) *F and G are n-times continuously Frechet-differentiable in a neighbourhood of* u_0, *with an even integer* $n \geqslant 2$;

(*b*) *there exist* $c > 0$ *and* $\Lambda \in Y^*$ *such that*:

$$F^{(r)}(u_0)k^r - \Lambda(G^{(r)}(u_0)k^r) = 0, \quad r = 1, \ldots, n-1,$$

$$F^{(n)}(u_0)h^n - \Lambda(G^{(n)}(u_0)h^n) \geqslant c\,\|h\|^n,$$

for all $k \in X, h \in X$ *such that* $G'(u_0)h = 0$.

To apply Theorem 2.5 to prove the existence of Lagrange multipliers for the problem (2.10), we make the change of variable:

$$\omega(s) = \theta(s) - L(s) \tag{2.32}$$

where

$$L(s) = \alpha_1\left(1 - \frac{s}{l}\right) + \alpha_2\frac{s}{l}.$$

Then, $\omega \in \bar{H}(P_1, P_2; 0, 0; l)$ iff $\theta \in H(P_1, P_2; \alpha_1, \alpha_2; l)$, where:

$$\bar{H}(P_1, P_2; 0, 0; l) = \left\{\omega \in H_0^1(0, l)\,\middle|\, \int_0^l \cos[\omega(s) + L(s)]\,ds = x_2 - x_1,\right.$$

$$\left.\int_0^l \sin[\omega(s) + L(s)]\,ds = y_2 - y_1\right\}, \tag{2.33}$$

and, the minimization problem (2.10) is equivalent to the following.

Find $\omega^* \in \bar{H}(P_1, P_2; 0, 0; l)$, such that

$$F(\omega^*) = \inf\{F(\omega) \mid \omega \in \bar{H}(P_1, P_2; 0, 0; l)\} \tag{2.34}$$

where

$$F(\omega) = E(\omega + L) = \int_0^l \tfrac{1}{2}[\omega'(s) + L'(s)]^2\,ds. \tag{2.35}$$

Now, we take $U = X = H_0^1(0, l)$, with the inner product

$$(\phi, \psi) = \int_0^l \phi'\psi'\,ds;$$

$Y = \mathbb{R}^2$, with the usual Euclidean inner product,

$$G(\omega) = \begin{pmatrix} \int_0^l \cos[\omega(s) + L(s)]\,ds - (x_2 - x_1) \\ \int_0^l \sin[\omega(s) + L(s)]\,ds - (y_2 - y_1) \end{pmatrix}.$$

Then the constraint set $\bar{H}(P_1, P_2; 0, 0; l) = \{\omega \in X \mid G(\omega) = 0\}$.

Lemma 2.6. *F is Frechet-differentiable at any $\omega \in X$.*

Proof. From the definition of a Frechet-differentiation, we have:

$$\langle F'(\omega), \psi \rangle = \int_0^l [\omega'(s) + L'(s)]\psi'(s)\,\mathrm{d}s, \quad \forall \psi \in X.$$

Thus, F is Frechet-differentiable at ω. □

Lemma 2.7. *G is a submersion at any $\omega \in \bar{H}(P_1, P_2; 0, 0; l)$.*

Proof.

(i) $$\langle G'(\phi), \psi \rangle = \begin{pmatrix} \int_0^l -\sin(\phi + L)\psi\,\mathrm{d}s \\ \int_0^l \cos(\phi + L)\psi\,\mathrm{d}s \end{pmatrix}, \quad \forall \phi, \psi \in X.$$

Thus, G is continuously Frechet-differentiable on X.

(ii) Since $\omega \in \bar{H}(P_1, P_2; 0, 0; l)$, $\omega + L$ is not a constant. By the continuity of ω, there exist some $\delta > 0, s_1, s_2 \in (0, l)$, such that $I_1 = (s_1 - \delta, s_1 + \delta) \subset (0, l)$, $I_2 = (s_2 - \delta, s_2 + \delta) \subset (0, l)$, $I_1 \cap I_2 = \varnothing$, and:

$$\begin{vmatrix} \sin[\omega(t_1) + L(t_1)] & \sin[\omega(t_3) + L(t_3)] \\ \cos[\omega(t_2) + L(t_2)] & \cos[\omega(t_4) + L(t_4)] \end{vmatrix} \neq 0, \quad \forall t_1, t_2 \in I_1, \quad \forall t_3, t_4 \in I_2.$$

Define:

$$\eta_1(s) = \begin{cases} |s - s_1|/\delta, & \text{if } |s - s_1| \leqslant \delta, \\ 0, & \text{otherwise}; \end{cases}$$

$$\eta_2(s) = \begin{cases} |s - s_2|/\delta, & \text{if } |s - s_2| \leqslant \delta, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\eta_1, \eta_2 \in X$.

By the mean-value theorem, there exist $t_1, t_2 \in I_1, t_3, t_4 \in I_2$, such that:

$$\int_0^l \eta_1 \sin[\omega(s) + L(s)]\,\mathrm{d}s = \int_{I_1} \eta_1(s)\sin[\omega(s) + L(s)]\,\mathrm{d}s = J\sin[\omega(t_1) + L(t_1)],$$

$$\int_0^l \eta_1 \cos[\omega(s) + L(s)]\,\mathrm{d}s = \int_{I_1} \eta_1(s)\cos[\omega(s) + L(s)]\,\mathrm{d}s = J\cos[\omega(t_2) + L(t_2)],$$

$$\int_0^l \eta_2 \sin[\omega(s) + L(s)]\,\mathrm{d}s = \int_{I_2} \eta_2(s)\sin[\omega(s) + L(s)]\,\mathrm{d}s = J\sin[\omega(t_3) + L(t_3)],$$

$$\int_0^l \eta_2 \cos[\omega(s) + L(s)]\,\mathrm{d}s = \int_{I_2} \eta_2(s)\cos[\omega(s) + L(s)]\,\mathrm{d}s = J\cos[\omega(t_4) + L(t_4)],$$

where $J = \int_{I_1} \eta_1(s)\,\mathrm{d}s = \int_{I_2} \eta_2(s)\,\mathrm{d}s = \delta$.

