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Numerical solution of an $H(\mathbf{curl})$-elliptic hemivariational inequality

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This paper is concerned with the analysis and numerical solution of an $\boldsymbol{H}(\mathbf{curl})$-elliptic hemivariational inequality (HVI). One source of the HVI is through a temporal semidiscretization of a related hyperbolic Maxwell equation problem. An equivalent minimization principle is introduced, and the solution existence and uniqueness of the $\boldsymbol{H}(\mathbf{curl})$-elliptic HVI are proved. Numerical analysis of the HVI is provided with a general Galerkin approximation, including a Céa's inequality for convergence and error estimation. When the linear edge finite element method is employed, an optimal-order error estimate is derived under a suitable solution regularity assumption. A fully discrete scheme based on the backward Euler difference in time and a mixed finite element method in space is also analyzed, and stability estimates are derived for first-order terms of the fully discrete solution. Numerical results are reported on linear edge finite element solutions of the $\boldsymbol{H}(\mathbf{curl})$-elliptic HVI for numerical evidence of the theoretically predicted convergence order.

Keywords: $\boldsymbol{H}(\mathbf{curl})$-elliptic hemivariational inequality; minimization principle; finite element method; optimal-order error estimate.

1. Introduction

Bean (1962, 1964) proposed a critical state model that describes an irreversible and hysteretic magnetization process of high-temperature superconductors in a temporally varying applied magnetic field. The Bean model employs a nonsmooth constitutive relation between the current density $\boldsymbol{J}$ and the electric field $\boldsymbol{E}$: the current density strength cannot exceed a critical value; the electric field vanishes if the current density strength is strictly less than the critical value; otherwise, the electric field is parallel to the current density. Denoting by g the critical value, the constitutive relation can be stated mathematically as follows:

$$|\boldsymbol{J}| \le g; \quad |\boldsymbol{J}| < g \;\Rightarrow\; \boldsymbol{E} = \boldsymbol{0}; \quad |\boldsymbol{J}| = g \;\Rightarrow\; \boldsymbol{J} = \kappa\,\boldsymbol{E} \text{ for some } \kappa \ge 0, \tag{1.1}$$

valid in the domain where the electromagnetism process is studied. It is possible to eliminate the unknown parameter κ. Indeed, (1.1) can be equivalently written as

$$|\boldsymbol{J}| \le g, \quad \boldsymbol{J} \cdot \boldsymbol{E} = g\,|\boldsymbol{E}|. \tag{1.2}$$

Using the notion of the convex subdifferential we can further express (1.2) concisely as

$$\boldsymbol{J} \in \partial\,(g\,|\boldsymbol{E}|)\,. \tag{1.3}$$

Note that the function $\boldsymbol{E} \mapsto g\,|\boldsymbol{E}|$ is nonsmooth and convex. In the eddy current approximation that eliminates the displacement current, the Bean critical state model (1.1) with magnetic induction dependence $g = g(\boldsymbol{B})$ leads to a parabolic quasi-variational inequality (Prigozhin, 1996a,b; Barrett & Prigozhin, 2006, 2015; Elliott & Kashima, 2007). In the full three-dimensional Maxwell system the Bean critical state model (1.1) with $g = g(x)$ leads to a hyperbolic Maxwell variational inequality of the second kind (Yousept, 2017, 2020a; Winckler & Yousept, 2019).

In Winckler *et al.* (2020) an adaptive edge element method is developed to solve a corresponding $\boldsymbol{H}(\mathbf{curl})$-elliptic variational inequality. To describe the $\boldsymbol{H}(\mathbf{curl})$-elliptic variational inequality and for use later in this paper we need to introduce some function spaces. Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. Denote by Γ the boundary of Ω. Let

$$\boldsymbol{H}(\mathbf{curl},\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\},$$

which is a Hilbert space with the norm $\|\cdot\|_{\mathbf{curl},\Omega}$ defined through

$$\|\boldsymbol{v}\|^2_{\mathbf{curl},\Omega} = \|\boldsymbol{v}\|^2_{0,\Omega} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega},$$

where the **curl**-operator is understood in the sense of distributions. Let $\boldsymbol{C}_0^\infty(\Omega)$ be the space of all infinitely differentiable functions with compact support contained in Ω. Then define

$$\boldsymbol{V} = \boldsymbol{H}_0(\mathbf{curl},\Omega)$$

to be the closure of $\boldsymbol{C}_0^\infty(\Omega)$ with respect to the $\boldsymbol{H}(\mathbf{curl},\Omega)$ norm. The following characterization holds:

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\Omega) : \boldsymbol{n} \times \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma\}\,.$$

The $\boldsymbol{H}(\mathbf{curl})$-elliptic variational inequality studied in Winckler *et al.* (2020) is the following: find $\boldsymbol{E} \in \boldsymbol{V}$ such that

$$a(\boldsymbol{E},\boldsymbol{v}-\boldsymbol{E}) + \int_\Omega g\,(|\boldsymbol{v}| - |\boldsymbol{E}|)\,\mathrm{d}x \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{E})\,\mathrm{d}x \quad \forall\,\boldsymbol{v} \in \boldsymbol{V}, \tag{1.4}$$

where the bilinear form is

$$a(\boldsymbol{E},\boldsymbol{v}) = \int_\Omega \left(\epsilon\,\boldsymbol{E}\cdot\boldsymbol{v} + \mu^{-1}\mathbf{curl}\boldsymbol{E}\cdot\mathbf{curl}\boldsymbol{v}\right)\mathrm{d}x, \tag{1.5}$$

ϵ, μ are material property parameters and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$.

