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Numerical analysis of the diffusive-viscous wave equation

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ABSTRACT

The diffusive-viscous wave equation arises in a variety of applications in geophysics, and it plays an important role in seismic exploration. In this paper, semi-discrete and fully discrete numerical methods are introduced to solve a general initial-boundary value problem of the diffusive-viscous wave equation. The spatial discretization is carried out through the finite element method, whereas the time derivatives are approximated by finite differences. Optimal order error estimates are derived for the numerical methods. Numerical results on a test problem are reported to illustrate the numerical convergence orders.

1. Introduction

The goal of petroleum exploration is to identify and locate potential oil/gas reservoirs. Engineers need to assess the quantity of hydrocarbon which might be contained in the reservoirs. To increase the success rate in locating a prospective reservoir and evaluating the expected volume of petroleum, numerical simulations based on seismic wave equations have become a valuable technique. In order to simulate wave propagation in practical seismic exploration, an appropriate wave equation should be chosen. Compared with the reflection from the gas-saturated layer, the seismic response is more complicated in the fluid-saturated layer. From laboratory and field data, it is observed that the reflection from the fluid-saturated layer causes an increase in amplitude and delayed propagation time at low frequencies, which can be used to detect fluid-saturated layers. The diffusive-viscous wave equation was introduced to describe this phenomenon ([14,25]). In addition, numerical simulation of the diffusive-viscous wave equation was applied to address practical issues ([27,7,23,9,10]). In [29–31], the finite difference method was used to solve the diffusive-viscous wave equation numerically. A finite volume method was studied to simulate seismic wave propagation in a fluid-saturated medium using the diffusive-viscous wave equation ([26]). It appears that no rigorous numerical analysis can be found in the literature for solving the initial-boundary value problem of the diffusive-viscous wave equation. In this paper, we fill this gap by developing and rigorously analyzing a numerical method for a general initial-boundary value problem of the diffusive-viscous wave equation. More precisely, the numerical method is designed so that it is of second-order accurate with respect to the time-step and the standard finite element method is used to discretize the spatial variable. The framework developed in this paper can be applied for analysis of other numerical methods with different discretizations in time and in space. A somewhat related equation is the second-order wave equation, and a rich literature is available on its numerical solution, e.g. [13,5] with finite element methods, [12,6] on mixed finite element methods, [24,15,16,2,4,28,18,20–22] on discontinuous Galerkin methods, and so on.

The initial-boundary value problem (IBVP) of the diffusive-viscous wave equation will be considered in any spatial dimension d; the cases with $d = 2$ and 3 are more relevant to applications. A mixture of Dirichlet, Neumann, and Robin boundary conditions is allowed. Assume the spatial

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domain $\Omega \subset \mathbb{R}^d$ has a Lipschitz continuous boundary which is split into three parts: $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_R}$ such that Γ_D, Γ_N, and Γ_R are relatively open, and are mutually disjoint. We allow the possibility for one or two of the three subsets Γ_D, Γ_N, and Γ_N to be empty. Since $\partial\Omega$ is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu}$ is defined a.e. on $\partial\Omega$. Let $[0,T]$ be the time interval of interest, $T>0$. We use the notation $Q=\Omega\times(0,T)$.

The pointwise formulation of the IBVP of the diffusive-viscous wave equation is as follows:

$$\frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial u}{\partial t} - \frac{\partial}{\partial t}\mathrm{div}\,(\eta\nabla u) - \mathrm{div}\left(\zeta^2\nabla u\right) = f \quad \text{in } Q, \tag{1.1}$$

$$u = g_1 \quad \text{on } \Gamma_D \times [0,T], \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} = g_2 \quad \text{on } \Gamma_N \times [0,T], \tag{1.3}$$

$$\frac{\partial u}{\partial \nu} + \kappa\, u = g_3 \quad \text{on } \Gamma_R \times [0,T], \tag{1.4}$$

$$u = u_0,\ \frac{\partial u}{\partial t} = w_0 \quad \text{on } \Omega\times\{0\}, \tag{1.5}$$

where $u=u(\boldsymbol{x},t)$ is the unknown wave field and $f=f(\boldsymbol{x},t)$ is the source function. Here, $\gamma>0$ is the diffusive attenuation parameter, $\eta>0$ is the viscous attenuation parameter, $\zeta>0$ is the wave propagation speed in the non-dispersive medium, $\kappa=\kappa(\boldsymbol{x})>0$ is the coefficient for the Robin boundary condition, $g_1=g_1(\boldsymbol{x},t)$ is the wave field on the Dirichlet boundary Γ_D, $g_2=g_2(\boldsymbol{x},t)$ is the outward normal derivative of the wave field on the Neumann boundary Γ_N, and $g_3=g_3(\boldsymbol{x},t)$ is the Robin boundary data. In addition, $u_0=u_0(\boldsymbol{x})$ is the initial wave field, and $w_0=w_0(\boldsymbol{x})$ is the initial wave field velocity. The expression $\partial u/\partial\nu$ denotes the normal derivative of u on $\partial\Omega$. The parameters γ, η, ζ and κ can depend on the spatial variable $\boldsymbol{x}$, and they are assumed to be bounded functions of $\boldsymbol{x}$ and are bounded below away from zero.

The rest of the paper is organized as follows. The weak formulation of the IBVP of the diffusive-viscous wave equation is recalled in Section 2. In Section 3, a semi-discrete numerical method is introduced with the spatial variable discretized by the finite element method; optimal order error estimates are derived under appropriate solution regularity assumptions. In Section 4, a fully discrete numerical method is studied, where in addition to the finite element method for the spatial variable, finite differences are used to approximate the time derivatives. Again, optimal order error estimates are derived under appropriate solution regularity assumptions. In Section 5, simulation results from a numerical example are reported, illustrating the numerical convergence orders that are in agreement with the theoretical predictions.

2. Weak formulation

We first introduce some function spaces. Given a bounded domain $D\subset\mathbb{R}^d$ and an integer $m\ge 0$, $W^{m,p}(D)$ is the Sobolev space with the corresponding usual norm $\|\cdot\|_{W^{m,p}(D)}$ and semi-norm $|\cdot|_{W^{m,p}(D)}$. When $p=2$, the space $W^{m,2}(D)$ is written as $H^m(D)$, and the associated norm and semi-norm are denoted by $\|\cdot\|_{H^m(D)}$ and $|\cdot|_{H^m(D)}$, respectively. For the Lebesgue space $L^2(D)\equiv H^0(D)$, the norm and inner product are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$. In particular, we set

$$V = \left\{ v\in H^1(\Omega) \mid v=0 \text{ on } \Gamma_D \right\}. \tag{2.1}$$

Furthermore, for the time dependent functions, we introduce the space

$$W^{m,p}(0,T;V) = \{ v\in L^p(0,T;V) \mid \|\partial_t^l v\|_{L^p(0,T;V)} < \infty \ \forall\, l\le m\}$$

with the norm

$$\|v\|_{W^{m,p}(0,T;V)} = \begin{cases} \left(\displaystyle\int_0^T \sum_{0\le l\le m} \|\partial_t^l v\|_V^p\, dt \right)^{1/p} & \text{if } 1\le p<\infty, \\ \max_{0\le l\le m} \operatorname{ess\,sup}_{0\le t\le T} \|\partial_t^l v\|_V & \text{if } p=\infty. \end{cases}$$

For a function $u(\boldsymbol{x},t)$, we will also use the notation $u(t)$, and will write $\dot u(t)$ for $\partial u(\boldsymbol{x},t)/\partial t$, and $\ddot u(t)$ for $\partial^2 u(\boldsymbol{x},t)/\partial t^2$. A standard reference on Sobolev and Lebesgue spaces is [1].