Since

$$\begin{vmatrix} -J\sin[\omega(t_1)+L(t_1)] & -J\sin[\omega(t_3)+L(t_3)] \\ J\cos[\omega(t_2)+L(t_2)] & J\cos[\omega(t_4)+L(t_4)] \end{vmatrix}$$

$$= -J^2\begin{vmatrix} \sin[\omega(t_1)+L(t_1)] & \sin[\omega(t_3)+L(t_3)] \\ \cos[\omega(t_2)+L(t_2)] & \cos[\omega(t_4)+L(t_4)] \end{vmatrix} \neq 0$$

we have

$$\operatorname{span}\{\langle G'(\omega),\eta_1\rangle, \langle G'(\omega),\eta_2\rangle\} = Y.$$

Therefore, $R(G'(\omega)) = Y$.

(iii) Since $N(G'(\omega)) = \{\eta\in X \mid \langle G'(\omega),\eta\rangle = 0\}$ is closed in X, there exists a continuous projection operator of X on $N(G'(\omega))$.

Hence, G is a submersion at any $\omega\in \bar{H}(P_1, P_2; 0, 0; l)$. □

Thus, for the problem (2.34), the assumptions of Theorem 2.5 hold.

Theorem 2.8. *If ω^* is a solution of the problem* (2.34), *then there exist $\lambda_1, \lambda_2\in\mathbb{R}$, such that*

$$\int_0^l \{(\omega^{*\prime} + L')\psi' - [\lambda_1\sin(\omega^* + L) + \lambda_2\cos(\omega^* + L)]\psi\}\,\mathrm{d}s = 0,$$

$$\forall\psi\in H_0^1(0,l) \tag{2.36}$$

or, in the sense of distribution,

$$\omega^{*\prime\prime} + \lambda_1\sin(\omega^* + L) + \lambda_2\cos(\omega^* + L) = 0.$$

Conversely, if $\omega^*\in\bar{H}(P_1, P_2; 0, 0; l)$ satisfies (2.36), *and*

$$\sqrt{(\lambda_1^2 + \lambda_2^2)} < (\pi/l)^2 \tag{2.37}$$

then ω^* is a local strict minimum point of the problem (2.34).

To prove the theorem, we need the following:

Lemma 2.9. *For any $\psi\in H_0^1(0,l)$,*

$$\int_0^l \psi^2\,\mathrm{d}s \leqslant \left(\frac{l}{\pi}\right)^2 \int_0^l \psi'^2\,\mathrm{d}s. \tag{2.38}$$

Proof. We expand ψ in a sine series:

$$\psi(s) \sim \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}s,$$

then

$$\psi'(s) \sim \sum_{n=1}^{\infty} a_n \frac{n\pi}{l}\cos\frac{n\pi}{l}s.$$

By Parseval's equality, we have:

$$\int_0^l \psi^2\,\mathrm{d}s = \frac{l}{2}\sum_{n=1}^{\infty} a_n^2,$$

$$\int_0^l \psi'^2\,\mathrm{d}s = \frac{l}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{l}\right)^2 a_n^2.$$

Thus

$$\int_0^l \psi'^2\,ds \geqslant \left(\frac{\pi}{l}\right)^2 \frac{l}{2}\sum_{n=1}^{\infty} a_n^2 = \left(\frac{\pi}{l}\right)^2 \int_0^l \psi^2\,ds.$$

□

Proof of Theorem 2.8. We apply Theorem 2.5 to the problem (2.34).

Note that $Y^* = \mathbb{R}^2$. From the necessary condition part of Theorem 2.5, we have the existence of a

$$\Lambda = \begin{pmatrix} -\lambda_1 \\ \lambda_2 \end{pmatrix} \in Y^*,$$

such that

$$\int_0^l (\omega^{*\prime} + L')\psi'\,ds - (-\lambda_1, \lambda_2)\left(\int_0^l [-\sin(\omega^* + L)]\psi\,ds,\right.$$

$$\left.\int_0^l \cos(\omega^* + L)\psi\,ds\right)^{\mathrm{T}} = 0, \quad \forall \psi \in X,$$

that is (2.36) holds.

Now assume $\omega^* \in H_0^1(0, l)$ satisfies (2.36) with λ_1, λ_2 verifying (2.37). We apply the sufficient condition part (for $n = 2$) of Theorem 2.5. F, G are twice continuously Frechet-differentiable at any $\omega \in X$, with

$$F''(\omega)\phi\psi = \int_0^l \phi'\psi'\,ds, \quad \forall \phi, \psi \in X,$$

$$G''(\omega)\phi\psi = \begin{pmatrix} \int_0^l [-\cos(\omega + L)]\phi\psi\,ds \\ \int_0^l [-\sin(\omega + L)]\phi\psi\,ds \end{pmatrix}, \quad \forall \phi, \psi \in X.$$

Then, for any $\psi \in X$,

$$F''(\omega^*)\psi^2 - \Lambda(G''(\omega^*)\psi^2)$$

$$= \int_0^l \psi'^2\,ds - (-\lambda_1, \lambda_2)\left(\int_0^l [-\cos(\omega^* + L)]\psi^2\,ds,\right.$$

$$\left.\int_0^l [-\sin(\omega^* + L)]\psi^2\,ds\right)^{\mathrm{T}}$$

$$= \int_0^l \psi'^2\,ds + \int_0^l [-\lambda_1\cos(\omega^* + L) + \lambda_2\sin(\omega^* + L)]\psi^2\,ds$$

$$\geqslant \int_0^l \psi'^2\,ds - \sqrt{(\lambda_1^2 + \lambda_2^2)} \int_0^l \psi^2\,ds$$

$$\geqslant \left[1 - \left(\frac{l}{\pi}\right)^2 \sqrt{(\lambda_1^2 + \lambda_2^2)}\right]^{1/2} \int_0^l \psi'^2\,ds.$$

By Theorem 2.5, ω^* is a local strict minimum point of the problem (2.36). □

Back to the original problem (2.10), recalling the variable change (2.32) we made, from Theorem 2.8, we obtain:

Theorem 2.10. *If θ^* is a solution of the problem* (2.10), *then there exist $\lambda_1, \lambda_2 \in \mathbb{R}$, such that*:

$$\int_0^l [\theta^{*\prime}\psi' - (\lambda_1 \sin\theta^* + \lambda_2 \cos\theta^*)\psi]\,ds = 0, \quad \forall\psi \in H_0^1(0,l) \tag{2.39}$$

or, in the sense of distribution,

$$\theta^{*\prime\prime} + \lambda_1 \sin\theta^* + \lambda_2 \cos\theta^* = 0.$$

Conversely, if $\theta^* \in H(P_1, P_2; \alpha_1, \alpha_2; l)$ satisfies (2.39), *and*

$$\sqrt{(\lambda_1^2 + \lambda_2^2)} < \left(\frac{\pi}{l}\right)^2$$

then θ^* is a local strict minimum point of the problem (2.10).

