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Minimax principles for elliptic mixed hemivariational–variational inequalities[☆]

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ABSTRACT

In this paper, minimax principles are explored for elliptic mixed hemivariational–variational inequalities. Under certain conditions, a saddle-point formulation is shown to be equivalent to a mixed hemivariational–variational inequality. While the minimax principle is of independent interest, it is employed in this paper to provide an elementary proof of the solution existence of the mixed hemivariational–variational inequality. Theoretical results are illustrated in the applications of two contact problems.

1. Introduction

Since the pioneering work of Panagiotopoulos in early 1980s [1], the area of hemivariational inequalities, including modeling, well-posedness analysis, numerical approximation and simulation, and applications, has attracted steady attention from the research community. In this paper, we use the terms "hemivariational inequalities" and the more general "variational–hemivariational inequalities" interchangeably. In recent years, there has been an explosive growth in the literature on hemivariational inequalities, cf. e.g. [2] for recent results on theoretical analysis of hemivariational inequalities, and [3] for a survey of numerical analysis of hemivariational inequalities.

In dealing with constraints in application problems, e.g. the incompressibility constraint for fluid flows, mixed hemivariational inequalities arise naturally where a hemivariational inequality is coupled with a variational equation or inequality that reflects the constraint, e.g., [4,5], and the mixed formulations are the foundation for developing efficient numerical methods such as the mixed finite element method. More generally, in problems involving inequality relations, especially for applications in contact mechanics, it is possible

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to introduce Lagrange multipliers so that the weak formulation is in the form of a mixed hemivariational–variational inequality where a hemivariational inequality is supplemented by a variational equation or inequality. Elliptic mixed hemivariational–variational inequalities have been studied in a number of papers, e.g., [6–10]. In these references, the main theoretical tools for proving solution existence are fixed-point results for set-valued mappings. In this paper, we provide a direct approach for the study of a family of elliptic mixed hemivariational–variational inequalities through consideration of a Lagrangian functional. More precisely, we introduce a saddle-point formulation corresponding to a mixed hemivariational–variational inequality and explore the existence of a saddle-point of the Lagrangian functional. We prove the equivalence between the saddle-point formulation and the mixed hemivariational–variational inequality. In this way, a minimax principle on the saddle-point formulation provides an elementary proof of the solution existence on the mixed hemivariational–variational inequality.

The rest of the paper is organized as follows. In Section 2, we review preliminary materials needed later. In Section 3, we present minimax principles for a general elliptic mixed hemivariational–variational inequality. In Section 4, we apply the theoretical results in the study of two contact problems.

2. Preliminaries

We first recall the definition of a saddle point for a functional of two arguments and an existence result on the saddle point. Detailed discussions on saddle points can be found in [11, Chapter VI].

Definition 2.1. Let V and Λ be two linear spaces, let $K_V \subset V$ and $K_\Lambda \subset \Lambda$ be non-empty subsets of the spaces, and let $L\colon K_V \times K_\Lambda \to \mathbb{R}$ be a given functional. A pair $(u,\lambda) \in K_V \times K_\Lambda$ is said to be a saddle point of L if

$$L(u,\mu) \le L(u,\lambda) \le L(v,\lambda) \quad \forall\, v \in K_V,\ \mu \in K_\Lambda.$$

It can be shown (e.g., [11, Chapter VI, Proposition 1.2]) that $(u,\lambda) \in K_V \times K_\Lambda$ is a saddle point of L if and only if

$$L(u,\lambda) = \max_{\mu\in K_\Lambda} \inf_{v\in K_V} L(v,\mu) = \min_{v\in K_V} \sup_{\mu\in K_\Lambda} L(v,\mu)$$

We will apply the following existence result (cf. [11, Chapter VI, Proposition 2.4]) in our studies of elliptic mixed hemivariational–variational inequalities.

Theorem 2.2. *Let V and Λ be two Hilbert spaces, let K_V be a non-empty, closed, convex subset of V, and let K_Λ be a non-empty, closed, convex subset of Λ. Assume a functional $L\colon K_V \times K_\Lambda \to \mathbb{R}$ has the following properties:*

$$\begin{aligned}
&\forall\, \mu \in K_\Lambda,\ v \mapsto L(v,\mu) \text{ is convex and l.s.c.;}\\
&\forall\, v \in K_V,\ \mu \mapsto L(v,\mu) \text{ is concave and u.s.c.;}\\
&\text{either } K_V \text{ is bounded or } \lim_{\|v\|_V\to\infty,\, v\in K_V} L(v,\mu_*) = \infty \text{ for some } \mu_* \in K_\Lambda;\\
&\text{either } K_\Lambda \text{ is bounded or } \lim_{\|\mu\|_\Lambda\to\infty,\, \mu\in K_\Lambda} \inf_{v\in K_V} L(v,\mu) = -\infty.
\end{aligned}$$

Then, L has at least one saddle point over $K_V \times K_\Lambda$.

Next, we recall definitions of the generalized directional derivative and generalized subdifferential in the sense of Clarke for a locally Lipschitz continuous function [12]; these notions and their basic properties are needed in studies of hemivariational inequalities. Let $\Psi\colon V \to \mathbb{R}$ be a locally Lipschitz continuous functional defined on a real Banach space V. Then its generalized (Clarke) directional derivative at $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u;v) := \limsup_{w\to u,\, \lambda\downarrow 0} \frac{\Psi(w+\lambda v) - \Psi(w)}{\lambda},$$

whereas the generalized subdifferential of Ψ at $u \in V$ is

$$\partial\Psi(u) := \left\{\eta \in V^* \mid \Psi^0(u;v) \geq \langle \eta, v\rangle \ \forall\, v \in V\right\}.$$

Detailed discussions of the generalized directional derivative and the generalized subdifferential, including their properties, can be found in [12]. We mention some of the basic properties that will be needed in this paper.

Proposition 2.3. *Assume V is a real Banach space. If $\Psi\colon V \to \mathbb{R}$ is locally Lipschitz continuous and convex, then the subdifferential $\partial\Psi(u)$ at any $u \in V$ in the sense of Clarke coincides with the convex subdifferential $\partial\Psi(u)$.*

Let $\Psi, \Psi_1, \Psi_2\colon V \to \mathbb{R}$ be locally Lipschitz functions. Then $\partial(\lambda\,\Psi)(u) = \lambda\,\partial\Psi(u)$ for all $\lambda \in \mathbb{R}$ and all $u \in V$. Moreover, the inclusion

$$\partial(\Psi_1 + \Psi_2)(u) \subset \partial\Psi_1(u) + \partial\Psi_2(u) \quad \forall\, u \in V \tag{2.1}$$

holds, or equivalently,

$$(\Psi_1 + \Psi_2)^0(u;v) \leq \Psi_1^0(u;v) + \Psi_2^0(u;v) \quad \forall\, u, v \in V. \tag{2.2}$$

In our analysis of saddle-point formulations, we will need the notion of strong convexity. A function $\Phi\colon V \to \mathbb{R}$ is said to be strongly convex on V with a constant $\alpha > 0$ if

$$\Phi(\lambda\, u + (1-\lambda)\, v) \leq \lambda\,\Phi(u) + (1-\lambda)\,\Phi(v) - \alpha\,\lambda\,(1-\lambda)\,\|u-v\|_V^2 \quad \forall\, u, v \in V,\ \forall\, \lambda \in [0,1].$$

Obviously, strong convexity of a functional implies its strict convexity. The next result follows from Proposition 3.1 and Theorem 3.4 in [13], which will be applied later.