The weak formulation of the IBVP (1.1)–(1.5) is as follows: Find a function u defined on Q such that for a.e. $t\in(0,T)$, $u(\cdot,t)\in H^1(\Omega)$, $u(\boldsymbol{x},t)=g_1(\boldsymbol{x},t)$ on Γ_D, and

$$\begin{aligned}&(\ddot u(t),v)_\Omega + (\gamma\,\dot u(t),v)_\Omega + (\eta\,\nabla\dot u(t),\nabla v)_\Omega + (\zeta^2\nabla u(t),\nabla v)_\Omega + \left(\eta\,\kappa\,\dot u(t) + \zeta^2\kappa\, u(t), v\right)_{\Gamma_R} \\ &\quad = (f(t),v)_\Omega + \left(\eta\,\dot g_2(t) + \zeta^2 g_2(t), v\right)_{\Gamma_N} + \left(\eta\,\dot g_3(t) + \zeta^2 g_3(t), v\right)_{\Gamma_R} \quad \forall\, v\in V,\end{aligned} \tag{2.2}$$

$$u(0)=u_0,\ \dot u(0)=w_0. \tag{2.3}$$

To simplify the notation, we use $(\cdot,\cdot)_\Omega$ for the inner product also in $L^2(\Omega)^d$, e.g. $(\eta\,\nabla\dot u(t),\nabla v)_\Omega$. Similarly, in the following, the notation $\|\cdot\|_\Omega$ will also stand for the $L^2(\Omega)^d$-norm. Strictly speaking, the first term $(\ddot u(t),v)_\Omega$ in (2.2) should be understood in the sense of a duality pairing, but this will not have an impact on the development of the numerical method. Well-posedness of this problem is addressed in [17]. This paper is devoted to the numerical solution of the problem. For simplicity in writing, we will only consider the case where $g_1=0$. The case of a non-homogeneous Dirichlet boundary condition can be converted to the one with homogeneous Dirichlet boundary condition through a standard technique (cf. [3, Chapter 8]). Thus, defining

$$\ell(t,v) = (f(t),v)_\Omega + \left(\eta\,\dot g_2(t) + \zeta^2 g_2(t), v\right)_{\Gamma_N} + \left(\eta\,\dot g_3(t) + \zeta^2 g_3(t), v\right)_{\Gamma_R} \quad \forall\, v\in V, \tag{2.4}$$

we have the continuous level problem.

Problem 2.1. Find u such that for a.e. $t \in (0,T)$, $u(\cdot,t) \in V$ and

$$(\ddot{u}(t), v)_\Omega + (\gamma\, \dot{u}(t), v)_\Omega + (\eta\, \nabla \dot{u}(t), \nabla v)_\Omega + (\zeta^2 \nabla u(t), \nabla v)_\Omega$$

$$+ \left(\eta\, \kappa\, \dot{u}(t) + \zeta^2 \kappa\, u(t), v\right)_{\Gamma_R} = \ell(t, v) \quad \forall\, v \in V, \tag{2.5}$$

$$u(0) = u_0,\ \dot{u}(0) = w_0. \tag{2.6}$$

We now introduce assumptions on the problem data. For some positive constants $\eta_1, \eta_2, \zeta_1, \zeta_2, \kappa_1$ and κ_2, we assume

$$\gamma \in L^\infty(\Omega),\ 0 < \gamma_1 \le \gamma \le \gamma_2 < \infty \text{ a.e. on } \Omega, \tag{2.7}$$

$$\eta \in L^\infty(\Omega) \text{ and } \eta \in L^\infty(\Gamma_N \cup \Gamma_R),\ 0 < \eta_1 \le \eta \le \eta_2 < \infty \text{ a.e. on } \Omega \text{ and } \Gamma_N \cup \Gamma_R, \tag{2.8}$$

$$\zeta \in L^\infty(\Omega) \text{ and } \zeta \in L^\infty(\Gamma_N \cup \Gamma_R),\ 0 < \zeta_1 \le \zeta \le \zeta_2 < \infty \text{ a.e. on } \Omega \text{ and } \Gamma_N \cup \Gamma_R, \tag{2.9}$$

$$\kappa \in L^\infty(\Gamma_R),\ 0 < \kappa_1 \le \kappa \le \kappa_2 < \infty \text{ a.e. on } \Gamma_R. \tag{2.10}$$

Moreover, we assume

$$f \in L^2(0,T;V^*), \quad g_2 \in W^{1,2}(0,T;L^2(\Gamma_N)), \quad g_3 \in W^{1,2}(0,T;L^2(\Gamma_R)), \tag{2.11}$$

$$u_0 \in V, \quad w_0 \in L^2(\Omega), \tag{2.12}$$

and

$$\text{meas}(\Gamma_D) + \text{meas}(\Gamma_R) > 0. \tag{2.13}$$

Here, V^* denotes the dual space of V. We comment that under the assumption (2.13), the expression $(\|\nabla v\|_\Omega^2 + \|v\|_{\Gamma_R}^2)^{1/2}$ defines a norm over the space V, which is equivalent to the norm $\|v\|_{H^1(\Omega)}$, i.e., there exist positive constants $\tilde{c}_1$ and $\tilde{c}_2$ such that

$$\tilde{c}_1 \|v\|_{H^1(\Omega)} \le \left(\|\nabla v\|_\Omega^2 + \|v\|_{\Gamma_R}^2\right)^{1/2} \le \tilde{c}_2 \|v\|_{H^1(\Omega)} \quad \forall\, v \in H^1(\Omega). \tag{2.14}$$

The following solution existence and uniqueness result is proved in [17] for Problem 2.1.

Theorem 2.2. *Assume* (2.7)–(2.13). *Then Problem 2.1 has a unique solution u with*

$$u \in L^\infty(0,T;V),\ \dot{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V),\ \ddot{u} \in L^2(0,T;V^*).$$

We will use the following elementary inequality repeatedly without explicitly mentioning it:

$$a\, b \le \epsilon\, a^2 + \frac{1}{4\epsilon} b^2 \quad \forall\, \epsilon > 0,\ a, b \in \mathbb{R}.$$

3. A spatially semi-discrete method

For simplicity, we assume Ω is a polygonal/polyhedral domain. Let $\{\mathcal{T}^h\}_{h>0}$ be a regular family of finite element partitions of $\overline{\Omega}$ into triangles ($d = 2$) or tetrahedrons ($d = 3$). Corresponding to each finite element partition $\mathcal{T}^h$, let $V^h \subset V$ be a finite element space of continuous piecewise polynomials of a degree less than or equal to p.

Let us introduce a V-orthogonal projection operator $\Pi^h : V \to V^h$. In the case $|\Gamma_D| > 0$, this projection is defined by the following relation: for $u \in V$,

$$\Pi^h u \in V^h, \quad (\nabla \Pi^h u, \nabla v^h)_\Omega = (\nabla u, \nabla v^h)_\Omega \quad \forall\, v^h \in V^h, \tag{3.1}$$

and in case $|\Gamma_D| = 0$, (3.1) is to be supplemented by the condition

$$\int_\Omega \Pi^h u\, dx = \int_\Omega u\, dx.$$

We recall the following error bounds ([8,11]):

$$\|u - \Pi^h u\|_\Omega + h\, \|\nabla(u - \Pi^h u)\|_\Omega \le c\, h^{p+1} \|u\|_{H^{p+1}(\Omega)} \quad \text{if } u \in H^{p+1}(\Omega). \tag{3.2}$$

The spatially semi-discrete method for solving Problem 2.1 is the following.

Problem 3.1. Find u^h such that for a.e. $t \in (0,T)$, $u^h(\cdot,t) \in V^h$ and

$$\left(\ddot{u}^h(t), v^h\right)_\Omega + (\gamma\, \dot{u}^h(t), v^h)_\Omega + (\eta\, \nabla \dot{u}^h(t), \nabla v^h)_\Omega + (\zeta^2 \nabla u^h(t), \nabla v^h)_\Omega + \left(\eta\, \kappa\, \dot{u}^h(t) + \zeta^2 \kappa\, u^h(t), v^h\right)_{\Gamma_R} = \ell(t, v^h) \quad \forall\, v^h \in V^h, \tag{3.3}$$

$$u^h(0) = \Pi^h u_0,\ \dot{u}^h(0) = \Pi^h w_0. \tag{3.4}$$

Similar to Problem 2.1, Problem 3.1 admits a unique solution. The rest of the section is devoted to a derivation of error estimates for the numerical solution defined by Problem 3.1. We will use the norm equivalence (2.14) repeatedly. We split the error into two parts:

$$u^h(t) - u(t) = e^h(t) + e_0^h(t), \tag{3.5}$$

where

$$e^h(t) = u^h(t) - \Pi^h u(t), \tag{3.6}$$

$$e_0^h(t) = \Pi^h u(t) - u(t). \tag{3.7}$$

Note that we have the error bound (3.2) for $e_0^h(t)$. In addition, similar to (3.2),

$$\|\dot u(t) - \Pi^h \dot u(t)\|_\Omega + h\,\|\nabla(\dot u(t) - \Pi^h \dot u(t))\|_\Omega \le c\,h^{p+1}\|\dot u(t)\|_{H^{p+1}(\Omega)} \quad \text{if } \dot u(t) \in H^{p+1}(\Omega), \tag{3.8}$$

$$\|\ddot u(t) - \Pi^h \ddot u(t)\|_\Omega + h\,\|\nabla(\ddot u(t) - \Pi^h \ddot u(t))\|_\Omega \le c\,h^{p+1}\|\ddot u(t)\|_{H^{p+1}(\Omega)} \quad \text{if } \ddot u(t) \in H^{p+1}(\Omega). \tag{3.9}$$

Thus, in the following, we focus on estimating the error component $e^h(t)$.