2.3. *Some basic properties of a minimal-energy spline segment*

In this section we study the smoothness, geometric invariance and curvature relationship of minimal-energy spline segments.

2.3.1. The smoothness. The smoothness of a minimal-energy spline segment is determined by the smoothness of a minimizer θ^*. We show that $\theta^* \in C^\infty([0,l])$, so a minimal-energy spline segment is infinitely smooth.

We need the following result, which is a special case of Theorem 8.13 in [4].

Lemma 2.11. *Let $\theta \in H^1(0,l)$ be a weak solution of the problem*:

$$-\theta''(s) = f(s) \text{ in } (0,l), \quad \theta(0) = \alpha_1, \quad \theta(l) = \alpha_2. \tag{2.40}$$

If $f \in H^k(0,l)$, $k \geq 0$, then

$$\theta \in H^{k+2}(0,l), \text{ and } \|\theta\|_{H^{k+2}(0,l)} \leq (\|\theta\|_{L^2(0,l)} + \|f\|_{H^k(0,l)} + |\alpha_1| + |\alpha_2|).$$

Let us apply the lemma to a minimizer θ^* of problem (2.10).

By Theorem 2.10, $\theta^* \in H^1(0,l)$ is a weak solution of (2.40) with

$$f(s) = \lambda_1 \sin\theta^*(s) + \lambda_2 \cos\theta^*(s).$$

Now $\theta^* \in H^1(0,l)$ implies $f \in H^1(0,l)$, thus by Lemma 2.11, $\theta^* \in H^3(0,l)$. By Sobolev's imbedding theorem, $\theta^* \in C^2([0,L])$. Hence, θ^* satisfies the equation pointwisely:

$$-\theta^{*\prime\prime}(s) = \lambda_1 \sin\theta^*(s) + \lambda_2 \cos\theta^*(s). \tag{2.41}$$

Since the right-hand side is a $C^2([0,l])$ function, we obtain from (2.41) that $\theta^* \in C^4([0,l])$. Once more using (2.41), we have $\theta^* \in C^6([0,l])$. We continue this procedure, and arrive at the conclusion: $\theta^* \in C^\infty([0,l])$.

2.3.2. Translational and rotational invariance. From the first set of input data $\{(x_1, y_1), (x_2, y_2); \alpha_1, \alpha_2; l\}$ and the second set of input data $\{(\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2); \hat{\alpha}_1, \hat{\alpha}_2; l\}$, we formulate the following two problems.

(a) Find $\theta \in H$, such that

$$J(\theta) = \inf_{\psi \in H} J(\psi) = \int_0^l \tfrac{1}{2}\psi'(s)^2\,ds$$

with

$$H = \left\{ \psi \in H^1(0,l) \mid \psi(0) = \alpha_1, \psi(l) = \alpha_2, \right.$$

$$\left. \int_0^l \cos\psi(s)\,ds = x_2 - x_1, \qquad \int_0^l \sin\psi(s)\,ds = y_2 - y_1 \right\}. \tag{2.42}$$

Then define a curve $\{(x(s), y(s)) \mid 0 \leqslant s \leqslant l\}$ through:

$$x(s) = x_1 + \int_0^s \cos\theta(s)\,ds, \qquad y(s) = y_1 + \int_0^s \sin\theta(s)\,ds.$$

(b) Find $\hat{\theta} \in \hat{H}$, such that

$$J(\hat{\theta}) = \inf_{\hat{\psi}\in\hat{H}} J(\hat{\psi}) = \int_0^l \tfrac{1}{2}\hat{\psi}'(s)^2\,ds$$

with

$$\hat{H} = \left\{ \hat{\psi} \in H^1(0,l) \mid \hat{\psi}(0) = \hat{\alpha}_1, \hat{\psi}(l) = \hat{\alpha}_2, \right.$$

$$\left. \int_0^l \cos\hat{\psi}(s)\,ds = \hat{x}_2 - \hat{x}_1, \qquad \int_0^l \sin\hat{\psi}(s)\,ds = \hat{y}_2 - \hat{y}_1 \right\}. \tag{2.43}$$

Then define a curve $\{(\hat{x}(s), \hat{y}(s)) \mid 0 \leqslant s \leqslant l\}$ through:

$$\hat{x}(s) = \hat{x}_1 + \int_0^s \cos\hat{\theta}(s)\,ds, \qquad \hat{y}(s) = \hat{y}_1 + \int_0^s \sin\hat{\theta}(s)\,ds.$$

Now assume the following relations between the two sets of input data:

$$\begin{pmatrix} \hat{x}_1 \\ \hat{y}_1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

$$\begin{pmatrix} \hat{x}_2 \\ \hat{y}_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

$$\hat{\alpha}_1 = \alpha_1 + \phi,$$

$$\hat{\alpha}_2 = \alpha_2 + \phi,$$

Obviously, if $\theta \in H$ is a solution of (a), then $\hat{\theta} = \theta + \phi \in \hat{H}$ is a solution of (b), and vice versa. Thus we may take $\hat{\theta}(s) = \theta(s) + \phi$, $0 \leqslant s \leqslant l$. Therefore:

$$\hat{x}(s) = \hat{x}_1 + \int_0^l \cos[\theta(s) + \phi]\,ds,$$

$$\hat{y}(s) = \hat{y}_1 + \int_0^l \sin[\theta(s) + \phi]\,ds,$$

that is

$$\begin{pmatrix} \hat{x}(s) \\ \hat{y}(s) \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

2.3.3. Curvature relationships. By (2.41), a minimizer θ^* of problem (2.10) satisfies

$$\theta^{*''} + \lambda_1 \sin\theta^* + \lambda_2 \cos\theta^* = 0. \tag{2.44}$$

Note that $\kappa = \theta^{*'}$, $x' = \cos\theta^*$, and $y' = \sin\theta^*$. Integrating (2.44), we obtain the linear curvature relation:

$$\kappa = -\lambda_1 y - \lambda_2 x - C \tag{2.45}$$

where C is a constant.