Lemma 2.4. *Let V be a real Banach space and let Φ be locally Lipschitz continuous on V. Then the following statements are equivalent.*

(i) *The function Φ is strongly convex on V with a constant $\alpha > 0$.*

(ii) *The generalized subdifferential $\partial\Phi$ is strongly monotone on V with a constant $2\,\alpha$, i.e.,*

$$\langle \xi - \eta, u - v\rangle \geq 2\,\alpha\,\|u-v\|_V^2 \quad \forall\, u, v \in V,\ \xi \in \partial\Phi(u),\ \eta \in \partial\Phi(v).$$

(iii) *There exists a constant $\alpha > 0$ such that*

$$\Phi(v) \geq \Phi(u) + \langle \xi, v-u\rangle + \alpha\,\|v-u\|_V^2 \quad \forall\, u, v \in V,\ \xi \in \partial\Phi(u).$$

Finally, we recall the concept of a potential operator and some basic facts. Detailed discussion on potential operators can be found in [14, Section 41.3]. From now on, we will assume V to be a real Hilbert space. An operator $A\colon V \to V^*$ is called a potential operator if there exists a Gâteaux differentiable functional $F_A\colon V \to \mathbb{R}$ such that $A = F_A'$; the functional F_A is called a potential of A.

If A is hemicontinuous, i.e., the function $t \mapsto \langle A(v_1 + t\, v_2), v_3\rangle$ is continuous on $[0,1]$ for all $v_1, v_2, v_3 \in V$, then A is a potential operator if and only if

$$\int_0^1 [\langle A(t\,u), u\rangle - \langle A(t\,v), v\rangle]\, dt = \int_0^1 \langle A(v + t\,(u-v)), u-v\rangle\, dt \quad \forall\, u, v \in V.$$

A formula to compute a potential of A is

$$F_A(v) = \int_0^1 \langle A(t\,v), v\rangle\,dt. \tag{2.3}$$

Any other potential of A differs from F_A by a constant.

If A is Gâteaux differentiable and the mapping

$$(t,s) \mapsto \langle A'(v_1 + t\,v_2 + s\,v_3)\,v_4, v_5\rangle$$

is continuous on $[0,1]\times[0,1]$ for all $v_i \in V$, $1 \le i \le 5$, then the operator A is potential if and only if

$$\langle A'(u)\,v, w\rangle = \langle A'(u)\,w, v\rangle \quad \forall\, u, v, w \in V.$$

In particular, $A \in \mathcal{L}(V, V^*)$ is a potential operator if and only if it is symmetric:

$$\langle Av_1, v_2\rangle = \langle Av_2, v_1\rangle \quad \forall\, v_1, v_2 \in V. \tag{2.4}$$

Moreover, under the symmetry condition (2.4), a potential functional is

$$F_A(v) = \frac{1}{2}\,\langle Av, v\rangle, \quad v \in V.$$

Throughout the paper, we will use c to denote a generic positive constant whose value may change from one place to another but it is independent of other quantities of concern in the context.

3. Minimax principles for a general elliptic mixed hemivariational–variational inequality

In this section, we consider an abstract elliptic mixed hemivariational–variational inequality and a corresponding saddle-point formulation. We show the equivalence between the two formulations, and explore the solution existence and uniqueness.

Let V and Λ be two real Hilbert spaces. We write V^* and Λ^* for their dual spaces, and use $\langle\cdot,\cdot\rangle$ to denote the duality pairing between V^* and V, or between Λ^* and Λ; it should be clear from the context which duality pairing is meant by $\langle\cdot,\cdot\rangle$. Let $K_V \subset V$ and $K_\Lambda \subset \Lambda$.

Given operators and functionals $A\colon V \to V^*$, $b\colon V\times\Lambda \to \mathbb{R}$, $\Phi\colon V \to \mathbb{R}$, $\Psi\colon V \to \mathbb{R}$, and $f \in V^*$, we consider the following mixed inequality problem.

Problem 3.1. Find $(u,\lambda) \in K_V \times K_\Lambda$ such that

$$\langle Au, v-u\rangle + b(v-u,\lambda) + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \ge \langle f, v-u\rangle \quad \forall\, v \in K_V, \tag{3.1}$$

$$b(u, \mu-\lambda) \le 0 \quad \forall\, \mu \in K_\Lambda. \tag{3.2}$$

We will focus on the case where A is a potential operator and note that this is the case for majority of applications in mechanics and engineering. We denote by F_A the potential of the operator A and introduce a Lagrangian functional $L\colon V_K \times V_\Lambda \to \mathbb{R}$ by the formula

$$L(v,\mu) = F_A(v) + \Phi(v) + \Psi(v) - \langle f, v\rangle + b(v,\mu), \quad v \in K_V,\ \mu \in K_\Lambda. \tag{3.3}$$

Then, we consider a saddle-point problem corresponding to Problem 3.1.

Problem 3.2. Find $(u,\lambda) \in K_V \times K_\Lambda$ such that

$$L(u,\mu) \le L(u,\lambda) \le L(v,\lambda) \quad \forall\, v \in K_V,\ \mu \in K_\Lambda. \tag{3.4}$$

Let us introduce conditions on the problem data.

- $H(K_V)$ V is a real Hilbert space, $K_V \subset V$ is non-empty, closed and convex.
- $H(K_\Lambda)$ Λ is a real Hilbert space, $K_\Lambda \subset \Lambda$ is non-empty, closed and convex.
- $H(A)$ $A\colon V \to V^*$ is a Lipschitz continuous, strongly monotone potential operator.
- $H(b)$ $b : V \times \Lambda \to \mathbb{R}$ is bilinear and bounded.
- $H(\Phi)$ $\Phi\colon V \to \mathbb{R}$ is convex and continuous.
- $H(\Psi)$ $\Psi\colon V \to \mathbb{R}$ is locally Lipschitz continuous, and there exists a constant $\alpha_\Psi \ge 0$ such that

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \le \alpha_\Psi \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V. \tag{3.5}$$

- $H(f)$ $f \in V^*$.

Related to the condition $H(A)$, we will use M_A for the Lipschitz constant:

$$\|Av_1 - Av_2\|_{V^*} \le M_A \|v_1 - v_2\|_V \quad \forall\, v_1, v_2 \in V, \tag{3.6}$$

and use m_A for the strong monotonicity constant:

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \ge m_A \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V. \tag{3.7}$$

Related to the condition $H(b)$, we will use $M_b > 0$ for the boundedness constant:

$$|b(v, \mu)| \le M_b \|v\|_V \|\mu\|_\Lambda \quad \forall\, v \in V,\ \mu \in \Lambda. \tag{3.8}$$

Assumption $H(b)$ allows us to define an operator $B \in \mathcal{L}(V; \Lambda^*)$ by the relation

$$\langle Bv, \mu\rangle = b(v, \mu) \quad \forall\, v \in V,\ \mu \in \Lambda.$$

In $H(\Phi)$, the convex function $\Phi\colon V \to \mathbb{R}$ is assumed to be continuous, instead of l.s.c. As is explained in [15,16], there is no loss of generality with the stronger assumption of continuity for a vast majority of applications.