For any $v^h \in V^h$, by subtracting (2.5) with $v = v^h$ from (3.3) and by using the splitting (3.5), we have

$$\begin{aligned}
&\left(\ddot e^h(t), v^h\right)_\Omega + (\gamma\,\dot e^h(t), v^h)_\Omega + (\eta\,\nabla \dot e^h(t), \nabla v^h)_\Omega + (\zeta^2 \nabla e^h(t), \nabla v^h)_\Omega + \left(\eta\,\kappa\,\dot e^h(t) + \zeta^2\kappa\,e^h(t), v^h\right)_{\Gamma_R} \\
&\quad = -\left(\ddot e_0^h(t), v^h\right)_\Omega - (\gamma\,\dot e_0^h(t), v^h)_\Omega - (\eta\,\nabla \dot e_0^h(t), \nabla v^h)_\Omega - (\zeta^2 \nabla e_0^h(t), \nabla v^h)_\Omega - \left(\eta\,\kappa\,\dot e_0^h(t) + \zeta^2\kappa\,e_0^h(t), v^h\right)_{\Gamma_R} \quad \forall\, v^h \in V^h,
\end{aligned} \tag{3.10}$$

$$e^h(0) = 0, \;\; \dot e^h(0) = 0. \tag{3.11}$$

Let $v^h = \dot e^h(t)$ in (3.10) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|\dot e^h(t)\|_\Omega^2 + \|\zeta \nabla e^h(t)\|_\Omega^2 + \|\zeta\,\kappa^{1/2} e^h(t)\|_{\Gamma_R}^2\right] + \|\gamma^{1/2}\dot e^h(t)\|_\Omega^2 + \|\eta^{1/2}\nabla \dot e^h(t)\|_\Omega^2 + \|\eta^{1/2}\kappa^{1/2}\dot e^h(t)\|_{\Gamma_R}^2 \\
&\quad = -\left(\ddot e_0^h(t), \dot e^h(t)\right)_\Omega - (\gamma\,\dot e_0^h(t), \dot e^h(t))_\Omega - (\eta\,\nabla \dot e_0^h(t), \nabla \dot e^h(t))_\Omega - (\zeta^2 \nabla e_0^h(t), \nabla \dot e^h(t))_\Omega - \left(\eta\,\kappa\,\dot e_0^h(t) + \zeta^2\kappa\,e_0^h(t), \dot e^h(t)\right)_{\Gamma_R}.
\end{aligned}$$

Integrate the above equality from 0 to t and make use of the initial conditions (3.11),

$$\begin{aligned}
&\frac{1}{2}\left[\|\dot e^h(t)\|_\Omega^2 + \|\zeta \nabla e^h(t)\|_\Omega^2 + \|\zeta\,\kappa^{1/2} e^h(t)\|_{\Gamma_R}^2\right] + \int_0^t \left[\|\gamma^{1/2}\dot e^h(s)\|_\Omega^2 + \|\eta^{1/2}\nabla \dot e^h(s)\|_\Omega^2 + \|\eta^{1/2}\kappa^{1/2}\dot e^h(s)\|_{\Gamma_R}^2\right] ds \\
&\quad = -\int_0^t \left[\left(\ddot e_0^h(s), \dot e^h(s)\right)_\Omega + (\gamma\,\dot e_0^h(s), \dot e^h(s))_\Omega + (\eta\,\nabla \dot e_0^h(s), \nabla \dot e^h(s))_\Omega\right] ds - \int_0^t \left[(\zeta^2 \nabla e_0^h(s), \nabla \dot e^h(s))_\Omega + \left(\eta\,\kappa\,\dot e_0^h(s) + \zeta^2\kappa\,e_0^h(s), \dot e^h(s)\right)_{\Gamma_R}\right] ds.
\end{aligned}$$

By the assumptions on coefficients and the norm equivalence (2.14), we derive from the above inequality that for a constant $c_1 > 0$,

$$\begin{aligned}
\|\dot e^h(t)\|_\Omega^2 + \|e^h(t)\|_{H^1(\Omega)}^2 + \int_0^t \|\dot e^h(s)\|_{H^1(\Omega)}^2 ds &\le -c_1 \int_0^t \left[\left(\ddot e_0^h(s), \dot e^h(s)\right)_\Omega + (\gamma\,\dot e_0^h(s), \dot e^h(s))_\Omega + (\eta\,\nabla \dot e_0^h(s), \nabla \dot e^h(s))_\Omega\right] ds \\
&\quad - c_1 \int_0^t \left[(\zeta^2 \nabla e_0^h(s), \nabla \dot e^h(s))_\Omega + \left(\eta\,\kappa\,\dot e_0^h(s) + \zeta^2\kappa\,e_0^h(s), \dot e^h(s)\right)_{\Gamma_R}\right] ds.
\end{aligned} \tag{3.12}$$

Let us bound each term on the right side of (3.12). Take the first term as an example,

$$-c_1 \int_0^t \left(\ddot e_0^h(s), \dot e^h(s)\right)_\Omega ds \le c_1 \int_0^t \|\dot e^h(s)\|_\Omega \|\ddot e_0^h(s)\|_\Omega ds \le c_1 \int_0^t \|\dot e^h(s)\|_{H^1(\Omega)} \|\ddot e_0^h(s)\|_\Omega ds.$$

Thus, for any (small) $\epsilon > 0$, there exists an ϵ-dependent constant $c > 0$ such that

$$-c_1 \int_0^t \left(\ddot e_0^h(s), \dot e^h(s)\right)_\Omega ds \le \epsilon \int_0^t \|\dot e^h(s)\|_{H^1(\Omega)}^2 ds + c \int_0^t \|\ddot e_0^h(s)\|_\Omega^2 ds.$$

Similarly, for the other five terms on the right side of (3.12), we have

$$-c_1 \int_0^t (\gamma\,\dot e_0^h(s), \dot e^h(s))_\Omega ds \le \epsilon \int_0^t \|\dot e^h(s)\|_{H^1(\Omega)}^2 ds + c \int_0^t \|\dot e_0^h(s)\|_\Omega^2 ds,$$

$$-c_1 \int_0^t (\eta\,\nabla \dot e_0^h(s), \nabla \dot e^h(s))_\Omega ds \le \epsilon \int_0^t \|\dot e^h(s)\|_{H^1(\Omega)}^2 ds + c \int_0^t \|\nabla \dot e_0^h(s)\|_\Omega^2 ds,$$

$$-c_1 \int_0^t (\zeta^2 \nabla e_0^h(s), \nabla \dot e^h(s))_\Omega ds \le \epsilon \int_0^t \|\dot e^h(s)\|_{H^1(\Omega)}^2 ds + c \int_0^t \|\nabla e_0^h(s)\|_\Omega^2 ds,$$

$$-c_1\int_0^t \left(\eta\,\kappa\,\dot e_0^h(s),\dot e^h(s)\right)_{\Gamma_R} ds \le \epsilon\int_0^t \|\dot e^h(s)\|^2_{H^1(\Omega)} ds + c\int_0^t \|\dot e_0^h(s)\|^2_{\Gamma_R} ds,$$

$$-c_1\int_0^t \left(\zeta^2\kappa\,e_0^h(s),\dot e^h(s)\right)_{\Gamma_R} ds \le \epsilon\int_0^t \|\dot e^h(s)\|^2_{H^1(\Omega)} ds + c\int_0^t \|e_0^h(s)\|^2_{\Gamma_R} ds.$$

Here, on the right sides of these inequalities, $\|\nabla \dot e_0^h(s)\|^2_\Omega$ and $\|\dot e_0^h(s)\|^2_{\Gamma_R}$ are to be bounded by a constant multiple of $\|\dot e_0^h(s)\|^2_{H^1(\Omega)}$, whereas $\|\nabla e_0^h(s)\|^2_\Omega$ and $\|e_0^h(s)\|^2_{\Gamma_R}$ are to be bounded by a constant multiple of $\|e_0^h(s)\|^2_{H^1(\Omega)}$. Using these lower bounds in (3.12) and taking $\epsilon < 1/6$, we obtain

$$\|\dot e^h(t)\|^2_\Omega + \|e^h(t)\|^2_{H^1(\Omega)} + \int_0^t \|\dot e^h(s)\|^2_{H^1(\Omega)} ds \le c\int_0^t \left[\|\ddot e_0^h(s)\|^2_\Omega + \|\dot e_0^h(s)\|^2_{H^1(\Omega)} + \|e_0^h(s)\|^2_{H^1(\Omega)}\right] ds. \tag{3.13}$$