Differentiating (2.44), we get

$$\theta^{*'''} + \lambda_1(\cos\theta^*)\theta^{*'} - \lambda_2(\sin\theta^*)\theta^{*'} = 0. \tag{2.46}$$

Multiplying (2.44) by $\theta^{*'}$ and integrating, we have

$$\tfrac{1}{2}\theta^{*'2} - \lambda_1 \cos\theta^* + \lambda_2 \sin\theta^* = K \tag{2.47}$$

where K is a constant.

Eliminating λ_1 and λ_2 from (2.46) and (2.47), we obtain

$$\theta^{*'''} + \tfrac{1}{2}\theta^{*'3} = K\theta^{*'}, \tag{2.48}$$

that is

$$\kappa'' + \tfrac{1}{2}\kappa^3 = K\kappa. \tag{2.49}$$

This relation was obtained by Birkhoff, Burchard and Thomas [2] under a certain assumption.

3. Minimal-energy splines

3.1. *Formulation of the problem and the existence of a solution*

Given $N+1$ points $P_i = (x_i, y_i)$, $i = 0, 1, \ldots, N$, $\alpha_0, \alpha_N \in \mathbb{R}$, and N positive numbers l_i, $i = 1, \ldots, N$, we try to find a curve connecting $P_0, P_1, \ldots, P_N$, being of length l_i between P_{i-1} and P_i ($i = 1, \ldots, N$), having direction angles α_0 at P_0 and α_N at P_N, such that the energy function:

$$\int_0^l \tfrac{1}{2}\kappa(s)^2\, \mathrm{d}s \tag{3.1}$$

is minimized, where $\kappa(s)$ is the curvature of the curve,

$$l = \sum_{i=1}^{N} l_i.$$

Let us denote:

$$\mathbf{P} = (P_0, P_1, \ldots, P_N);$$

$$\mathbf{l} = (l_1, \ldots, l_N);$$

$$L_0 = 0, \qquad L_i = \sum_{j=1}^{i} l_j, \quad i = 1, \ldots, N.$$

Definition 3.1. *A plane curve* $\{(x^*(s), y^*(s)) | 0 \leqslant s \leqslant l\}$ *is said to be a minimal-energy spline of length* l*, passing through* $\mathbf{P}$ *with length* l_i *between* P_{i-1} *and* $P_i (i = 1, \ldots, N)$, *having direction angles α_0 at P_0 and α_N at P_N, if*:

$$x^*(s) = x_0 + \int_0^s \cos\theta^*(s)\,ds, \tag{3.2}$$

$$y^*(s) = y_0 + \int_0^s \sin\theta^*(s)\,ds, \tag{3.3}$$

where $\theta^* \in H(\mathbf{P}; \alpha_0, \alpha_N; \mathbf{l})$ is such that:

$$E(\theta^*) = \inf\{E(\theta) | \theta \in H(\mathbf{P}; \alpha_0, \alpha_N; \mathbf{l})\} \tag{3.4}$$

with the energy function:

$$E(\theta) = \int_0^l \tfrac{1}{2}\theta'(s)^2\,ds, \tag{3.5}$$

and the constraint set:

$$H(\mathbf{P}; \alpha_0, \alpha_N; \mathbf{l}) = \Bigg\{\theta \in H^1(0, l) | \theta(0) = \alpha_0, \theta(l) = \alpha_N, \int_{L_{i-1}}^{L_i} \cos\theta(s)\,ds = x_i - x_{i-1},$$
$$\int_{L_{i-1}}^{L_i} \sin\theta(s)\,ds = y_i - y_{i-1}, \quad i = 1, \ldots, N\Bigg\}. \tag{3.6}$$

As for the question when the constraint set is non-empty, we have

Lemma 3.2. *Assume*:

$$l_i^2 > (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2, \quad i = 1, \ldots, N, \tag{3.7}$$

then $H(\mathbf{P}; \alpha_0, \alpha_N; \mathbf{l}) \neq \varnothing$.

Proof. We take $N - 1$ real numbers $\alpha_1, \ldots, \alpha_{N-1}$. By Lemma 2.2, under the assumption (3.7), for each i, there is a function $\theta_i(s)$ on $[L_{i-1}, L_i]$, such that:

$$\theta_i \in H^1(L_{i-1}, L_i), \tag{3.8}$$

$$\theta_i(L_{i-1}) = \alpha_{i-1}, \qquad \theta_i(L_i) = \alpha_i, \tag{3.9}$$

$$\int_{L_{i-1}}^{L_i} \cos\theta_i(s)\,ds = x_i - x_{i-1}, \qquad \int_{L_{i-1}}^{L_i} \sin\theta_i(s)\,ds = y_i - y_{i-1}. \tag{3.10}$$

Now we define a function on $[0, l]$

$$\theta(s) = \theta_i(s), \quad \text{if } s \in [L_{i-1}, L_i], \quad i = 1, \ldots, N.$$

By conditions (3.8, 3.9), we then have $\theta \in H^1(0, l)$. By (3.10), we obtain:

$$\int_{L_{i-1}}^{L_i} \cos\theta(s)\,ds = x_i - x_{i-1}, \qquad \int_{L_{i-1}}^{L_i} \sin\theta(s)\,ds = y_i - y_{i-1}.$$

Hence, $\theta \in H(\mathbf{P}; \alpha_0, \alpha_N; \mathbf{l})$. □

With a few obvious modifications of the proof of Theorem 2.3, we have the existence of a minimal-energy spline:

Theorem 3.3. *Under the assumption* (3.7), *there is a $\theta^* \in H(\mathbf{P}; \alpha_0, \alpha_N; \mathbf{l})$, such that*

$$E(\theta^*) = \inf\{E(\theta) | \theta \in H(\mathbf{P}; \alpha_0, \alpha_N; \mathbf{l})\}. \tag{3.11}$$

In the remaining part of this section, we assume (3.7) holds.