In Bean's original model the critical value g is a positive constant and it is a physically reasonable assumption where the magnetic field is not strong. However, experiments show that in the presence of strong external fields, the critical value can depend on the magnetic field (Kim *et al.*, 1963). In the case where the critical value depends on the strength of the magnetic field, the mathematical models may be quasi-variational inequalities; see Barrett & Prigozhin (2006) for analysis of an associated parabolic quasi-variational inequality in a scalar two-dimensional setting, and Barrett & Prigozhin (2015) on a nonconforming finite element method for the problem. In this paper we consider an $\boldsymbol{H}(\mathbf{curl})$-elliptic problem in which the constitutive relation is a generalization of (1.3), i.e.,

$$\boldsymbol{J} \in \partial\psi(\boldsymbol{E}), \tag{1.6}$$

where the function ψ is again nonsmooth but is allowed to be nonconvex. For a locally Lipschitz continuous function ψ, $\partial\psi$ stands for the subdifferential in the sense of Clarke (cf. Section 2). Instead of a variational inequality of the form (1.4) the weak formulation of the $\boldsymbol{H}(\mathbf{curl})$-elliptic problem subject to the constitutive relation (1.6) is a hemivariational inequality (HVI): find $\boldsymbol{E} \in \boldsymbol{V}$ such that

$$a(\boldsymbol{E}, \boldsymbol{v}) + \int_\Omega \psi^0(\boldsymbol{E}; \boldsymbol{v})\,\mathrm{d}x \geq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{1.7}$$

where the bilinear form $a(\cdot,\cdot)$ is defined by (1.5), and $\psi^0(\boldsymbol{E}; \boldsymbol{v})$ denotes the generalized directional derivative of ψ at $\boldsymbol{E}$ in the direction $\boldsymbol{v}$ (cf. Section 2). In the recent papers by Yousept (2020a,b), well-posedness results are proved for hyperbolic Maxwell variational inequalities of the second kind and obstacle-type hyperbolic Maxwell variational inequalities. In this paper we will present a solution existence and uniqueness result for the $\boldsymbol{H}(\mathbf{curl})$-elliptic HVI and apply the edge finite element method to solve it.

The framework of HVIs is useful for analyzing and solving some families of nonsmooth and nonconvex problems. It is closely related to the development of the concept of the generalized gradient of a locally Lipschitz function provided by Clarke (1975, 1983). Studies of HVIs started in the early 1980s (Panagiotopoulos, 1983), responding to needs in engineering applications. Through HVIs, problems involving nonsmooth, nonmonotone and set-valued relations among physical quantities can be modeled and analyzed. Comprehensive references on modeling and mathematical analysis of HVIs include Carl *et al.* (2007), Migórski *et al.* (2013), Naniewicz & Panagiotopoulos (1995), Panagiotopoulos (1993), Sofonea & Migórski (2018). A comprehensive reference on the finite element method for HVIs is Haslinger *et al.* (1999). In recent papers, e.g., Han (2018), Han & Sofonea (2019), Han *et al.* (2017, 2018), optimal-order error estimates are derived for the linear finite element solutions of some hemivariational inequalities.

The rest of the paper is organized as follows. In Section 2 we review the notions of the generalized directional derivative and the subdifferential in the sense of Clarke, as well as that of strongly convex functions. In Section 3 we introduce the $\boldsymbol{H}(\mathbf{curl})$-elliptic HVI as a result of a temporally semidiscrete approximation of a hyperbolic Maxwell equation problem, (3.1)–(3.2). In Section 4 we establish a minimization principle and prove the solution existence and uniqueness of the $\boldsymbol{H}(\mathbf{curl})$-elliptic HVI. In Section 5 we introduce a Galerkin approximation of the $\boldsymbol{H}(\mathbf{curl})$-elliptic HVI; the existence and uniqueness of a discrete solution follow from the argument in Section 4 for the finite-dimensional case. We establish Céa's inequality, which is the basis for convergence analysis and error estimation of numerical solutions. In Section 6 we use the linear edge finite elements for numerical approximation and derive an optimal-order error estimate for the linear edge finite element solution under certain solution regularity assumptions. In Section 7 we consider a fully discrete scheme based on the backward Euler difference in time and a mixed finite element method in space consisting of Nédélec's edge elements for the electric field and piecewise constant elements for the magnetic induction. We also derive stability estimates for the first-order terms of the fully discrete solution. In Section 8 we report simulation results on linear edge finite element solutions of the $\boldsymbol{H}(\mathbf{curl})$-elliptic HVI, illustrating numerical evidence of the theoretically predicted convergence order.