Now we use (3.2), (3.8) and (3.9) to bound the right side of (3.13). Under the solution regularity assumptions

$$u \in H^1(0,T;H^{p+1}(\Omega)), \quad \ddot u \in L^2(0,T;H^p(\Omega)), \tag{3.14}$$

we can derive from (3.13) that

$$\|\dot e^h\|_{L^\infty(0,T;L^2(\Omega))} + \|e^h\|_{L^\infty(0,T;H^1(\Omega))} + \|\dot e^h\|_{L^2(0,T;H^1(\Omega))} \le c\,h^p, \tag{3.15}$$

with a constant c depending on the norm of u in $H^1(0,T;H^{p+1}(\Omega))$ and the norm of $\ddot u$ in $L^2(0,T;H^p(\Omega))$. Combining (3.15) with the bounds on e_0^h from (3.2), (3.8) and (3.9), we finally obtain the following result.

Theorem 3.2. *Assume the solution of the Problem 2.1 has the regularity (3.14). Then we have the optimal order error estimate for the solution u^h of Problem 3.1:*

$$\|\dot u^h - \dot u\|_{L^\infty(0,T;L^2(\Omega))} + \|u^h - u\|_{L^\infty(0,T;H^1(\Omega))} + \|\dot u^h - \dot u\|_{L^2(0,T;H^1(\Omega))} \le c\,h^p. \tag{3.16}$$

4. A fully discrete method

For the fully discrete scheme, in addition to the finite element method for spatial discretization, we use finite differences to approximate the time derivatives in the equation. For simplicity, we use uniform partitions of the time interval, although all the discussions in the rest of the section can be extended to finite difference approximations based on general non-uniform partitions. Given a (large) positive integer N, we denote by $k = T/N$ the time step-size, and $t_n = n\,k$, $0 \le n \le N$, the node points. Then the time interval $[0,T]$ is split into sub-intervals of equal length:

$$[0,T] = \cup_{n=0}^{N-1}[t_n,t_{n+1}].$$

Let $w = \dot u$, and rewrite (2.5)–(2.6) as

$$(\dot w(t),v)_\Omega + (\gamma\,w(t),v)_\Omega + (\eta\,\nabla w(t),\nabla v)_\Omega + (\zeta^2\nabla u(t),\nabla v)_\Omega + \left(\eta\,\kappa\,w(t) + \zeta^2\kappa\,u(t),v\right)_{\Gamma_R} = \ell(t,v) \quad \forall\,v \in V, \tag{4.1}$$

$$u(t) = u_0 + \int_0^t w(s)\,ds, \tag{4.2}$$

$$w(0) = w_0. \tag{4.3}$$

For fully discrete numerical method, we use the notation $w^{hk} = (w_n^{hk})_{0\le n\le N}$ and $u^{hk} = (u_n^{hk})_{0\le n\le N}$, $w_n^{hk}(\cdot) \in V^h$ being an approximation of $w(\cdot,t_n)$, and $u_n^{hk}(\cdot) \in V^h$ being an approximation of the solution $u(\cdot,t_n)$. We define

$$\partial_k u_n^{hk} = \frac{u_{n+1}^{hk} - u_n^{hk}}{k}.$$

We approximate (4.1) at $t = t_{n+1/2}$, and the numerical method for (4.1)–(4.3) is:

$$\begin{aligned}&\left(\partial_k w_n^{hk},v^h\right)_\Omega + \left(\frac{\gamma}{2}(w_{n+1}^{hk} + w_n^{hk}),v^h\right)_\Omega + \left(\frac{\eta}{2}(\nabla w_{n+1}^{hk} + \nabla w_n^{hk}),\nabla v^h\right)_\Omega + \left(\frac{\zeta^2}{2}(\nabla u_{n+1}^{hk} + \nabla u_n^{hk}),\nabla v^h\right)_\Omega\\ &\quad + \left(\frac{\eta\,\kappa}{2}(w_{n+1}^{hk} + w_n^{hk}) + \frac{\zeta^2\kappa}{2}(u_{n+1}^{hk} + u_n^{hk}),v^h\right)_{\Gamma_R} = \ell(t_{n+1/2},v^h) \quad \forall\,v^h \in V^h,\end{aligned} \tag{4.4}$$

$$u_n^{hk} = u_0^h + k\sum_{0\le j\le n}{}' w_j^{hk}, \tag{4.5}$$

$$w_0^{hk} = w_0^h, \tag{4.6}$$

where

$$\sum_{0\le j\le n}{}' w_j^{hk} = \frac{1}{2}\left(w_0^{hk} + w_n^{hk}\right) + \sum_{1\le j\le n-1} w_j^{hk}.$$

By an inductive argument with an application of the Lax-Milgram lemma ([3, Section 8.3]), it can be shown that the discrete problem defined by (4.4)–(4.6) has a unique solution. In the following, we derive an error bound for the numerical solution. We will use the V-projection operator defined in (3.1) and express the total errors as

$$w_n^{hk} - w_n = e_n^w + e_{n,0}^w, \tag{4.7}$$

$$u_n^{hk} - u_n = e_n^u + e_{n,0}^u, \tag{4.8}$$

where

$$e_n^w = w_n^{hk} - \Pi^h w_n, \quad e_{n,0}^w = \Pi^h w_n - w_n, \tag{4.9}$$

$$e_n^u = u_n^{hk} - \Pi^h u_n, \quad e_{n,0}^u = \Pi^h u_n - u_n. \tag{4.10}$$

By (3.2),

$$\|e_{n,0}^w\|_\Omega + h\,\|e_{n,0}^w\|_{H^1(\Omega)} \le c\,h^{p+1}\|w_n\|_{H^{p+1}(\Omega)} \quad \text{if } w_n \in H^{p+1}(\Omega), \tag{4.11}$$

$$\|e_{n,0}^u\|_\Omega + h\,\|e_{n,0}^u\|_{H^1(\Omega)} \le c\,h^{p+1}\|u_n\|_{H^{p+1}(\Omega)} \quad \text{if } u_n \in H^{p+1}(\Omega). \tag{4.12}$$

The equation (4.1) at $t = t_{n+1/2}$ is

$$\begin{aligned}&\left(\dot w_{n+1/2}, v\right)_\Omega + (\gamma\, w_{n+1/2}, v)_\Omega + (\eta\,\nabla w_{n+1/2}, \nabla v)_\Omega + (\zeta^2 \nabla u_{n+1/2}, \nabla v)_\Omega \\ &\quad + \left(\eta\,\kappa\, w_{n+1/2} + \zeta^2 \kappa\, u_{n+1/2}, v\right)_{\Gamma_R} = \ell(t_{n+1/2}, v) \quad \forall\, v \in V.\end{aligned} \tag{4.13}$$

Consider the quantity

$$\begin{aligned}L_{n+1/2}(v^h) = {}& \left(\partial_k e_n^w, v^h\right)_\Omega + \left(\frac{\gamma}{2}\,(e_{n+1}^w + e_n^w), v^h\right)_\Omega + \left(\frac{\eta}{2}\,(\nabla e_{n+1}^w + \nabla e_n^w), \nabla v^h\right)_\Omega \\ & + \left(\frac{\zeta^2}{2}\,(\nabla e_{n+1}^u + \nabla e_n^u), \nabla v^h\right)_\Omega + \left(\frac{\eta\,\kappa}{2}\,(e_{n+1}^w + e_n^w) + \frac{\zeta^2\kappa}{2}\,(e_{n+1}^u + e_n^u), v^h\right)_{\Gamma_R}.\end{aligned}$$