3.2. *Lagrange multipliers*

Now let us use Theorem 2.5 to justify the Lagrange multiplier rule for the constrained minimization problem (3.4). Again, we make the change of variable:

$$\omega(s) = \theta(s) - L(s) \tag{3.12}$$

where

$$L(s) = \alpha_0\left(1 - \frac{s}{l}\right) + \alpha_N \frac{s}{l}. \tag{3.13}$$

Then, $\theta \in H(\mathbf{P}; \alpha_0, \alpha_N; \mathbf{l})$ iff $\omega \in \bar{H}(\mathbf{P}; 0, 0; \mathbf{l})$, where:

$$\bar{H}(\mathbf{P}; 0, 0; \mathbf{l}) = \left\{ \omega \in H_0^1(0, l) \,\middle|\, \int_{L_{i-1}}^{L_i} \cos[\omega(s) + L(s)]\,\mathrm{d}s = x_i - x_{i-1}, \right.$$
$$\left. \int_{L_{i-1}}^{L_i} \sin[\omega(s) + L(s)]\,\mathrm{d}s = y_i - y_{i-1}, \quad i = 1, \ldots, N \right\}. \tag{3.14}$$

And, the constrained minimization problem (3.4) is equivalent to the following.
Find $\omega^* \in \bar{H}(\mathbf{P}; 0, 0; \mathbf{l})$, such that

$$F(\omega^*) = \inf\{F(\omega) | \omega \in \bar{H}(\mathbf{P}; 0, 0; \mathbf{l})\} \tag{3.15}$$

where

$$F(\omega) = E(\omega + L) = \int_0^l \tfrac{1}{2}[\omega'(s) + L'(s)]^2\,\mathrm{d}s. \tag{3.16}$$

We set

$U = X = H_0^1(0, l)$, with the inner product $(\phi, \psi) = \int_0^l \phi' \psi'\,\mathrm{d}s$;

$Y = R^{2N}$, with the usual scalar product in R^{2N};

$G(\omega) = (G_1(\omega)^{\mathrm{T}}, \ldots, G_N(\omega)^{\mathrm{T}})^{\mathrm{T}}$,

$$G_i(\omega) = \begin{pmatrix} \int_{L_{i-1}}^{L_i} \cos[\omega(s) + L(s)]\,\mathrm{d}s - (x_i - x_{i-1}) \\ \int_{L_{i-1}}^{L_i} \sin[\omega(s) + L(s)]\,\mathrm{d}s - (y_i - y_{i-1}) \end{pmatrix}, \quad i = 1, \ldots, N.$$

Then the constraint set $\bar{H}(\mathbf{P}; 0, 0; \mathbf{l}) = \{\omega \in X | G(\omega) = 0\}$. Let us verify the assumptions of Theorem 2.5 for the problem (3.15).

F is Frechet-differentiable at any $\omega \in X$. Indeed, we have

$$\langle F'(\omega), \psi \rangle = \int_0^l [\omega'(s) + L'(s)]\psi'(s)\mathrm{d}s, \quad \forall \psi \in X.$$

G is a submersion at any $\omega \in \bar{H}(\mathbf{P}; 0, 0; \mathbf{l})$, for:

(i) G is continuously Frechet-differentiable on X, with

$$\langle G'(\phi), \psi \rangle = (\langle G_1'(\phi), \psi \rangle^{\mathrm{T}}, \ldots, \langle G_N'(\phi), \psi \rangle^{\mathrm{T}})^{\mathrm{T}},$$

$$\langle G_i'(\phi), \psi \rangle = \begin{pmatrix} \int_{L_{i-1}}^{L_i} [-\sin(\phi + L)\psi]\mathrm{d}s \\ \int_{L_{i-1}}^{L_i} \cos(\phi + L)\psi\,\mathrm{d}s \end{pmatrix}, \quad i = 1, \ldots, N, \quad \forall \phi, \psi \in X.$$

(ii) Recall the assumption (3.7). If $\omega \in \bar{H}(\mathbf{P}; 0, 0; \mathbf{l})$, then $\omega + L$ is not a constant on $[L_{i-1}, L_i]$, for each i. By the continuity of ω, for each i, there is a $\delta_i > 0, s_{i,1}, s_{i,2} \in (L_{i-1}, L_i)$, such that $I_{i,1} = (s_{i,1} - \delta_i, s_{i,1} + \delta_i) \subset (L_{i-1}, L_i)$, $I_{i,2} = (s_{i,2} - \delta_i, s_{i,2} + \delta_i) \subset (L_{i-1}, L_i)$, $I_{i,1} \cap I_{i,2} = \varnothing$, and:

$$\begin{vmatrix} \sin[\omega(t_{i,1}) + L(t_{i,1})] & \sin[\omega(t_{i,3}) + L(t_{i,3})] \\ \cos[\omega(t_{i,2}) + L(t_{i,2})] & \cos[\omega(t_{i,4}) + L(t_{i,4})] \end{vmatrix} \neq 0,$$

$$\forall t_{i,1}, t_{i,2} \in I_{i,1}, \quad \forall t_{i,3}, t_{i,4} \in I_{i,2}.$$

Define two functions in X:

$$\eta_{i,1}(s) = \begin{cases} |s - s_{i,1}|/\delta_i, & \text{if } |s - s_{i,1}| \leqslant \delta_i, \\ 0, & \text{otherwise;} \end{cases}$$

$$\eta_{i,2}(s) = \begin{cases} |s - s_{i,2}|/\delta_i, & \text{if } |s - s_{i,2}| \leqslant \delta_i, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\eta_1, \eta_2 \in X$.