2. Preliminaries

In this section we review some basic notions and results that will be needed later.

We first recall the definitions of the generalized directional derivative and generalized subdifferential in the sense of Clarke for a locally Lipschitz function and quote some properties. For details the reader is referred to Clarke (1983), Migórski *et al.* (2013).

DEFINITION 2.1 Let V be a Banach space and denote by V^* its dual. Let $\psi : V \to \mathbb{R}$ be a locally Lipschitz functional. The generalized (Clarke) directional derivative of ψ at $u \in V$ in the direction $v \in V$ is defined by

$$\psi^0(u;v) = \limsup_{w\to u,\, \lambda\downarrow 0} \frac{\psi(w+\lambda v) - \psi(w)}{\lambda}.$$

The generalized gradient (subdifferential) of ψ at u is defined by

$$\partial\psi(u) = \{\zeta \in V^* : \psi^0(u;v) \geq \langle \zeta, v\rangle \ \forall\, v \in V\}.$$

Given the generalized subdifferential we can compute the generalized directional derivative through the formula (Clarke, 1983, Proposition 2.1.2)

$$\psi^0(u;v) = \max\{\langle \xi, v\rangle : \xi \in \partial\psi(u)\} \quad \forall\, u, v \in V. \tag{2.8}$$

The following subadditivity rule holds (Clarke, 1983, Proposition 2.1.1):

$$\psi^0(u;v_1+v_2) \leq \psi^0(u;v_1) + \psi^0(u;v_2) \quad \forall\, u, v_1, v_2 \in V. \tag{2.9}$$

Moreover, if $\psi_1, \psi_2 \colon V \to \mathbb{R}$ are locally Lipschitz continuous, then (Clarke, 1983, Proposition 2.3.3)

$$\partial(\psi_1+\psi_2)(u) \subset \partial\psi_1(u) + \partial\psi_2(u) \quad \forall\, u \in V, \tag{2.10}$$

or equivalently,

$$(\psi_1+\psi_2)^0(u;v) \leq \psi_1^0(u;v) + \psi_2^0(u;v) \quad \forall\, u, v \in V.$$

Then we recall that a function $g\colon V \to \mathbb{R}$ defined on a normed space V is said to be strongly convex on V with a constant $\alpha > 0$ if

$$g(\lambda\, u + (1-\lambda)\, v) \leq \lambda\, g(u) + (1-\lambda)\, g(v) - \alpha\, \lambda\, (1-\lambda)\, \|u-v\|_V^2 \quad \forall\, u, v \in V,\ \forall\, \lambda \in [0,1].$$

In the analysis of the HVI in this paper we will need the following result (Fan *et al.*, 2003, Theorem 3.4).

LEMMA 2.2 Let V be a real Banach space, and let $g\colon V \to \mathbb{R}$ be locally Lipschitz continuous. Then g is strongly convex on V with a constant $\alpha > 0$ if and only if ∂g is strongly monotone on V with a constant 2α, i.e.,

$$\langle \xi - \eta, u - v\rangle \geq 2\,\alpha\,\|u - v\|_V^2 \quad \forall\, u, v \in V,\ \xi \in \partial g(u),\ \eta \in \partial g(v).$$

A proof of the next result can be found in Han (2020, Proposition 2.5).

PROPOSITION 2.3 Let V be a real Hilbert space, and let $g\colon V \to \mathbb{R}$ be a locally Lipschitz continuous and strongly convex functional on V with a constant $\alpha > 0$. Then there exist two constants $\overline{c}_0$ and $\overline{c}_1$ such that

$$g(v) \geq \alpha\,\|v\|_V^2 + \overline{c}_0 + \overline{c}_1\|v\|_V \quad \forall\, v \in V. \tag{2.11}$$

Consequently, $g(\cdot)$ is coercive on V.

3. $\boldsymbol{H}$(curl)-elliptic hemivariational inequality

In this section we derive the $\boldsymbol{H}(\mathbf{curl})$-elliptic HVI (1.7) as a result of a temporal semidiscretization of a hyperbolic Maxwell equation problem.

Let the Lipschitz domain $\Omega \subset \mathbb{R}^3$ be the region for the electromagnetism process. Denote by $\boldsymbol{n}$ the unit outward normal on the boundary Γ. Let the time interval of interest be $[0, T]$. Consider an initial-boundary value problem of the Maxwell equations:

$$\tilde{\epsilon}\boldsymbol{E}_t - \mathbf{curl}\,(\tilde{\mu}^{-1}\boldsymbol{B}) + \boldsymbol{J} = \boldsymbol{l} \qquad \text{in } \Omega \times (0, T), \tag{3.1a}$$

$$\boldsymbol{B}_t + \mathbf{curl}\,\boldsymbol{E} = \boldsymbol{0} \qquad \text{in } \Omega \times (0, T), \tag{3.1b}$$

$$\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0} \qquad \text{on } \Gamma \times (0, T), \tag{3.1c}$$

$$\boldsymbol{E}(\cdot, 0) = \boldsymbol{E}_0 \qquad \text{in } \Omega, \tag{3.1d}$$

$$\boldsymbol{B}(\cdot, 0) = \boldsymbol{B}_0 \qquad \text{in } \Omega, \tag{3.1e}$$

with the current density

$$\boldsymbol{J} \in \partial\psi(\boldsymbol{E}) \quad \text{in } \Omega \times (0, T). \tag{3.2}$$