For α_Ψ from (3.5) and m_A from (3.7), we will assume

- $H(s)$ $\alpha_\Psi < m_A$.

The condition $H(s)$ is known as a smallness condition in the literature.

In case where K_V is a subspace of V, we will assume the bilinear form $b(\cdot, \cdot)$ satisfies an inf–sup condition: there exists a constant $\alpha_b > 0$ such that

$$\sup_{0 \ne v \in K_V} \frac{b(v, \mu)}{\|v\|_V} \ge \alpha_b \|\mu\|_\Lambda \quad \forall\, \mu \in \Lambda. \tag{3.9}$$

We will make repeated use of the next result on the Lagrangian L.

Proposition 3.3. *Assume $H(K_V)$, $H(K_\Lambda)$, $H(A)$, $H(b)$, $H(\Phi)$, $H(\Psi)$, $H(f)$, and $H(s)$. Then for each fixed $\mu \in \Lambda$, the mapping $v \mapsto L(v, \mu)$ is locally Lipschitz and strongly convex on V, and*

$$\langle v_1^* - v_2^*, v_1 - v_2\rangle \ge (m_A - \alpha_\Psi)\, \|v_1 - v_2\|_V^2 \quad \forall\, v_i \in V,\ v_i^* \in \partial L(v_i, \mu),\ i = 1, 2.$$

Moreover, the same statement is valid when $L(v, \mu)$ is replaced by $L(v, \mu) - \Phi(v)$, i.e., for any fixed element $\mu \in \Lambda$, the mapping $v \mapsto L(v, \mu) - \Phi(v)$ is locally Lipschitz and strongly convex on V.

The result follows from the proof of Proposition 3.4 in [15], and we skip its proof in this paper.

The relation between Problems 3.1 and 3.2 is presented next.

Theorem 3.4. *Assume $H(K_V)$, $H(K_\Lambda)$, $H(A)$, $H(b)$, $H(\Phi)$, $H(\Psi)$, $H(f)$, and $H(s)$. Then, Problems 3.1 and 3.2 are equivalent, i.e., $(u,\lambda) \in K_V \times K_\Lambda$ is a solution of Problem 3.1 if and only if it is a solution of Problem 3.2.*

Proof. First, it is easy to see that (3.2) and the first inequality in (3.4),

$$L(u,\mu) \le L(u,\lambda) \quad \forall\, \mu \in K_\Lambda$$

are equivalent.

It remains to prove the equivalence of (3.1) and the second inequality in (3.4):

$$L(u,\lambda) \le L(v,\lambda) \quad \forall\, v \in K_V. \tag{3.10}$$

Assume (3.1) is valid. By Proposition 3.3, $L_1(\cdot,\lambda) = L(\cdot,\lambda) - \Phi(\cdot)$ is convex. So for any $v \in K_V$ and any $t \in (0,1)$, we have

$$L_1(u + t\,(v-u),\lambda) \le t\,L_1(v,\lambda) + (1-t)\,L_1(u,\lambda).$$

Rewrite the inequality as

$$\frac{1}{t}\,[L_1(u+t\,(v-u),\lambda) - L_1(u,\lambda)] \le L_1(v,\lambda) - L_1(u,\lambda),$$

i.e., after using the definition of the functional L_1,

$$\begin{aligned} &\frac{1}{t}\,[F_A(u+t\,(v-u)) - F_A(u)] + \frac{1}{t}\,[\Psi(u+t\,(v-u)) - \Psi(u)] \\ &\quad - \langle f, v-u\rangle + b(v-u,\lambda) \le L_1(v,\lambda) - L_1(u,\lambda). \end{aligned}$$

Take the upper limit of both sides of the above inequality as $t \to 0+$ to obtain

$$\langle Au, v-u\rangle + \Psi^0(u; v-u) - \langle f, v-u\rangle + b(v-u,\lambda) \le L_1(v,\lambda) - L_1(u,\lambda).$$

i.e.,

$$\langle Au, v-u\rangle + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) - \langle f, v-u\rangle + b(v-u,\lambda) \le L(v,\lambda) - L(u,\lambda).$$

The left side of the inequality is non-negative by (3.1). Hence,

$$0 \le L(v,\lambda) - L(u,\lambda) \quad \forall\, v \in K_V,$$

which is (3.10).

Conversely, assume (3.10). Denote by $I_{K_V}(\cdot)$ the indicator function of the set K_V. Then, we have

$$0 \in \partial\left(L(u,\lambda) + I_{K_V}(u)\right) \subset \partial L(u,\lambda) + \partial I_{K_V}(u), \tag{3.11}$$

where (2.1) is applied for the second inclusion. Since

$$\partial L(u,\lambda) \subset Au + \partial\Phi(u) + \partial\Psi(u) - f + Bu,$$

we deduce from (3.11) that

$$\langle Au, v-u\rangle + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) - \langle f, v-u\rangle + b(v-u,\lambda) \ge 0 \quad \forall\, v \in K_V,$$

i.e., (3.1) holds. ■

We now present existence results on Problem 3.2. We distinguish two cases: first for the case where K_Λ is bounded (Theorem 3.5), and then for the case where K_Λ is unbounded (Theorem 3.6).

Theorem 3.5. *Assume $H(K_V)$, $H(K_\Lambda)$, $H(A)$, $H(b)$, $H(\Phi)$, $H(\Psi)$, $H(f)$, and $H(s)$. Moreover, assume K_Λ is bounded. Then Problem 3.2 has at least one solution $(u, \lambda) \in K_V \times K_\Lambda$.*

Proof. By Proposition 3.3, for any $\mu \in K_\Lambda$, the function $v \mapsto L(v, \mu)$ is strictly convex and continuous. Evidently, for any $v \in K_V$, the function $\mu \mapsto L(v, \mu)$ is concave and continuous. If K_V is bounded, we can apply Theorem 2.2 directly to conclude that L has at least one saddle point.

Assume now that K_V is unbounded. For any $\mu^* \in \Lambda$ chosen and fixed, by Proposition 3.3, $v \mapsto L(v, \mu^*)$ is locally Lipschitz and strongly convex on V. In particular, $L(\cdot, \mu^*)$ is coercive on K_V; thus,

$$\lim_{\|v\|_V \to \infty,\, v \in K_V} L(v, \mu^*) = \infty. \tag{3.12}$$

Again we can apply Theorem 2.2 to conclude that L has at least one saddle point. ■

Theorem 3.6. *Assume $H(K_V)$, $H(K_\Lambda)$, $H(A)$, $H(b)$, $H(\Phi)$, $H(\Psi)$, $H(f)$, and $H(s)$. Moreover, assume K_Λ is unbounded; K_V is a subspace of V; $\Phi\colon V \to \mathbb{R}$ is Lipschitz continuous on K_V; there exist non-negative constants c_0, c_1 and $\kappa \in [0, 2)$ such that*

$$\|\partial \Psi(v)\|_{V^*} \le c_0 + c_1 \|v\|_V^\kappa \quad \forall\, v \in V; \tag{3.13}$$

and the inf–sup condition (3.9) holds. Then Problem 3.2 has at least one solution $(u, \lambda) \in K_V \times K_\Lambda$.