Note that $\int_{I_{i,1}} \eta_{i,1}(s) ds = \int_{I_{i,2}} \eta_{i,2}(s) ds = \delta_i > 0$. By the mean-value theorem, there exist $t_{i,1}, t_{i,2} \in I_{i,1}, t_{i,3}, t_{i,4} \in I_{i,2}$, such that:

$$\det(\langle G_i'(\omega), \eta_{i,1} \rangle, \langle G_i'(\omega), \eta_{i,2} \rangle)$$

$$= \det\begin{pmatrix} -\int_{L_{i-1}}^{L_i} \eta_{i,1} \sin[\omega(s) + L(s)] ds & -\int_{L_{i-1}}^{L_i} \eta_{i,2} \sin[\omega(s) + L(s)] ds \\ \int_{L_{i-1}}^{L_i} \eta_{i,1} \cos[\omega(s) + L(s)] ds & \int_{L_{i-1}}^{L_i} \eta_{i,2} \cos[\omega(s) + L(s)] ds \end{pmatrix}$$

$$= \det\begin{pmatrix} -\int_{I_1} \eta_{i,1}(s) \sin[\omega(s) + L(s)] ds & -\int_{I_2} \eta_{i,2}(s) \sin[\omega(s) + L(s)] ds \\ \int_{I_1} \eta_{i,1}(s) \cos[\omega(s) + L(s)] ds & \int_{I_2} \eta_{i,2}(s) \cos[\omega(s) + L(s)] ds \end{pmatrix}$$

$$= \det\begin{pmatrix} -\delta_i \sin[\omega(t_{i,1}) + L(t_{i,1})] & -\delta_i \sin[\omega(t_{i,3}) + L(t_{i,3})] \\ \delta_i \cos[\omega(t_{i,2}) + L(t_{i,2})] & \delta_i \cos[\omega(t_{i,4}) + L(t_{i,4})] \end{pmatrix}$$

$$= -\delta_i^2 \det\begin{pmatrix} \sin[\omega(t_{i,1}) + L(t_{i,1})] & \sin[\omega(t_{i,3}) + L(t_{i,3})] \\ \cos[\omega(t_{i,2}) + L(t_{i,2})] & \cos[\omega(t_{i,4}) + L(t_{i,4})] \end{pmatrix}$$

$$\neq 0.$$

Noting that: $\langle G_j', \eta_{i,1} \rangle = \langle G_j', \eta_{i,2} \rangle = 0$ for $j \neq i$, we then have:

$$\text{span}\{\langle G'(\omega), \eta_{i,1} \rangle, \langle G'(\omega), \eta_{i,2} \rangle, i = 1, \ldots, N\} = Y.$$

Hence, $R(G'(\omega)) = Y$.

(iii) Since $N(G'(\omega)) = \{\eta \in X | \langle G'(\omega), \eta \rangle = 0\}$ is closed in X, there exists a continuous projection operator of X on $N(G'(\omega))$.

Therefore, we can apply Theorem 2.5 to the problem (3.15) and obtain the existence of

$$\Lambda = (-\lambda_{1,1}, \lambda_{1,2}, \ldots, -\lambda_{N,1}, \lambda_{N,2})^{\mathrm{T}} \in Y^* = R^{2N},$$

such that

$$\int_0^l (\omega^{*\prime} + L')\psi' ds - \sum_{i=1}^N (-\lambda_{i,1}, \lambda_{i,2}) \left(\int_{L_{i-1}}^{L_i} [-\sin(\omega^* + L)] \psi \, ds, \right.$$

$$\int_{L_{i-1}}^{L_i} \cos[(\omega^* + L)\psi]^{\mathrm{T}} \, \mathrm{d}s \Big) = 0, \quad \forall \psi \in X \tag{3.17}$$

that is

$$\int_0^l (\omega^{*\prime} + L')\psi' \, \mathrm{d}s - \sum_{i=1}^{N} \left(\int_{L_{i-1}}^{L_i} [\lambda_{i,1} \sin(\omega^* + L)\psi \right.$$

$$\left. + \lambda_{i,2} \cos(\omega^* + L)\psi] \, \mathrm{d}s \right) = 0, \quad \forall \psi \in X. \tag{3.18}$$

Back to the original problem (3.4), we then have:

Theorem 3.4. *If θ^* is a solution of the problem* (3.4), *then there exist $2N$ real numbers $\lambda_{i,1}, \lambda_{i,2}$, $i = 1, \ldots, N$, such that*

$$\int_0^l \theta^{*\prime}\psi' \, \mathrm{d}s - \sum_{i=1}^{N} \int_{L_{i-1}}^{L_i} (\lambda_{i,1} \sin\theta^* + \lambda_{i,2} \cos\theta^*)\psi \, \mathrm{d}s = 0, \quad \forall \psi \in H_0^1(0, l). \tag{3.19}$$

From (3.19), we see that on each subinterval (L_{i-1}, L_i), θ^* satisfies the equation:

$$\theta^{*\prime\prime} + \lambda_{i,1} \sin\theta^* + \lambda_{i,2} \cos\theta^* = 0, \tag{3.20}$$

in the sense of distribution.

3.3. *Smoothness of a minimal-energy spline*

We study the smoothness of a minimal-energy spline. Since $\theta^* \in H^1(0, l)$, θ^* is continuous on $[0, l]$. Since θ^* solves (3.20) on (L_{i-1}, L_i), by the same argument as that in section 2, we have $\theta^* \in C^\infty([L_{i-1}, L_i])$, $i = 1, \ldots, N$. In (3.19), let us take ψ as follows:

$$\psi_{i,h}(s) = \begin{cases} |s - L_i|/h, & \text{if } |s - L_i| \leqslant h, \\ 0, & \text{otherwise,} \end{cases}$$

and let $h \to 0+$, we then have:

$$\theta^{*\prime}(L_i+) = \theta^{*\prime}(L_i-), \quad i = 1, \ldots, N-1,$$

that is θ^* is continuously differentiable. In conclusion, we have

$$\theta^* \in C^1([0, l]) \cap \bigcap_{i=1}^{N} C^\infty([L_{i-1}, L_i]). \tag{3.21}$$

A trivial consequence of the smoothness (3.21) is that θ^* is a classical solution of the equation (3.20) on (L_{i-1}, L_i), $i = 1, \ldots, N$.

4. Computation of minimal-energy splines

One may derive various algorithms to compute minimal-energy splines with angle constraints on the basis of Definition 2.1 for spline segments and Definition 3.1 for spline curves. We state a simple algorithm to solve the constrained minimization problem (2.10).