Here, the unknowns are the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$; $\boldsymbol{l}$ is a given applied current source, $\tilde{\epsilon}$ and $\tilde{\mu}$ represent the electric permittivity and the magnetic permeability, respectively. The boundary condition (3.1c) is known as the perfectly conducting condition; more general boundary conditions can also be considered. The function $\psi : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is assumed to be locally Lipschitz continuous with respect to its second argument. To simplify the notation we write $\psi(\boldsymbol{E})$ for $\psi(\boldsymbol{x}, \boldsymbol{E})$ and denote by $\partial\psi$ Clarke's generalized subdifferential of ψ with respect to its second argument.

We assume $\tilde{\epsilon}$ and $\tilde{\mu}$ are of class $L^\infty(\Omega)$ and satisfy

$$\tilde{\epsilon}_0 \le \tilde{\epsilon} \le \tilde{\epsilon}_1 \quad \text{and} \quad \tilde{\mu}_0 \le \tilde{\mu} \le \tilde{\mu}_1 \quad \text{a.e. in } \Omega,$$

for some positive constants $0 < \tilde{\epsilon}_0 \le \tilde{\epsilon}_1 < \infty$ and $0 < \tilde{\mu}_0 \le \tilde{\mu}_1 < \infty$. For the function $\psi : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ we assume

$$\begin{cases} \text{(a) } \psi(\cdot, \boldsymbol{\xi}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^3 \text{ and } \psi(\cdot, \mathbf{0}) \in L^1(\Omega); \\ \text{(b) } \psi(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R}^3 \text{ for a.e. } \boldsymbol{x} \in \Omega; \\ \text{(c) there exist constants } c_0, c_1 > 0 \text{ such that} \\ \qquad |\partial \psi(\boldsymbol{x}, \boldsymbol{\xi})| \le c_0 + c_1 |\boldsymbol{\xi}| \quad \text{a.e. } \boldsymbol{x} \in \Omega,\ \forall\, \boldsymbol{\xi} \in \mathbb{R}^3; \\ \text{(d) there exists a constant } m \text{ such that for a.e. } \boldsymbol{x} \in \Omega, \\ \qquad \psi^0(\boldsymbol{x}, \boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + \psi^0(\boldsymbol{x}, \boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \le m |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 \quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^3. \end{cases} \tag{3.3}$$

The inequality in (3.3)(c) stands for

$$|\boldsymbol{\eta}| \le c_0 + c_1 |\boldsymbol{\xi}| \quad \text{a.e. } \boldsymbol{x} \in \Omega,\ \forall\, \boldsymbol{\xi} \in \mathbb{R}^3,\ \boldsymbol{\eta} \in \partial \psi(\boldsymbol{x}, \boldsymbol{\xi}).$$

By (2.8), (3.3)(c) implies

$$\left| \psi^0(\boldsymbol{\xi}_1; \boldsymbol{\xi}_2) \right| \le \left(c_0 + c_1 |\boldsymbol{\xi}_1| \right) |\boldsymbol{\xi}_2| \quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^3. \tag{3.4}$$

It is known (Migórski *et al.*, 2013) that (3.3)(d) is equivalent to

$$(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge -m |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 \quad \text{a.e. } \boldsymbol{x} \in \Omega,\ \forall\, \boldsymbol{\xi}_i \in \mathbb{R}^3,\ \boldsymbol{\eta}_i \in \partial \psi(\boldsymbol{x}, \boldsymbol{\xi}_i),\ i = 1, 2. \tag{3.5}$$

Let us consider a semidiscretization of problem (3.1)–(3.2) with respect to the temporal variable t. For simplicity in exposition we use a uniform partition of the time interval $[0, T]$, and we comment that the discussion can be directly extended to the case of general nonuniform partitions. Let $k = T/N$ be the time-step size for a positive integer N, and denote the node points by $t_n = nk$, $0 \le n \le N$. We assume

$$\boldsymbol{l} \in C([0, T]; \boldsymbol{L}^2(\Omega))$$

is a continuous function of $t \in [0, T]$, and write $\boldsymbol{l}^n$ for $\boldsymbol{l}(t_n)$, $0 \le n \le N$. Then a backward semidiscrete scheme for problem (3.1)–(3.2) is to find $\{(\boldsymbol{E}^n, \boldsymbol{B}^n)\}_{n=1}^N$ such that for $n = 1, \ldots, N$,