By (4.5) and (4.13),

$$\begin{aligned}&\left(\partial_k w_n^{hk}, v^h\right)_\Omega + \left(\frac{\gamma}{2}\,(w_{n+1}^{hk} + w_n^{hk}), v^h\right)_\Omega + \left(\frac{\eta}{2}\,(\nabla w_{n+1}^{hk} + \nabla w_n^{hk}), \nabla v^h\right)_\Omega \\ &\quad + \left(\frac{\zeta^2}{2}\,(\nabla u_{n+1}^{hk} + \nabla u_n^{hk}), \nabla v^h\right)_\Omega + \left(\frac{\eta\,\kappa}{2}\,(w_{n+1}^{hk} + w_n^{hk}) + \frac{\zeta^2\kappa}{2}\,(u_{n+1}^{hk} + u_n^{hk}), v^h\right)_{\Gamma_R} \\ &\quad = \ell(t_{n+1/2}, v^h) \\ &\quad = \left(\dot w_{n+1/2}, v^h\right)_\Omega + (\gamma\, w_{n+1/2}, v^h)_\Omega + (\eta\,\nabla w_{n+1/2}, \nabla v^h)_\Omega + (\zeta^2 \nabla u_{n+1/2}, \nabla v^h)_\Omega + \left(\eta\,\kappa\, w_{n+1/2} + \zeta^2\kappa\, u_{n+1/2}, v^h\right)_{\Gamma_R}.\end{aligned}$$

Then

$$L_{n+1/2}(v^h) = -R_{n+1/2,1}(v^h) - R_{n+1/2,2}(v^h), \tag{4.14}$$

where

$$\begin{aligned}R_{n+1/2,1}(v^h) = {}& \left(\partial_k w_n - \dot w_{n+1/2}, v^h\right)_\Omega + \left(\gamma\, E_{n+1/2}(w), v^h\right)_\Omega + \left(\eta\,\nabla E_{n+1/2}(w), \nabla v^h\right)_\Omega \\ & + \left(\zeta^2 \nabla E_{n+1/2}(u), \nabla v^h\right)_\Omega + \left(\eta\,\kappa\, E_{n+1/2}(w) + \zeta^2\kappa\, E_{n+1/2}(u), v^h\right)_{\Gamma_R},\end{aligned} \tag{4.15}$$

$$\begin{aligned}R_{n+1/2,2}(v^h) = {}& \left(\partial_k e_{n,0}^w, v^h\right)_\Omega + \left(\frac{\gamma}{2}\,(e_{n+1,0}^w + e_{n,0}^w), v^h\right)_\Omega + \left(\frac{\eta}{2}\,\nabla(e_{n+1,0}^w + e_{n,0}^w), \nabla v^h\right)_\Omega + \left(\frac{\zeta^2}{2}\,\nabla(e_{n+1,0}^u + e_{n,0}^u), \nabla v^h\right)_\Omega \\ & + \left(\frac{\eta\,\kappa}{2}\,(e_{n+1,0}^w + e_{n,0}^w) + \frac{\zeta^2\kappa}{2}\,(e_{n+1,0}^u + e_{n,0}^u), v^h\right)_{\Gamma_R},\end{aligned} \tag{4.16}$$

and for convenience, for a continuous function $v(t)$, we denote

$$E_{n+1/2}(v) = \frac{v_{n+1} + v_n}{2} - v_{n+1/2}, \quad v_n = v(t_n),\ v_{n+1/2} = v(t_{n+1/2}). \tag{4.17}$$

Now take $v^h = \frac{1}{2}\left(e_{n+1}^w + e_n^w\right)$ in (4.14). We have

$$\begin{aligned}L_{n+1/2}\left(\frac{1}{2}(e_{n+1}^w + e_n^w)\right) = {}& \frac{1}{2k}\left(e_{n+1}^w - e_n^w, e_{n+1}^w + e_n^w\right)_\Omega + \frac{1}{4}\,\|\gamma^{1/2}(e_{n+1}^w + e_n^w)\|_\Omega^2 + \frac{1}{4}\,\|\eta^{1/2}\nabla(e_{n+1}^w + e_n^w)\|_\Omega^2 + \frac{1}{4}\,(\zeta^2\nabla(e_{n+1}^u + e_n^u), \nabla(e_{n+1}^w + e_n^w))_\Omega \\ & + \frac{1}{4}\,\|\eta^{1/2}\kappa^{1/2}(e_{n+1}^w + e_n^w)\|_{\Gamma_R}^2 + \frac{1}{4}\left(\zeta^2\kappa\,(e_{n+1}^u + e_n^u), e_{n+1}^w + e_n^w\right)_{\Gamma_R}.\end{aligned} \tag{4.18}$$

Note that

$$\left(e_{n+1}^w - e_n^w, e_{n+1}^w + e_n^w\right)_\Omega = \|e_{n+1}^w\|_\Omega^2 - \|e_n^w\|_\Omega^2. \tag{4.19}$$

From (4.4),

$$u_{n+1}^{hk} - u_n^{hk} = \frac{k}{2}\left(w_{n+1}^{hk} + w_n^{hk}\right),$$

i.e.,

$$w_{n+1}^{hk} + w_n^{hk} = \frac{2}{k}\left(u_{n+1}^{hk} - u_n^{hk}\right). \tag{4.20}$$

Similarly,

$$e_{n+1}^{w} + e_n^{w} = \frac{2}{k}\left(e_{n+1}^{u} - e_n^{u}\right) + \frac{2}{k}\,\delta_{n+1/2}^{w},$$

where

$$\delta_{n+1/2}^{w} = \Pi^h\Big[\int_{t_n}^{t_{n+1}} w(s)\,ds - \frac{k}{2}\left(w_{n+1} + w_n\right)\Big]. \tag{4.21}$$

So

$$\begin{aligned}(\zeta^2\nabla(e_{n+1}^u + e_n^u), \nabla(e_{n+1}^w + e_n^w))_\Omega &= \frac{2}{k}\left(\zeta^2\nabla(e_{n+1}^u + e_n^u), \nabla(e_{n+1}^u - e_n^u)\right)_\Omega + \frac{2}{k}\left(\zeta^2\nabla(e_{n+1}^u + e_n^u), \delta_{n+1/2}^w\right)_\Omega\\ &= \frac{2}{k}\left(\|\zeta\,\nabla e_{n+1}^u\|_\Omega^2 - \|\zeta\,\nabla e_n^u\|_\Omega^2\right) + \frac{2}{k}\left(\zeta^2\nabla(e_{n+1}^u + e_n^u), \delta_{n+1/2}^w\right)_\Omega.\end{aligned} \tag{4.22}$$

Also,

$$\left(\zeta^2\kappa\,(e_{n+1}^u + e_n^u), e_{n+1}^w + e_n^w\right)_{\Gamma_R} = \frac{2}{k}\left(\|\zeta\,\kappa^{1/2}e_{n+1}^u\|_{\Gamma_R}^2 - \|\zeta\,\kappa^{1/2}e_n^u\|_{\Gamma_R}^2\right) + \frac{2}{k}\left(\zeta^2\kappa(e_{n+1}^u + e_n^u), \delta_{n+1/2}^w\right)_{\Gamma_R}. \tag{4.23}$$

Use (4.19), (4.22) and (4.23) in (4.18) to obtain

$$\begin{aligned}L_{n+1/2}\left(\frac{1}{2}(e_{n+1}^w + e_n^w)\right) &= \frac{1}{2\,k}\left(\|e_{n+1}^w\|_\Omega^2 - \|e_n^w\|_\Omega^2\right) + \frac{1}{4}\,\|\gamma^{1/2}(e_{n+1}^w + e_n^w)\|_\Omega^2 + \frac{1}{4}\,\|\eta^{1/2}\nabla(e_{n+1}^w + e_n^w)\|_\Omega^2\\ &\quad + \frac{1}{2\,k}\left(\|\zeta\,\nabla e_{n+1}^u\|_\Omega^2 - \|\zeta\,\nabla e_n^u\|_\Omega^2\right) + \frac{1}{4}\,\|\eta^{1/2}\kappa^{1/2}(e_{n+1}^w + e_n^w)\|_{\Gamma_R}^2\\ &\quad + \frac{1}{2\,k}\left(\|\zeta\,\kappa^{1/2}e_{n+1}^u\|_{\Gamma_R}^2 - \|\zeta\,\kappa^{1/2}e_n^u\|_{\Gamma_R}^2\right) + \frac{1}{2\,k}\left(\zeta^2\nabla(e_{n+1}^u + e_n^u), \delta_{n+1/2}^w\right)_\Omega + \frac{1}{2\,k}\left(\zeta^2\kappa\,(e_{n+1}^u + e_n^u), \delta_{n+1/2}^w\right)_{\Gamma_R}.\end{aligned} \tag{4.24}$$