Proof. We comment that the inequality (3.13) is understood in the sense that

$$\|\eta\|_{V^*} \le c_0 + c_1 \|v\|_V^\kappa \quad \forall\, v \in V,\ \forall\, \eta \in \partial \Psi(v).$$

To apply Theorem 2.2 on the existence of a solution to Problem 3.2, the only condition that remains to be verified is

$$\lim_{\|\mu\|_\Lambda \to \infty,\, \mu \in K_\Lambda} \inf_{v \in K_V} L(v, \mu) = -\infty. \tag{3.14}$$

Let $\mu \in \Lambda$. From Proposition 3.3, we know that the mapping $v \mapsto L(v, \mu)$ is locally Lipschitz and strongly convex. Therefore, there is a unique element $u_\mu \in K_V$ such that

$$L(u_\mu, \mu) = \inf_{v \in K_V} L(v, \mu).$$

In particular, this implies

$$\langle u_\mu^*, v - u_\mu \rangle \ge 0 \quad \forall\, v \in K_V,\ u_\mu^* \in \partial L(u_\mu, \mu). \tag{3.15}$$

Here, $\partial L(u_\mu, \mu)$ is the generalized gradient of $L(\cdot, \mu)$ at u_μ. We have the characterization (cf. [15, Theorem 3.5])

$$\langle A u_\mu, v - u_\mu \rangle + \Phi(v) - \Phi(u_\mu) + \Psi^0(u_\mu; v - u_\mu) \ge \langle f, v - u_\mu \rangle - b(v - u_\mu, \mu) \quad \forall\, v \in K_V. \tag{3.16}$$

Since K_V is a linear subspace of V, we can substitute v with $u_\mu + v$ in (3.16) to get

$$\langle A u_\mu, v \rangle + \Phi(u_\mu + v) - \Phi(u_\mu) + \Psi^0(u_\mu; v) \ge \langle f, v \rangle - b(v, \mu) \quad \forall\, v \in K_V.$$

Replacing v by $-v$ in the above inequality,

$$b(v, \mu) \le -\langle A u_\mu, v \rangle + \Phi(u_\mu - v) - \Phi(u_\mu) + \Psi^0(u_\mu; -v) + \langle f, v \rangle \quad \forall\, v \in K_V.$$

Write

$$\langle A u_\mu, v \rangle = \langle A u_\mu - A0, v \rangle + \langle A0, v \rangle.$$

By $H(A)$,

$$|\langle Au_\mu, v\rangle| \le (M_A\|u_\mu\|_V + \|A0\|_{V^*})\,\|v\|_V.$$

By applying (3.13),

$$\Psi^0(u_\mu; -v) \le (c_0 + c_1\|u_\mu\|_V^\kappa)\,\|v\|_V.$$

Denote by M_Φ the Lipschitz constant of Φ:

$$|\Phi(v_1) - \Phi(v_2)| \le M_\Phi\|v_1 - v_2\|_V \quad \forall\, v_1, v_2 \in K_V.$$

Then,

$$|\Phi(u_\mu - v) - \Phi(u_\mu)| \le M_\Phi\|v\|_V \quad \forall\, v \in K_V.$$

Hence,

$$b(v,\mu) \le (M_A\|u_\mu\|_V + c_1\|u_\mu\|_V^\kappa + \|A0\|_{V^*} + M_\Phi + c_0 + \|f\|_{V^*})\,\|v\|_V \quad \forall\, v \in K_V.$$

Apply the inf–sup condition (3.9),

$$\alpha_b\|\mu\|_\Lambda \le \sup_{0\ne v\in K_V}\frac{b(v,\mu)}{\|v\|_V} \le M_A\|u_\mu\|_V + c_1\|u_\mu\|_V^\kappa + \|A0\|_{V^*} + M_\Phi + c_0 + \|f\|_{V^*}.$$

Thus, for some constant $c > 0$,

$$\|u_\mu\|_V \ge c\left[\min(\|\mu\|_\Lambda, \|\mu\|_\Lambda^{1/\kappa}) - 1\right]. \tag{3.17}$$

Here, by convention, when $\kappa = 0$, the term $\|\mu\|_\Lambda^{1/\kappa}$ is dropped from (3.17):

$$\|u_\mu\|_V \ge c\,(\|\mu\|_\Lambda - 1)\,.$$

Now let us choose an arbitrary element $v_0 \in K_V$ and fix it. Then by Lemma 2.4 and Proposition 3.3, we have the relation

$$L(v_0,\mu) - L(u_\mu,\mu) \ge \langle u_\mu^*, v_0 - u_\mu\rangle + \frac{m_A - \alpha_\Psi}{2}\,\|v_0 - u_\mu\|^2, \quad u_\mu^* \in \partial L(u_\mu,\mu).$$

By making use of (3.15), we obtain

$$L(u_\mu,\mu) \le L(v_0,\mu) - \frac{m_A - \alpha_\Psi}{2}\,\|v_0 - u_\mu\|^2. \tag{3.18}$$

By the definition (3.3) of L,

$$L(v_0,\mu) = \bar{c}_0 + b(v_0,\mu),$$

where

$$\bar{c}_0 = F_A(v_0) + \Phi(v_0) + \Psi(v_0) - \langle f, v_0\rangle$$

is a constant independent of μ. Note that

$$|b(v_0,\mu)| \le M_b\|v_0\|\,\|\mu\|_\Lambda. \tag{3.19}$$

Combining (3.17), (3.18) and (3.19), we can derive the inequality

$$L(u_\mu,\mu) \le c\left[1 + \|\mu\|_\Lambda - \min(\|\mu\|_\Lambda^2, \|\mu\|_\Lambda^{2/\kappa})\right]$$

for some positive constant c. Since $\kappa \in [0,2)$, (3.14) holds. ■

Since Problems 3.1 and 3.2 are equivalent, we can state the following result on Problem 3.1.

Theorem 3.7. *Under the assumptions stated in Theorem 3.5 or in Theorem 3.6, Problem 3.1 has a solution.*

Finally, we present a uniqueness result on the first component u of the solution. Similar uniqueness result can be found in [6,7,10].