$$\tilde{\epsilon}\, \frac{\boldsymbol{E}^n - \boldsymbol{E}^{n-1}}{k} - \mathbf{curl}\, (\tilde{\mu}^{-1} \boldsymbol{B}^n) + \boldsymbol{J}^n = \boldsymbol{l}^n \quad \text{in } \Omega, \tag{3.6}$$

$$\frac{\boldsymbol{B}^n - \boldsymbol{B}^{n-1}}{k} + \mathbf{curl}\, \boldsymbol{E}^n = \mathbf{0} \quad \text{in } \Omega, \tag{3.7}$$

$$\boldsymbol{n} \times \boldsymbol{E}^n = \mathbf{0} \quad \text{on } \Gamma, \tag{3.8}$$

where

$$\boldsymbol{J}^n \in \partial\psi(\boldsymbol{E}^n) \quad \text{in } \Omega, \tag{3.9}$$

and

$$\boldsymbol{E}^0 = \boldsymbol{E}_0 \quad \text{in } \Omega, \tag{3.10}$$

$$\boldsymbol{B}^0 = \boldsymbol{B}_0 \quad \text{in } \Omega. \tag{3.11}$$

Assuming $(\boldsymbol{E}^{n-1}, \boldsymbol{B}^{n-1})$ is known we derive from (3.7) that

$$\boldsymbol{B}^n = \boldsymbol{B}^{n-1} - k\,\mathbf{curl}\,\boldsymbol{E}^n, \tag{3.12}$$

and use this relation in (3.6) to obtain

$$k^{-1}\tilde{\epsilon}\,\boldsymbol{E}^n + \mathbf{curl}\left(k\tilde{\mu}^{-1}\mathbf{curl}\,\boldsymbol{E}^n\right) + \boldsymbol{J}^n = \tilde{\boldsymbol{l}}^n \quad \text{in } \Omega, \tag{3.13}$$

where

$$\tilde{\boldsymbol{l}}^n = \boldsymbol{l}^n + \mathbf{curl}\,(\tilde{\mu}^{-1}\boldsymbol{B}^{n-1}) + k^{-1}\tilde{\epsilon}\,\boldsymbol{E}^{n-1}. \tag{3.14}$$

In the derivation of the weak formulation of the unknown $\boldsymbol{E}^n$ we assume the problem defined by (3.13) and (3.8)–(3.9) has a smooth solution $\boldsymbol{E}^n$ so that all the steps in the derivation below are meaningful. We multiply both sides of (3.13) by an arbitrary test function $\boldsymbol{v} \in \boldsymbol{V} = \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ and integrate over Ω:

$$\int_\Omega \left[k^{-1}\tilde{\epsilon}\,\boldsymbol{E}^n\cdot\boldsymbol{v} + \mathbf{curl}\left(k\tilde{\mu}^{-1}\mathbf{curl}\,\boldsymbol{E}^n\right)\cdot\boldsymbol{v} + \boldsymbol{J}^n\cdot\boldsymbol{v}\right]\mathrm{d}x = \int_\Omega \tilde{\boldsymbol{l}}^n\cdot\boldsymbol{v}\,\mathrm{d}x.$$

Perform an integration by parts on the integral of the second term on the left-hand side and make use of the inequality

$$\int_\Omega \boldsymbol{J}^n\cdot\boldsymbol{v}\,\mathrm{d}x \le \int_\Omega \psi^0(\boldsymbol{E}^n;\boldsymbol{v})\,\mathrm{d}x,$$

thanks to (3.9). Then we have the following weak formulation:

$$\begin{aligned}\boldsymbol{E}^n \in \boldsymbol{V}, \quad &\int_\Omega \left(k^{-1}\tilde{\epsilon}\,\boldsymbol{E}^n\cdot\boldsymbol{v} + k\tilde{\mu}^{-1}\mathbf{curl}\,\boldsymbol{E}^n\cdot\mathbf{curl}\,\boldsymbol{v}\right)\mathrm{d}x + \int_\Omega \psi^0(\boldsymbol{E}^n;\boldsymbol{v})\,\mathrm{d}x \\ &\ge \int_\Omega \tilde{\boldsymbol{l}}^n\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}.\end{aligned} \tag{3.15}$$

Once the electric field $\boldsymbol{E}^n$ is found we can compute the magnetic induction $\boldsymbol{B}^n$ by (3.12).

To simplify the notation in the study of (3.15) we define

$$\epsilon = k^{-1}\tilde{\epsilon}, \quad \mu = k^{-1}\tilde{\mu}. \tag{3.16}$$

We comment that these definitions depend on the time step k. We further use the symbol $\boldsymbol{E}$ for $\boldsymbol{E}^n$ and define $\boldsymbol{f} \in \boldsymbol{V}^*$ by

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int_\Omega \tilde{\boldsymbol{l}}^n \cdot \boldsymbol{v} \, \mathrm{d}x, \quad \boldsymbol{v} \in \boldsymbol{V}.$$