Now we bound each term in $-R_{n+1/2,1}(v^h)$ and in $-R_{n+1/2,2}(v^h)$. As an example,

$$-\left(\partial_k w_n - \dot{w}_{n+1/2}, v^h\right)_\Omega \le \|\partial_k w_n - \dot{w}_{n+1/2}\|_\Omega\|v^h\|_\Omega \le \|\partial_k w_n - \dot{w}_{n+1/2}\|_\Omega\|v^h\|_{H^1(\Omega)};$$

then for any small number $\bar{\epsilon} > 0$,

$$-\left(\partial_k w_n - \dot{w}_{n+1/2}, v^h\right)_\Omega \le \bar{\epsilon}\,\|v^h\|_{H^1(\Omega)}^2 + \bar{c}\,\|\partial_k w_n - \dot{w}_{n+1/2}\|_\Omega^2$$

for some constant $\bar{c}$ depending on $\bar{\epsilon} > 0$. Other terms are bounded similarly. As a result, for any $\epsilon > 0$, there exists a constant $c > 0$ depending on ϵ such that

$$\begin{aligned}-R_{n+1/2,1}(v^h) &\le \epsilon\,\|v^h\|_{H^1(\Omega)}^2 + c\Big[\|\partial_k w_n - \dot{w}_{n+1/2}\|_\Omega^2 + \|\gamma^{1/2}E_{n+1/2}(w)\|_\Omega^2 + \|\eta^{1/2}\nabla E_{n+1/2}(w)\|_\Omega^2\\ &\quad + \|\zeta\,\nabla E_{n+1/2}(u)\|_\Omega^2 + \|\eta^{1/2}\kappa^{1/2}E_{n+1/2}(w)\|_{\Gamma_R}^2 + \|\zeta\,\kappa^{1/2}E_{n+1/2}(u)\|_{\Gamma_R}^2\Big]\\ &\le \epsilon\,\|v^h\|_{H^1(\Omega)}^2 + c\Big[\|\partial_k w_n - \dot{w}_{n+1/2}\|_\Omega^2 + \|E_{n+1/2}(w)\|_{H^1(\Omega)}^2 + \|E_{n+1/2}(u)\|_{H^1(\Omega)}^2\Big],\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}-R_{n+1/2,2}(v^h) &\le \epsilon\,\|v^h\|_{H^1(\Omega)}^2 + c\Big[\|\partial_k e_{n,0}^w\|_\Omega^2 + \|\gamma^{1/2}(e_{n+1,0}^w + e_{n,0}^w)\|_\Omega^2 + \|\eta^{1/2}\nabla(e_{n+1,0}^w + e_{n,0}^w)\|_\Omega^2\\ &\quad + \|\zeta\,\nabla(e_{n+1,0}^u + e_{n,0}^u)\|_\Omega^2 + \|\eta^{1/2}\kappa^{1/2}(e_{n+1,0}^w + e_{n,0}^w)\|_{\Gamma_R}^2 + \|\zeta\,\kappa^{1/2}\,(e_{n+1,0}^u + e_{n,0}^u)\|_{\Gamma_R}^2\Big]\\ &\le \epsilon\,\|v^h\|_{H^1(\Omega)}^2 + c\Big[\|\partial_k e_{n,0}^w\|_\Omega^2 + \|e_{n+1,0}^w + e_{n,0}^w\|_{H^1(\Omega)}^2 + \|e_{n+1,0}^u + e_{n,0}^u\|_{H^1(\Omega)}^2\Big].\end{aligned} \tag{4.26}$$

We use (4.25) and (4.26) with $v^h = \frac{1}{2}\left(e_{n+1}^w + e_n^w\right)$, and combine the inequalities with (4.24) to get

$$\begin{aligned}&\left(\|e_{n+1}^w\|_\Omega^2 - \|e_n^w\|_\Omega^2\right) + \left(\|\zeta\,\nabla e_{n+1}^u\|_\Omega^2 - \|\zeta\,\nabla e_n^u\|_\Omega^2\right) + \left(\|\zeta\,\kappa^{1/2}e_{n+1}^u\|_{\Gamma_R}^2 - \|\zeta\,\kappa^{1/2}e_n^u\|_{\Gamma_R}^2\right)\\ &+ \frac{k}{2}\,\|\gamma^{1/2}(e_{n+1}^w + e_n^w)\|_\Omega^2 + \frac{k}{2}\,\|\eta^{1/2}\nabla(e_{n+1}^w + e_n^w)\|_\Omega^2 + \frac{k}{2}\,\|\eta^{1/2}\kappa^{1/2}(e_{n+1}^w + e_n^w)\|_{\Gamma_R}^2\\ &\quad \le \epsilon\,k\,\|e_{n+1}^w + e_n^w\|_{H^1(\Omega)}^2 - \left(\zeta^2\nabla(e_{n+1}^u + e_n^u), \delta_{n+1/2}^w\right)_\Omega - \left(\zeta^2\kappa\,(e_{n+1}^u + e_n^u), \delta_{n+1/2}^w\right)_{\Gamma_R}\\ &\qquad + c\,k\Big[\|\partial_k w_n - \dot{w}_{n+1/2}\|_\Omega^2 + \|E_{n+1/2}(w)\|_{H^1(\Omega)}^2 + \|E_{n+1/2}(u)\|_{H^1(\Omega)}^2\Big]\\ &\qquad + c\,k\Big[\|\partial_k e_{n,0}^w\|_\Omega^2 + \|e_{n+1,0}^w + e_{n,0}^w\|_{H^1(\Omega)}^2 + \|e_{n+1,0}^u + e_{n,0}^u\|_{H^1(\Omega)}^2\Big].\end{aligned} \tag{4.27}$$

Let us make the following solution regularity assumptions:

$$u \in W^{1,\infty}(0,T;H^{p+1}(\Omega)) \cap W^{3,\infty}(0,T;H^1(\Omega)), \tag{4.28}$$

$$\ddot{u} \in L^2(0,T;H^p(\Omega)), \quad u^{(4)} \in L^2(0,T;L^2(\Omega)), \tag{4.29}$$

where $u^{(4)}$ denotes the fourth time derivative of u. Since $w = \dot{u}$, we have the following regularity on w:

$$w \in L^\infty(0,T;H^{p+1}(\Omega)) \cap W^{2,\infty}(0,T;H^1(\Omega)),$$

$$\dot{w} \in L^2(0,T;H^p(\Omega)), \quad w^{(3)} \in L^2(0,T;L^2(\Omega)).$$

By [19, Lemma 11.5],

$$\|\partial_k w_n - \dot{w}_{n+1/2}\|_\Omega \le c\,k\,\|w^{(3)}\|_{L^1(t_n,t_{n+1};L^2(\Omega))}.$$

Thus,

$$\|\partial_k w_n - \dot{w}_{n+1/2}\|_\Omega^2 \le c\,k^3\|w^{(3)}\|^2_{L^2(t_n,t_{n+1};L^2(\Omega))}. \tag{4.30}$$

By [19, Lemma 11.2], we have

$$\|E_{n+1/2}(w)\|^2_{H^1(\Omega)} \le c\,k^4\|\ddot{w}\|^2_{L^\infty(0,T;H^1(\Omega))}, \tag{4.31}$$

$$\|E_{n+1/2}(u)\|^2_{H^1(\Omega)} \le c\,k^4\|\ddot{u}\|^2_{L^\infty(0,T;H^1(\Omega))}. \tag{4.32}$$

From (3.2), (3.8) and (3.9), we have

$$\|\partial_k e^w_{n,0}\|^2_\Omega \le c\,h^{2p}\|\partial_k w_n\|^2_{H^p(\Omega)} \le c\,k^{-1}h^{2p}\|\dot{w}\|^2_{L^2(t_n,t_{n+1};H^p(\Omega))}, \tag{4.33}$$

and

$$\|e^w_{n+1,0}\|^2_{H^1(\Omega)} \le c\,h^{2p}\|w\|^2_{L^\infty(0,T;H^{p+1}(\Omega))}, \tag{4.34}$$

$$\|e^w_{n,0}\|^2_{H^1(\Omega)} \le c\,h^{2p}\|w\|^2_{L^\infty(0,T;H^{p+1}(\Omega))}, \tag{4.35}$$

$$\|e^u_{n+1,0}\|^2_{H^1(\Omega)} \le c\,h^{2p}\|u\|^2_{L^\infty(0,T;H^{p+1}(\Omega))}, \tag{4.36}$$

$$\|e^u_{n,0}\|^2_{H^1(\Omega)} \le c\,h^{2p}\|u\|^2_{L^\infty(0,T;H^{p+1}(\Omega))}. \tag{4.37}$$