Let N be a positive integer. Denote $h = l/N$, $s_i = ih$, $0 \leqslant i \leqslant N$; $I_i = [s_{i-1}, s_i]$, $1 \leqslant i \leqslant N$; and the discrete constraint set:

$$H_h(P_1, P_2; \alpha_1, \alpha_2; l) = \begin{cases} \theta_h \in H^1(0, l) | \theta_h(0) = \alpha_1, \theta_h(l) = \alpha_2, \\ \theta_h \text{ is a polynomial of degree one, on } I_i, 1 \leqslant i \leqslant N, \\ \sum_{i=1}^{N} \frac{h}{2} [\cos \theta_h(s_{i-1}) + \cos \theta_h(s_i)] = x_2 - x_1, \\ \sum_{i=1}^{N} \frac{h}{2} [\sin \theta_h(s_{i-1}) + \sin \theta_h(s_i)] = y_2 - y_1 \end{cases}. \tag{4.1}$$

Then we use the following problem to approximate the problem (2.10):

find $\theta_h^* \in H_h(P_1, P_2; \alpha_1, \alpha_2; l)$ such that

$$E(\theta_h^*) = \inf\{E(\theta_h) | \theta_h \in H_h(P_1, P_2; \alpha_1, \alpha_2; l)\}. \tag{4.2}$$

Under the assumption (2.13), we have the following statements concerning the approximate problem (4.2).

(a) There exists $h_0 > 0$, such that for all $h \in (0, h_0]$, $H_h(P_1, P_2; \alpha_1, \alpha_2; l) \neq \varnothing$, and (4.2) has a solution θ_h^*.

(b) Each sequence $\{\theta_h^*\}$ contains a subsequence which converges to θ^* in $H^1(0, l)$ and in $C^0([0, l])$, for a solution θ^* of the problem (2.10).

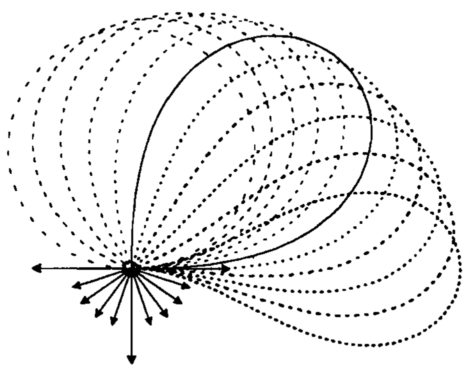
A similar algorithm and the same convergence results can be stated for the constrained minimization problem (3.4). The proof of these results will be published elsewhere.

Numerical results for two families of minimal-energy spline segments and two families of minimal-energy splines follow.

We compute a family of fixed-length *closed* minimal-energy spline segments with the constraint sets $H(P_1, P_2; \alpha_1, \alpha_2; l)$, where $P_1 = P_2 = (0, 0)$, $\alpha_1 = 0$, $\alpha_2 \in [\pi, 2\pi]$, and $l = \pi$. The 11 closed minimal-energy spline segments of Table 1 are shown in Figure 1. The solid curve corresponds to the case $\alpha_2 = 1.5\pi$. When $\alpha_2 = 2\pi$, the closed minimal-energy spline is a circle.

Table 1. Closed minimal-energy spline segments

P_1	P_2	α_1	α_2	Length	Energy
(0, 0)	(0, 0)	0.0	1.0π	π	5.8340
(0, 0)	(0, 0)	0.0	1.1π	π	5.2961
(0, 0)	(0, 0)	0.0	1.2π	π	4.8933
(0, 0)	(0, 0)	0.0	1.3π	π	4.6254
(0, 0)	(0, 0)	0.0	1.4π	π	4.4911
(0, 0)	(0, 0)	0.0	1.5π	π	4.4881
(0, 0)	(0, 0)	0.0	1.6π	π	4.6129
(0, 0)	(0, 0)	0.0	1.7π	π	4.8611
(0, 0)	(0, 0)	0.0	1.8π	π	5.2269
(0, 0)	(0, 0)	0.0	1.9π	π	5.7035
(0, 0)	(0, 0)	0.0	2.0π	π	6.2832



$P_1 = (0, 0)$, $P_2 = (0, 0)$, $\alpha_1 = 0$, $\alpha_2 \in [\pi, 2\pi]$, length $= \pi$

Fig. 1. Closed minimal-energy spline segments

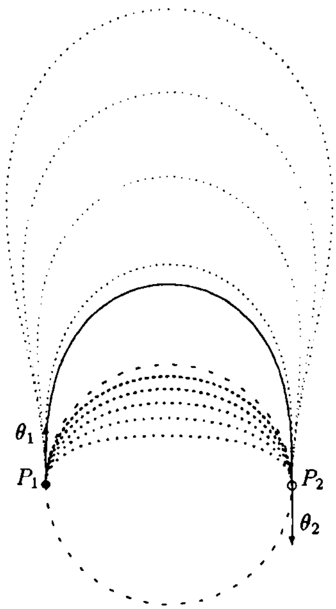
Table 2. Open minimal-energy spline segments

P_1	P_2	α_1	α_2	Length	Energy
(−1, 0)	(1, 0)	$\pi/2$	$-\pi/2$	0.75π	3.8612
(−1, 0)	(1, 0)	$\pi/2$	$-\pi/2$	0.80π	2.7186
(−1, 0)	(1, 0)	$\pi/2$	$-\pi/2$	0.85π	2.1572
(−1, 0)	(1, 0)	$\pi/2$	$-\pi/2$	0.90π	1.8504
(−1, 0)	(1, 0)	$\pi/2$	$-\pi/2$	0.95π	1.6743
(−1, 0)	(1, 0)	$\pi/2$	$-\pi/2$	1.00π	1.5708
(−1, 0)	(1, 0)	$\pi/2$	$-\pi/2$	1.39π	1.4354
(−1, 0)	(1, 0)	$\pi/2$	$-\pi/2$	1.50π	1.4347
(−1, 0)	(1, 0)	$\pi/2$	$-\pi/2$	2.00π	1.3899
(−1, 0)	(1, 0)	$\pi/2$	$-\pi/2$	2.50π	1.2994
(−1, 0)	(1, 0)	$\pi/2$	$-\pi/2$	3.00π	1.1996