Then we rewrite (3.15) as

$$\boldsymbol{E} \in \boldsymbol{V}, \quad a(\boldsymbol{E}, \boldsymbol{v}) + \int_\Omega \psi^0(\boldsymbol{E}; \boldsymbol{v}) \, \mathrm{d}x \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.17}$$

which is the HVI studied in this paper. Here, the bilinear form

$$a(\boldsymbol{E}, \boldsymbol{v}) = \int_\Omega \epsilon \boldsymbol{E} \cdot \boldsymbol{v} \, \mathrm{d}x + \int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}x.$$

From the assumptions on $\tilde{\epsilon}$ and $\tilde{\mu}$ we know that ϵ and μ are of class $L^\infty(\Omega)$ and

$$\epsilon_0 \leq \epsilon \leq \epsilon_1 \quad \text{and} \quad \mu_0 \leq \mu \leq \mu_1 \quad \text{a.e. in } \Omega,$$

for some positive constants $0 < \epsilon_0 \leq \epsilon_1 < \infty$ and $0 < \mu_0 \leq \mu_1 < \infty$. Indeed, due to (3.16), we may take

$$\epsilon_0 = k^{-1}\tilde{\epsilon}_0, \quad \epsilon_1 = k^{-1}\tilde{\epsilon}_1, \quad \mu_0 = k^{-1}\tilde{\mu}_0, \quad \mu_1 = k^{-1}\tilde{\mu}_1.$$

4. Existence and uniqueness

In this section we present a unique solvability result for the $\boldsymbol{H}(\mathbf{curl})$-elliptic HVI (3.17). We adopt the idea presented in Han (2020) and study the solution existence of (3.17) through consideration of an equivalent minimization problem. Define an energy functional

$$\mathcal{E}(\boldsymbol{v}) = \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) + \int_\Omega \psi(\boldsymbol{v}) \, \mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \quad \boldsymbol{v} \in \boldsymbol{V}. \tag{4.1}$$

We consider a corresponding minimization problem

$$\boldsymbol{E} \in \boldsymbol{V}, \quad \mathcal{E}(\boldsymbol{E}) = \inf \{\mathcal{E}(\boldsymbol{v}) : \boldsymbol{v} \in \boldsymbol{V}\}. \tag{4.2}$$

Let us explore properties of the energy functional.

LEMMA 4.1 Assume (3.3) and $m < \epsilon_0$. Then the functional $\mathcal{E}(\cdot)$ is locally Lipschitz continuous, strongly convex and coercive on $\boldsymbol{V}$.

Proof. The local Lipschitz continuity of $\mathcal{E}$ is obvious. Let us prove the strong convexity. For this purpose define a linear operator $A : \boldsymbol{V} \to \boldsymbol{V}^*$ by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = a(\boldsymbol{u}, \boldsymbol{v}) \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}. \tag{4.3}$$

Then $A \in \mathcal{L}(\boldsymbol{V}, \boldsymbol{V}^*)$ and it is strongly monotone. Define a functional $\Psi : \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ by

$$\Psi(\boldsymbol{v}) = \int_\Omega \psi(\boldsymbol{v})\,\mathrm{d}x \quad \forall\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega).$$

Then by Migórski *et al.* (2013, Theorem 3.47), under assumption (3.3), Ψ is well defined and locally Lipschitz continuous on $\boldsymbol{L}^2(\Omega)$, and

$$\partial\Psi(\boldsymbol{v}) \subset \int_\Omega \partial\psi(\boldsymbol{v})\,\mathrm{d}x \tag{4.4}$$

in the sense that for $\boldsymbol{\xi} \in \partial\Psi(\boldsymbol{v})$, there exists a function $\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Omega)$ such that $\boldsymbol{\zeta}(\boldsymbol{x}) \in \partial\psi(\boldsymbol{x}, \boldsymbol{v}(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Omega$ and

$$\langle \boldsymbol{\xi}, \boldsymbol{w}\rangle_{\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)} = \int_\Omega \boldsymbol{\zeta}(\boldsymbol{x}) \cdot \boldsymbol{w}(\boldsymbol{x})\,\mathrm{d}x \quad \forall\, \boldsymbol{w} \in \boldsymbol{L}^2(\Omega).$$

For $\boldsymbol{v} \in \boldsymbol{V}$ and $\boldsymbol{\eta} \in \partial\mathcal{E}(\boldsymbol{v})$, by (2.10) we can write

$$\boldsymbol{\eta} = A\boldsymbol{v} + \boldsymbol{\xi} - \boldsymbol{f}, \quad \boldsymbol{\xi} \in \partial\Psi(\boldsymbol{v}). \tag{4.5}$$

Thus, for $i = 1, 2$, with $\boldsymbol{v}_i \in \boldsymbol{V}$ and $\boldsymbol{\eta}_i \in \partial\mathcal{E}(\boldsymbol{v}_i)$, by (4.4) we have $\boldsymbol{\zeta}_i \in \boldsymbol{L}^2(\Omega)$ such that $\boldsymbol{\zeta}_i(\boldsymbol{x}) \in \partial\psi(\boldsymbol{x}, \boldsymbol{v}_i(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Omega$ and

$$\langle \boldsymbol{\eta}_i, \boldsymbol{w}\rangle = \langle A\boldsymbol{v}_i, \boldsymbol{w}\rangle + \int_\Omega \boldsymbol{\zeta}_i(\boldsymbol{x}) \cdot \boldsymbol{w}(\boldsymbol{x})\,\mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{w}\rangle \quad \forall\, \boldsymbol{w} \in \boldsymbol{L}^2(\Omega).$$