Moreover,

$$\begin{aligned}
-\left(\zeta^2\nabla(e^u_{n+1}+e^u_n),\delta^w_{n+1/2}\right)_\Omega - \left(\zeta^2\kappa\,(e^u_{n+1}+e^u_n),\delta^w_{n+1/2}\right)_{\Gamma_R} &\le \|\zeta^2\nabla(e^u_{n+1}+e^u_n)\|_\Omega\|\delta^w_{n+1/2}\|_\Omega + \|\zeta^2\kappa\,(e^u_{n+1}+e^u_n)\|_{\Gamma_R}\|\delta^w_{n+1/2}\|_{\Gamma_R}\\
&\le c\,\|e^u_{n+1}+e^u_n\|_{H^1(\Omega)}\|\delta^w_{n+1/2}\|_{H^1(\Omega)}.
\end{aligned}$$

Thus, for an arbitrary small $\tilde{\epsilon} > 0$,

$$-\left(\zeta^2\nabla(e^u_{n+1}+e^u_n),\delta^w_{n+1/2}\right)_\Omega - \left(\zeta^2\kappa\,(e^u_{n+1}+e^u_n),\delta^w_{n+1/2}\right)_{\Gamma_R} \le \tilde{\epsilon}\,k\,\|e^u_{n+1}+e^u_n\|^2_{H^1(\Omega)} + c\,k^{-1}\|\delta^w_{n+1/2}\|^2_{H^1(\Omega)}. \tag{4.38}$$

We use the bounds (4.30)–(4.38) for the right side of (4.27) and take ϵ small to get

$$\begin{aligned}
&\left(\|e^w_{n+1}\|^2_\Omega - \|e^w_n\|^2_\Omega\right) + \left(\|\zeta\,\nabla e^u_{n+1}\|^2_\Omega - \|\zeta\,\nabla e^u_n\|^2_\Omega\right) + \left(\|\zeta\,\kappa^{1/2}e^u_{n+1}\|^2_{\Gamma_R} - \|\zeta\,\kappa^{1/2}e^u_n\|^2_{\Gamma_R}\right) + k\,\|e^w_{n+1}+e^w_n\|^2_{H^1(\Omega)}\\
&\quad \le c\,\tilde{\epsilon}\,k\,\|e^u_{n+1}+e^u_n\|^2_{H^1(\Omega)} + c\,k^{-1}\|\delta^w_{n+1/2}\|^2_{H^1(\Omega)} + c\,k^4\left[\|u^{(4)}\|^2_{L^2(t_n,t_{n+1};L^2(\Omega))} + k\,\|u\|^2_{W^{3,\infty}(0,T;H^1(\Omega))}\right]\\
&\qquad + c\,h^{2p}\left[\|\ddot{u}\|^2_{L^2(t_n,t_{n+1};H^p(\Omega))} + k\,\|u\|^2_{W^{1,\infty}(0,T;H^{p+1}(\Omega))}\right].
\end{aligned} \tag{4.39}$$

We make a summation on (4.39) over the index n from 0 to $n \le N-1$,

$$\begin{aligned}
&\|e^w_{n+1}\|^2_\Omega + \|\zeta\,\nabla e^u_{n+1}\|^2_\Omega + \|\zeta\,\kappa^{1/2}e^u_{n+1}\|^2_{\Gamma_R} + k\sum_{j=0}^n\|e^w_{j+1}+e^w_j\|^2_{H^1(\Omega)}\\
&\quad \le \|e^w_0\|^2_\Omega + \|\zeta\,\nabla e^u_0\|^2_\Omega + \|\zeta\,\kappa^{1/2}e^u_0\|^2_{\Gamma_R} + c\,\tilde{\epsilon}k\sum_{j=0}^n\|e^u_{j+1}+e^u_j\|^2_{H^1(\Omega)} + c\,k^{-1}\sum_{j=0}^n\|\delta^w_{j+1/2}\|^2_{H^1(\Omega)}\\
&\qquad + c\,k^4\left[\|u^{(4)}\|^2_{L^2(0,T;L^2(\Omega))} + \|u\|^2_{W^{3,\infty}(0,T;H^1(\Omega))}\right] + c\,h^{2p}\left[\|\ddot{u}\|^2_{L^2(0,T;H^p(\Omega))} + \|u\|^2_{W^{1,\infty}(0,T;H^{p+1}(\Omega))}\right],
\end{aligned}$$

which implies

$$\begin{aligned}
\|e^w_{n+1}\|^2_\Omega + \|e^u_{n+1}\|^2_{H^1(\Omega)} + k\sum_{j=0}^n\|e^w_{j+1}+e^w_j\|^2_{H^1(\Omega)} &\le c\left(\|e^w_0\|^2_\Omega + \|e^u_0\|^2_{H^1(\Omega)}\right) + c\,\tilde{\epsilon}k\sum_{j=0}^n\|e^u_{j+1}+e^u_j\|^2_{H^1(\Omega)} + c\,k^{-1}\sum_{j=0}^n\|\delta^w_{j+1/2}\|^2_{H^1(\Omega)}\\
&\quad + c\,k^4\left[\|u^{(4)}\|^2_{L^2(0,T;L^2(\Omega))} + \|u\|^2_{W^{3,\infty}(0,T;H^1(\Omega))}\right] + c\,h^{2p}\left[\|\ddot{u}\|^2_{L^2(0,T;H^p(\Omega))} + \|u\|^2_{W^{1,\infty}(0,T;H^{p+1}(\Omega))}\right].
\end{aligned}$$

By taking $\tilde{\epsilon} > 0$ sufficiently small, we can deduce from the above inequality that

$$\max_{0\le n\le N}\|e_n^w\|_\Omega^2+\max_{0\le n\le N}\|e_n^u\|_{H^1(\Omega)}^2+k\sum_{n=0}^{N-1}\|e_{n+1}^w+e_n^w\|_{H^1(\Omega)}^2\le c\left(\|e_0^w\|_\Omega^2+\|e_0^u\|_{H^1(\Omega)}^2\right)+c\,k^{-1}\sum_{n=0}^{N-1}\|\delta_{n+1/2}^w\|_{H^1(\Omega)}^2$$
$$+c\,k^4\left[\|u^{(4)}\|_{L^2(0,T;L^2(\Omega))}^2+\|u\|_{W^{3,\infty}(0,T;H^1(\Omega))}^2\right]$$
$$+c\,h^{2p}\left[\|\ddot u\|_{L^2(0,T;H^p(\Omega))}^2+\|u\|_{W^{1,\infty}(0,T;H^{p+1}(\Omega))}^2\right]. \tag{4.40}$$

For the term $\delta_{n+1/2}^w$ defined by (4.21), first notice that

$$\int_{t_n}^{t_{n+1}} w(s)\,ds-\frac{k}{2}\left(w_{n+1}+w_n\right)=\int_{t_n}^{t_{n+1}}\left[w(s)-w_{n+1}\frac{s-t_n}{k}-w_n\frac{t_{n+1}-s}{k}\right]ds.$$

By the Taylor formula,

$$w_{n+1}=w(s)+\dot w(s)\left(t_{n+1}-s\right)+\int_0^1(1-\tau)\,\ddot w(s+\tau\,(t_{n+1}-s))\,d\tau\,(t_{n+1}-s)^2,$$
$$w_n=w(s)+\dot w(s)\left(t_n-s\right)+\int_0^1(1-\tau)\,\ddot w(s+\tau\,(t_n-s))\,d\tau\,(t_n-s)^2.$$

Thus,

$$\delta_{n+1/2}^w=-\Pi^h\int_{t_n}^{t_{n+1}}\int_0^1\frac{1-\tau}{k}\left[\ddot w(s+\tau\,(t_{n+1}-s))\,(t_{n+1}-s)^2(s-t_n)+\ddot w(s+\tau\,(t_n-s))\,(t_n-s)^2(t_{n+1}-s)\right]d\tau.$$

From this representation formula and the stability of the projection operator Π^h in $H^1(\Omega)$, we derive the bound

$$\|\delta_{n+1/2}^w\|_{H^1(\Omega)}^2\le c\,k^5\int_{t_n}^{t_{n+1}}\|\ddot w(s)\|_{H^1(\Omega)}^2\,ds.$$

Therefore,

$$\sum_{n=0}^{N-1}\|\delta_{n+1/2}^w\|_{H^1(\Omega)}^2\le c\,k^5\|\ddot w\|_{L^2(0,T;H^1(\Omega))}^2\,ds. \tag{4.41}$$

Then, we can derive the following inequality from (4.40):

$$\max_{0\le n\le N}\|e_n^w\|_\Omega^2+\max_{0\le n\le N}\|e_n^u\|_{H^1(\Omega)}^2+k\sum_{n=0}^{N-1}\|e_{n+1}^w+e_n^w\|_{H^1(\Omega)}^2\le c\left(k^4+h^{2p}\right), \tag{4.42}$$

where the constant c depends on the various norms related to the solution regularities (4.28)–(4.29), but does not depend h and k.