Theorem 3.8. *Keep the assumptions stated in Theorem 3.5 or in Theorem 3.6. Then, the first component u of a solution of Problem 3.1 or Problem 3.2 is unique.*

Proof. Assume Problem 3.1 has two solutions $(u_1, \lambda_1), (u_2, \lambda_2) \in K_V \times K_\Lambda$. Then in the defining inequalities (3.1) and (3.2) for the solution (u_1, λ_1), we take $v = u_2$ and $\mu = \lambda_2$ to obtain

$$\langle Au_1, u_2 - u_1\rangle + b(u_2 - u_1, \lambda_1) + \Phi(u_2) - \Phi(u_1) + \Psi^0(u_1; u_2 - u_1) \geq \langle f, u_2 - u_1\rangle, \tag{3.20}$$

$$b(u_1, \lambda_2 - \lambda_1) \leq 0. \tag{3.21}$$

Similarly,

$$\langle Au_2, u_1 - u_2\rangle + b(u_1 - u_2, \lambda_2) + \Phi(u_1) - \Phi(u_2) + \Psi^0(u_2; u_1 - u_2) \geq \langle f, u_1 - u_2\rangle, \tag{3.22}$$

$$b(u_2, \lambda_1 - \lambda_2) \leq 0. \tag{3.23}$$

Add (3.20) and (3.22) to get

$$\langle Au_1 - Au_2, u_1 - u_2\rangle \leq b(u_2 - u_1, \lambda_1 - \lambda_2) + \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2). \tag{3.24}$$

Note that by (3.21) and (3.23),

$$b(u_2 - u_1, \lambda_1 - \lambda_2) = b(u_2, \lambda_1 - \lambda_2) + b(u_1, \lambda_2 - \lambda_1) \leq 0.$$

Then from (3.24),

$$\langle Au_1 - Au_2, u_1 - u_2\rangle \leq \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2).$$

Apply the conditions $H(A)$ and $H(\Psi)$,

$$m_A \|u_1 - u_2\|_V^2 \leq \alpha_\Psi \|u_1 - u_2\|_V^2.$$

By the condition $H(s)$, this inequality implies $\|u_1 - u_2\|_V = 0$, i.e., $u_1 = u_2$. ■

4. Examples arising from contact mechanics

In this section we discuss two examples from contact mechanics to illustrate the application of the theoretical results shown in Section 3.

We will consider contact problems for the deformation of a deformable body whose initial configuration is a Lipschitz domain $\Omega \subset \mathbb{R}^d$ $(d \leq 3)$. Since $\partial\Omega$ is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d)^T$ is defined a.e. on $\partial\Omega$. For an $\mathbb{R}^d$-valued function $\boldsymbol{u}$ on the boundary, its normal and tangential components are $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, respectively. Denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. For an $\mathbb{S}^d$-valued function $\boldsymbol{\sigma}$ on the boundary, we call $\sigma_\nu = \boldsymbol{\nu} \cdot \boldsymbol{\sigma\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$ the normal and tangential components of $\boldsymbol{\sigma}$ on the boundary.

We adopt the summation convention over a repeated index. The indices i and j run between 1 and d. The canonical inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \text{for all } \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\sigma}\|_{\mathbb{S}^d} = (\boldsymbol{\sigma} : \boldsymbol{\sigma})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

Example 4.1. We split the boundary $\partial\Omega$ into four mutually disjoint parts: $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3} \cup \overline{\Gamma_4}$ with $|\Gamma_1| > 0$, $|\Gamma_3| + |\Gamma_4| > 0$. The classical formulation of the contact problem is to find $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$ such that

$$\operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{4.1}$$

$$\boldsymbol{\sigma} \in \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \partial\psi_\Omega(\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in } \Omega, \tag{4.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{4.3}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{4.4}$$

$$u_\nu = 0,\ \|\boldsymbol{\sigma}_\tau\| \le g,\ \boldsymbol{\sigma}_\tau = -g\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0}, \quad \text{on } \Gamma_3, \tag{4.5}$$

$$-\sigma_\nu = F,\ \|\boldsymbol{\sigma}_\tau\| \le k|\sigma_\nu|,\ \boldsymbol{\sigma}_\tau = -k|\sigma_\nu|\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0}, \quad \text{on } \Gamma_4. \tag{4.6}$$

Let us briefly comment on the equations and conditions in this problem. The equilibrium equation is represented by (4.1) where $\boldsymbol{f}_0$ is the density of the body force. The inclusion (4.2) is the constitutive law for elastic materials in which $\mathcal{E}$ represents the elasticity operator and $\partial\psi_\Omega$ stands for the generalized gradient of a superpotential function ψ_Ω (cf. [10]). The condition (4.3) means that the body is fixed on Γ_1. The condition (4.4) gives a surface traction condition on Γ_2, where $\boldsymbol{f}_2$ is the density of the applied surface traction. On Γ_3 the contact is bilateral, signified by the first condition in (4.5). The second and third relations in (4.5) represent a version of static Coulomb's law of dry friction, $g > 0$ being a given friction bound. It can be verified that these two relations are equivalent to

$$\|\boldsymbol{\sigma}_\tau\| \le g, \quad \boldsymbol{\sigma}_\tau \cdot \boldsymbol{u}_\tau + g\,\|\boldsymbol{u}_\tau\| = 0 \quad \text{on } \Gamma_3. \tag{4.7}$$

On Γ_4, we have a frictional contact condition with prescribed normal stress, see (4.6). For more details on the lines (4.5) and (4.6) the reader may consult, for instance, [17,18]. Note that the assumption "$|\Gamma_3|+|\Gamma_4| > 0$" allows the possibility that $|\Gamma_3| > 0$ and $\Gamma_4 = \emptyset$, or $|\Gamma_4| > 0$ and $\Gamma_3 = \emptyset$; in the former case, the contact problem contains the contact condition (4.5) only, whereas in the latter case, the contact problem contains the contact condition (4.6) only.

For simplicity in writing, we assume the elasticity operator $\mathcal{E}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ to be linear so that $\mathcal{E} = (\mathcal{E}_{ijkl})_{1\le i,j,k,l\le d}$ is symmetric, bounded, and pointwise stable:

$$\mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij}, \quad 1 \le i,j,k,l \le d, \tag{4.8}$$

$$\mathcal{E}_{ijkl} \in L^\infty(\Omega), \quad 1 \le i,j,k,l \le d, \tag{4.9}$$

$$(\mathcal{E}_{ijkl}\boldsymbol{\epsilon}) : \boldsymbol{\epsilon} \ge m_{\mathcal{E}}\|\boldsymbol{\epsilon}\|^2_{\mathbb{S}^d}, \quad m_{\mathcal{E}} > 0,\ \forall\, \boldsymbol{\epsilon} \in \mathbb{S}^d. \tag{4.10}$$

For the superpotential $\psi_\Omega\colon \Omega \times \mathbb{S}^d \to \mathbb{R}$, we assume

$$\psi_\Omega(\cdot,\boldsymbol{\epsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\epsilon} \in \mathbb{S}^d,\ \psi_\Omega(\cdot,\boldsymbol{0}) \in L^1(\Omega); \tag{4.11}$$

$$\psi_\Omega(\boldsymbol{x},\cdot) \text{ is locally Lipschitz continuous on } \mathbb{S}^d \text{ for a.e. } \boldsymbol{x} \in \Omega; \tag{4.12}$$

$$\psi^0_\Omega(\boldsymbol{x},\boldsymbol{\epsilon}_1;\boldsymbol{\epsilon}_2-\boldsymbol{\epsilon}_1) + \psi^0_\Omega(\boldsymbol{x},\boldsymbol{\epsilon}_2;\boldsymbol{\epsilon}_1-\boldsymbol{\epsilon}_2) \le \alpha_{\psi_\Omega}\|\boldsymbol{\epsilon}_1-\boldsymbol{\epsilon}_2\|^2_{\mathbb{S}^d}\ \forall\,\boldsymbol{\epsilon}_1,\boldsymbol{\epsilon}_2 \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega, \tag{4.13}$$

where $\alpha_{\psi_\Omega} \ge 0$ is a constant. On the external force densities, we assume

$$\boldsymbol{f}_0 \in L^2(\Omega;\mathbb{R}^d), \quad \boldsymbol{f}_2 \in L^2(\Gamma_2;\mathbb{R}^d). \tag{4.14}$$