Thus from (3.5),

$$\begin{aligned}
\langle \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle &= \langle A\boldsymbol{v}_1 - A\boldsymbol{v}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle + \int_\Omega \left(\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\right) \cdot \left(\boldsymbol{v}_1 - \boldsymbol{v}_2\right)\mathrm{d}x \\
&\geq \int_\Omega \left[(\epsilon - m)\,|\boldsymbol{v}_1 - \boldsymbol{v}_2|^2 + \mu^{-1}|\mathbf{curl}(\boldsymbol{v}_1 - \boldsymbol{v}_2)|^2\right]\mathrm{d}x \\
&\geq \min\{\epsilon_0 - m, \mu_1^{-1}\}\,\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2_{\mathbf{curl},\Omega}.
\end{aligned}$$

Thus, by Lemma 2.2, $\mathcal{E}(\cdot)$ is strongly convex. Moreover, by Proposition 2.3, $\mathcal{E}(\cdot)$ is coercive on $\boldsymbol{V}$. □

An assumption of the form $m < \epsilon_0$ is called a smallness condition in the literature (e.g., Migórski *et al.*, 2013). This condition is equivalent to $km < \tilde{\epsilon}_0$, which is always satisfied if the step size k is sufficiently small. In this sense the assumption $m < \epsilon_0$ in Lemma 4.1 and in the other results of the paper does not impose a constraint of practical significance.

PROPOSITION 4.2 Assume (3.3) and $m < \epsilon_0$. Then the minimization problem (4.2) has a unique solution $\boldsymbol{E} \in \boldsymbol{V}$.

Proof. Since $\mathcal{E}(\cdot)$ is continuous, strictly convex and coercive on V, the minimization problem (4.2) has a unique solution $\boldsymbol{E} \in \boldsymbol{V}$ (cf. Atkinson & Han, 2009, §3.3.2). □

We are ready to prove an existence and uniqueness result for (3.17).

THEOREM 4.3 Assume (3.3) and $m < \epsilon_0$. Then for any $\boldsymbol{f} \in \boldsymbol{V}^*$, the HVI (3.17) has a unique solution $\boldsymbol{E} \in \boldsymbol{V}$, which is also the unique solution of the minimization problem (4.2).

Proof. Applying (4.5) we see that $\boldsymbol{E}$ satisfies the relation

$$\langle A\boldsymbol{E}, \boldsymbol{v}\rangle + \int_\Omega \boldsymbol{\zeta}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\,\mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{v}\rangle \geq 0$$

for a function $\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Omega)$ such that $\boldsymbol{\zeta}(\boldsymbol{x}) \in \partial\psi(\boldsymbol{x}, \boldsymbol{E}(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Omega$. Since

$$\psi^0(\boldsymbol{x}, \boldsymbol{E}(\boldsymbol{x}); \boldsymbol{v}(\boldsymbol{x})) \geq \boldsymbol{\zeta}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \quad \text{a.e. } \boldsymbol{x} \in \Omega,$$

we see that $\boldsymbol{E}$ is a solution of problem (3.17).

The uniqueness of a solution to problem (3.17) is proved by a standard approach. Assume $\tilde{\boldsymbol{E}} \in \boldsymbol{V}$ is another solution of problem (3.17). Then

$$a(\tilde{\boldsymbol{E}}, \boldsymbol{v}) + \int_\Omega \psi^0(\tilde{\boldsymbol{E}}; \boldsymbol{v})\,\mathrm{d}x \geq \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{4.6}$$

Take $\boldsymbol{v} = \tilde{\boldsymbol{E}} - \boldsymbol{E}$ in (3.17), $\boldsymbol{v} = \boldsymbol{E} - \tilde{\boldsymbol{E}}$ in (4.6) and add the two resulting inequalities to get

$$\begin{aligned}
\epsilon_0 \|\tilde{\boldsymbol{E}} - \boldsymbol{E}\|_{0,\Omega}^2 + \mu_1^{-1} \|\mathrm{curl}(\tilde{\boldsymbol{E}} - \boldsymbol{E})\|_{0,\Omega}^2 &\leq a(\tilde{\boldsymbol{E}} - \boldsymbol{E}, \tilde{\boldsymbol{E}} - \boldsymbol{E}) \\
&\leq \int_\Omega \left[\psi^0(\boldsymbol{E}; \tilde{\boldsymbol{E}} - \boldsymbol{E}) + \psi^0(\tilde{\boldsymbol{E}}; \boldsymbol{E} - \tilde{\boldsymbol{E}})\right] \mathrm{d}x \\
&\leq m\, \|\tilde{\boldsymbol{E}} - \boldsymbol{E}\|_{0,\Omega}^2.
\end{aligned}$$