Finally, from (4.7) and (4.8) for the total error decompositions, (4.42) for bounds on e_n^w and e_n^u, and (3.2) for bounds on $\left(\Pi^h w_n-w_n\right)$ and $\left(\Pi^h u_n-u_n\right)$, we get the optimal order error estimate.

Theorem 4.1. *Assume the solution regularity* (4.28) *and* (4.29)*. Then we have the following error bound for the numerical solutions of the fully discrete scheme* (4.4)–(4.6)*:*

$$\max_{0\le n\le N}\|w_n^{hk}-w_n\|_\Omega^2+\max_{0\le n\le N}\|u_n^{hk}-u_n\|_{H^1(\Omega)}^2+k\sum_{n=0}^{N-1}\|(w_{n+1}^{hk}-w_{n+1})+(w_n^{hk}-w_n)\|_{H^1(\Omega)}^2\le c\left(k^4+h^{2p}\right). \tag{4.43}$$

5. Numerical example

In this section, we report numerical results on a test problem (2.5)–(2.6) of the diffusive-viscous wave equation solved by the fully discrete numerical scheme (4.4)–(4.6). When w_i^{hk} and u_i^{hk} are known for $0\le i\le n$, we solve the discrete problem (4.4)–(4.6) by the following procedures:

1. Replace u_{n+1}^{hk} in (4.4) by (4.5), then solve (4.4) to get w_{n+1}^{hk};
2. Compute u_{n+1}^{hk} by formula (4.5).

We use [km] for the length unit. Let $\Omega=(0,1)\times(0,1)$, $\Gamma_D=\left(\{0\}\times(0,1)\right)\cup\left((0,1)\times\{0\}\right)$, $\Gamma_N=\{1\}\times(0,1)$, and $\Gamma_R=(0,1)\times\{1\}$. For different mediums, the parameters are different. In the numerical experiment, we consider three kinds of mediums, with values of the parameters listed in Table 1 ([29]). The source function f, the boundary conditions, and the initial conditions are chosen so that the exact solution of the test problem is

$$u(x,y,t)=t^2\sin(\pi x)\,\sin(\pi y).$$

For the numerical simulation, we use uniform meshes to discretize $\overline{\Omega}$. In computing numerical convergence orders with respect to mesh size h, we fix $k=10^{-2}$ for $p=1$, and fix $k=10^{-4}$ for $p=2$. The errors and numerical convergence orders at $T=1$ are reported in Table 2–10. From Table 2,

Table 1
Parameters for different mediums.

Medium	γ [Hz]	η [km^2/s]	ζ [km/s]
Water-saturated rock	90	2×10^{-7}	1.470
Dry sandstone	56	5.6×10^{-8}	1.190
Oil-saturated rock	65.4	1.47×10^{-8}	1.015

Table 2
Numerical errors of u^h in H^1 norm at $T = 1$ for water-saturated rock.

h	Errors for $p = 1$	Order	Errors for $p = 2$	Order
2^{-2}	8.4261e-1	-	1.7315e-1	-
2^{-3}	4.3300e-1	0.96050	5.4892e-2	1.6574
2^{-4}	2.1766e-1	0.99229	1.5585e-2	1.8164
2^{-5}	1.0898e-1	0.99801	4.1130e-3	1.9219
2^{-6}	5.4512e-2	0.99942	1.0494e-3	1.9706

Table 3
Numerical errors of w^h in L^2 norm at $T = 1$ for water-saturated rock.

h	Errors for $p = 1$	Order	Errors for $p = 2$	Order
2^{-2}	8.7137e-2	-	2.4716e-2	-
2^{-3}	2.1128e-2	2.0441	5.2616e-3	2.2319
2^{-4}	5.1337e-3	2.0411	9.3290e-4	2.4957
2^{-5}	1.2599e-3	2.0267	1.4732e-4	2.6628
2^{-6}	3.1235e-4	2.0121	2.1767e-5	2.7587

Table 4
Numerical errors at $T = 1$ with fixed $h = 1/150$ and $p = 2$ for water-saturated rock.

k	u^h in H^1 norm	Order	w^h in L^2 norm	Order
2^{-1}	1.7812e-2	-	8.1496e-2	-
2^{-2}	4.4070e-3	2.0150	2.1096e-2	1.9498
2^{-3}	8.9430e-4	2.3010	5.2877e-3	1.9963
2^{-4}	1.7563e-4	2.3482	1.3288e-3	1.9925

Table 5
Numerical errors of u^h in H^1 norm at $T = 1$ for dry sandstone.

h	Errors for $p = 1$	Order	Errors for $p = 2$	Order
2^{-2}	8.4151e-1	-	1.7408e-1	-
2^{-3}	4.3285e-1	0.9591	5.5139e-2	1.6586
2^{-4}	2.1764e-1	0.9919	1.5620e-2	1.8197
2^{-5}	1.0898e-1	0.9979	4.1169e-3	1.9238
2^{-6}	5.4511e-2	0.9994	1.0497e-3	1.9716

Table 6
Numerical errors of w^h in L^2 norm at $T = 1$ for dry sandstone.

h	Errors for $p = 1$	Order	Errors for $p = 2$	Order
2^{-2}	8.8094e-2	-	2.5216e-2	-
2^{-3}	2.1404e-2	2.0412	5.3386e-3	2.2398
2^{-4}	5.2013e-3	2.0409	9.4201e-4	2.5026
2^{-5}	1.2763e-3	2.0269	1.4833e-4	2.6669
2^{-6}	3.1642e-3	2.0121	2.1877e-5	2.7613

Table 7
Numerical errors at $T = 1$ with fixed $h = 1/150$ and $p = 2$ for dry sandstone.

k	u^h in H^1 norm	Order	w^h in L^2 norm	Order
2^{-1}	1.7812e-2	-	8.1496e-2	-
2^{-2}	4.4070e-3	2.0150	2.1096e-2	1.9498
2^{-3}	8.9430e-4	2.3010	5.2877e-3	1.9963
2^{-4}	1.7563e-4	2.3482	1.3288e-3	1.9925

Table 8
Numerical errors of u^h in H^1 norm at $T = 1$ for oil-saturated rock.

h	Errors for $p = 1$	Order	Errors for $p = 2$	Order
2^{-2}	8.5582e-1	-	1.6523e-1	-
2^{-3}	4.3481e-1	0.97692	5.2656e-2	1.6498
2^{-4}	2.1789e-1	0.99679	1.5253e-2	1.7875
2^{-5}	1.0901e-1	0.99914	4.0770e-3	1.9035
2^{-6}	5.4516e-2	0.99971	1.0460e-3	1.9626

Table 9
Numerical errors of w^h in L^2 norm at $T = 1$ for oil-saturated rock.

h	Errors for $p = 1$	Order	Errors for $p = 2$	Order
2^{-2}	7.9121e-2	-	2.0916e-2	-
2^{-3}	1.8805e-2	2.0729	4.6687e-3	2.1635
2^{-4}	4.5663e-3	2.0420	8.6287e-4	2.4358
2^{-5}	1.219e-3	1.9050	1.3965e-4	2.6273
2^{-6}	2.7824e-4	2.1313	2.0929e-5	2.7382

Table 10
Numerical errors at $T = 1$ with fixed $h = 1/150$ and $p = 2$ for oil-saturated rock.

k	u^h in H^1 norm	Order	w^h in L^2 norm	Order
2^{-1}	1.2084e-2	-	6.6869e-2	-
2^{-2}	3.2733e-3	1.8843	1.7496e-2	1.9343
2^{-3}	7.1354e-4	2.1977	4.4177e-3	1.9857
2^{-4}	1.5897e-4	2.1662	1.1116e-3	1.9856

Table 5, and Table 8, we observe that the numerical convergence orders for the wave field u^h in H^1 norm are around p with $p = 1, 2$, which is consistent with the theoretical prediction from Theorem 4.1. Moreover, from Table 3, Table 6, and Table 9, we see that the numerical convergence orders for w^h in the L^2 norm are around $(p + 1)$ with $p = 1, 2$, which is one order higher than that from the error bound (4.43). We let $p = 2$ and $h = 1/150$ for considering the convergence orders with respect to time step k. In Table 4, Table 7, and Table 10, for both errors, the convergence orders are around 2, which supports the theoretical analysis.

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