In addition, we assume that

$$g \in L^2(\Gamma_3),\ g(\boldsymbol{x}) \ge 0 \text{ a.e. } \boldsymbol{x} \in \Gamma_3; \tag{4.15}$$

$$F \in L^2(\Gamma_4),\ F(\boldsymbol{x}) \geq 0 \text{ a.e. } \boldsymbol{x} \in \Gamma_4;$$
$$k \in L^\infty(\Gamma_4),\ k(\boldsymbol{x}) \geq 0 \text{ a.e. } \boldsymbol{x} \in \Gamma_4.$$

We use the space

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{\gamma v} = \boldsymbol{0} \text{ on } \Gamma_1,\ v_\nu = 0 \text{ on } \Gamma_3 \right\} \tag{4.16}$$

for the displacement variable. Here and below, for all $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$, $v_\nu = \boldsymbol{\gamma v} \cdot \boldsymbol{\nu}$ denotes the normal component of the trace $\boldsymbol{\gamma v}$ of $\boldsymbol{v}$ on Γ. Since $|\Gamma_1| > 0$, Korn's inequality holds: for some constant $c > 0$,

$$\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)} \leq c\, \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)} \quad \forall\, \boldsymbol{v} \in V$$

(cf. [19, p. 79]), and we can define the norm $\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)}$, which is equivalent to the standard norm $\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)}$ on V. For the Lagrange multipliers we use the space

$$\Lambda = Z^*, \tag{4.17}$$

where Z is the closed subspace of $H^{1/2}(\Gamma;\mathbb{R}^d)$ defined as

$$Z = \left\{ \boldsymbol{z} \in H^{1/2}(\Gamma;\mathbb{R}^d) \mid \boldsymbol{z} = \boldsymbol{\gamma v} \text{ for some } \boldsymbol{v} \in V \right\}; \tag{4.18}$$

see, e.g., [20] (page 4) for some details. Then we introduce the bilinear form

$$b(\boldsymbol{v},\boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma v}\rangle, \quad \boldsymbol{v} \in V,\ \boldsymbol{\mu} \in \Lambda, \tag{4.19}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\Lambda = Z^*$ and Z.

We also define $\Phi : V \to \mathbb{R}$,

$$\Phi(\boldsymbol{v}) = \int_{\Gamma_4} k\, F\, \|\boldsymbol{v}_\tau\|\, da \quad \forall\, \boldsymbol{v} \in V \tag{4.20}$$

and introduce the set

$$K_\Lambda = \left\{ \boldsymbol{\mu} \in \Lambda \mid \langle \boldsymbol{\mu}, \boldsymbol{z}\rangle \leq \int_{\Gamma_3} g\, \|\boldsymbol{z}\|\, da\ \forall\, \boldsymbol{z} \in Z \right\}. \tag{4.21}$$

Next, we define a Lagrange multiplier $\boldsymbol{\lambda} \in K_\Lambda$ as follows:

$$\langle \boldsymbol{\lambda}, \boldsymbol{z}\rangle = -\int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{z}\, da. \tag{4.22}$$

Let us derive a mixed weak formulation for the contact problem (4.1)–(4.6) for the variables $\boldsymbol{u}$ and $\boldsymbol{\lambda}$. For this purpose, we assume the problem has a smooth solution $(\boldsymbol{u},\boldsymbol{\sigma})$ so that each step in the derivation below is meaningful. We multiply (4.1) by $\boldsymbol{v} - \boldsymbol{u}$ with $\boldsymbol{v} \in V$ smooth, integrate over Ω, perform an integration by parts, and apply (4.4) and boundary conditions of $\boldsymbol{v}$ due to $\boldsymbol{v} \in V$ to obtain

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\, dx - \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, da - \int_{\Gamma_4} \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, da = \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle, \tag{4.23}$$

where

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{\gamma v}\, da, \quad \boldsymbol{v} \in V. \tag{4.24}$$

Note that here and below, for all $\boldsymbol{v} \in V$, $\boldsymbol{v}_\tau = \boldsymbol{\gamma v} - v_\nu \boldsymbol{\nu}$.

By the constitutive relation (4.2),

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\, dx \leq \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\, dx + \int_\Omega \psi_\Omega^0(\boldsymbol{\varepsilon}(\boldsymbol{u});\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}))\, dx.$$

From the definition (4.19),

$$-\int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, da = b(\boldsymbol{v}, \boldsymbol{\lambda}) - b(\boldsymbol{u}, \boldsymbol{\lambda}).$$

Hence, from (4.23), we deduce that

$$\int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\, dx + \int_\Omega \psi_\Omega^0(\boldsymbol{\varepsilon}(\boldsymbol{u}); \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}))\, dx \\ + \Phi(\boldsymbol{v}) - \Phi(\boldsymbol{u}) + b(\boldsymbol{v}, \boldsymbol{\lambda}) - b(\boldsymbol{u}, \boldsymbol{\lambda}) \geq \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle.$$

From (4.7), we have

$$\langle \boldsymbol{\lambda}, \gamma \boldsymbol{u}\rangle = \int_{\Gamma_3} g\, \|\boldsymbol{u}_\tau\|\, da.$$

Hence, for $\boldsymbol{\mu} \in K_\Lambda$,

$$b(\boldsymbol{u}, \boldsymbol{\mu}-\boldsymbol{\lambda}) = \langle \boldsymbol{\mu}, \gamma\boldsymbol{u}\rangle - \langle \boldsymbol{\lambda}, \gamma\boldsymbol{u}\rangle \leq 0.$$

In summary, the mixed weak formulation of the problem (4.1)–(4.6) is to find $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times K_\Lambda$ such that for all $\boldsymbol{v} \in V$ and $\boldsymbol{\mu} \in K_\Lambda$,

$$\langle A\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}\rangle + b(\boldsymbol{v}-\boldsymbol{u}, \boldsymbol{\lambda}) + \Phi(\boldsymbol{v}) - \Phi(\boldsymbol{u}) + \int_\Omega \psi_\Omega^0(\boldsymbol{\varepsilon}(\boldsymbol{u}); \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}))\, dx \geq \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle, \tag{4.25}$$

$$b(\boldsymbol{u}, \boldsymbol{\mu}-\boldsymbol{\lambda}) \leq 0, \tag{4.26}$$

where

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \quad \boldsymbol{u}, \boldsymbol{v} \in V. \tag{4.27}$$