Table 3. Closed minimal-energy spline

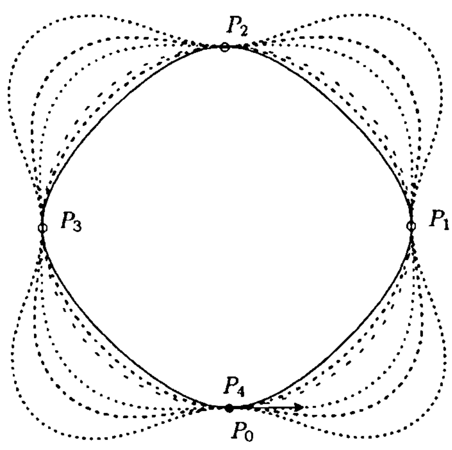
P_0	P_1	P_2	P_3	P_4	α_0	α_4	$l_1 = l_2 = l_3 = l_4$	Energy
(0, −1)	(1, 0)	(0, 1)	(−1, 0)	(0, −1)	0.0	2π	1.4994	4.4507
(0, −1)	(1, 0)	(0, 1)	(−1, 0)	(0, −1)	0.0	2π	1.5351	3.4301
(0, −1)	(1, 0)	(0, 1)	(−1, 0)	(0, −1)	0.0	2π	1.5708	3.1416
(0, −1)	(1, 0)	(0, 1)	(−1, 0)	(0, −1)	0.0	2π	1.7491	4.7859
(0, −1)	(1, 0)	(0, 1)	(−1, 0)	(0, −1)	0.0	2π	1.8921	6.7961
(0, −1)	(1, 0)	(0, 1)	(−1, 0)	(0, −1)	0.0	2π	2.2134	10.521

We also compute a family of *open* minimal-energy spline segments with the constraint sets $H(P_1, P_2; \alpha_1, \alpha_2; l)$, where $P_1 = (-1, 0)$, $P_2 = (1, 0)$, $\alpha_1 = \pi/2$, $\alpha_2 = -\pi/2$ and $l \in [0.75\pi, 3.0\pi]$. The 11 open minimal-energy spline segments of Table 2 are shown in Figure 2. The solid curve corresponds to the case $l = 1.39\pi$. When $l = \pi$, the open minimal-energy spline is a semi-circle.



$P_1 = (-1, 0)$, $P_2 = (1, 0)$, $\alpha_1 = 0.5\pi$, $\alpha_2 = -0.5\pi$, length $\in [0.75\pi, 3\pi]$

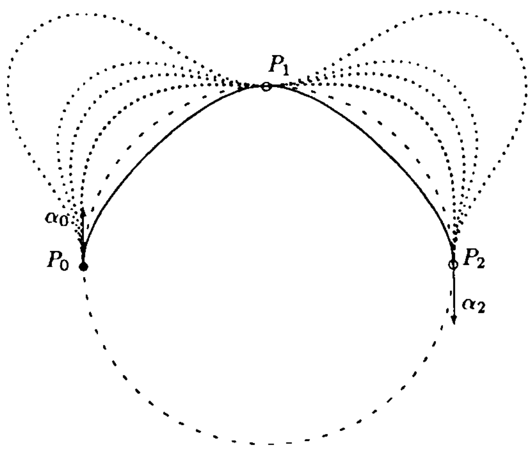
Fig. 2. Open minimal-energy spline segments



$P_0 = (0, -1)$, $P_1 = (1, 0)$, $P_2 = (0, 1)$, $P_3 = (-1, 0)$, $P_4 = (0, -1)$
$\alpha_0 = 0$, $\alpha_4 = 2\pi$, length $\in [1.91\pi, 2.82\pi]$

Fig. 3. Closed minimal-energy spline

Now, we compute a family of five-point *closed* minimal-energy splines with the constraint sets $H(P_0, P_1, P_2, P_3, P_4; \alpha_0, \alpha_4; l)$, where $P_0 = P_4 = (0, -1)$, $P_1 = (1, 0)$, $P_2 = (0, 1)$, $P_3 = (-1, 0)$, $\alpha_0 = 0$, $\alpha_4 = 2\pi$ and $l = 4s$, $s \in \{1.4994, 1.5351, 0.5\pi, 1.7493, 1.8921, 2.2134\}$. The six closed minimal-energy splines of Table 3 are shown in



$P_0 = (-1, 0)$, $P_1 = (0, 1)$, $P_2 = (1, 0)$
$\alpha_0 = 0.5\pi$, $\alpha_2 = -0.5\pi$, length $\in [1.91\pi, 3.59\pi]$

Fig. 4. Open minimal-energy spline

Table 4. Open minimal-energy spline

P_0	P_1	P_2	α_0	α_2	$l_1 = l_2$	Energy
$(-1, 0)$	$(0, 1)$	$(1, 0)$	$\pi/2$	$-\pi/2$	2.8203	7.0750
$(-1, 0)$	$(0, 1)$	$(1, 0)$	$\pi/2$	$-\pi/2$	2.1063	4.7245
$(-1, 0)$	$(0, 1)$	$(1, 0)$	$\pi/2$	$-\pi/2$	1.9278	3.6397
$(-1, 0)$	$(0, 1)$	$(1, 0)$	$\pi/2$	$-\pi/2$	1.7493	2.3932
$(-1, 0)$	$(0, 1)$	$(1, 0)$	$\pi/2$	$-\pi/2$	1.5708	1.5708
$(-1, 0)$	$(0, 1)$	$(1, 0)$	$\pi/2$	$-\pi/2$	1.4994	2.2254

Figure 3. The solid curve corresponds to the case $l = 5.9976$. When $l = 2\pi$, the closed minimal-energy spline is a circle.

Finally, we compute a family of three-point *open* minimal-energy splines with the constraint sets $H(P_0, P_1, P_2; \alpha_1, \alpha_2; l)$, where $P_0 = (-1, 0)$, $P_1 = (0, 1)$, $P_2 = (1, 0)$, $\alpha_0 = \pi/2, \alpha_2 = -\pi/2$, and $l = 2s$, $s \in \{1.4994, 0.5\pi, 1.7493, 1.9278, 2.1063, 2.8203\}$. The six open minimal-energy splines of Table 4 are shown in Figure 4. The solid curve corresponds to the case $l = 2.9988$. When $l = \pi$, the open minimal-energy spline is a semi-circle.

Acknowledgement

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