We consider an auxiliary problem: find $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times K_\Lambda$ such that

$$\langle A\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}\rangle + b(\boldsymbol{v}-\boldsymbol{u}, \boldsymbol{\lambda}) + \Phi(\boldsymbol{v}) - \Phi(\boldsymbol{u}) + \Psi^0(\boldsymbol{u}; \boldsymbol{v}-\boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle \quad \forall\, \boldsymbol{v} \in V, \tag{4.28}$$

$$b(\boldsymbol{u}, \boldsymbol{\mu}-\boldsymbol{\lambda}) \leq 0 \quad \forall\, \boldsymbol{\mu} \in K_\Lambda, \tag{4.29}$$

where

$$\Psi(\boldsymbol{v}) = \int_\Omega \psi_\Omega(\boldsymbol{\varepsilon}(\boldsymbol{v}))\, dx, \quad \boldsymbol{v} \in V. \tag{4.30}$$

The operator A is potential with

$$F_A(\boldsymbol{v}) = \frac{1}{2}\int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx.$$

We introduce the Lagrangian functional

$$L(\boldsymbol{v}, \boldsymbol{\mu}) = F_A(\boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{\mu}) + \Phi(\boldsymbol{v}) + \Psi(\boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \quad \boldsymbol{v} \in V,\ \boldsymbol{\mu} \in \Lambda$$

and the related saddle-point problem

$$(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times K_\Lambda, \quad L(\boldsymbol{u}, \boldsymbol{\mu}) \leq L(\boldsymbol{u}, \boldsymbol{\lambda}) \leq L(\boldsymbol{v}, \boldsymbol{\lambda}) \quad \forall\, (\boldsymbol{v}, \boldsymbol{\mu}) \in V \times K_\Lambda. \tag{4.31}$$

The problem (4.28)–(4.29) is a particular case of Problem 3.1, whereas the problem (4.31) is a particular case of Problem 3.2, with $K_V = V$ defined in (4.16), Λ defined in (4.17), K_Λ defined by (4.21), A defined by (4.27), b defined by (4.19), Φ defined by (4.20), Ψ defined by (4.30), and $f = \boldsymbol{f}$ defined by (4.24). Note that K_Λ is a bounded set in Λ, and we are in a position to apply Theorems 3.4, 3.5, 3.7 and 3.8. The conditions $H(K_V)$, $H(K_\Lambda)$, $H(A)$, $H(b)$, $H(\Phi)$, $H(\Psi)$ and $H(f)$ can be verified straightforward. It is easy to see that (3.7) holds with $m_A = m_{\mathcal{E}}$.

According to Proposition 3.37 (i) and Theorem 3.47 (iv) in [21], the following inequality holds true:

$$\int_\Omega \psi_\Omega^0(\boldsymbol{\varepsilon}(\boldsymbol{u}); \boldsymbol{\varepsilon}(\boldsymbol{v}))\, dx \geq \Psi^0(\boldsymbol{u}; \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V. \tag{4.32}$$

By using (4.13), we have, for any $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$,

$$\begin{aligned}\Psi^0(\boldsymbol{v}_1; \boldsymbol{v}_2 - \boldsymbol{v}_1) + \Psi^0(\boldsymbol{v}_2; \boldsymbol{v}_1 - \boldsymbol{v}_2) &\le \int_\Omega \left[\psi_\Omega^0(\boldsymbol{\varepsilon}(\boldsymbol{v}_1); \boldsymbol{\varepsilon}(\boldsymbol{v}_2 - \boldsymbol{v}_1)) + \psi_\Omega^0(\boldsymbol{\varepsilon}(\boldsymbol{v}_2); \boldsymbol{\varepsilon}(\boldsymbol{v}_1 - \boldsymbol{v}_2))\right] dx \\ &\le \alpha_{\psi_\Omega} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^2.\end{aligned}$$

Thus, (3.5) holds with α_Ψ equals α_{ψ_Ω} from (4.13). Hence, for the problem (4.28)–(4.29) and the problem (4.31), the smallness condition $H(s)$ becomes

$$\alpha_{\psi_\Omega} < m_{\mathcal{E}}. \tag{4.33}$$

By applying Theorems 3.4, 3.5, 3.7 and 3.8, we know that under the additional condition (4.33), the problem (4.31) and the problem (4.28)–(4.29) are equivalent, each problem has a solution $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times K_\Lambda$ and the first component $\boldsymbol{u}$ of the solution is unique.

Because of (4.32), we know that a solution $(\boldsymbol{u}, \lambda) \in V \times K_\Lambda$ of the problem (4.28)–(4.29) is also a solution of the problem (4.25)–(4.26). Moreover, the uniqueness of $\boldsymbol{u}$ for the problem (4.25)–(4.26) can be verified as in the proof of Theorem 3.8. ■

Example 4.2. We split the boundary $\partial\Omega$ into five mutually disjoint parts: $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3} \cup \overline{\Gamma_4} \cup \overline{\Gamma_5}$ with $|\Gamma_1| > 0$ and $|\Gamma_3| + |\Gamma_4| + |\Gamma_5| > 0$. The classical formulation of the contact problem is to find $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$ such that

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{4.34}$$

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{4.35}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{4.36}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{4.37}$$

$$u_\nu = 0,\ -\boldsymbol{\sigma}_\tau \in \partial\psi_\Gamma(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_3, \tag{4.38}$$

$$u_\nu = 0,\ \|\boldsymbol{\sigma}_\tau\| \le g,\ \boldsymbol{\sigma}_\tau = -g\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0}, \quad \text{on } \Gamma_4, \tag{4.39}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0},\ \sigma_\nu \le 0,\ u_\nu \le 0,\ \sigma_\nu u_\nu = 0 \quad \text{on } \Gamma_5. \tag{4.40}$$

The mechanical significance of (4.34), (4.36), (4.37) and (4.39) is like in the previous example and the constitutive law (4.35) is a classical one governed by the elastic operator $\mathcal{E}$. In addition, we use a bilateral frictional contact condition governed by the superpotential ψ_Γ, on Γ_3, see (4.38). Moreover, we use a frictionless unilateral contact condition on Γ_5, see (4.40); for more details on such a contact condition we refer to e.g. [17] and the references therein. Similar to Example 4.1, the assumption "$|\Gamma_1| > 0$ and $|\Gamma_3|+|\Gamma_4|+|\Gamma_5| > 0$" allows the possibility that only one or two of the three contact conditions (4.38)–(4.40) are present.

As working hypotheses we keep (4.8)–(4.10), (4.14), (4.15), and for the superpotential $\psi_\Gamma \colon \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$, we assume

$$\begin{aligned}&\psi_\Gamma(\cdot, \boldsymbol{v}) \text{ is measurable on } \Gamma_3 \text{ for all } \boldsymbol{v} \in \mathbb{R}^d,\ \psi_\Gamma(\cdot, \boldsymbol{0}) \in L^1(\Gamma_3);\\ &\psi_\Gamma(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz continuous on } \mathbb{R}^d \text{ for a.e. } \boldsymbol{x} \in \Gamma_3;\\ &\psi_\Gamma^0(\boldsymbol{x}, \boldsymbol{v}_1; \boldsymbol{v}_2 - \boldsymbol{v}_1) + \psi_\Gamma^0(\boldsymbol{x}, \boldsymbol{v}_2; \boldsymbol{v}_1 - \boldsymbol{v}_2) \le \alpha_\psi \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\mathbb{R}^d}^2\ \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3,\end{aligned} \tag{4.41}$$

where $\alpha_{\psi_\Gamma} \ge 0$ is a constant.

For this example, we use the space

$$V = \left\{\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{\gamma}\boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1,\ v_\nu = 0 \text{ on } \Gamma_3 \cup \Gamma_4\right\}, \quad K_V = V$$

for the displacement variable, and define an operator $A: V \to V$ and functionals $\Phi, \boldsymbol{f}: V \to \mathbb{R}$ by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V,$$

$$\Phi(\boldsymbol{v}) = \int_{\Gamma_4} g\,\|\boldsymbol{v}_\tau\|\,da \quad \forall\, \boldsymbol{v} \in V,$$

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{\gamma}\boldsymbol{v}\,da \quad \forall\, \boldsymbol{v} \in V.$$

For the Lagrange variable, we use the space

$$\Lambda = Z^*$$

where

$$Z = \left\{ \boldsymbol{z} \in H^{1/2}(\Gamma; \mathbb{R}^d) \mid \boldsymbol{z} = \boldsymbol{\gamma}\boldsymbol{v} \text{ for some } \boldsymbol{v} \in V \right\}.$$

Define the bilinear form $b: V \times \Lambda \to \mathbb{R}$ by

$$b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma}\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in V,\ \boldsymbol{\mu} \in \Lambda,$$

and the set

$$K_\Lambda = \{\boldsymbol{\mu} \in \Lambda \mid \langle \boldsymbol{\mu}, \boldsymbol{\gamma}\boldsymbol{v}\rangle \le 0\ \forall\, \boldsymbol{v} \in \mathcal{K}\},$$

where

$$\mathcal{K} = \{\boldsymbol{v} \in V \mid v_\nu \le 0 \text{ a.e. on } \Gamma_5\}.$$

Let us define a Lagrange multiplier $\boldsymbol{\lambda} \in \Lambda$ as follows:

$$\langle \boldsymbol{\lambda}, \boldsymbol{z}\rangle = -\int_{\Gamma_5} \sigma_\nu \boldsymbol{\nu} \cdot \boldsymbol{z}\,da.$$

Similar to Example 4.1, we arrive at the following mixed weak formulation for the problem (4.34)–(4.40): find $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times K_\Lambda$ such that for all $\boldsymbol{v} \in V$ and $\boldsymbol{\mu} \in K_\Lambda$,

$$\langle A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle + b(\boldsymbol{v} - \boldsymbol{u}, \boldsymbol{\lambda}) + \Phi(\boldsymbol{v}) - \Phi(\boldsymbol{u}) + \int_{\Gamma_3} \psi_\Gamma^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\,da \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle, \tag{4.42}$$

$$b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \le 0. \tag{4.43}$$

In relation to the above mixed weak formulation, we consider an auxiliary problem as follows: find $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times K_\Lambda$ such that

$$\langle A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle + b(\boldsymbol{v} - \boldsymbol{u}, \boldsymbol{\lambda}) + \Phi(\boldsymbol{v}) - \Phi(\boldsymbol{u}) + \Psi^0(\boldsymbol{u}; \boldsymbol{v} - \boldsymbol{u}) \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle \quad \forall\, \boldsymbol{v} \in V, \tag{4.44}$$

$$b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \le 0 \quad \forall\, \boldsymbol{\mu} \in K_\Lambda, \tag{4.45}$$

where

$$\Psi(\boldsymbol{v}) = \int_{\Gamma_3} \psi_\Gamma(\boldsymbol{v}_\tau)\,da.$$

The corresponding saddle-point problem is

$$(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times K_\Lambda, \quad L(\boldsymbol{u}, \boldsymbol{\mu}) \le L(\boldsymbol{u}, \boldsymbol{\lambda}) \le L(\boldsymbol{v}, \boldsymbol{\lambda}) \quad \forall\, (\boldsymbol{v}, \boldsymbol{\mu}) \in V \times K_\Lambda. \tag{4.46}$$

with the Lagrangian functional

$$L(\boldsymbol{v}, \boldsymbol{\mu}) = \frac{1}{2}\int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + b(\boldsymbol{v}, \boldsymbol{\mu}) + \int_{\Gamma_3} \psi_\Gamma(\boldsymbol{v}_\tau)\,da + \Phi(\boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v}\rangle.$$

Similar to (4.32), we have

$$\int_{\Gamma_3} \psi_\Gamma^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, da \ge \Psi^0(\boldsymbol{u}; \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V. \tag{4.47}$$

Then by using (4.41), for any $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$,

$$\begin{aligned}\Psi^0(\boldsymbol{v}_1; \boldsymbol{v}_2 - \boldsymbol{v}_1) + \Psi^0(\boldsymbol{v}_2; \boldsymbol{v}_1 - \boldsymbol{v}_2) &\le \int_{\Gamma_3} \left[\psi_\Gamma^0(\boldsymbol{v}_{1,\tau}; \boldsymbol{v}_{2,\tau} - \boldsymbol{v}_{1,\tau}) + \psi_\Gamma^0(\boldsymbol{v}_{2,\tau}; \boldsymbol{v}_{1,\tau} - \boldsymbol{v}_{2,\tau})\right] da \\ &\le \alpha_{\psi_\Gamma} \lambda_{\Gamma_3}^{-1} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^2,\end{aligned}$$

where $\lambda_{\Gamma_3} > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{w} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{w}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_3} \boldsymbol{w}_\tau \cdot \boldsymbol{v}_\tau\, da \quad \forall\, \boldsymbol{v} \in V.$$

Thus, (3.5) holds with α_Ψ equals $\alpha_{\psi_\Gamma} \lambda_{\Gamma_3}^{-1}$. Hence, for the problem (4.44)–(4.45) and the problem (4.46), the smallness condition $H(s)$ becomes

$$\alpha_{\psi_\Gamma} < m_{\mathcal{E}} \lambda_{\Gamma_3}. \tag{4.48}$$

Since the conditions $H(K_V)$, $H(K_\Lambda)$, $H(A)$, $H(b)$, $H(\Phi)$, $H(\Psi)$ and $H(f)$ are fulfilled, keeping in mind (4.48), we apply Theorems 3.4, 3.6, 3.7, 3.8. Note that the inf–sup property of the form b, (3.9), can be verified as in [20] (see pages 16–17). Also, it is worth underlining that in this example K_Λ is an unbounded set.

Hence, the saddle point problem (4.46) and the auxiliary problem (4.44)–(4.45) are equivalent, each problem has a solution $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times K_\Lambda$ and the first component $\boldsymbol{u}$ of the solution is unique.

Because of (4.47), we know that a solution $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times K_\Lambda$ of the problem (4.44)–(4.45) is also a solution of the problem (4.42)–(4.43). The uniqueness of $\boldsymbol{u}$ for the problem (4.42)–(4.43) can be verified by using a similar technique with that one used in the proof of Theorem 3.8. ■

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