By the smallness condition $m < \epsilon_0$ we deduce that $\tilde{\boldsymbol{E}} = \boldsymbol{E}$. □

5. Galerkin approximation

We consider the Galerkin approximation for problem (3.17). Let $\{\boldsymbol{V}_h\}_{h>0}$ be a family of finite-dimensional subspaces of $\boldsymbol{V}$ that approximates $\boldsymbol{V}$ in the sense that for any $\boldsymbol{v} \in \boldsymbol{V}$, there exists $\boldsymbol{v}_h \in \boldsymbol{V}_h$ such that

$$\|\boldsymbol{v}_h - \boldsymbol{v}\|_{\mathbf{curl},\Omega} \to 0 \quad \text{as } h \to 0.$$

The discretization parameter h can be identified with the mesh size if $\boldsymbol{V}_h$ is a finite element space. Then the Galerkin approximation of the $\boldsymbol{H}(\mathbf{curl})$-elliptic HVI problem (3.17) is

$$\boldsymbol{E}_h \in \boldsymbol{V}_h, \quad a(\boldsymbol{E}_h, \boldsymbol{v}_h) + \int_\Omega \psi^0(\boldsymbol{E}_h; \boldsymbol{v}_h)\,\mathrm{d}x \geq \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{5.1}$$

Similar to (4.2) we can introduce a corresponding minimization problem for (5.1):

$$\boldsymbol{E}_h \in \boldsymbol{V}_h, \quad \mathcal{E}(\boldsymbol{E}_h) = \inf\left\{\mathcal{E}(\boldsymbol{v}_h) : \boldsymbol{v}_h \in \boldsymbol{V}_h\right\}, \tag{5.2}$$

where the energy functional $\mathcal{E}$ is defined in (4.1).

The discrete analogue of Theorem 4.3 and Proposition 4.2 is the following.

THEOREM 5.1 Assume (3.3) and $m < \epsilon_0$. Then problem (5.1) is equivalent to problem (5.2), and both problems admit the same unique solution $\boldsymbol{E}_h \in \boldsymbol{V}_h$.

Let us show that the solution $\boldsymbol{E}_h$ is uniformly bounded in h.

LEMMA 5.2 Keep the assumptions of Theorem 5.1. Then the solution $\boldsymbol{E}_h$ of problem (5.1) is uniformly bounded independent of h.

Proof. We let $\boldsymbol{v}_h = -\boldsymbol{E}_h$ in (5.1) to get

$$a(\boldsymbol{E}_h, \boldsymbol{E}_h) \leq \int_\Omega \psi^0(\boldsymbol{E}_h; -\boldsymbol{E}_h)\,\mathrm{d}x + \langle \boldsymbol{f}, \boldsymbol{E}_h \rangle. \tag{5.3}$$

From (3.3)(d),

$$\psi^0(\boldsymbol{E}_h; -\boldsymbol{E}_h) \leq m\,|\boldsymbol{E}_h|^2 - \psi^0(\mathbf{0}; \boldsymbol{E}_h).$$

From (3.4),

$$-\,\psi^0(\mathbf{0}; \boldsymbol{E}_h) \leq c_0|\boldsymbol{E}_h|.$$

So

$$\int_\Omega \psi^0(\boldsymbol{E}_h; -\boldsymbol{E}_h)\,\mathrm{d}x \leq \int_\Omega \left(m\,|\boldsymbol{E}_h|^2 + c_0|\boldsymbol{E}_h|\right)\mathrm{d}x.$$

Thus from (5.3),

$$\int_\Omega \left[\epsilon\,|\boldsymbol{E}_h|^2 + \mu^{-1}|\mathbf{curl}\boldsymbol{E}_h|^2\right]\mathrm{d}x \leq \int_\Omega \left(m\,|\boldsymbol{E}_h|^2 + c_0|\boldsymbol{E}_h|\right)\mathrm{d}x + \|\boldsymbol{f}\|_{\boldsymbol{V}^*}\|\boldsymbol{E}_h\|_{\mathbf{curl},\Omega},$$

which is rewritten as

$$\int_\Omega \left[(\epsilon - m)\,|\boldsymbol{E}_h|^2 + \mu^{-1}|\mathbf{curl}\boldsymbol{E}_h|^2\right]\mathrm{d}x \leq \int_\Omega c_0|\boldsymbol{E}_h|\,\mathrm{d}x + \|\boldsymbol{f}\|_{\boldsymbol{V}^*}\|\boldsymbol{E}_h\|_{\mathbf{curl},\Omega}.$$

Hence,

$$\min\{\epsilon_0 - m, \mu_1^{-1}\} \, \|\boldsymbol{E}_h\|^2_{\mathbf{curl},\Omega} \le \left(c_0|\Omega|^{1/2} + \|\boldsymbol{f}\|_{\boldsymbol{V}^*}\right) \|\boldsymbol{E}_h\|_{\mathbf{curl},\Omega}$$

and then

$$\|\boldsymbol{E}_h\|_{\mathbf{curl},\Omega} \le \frac{c_0|\Omega|^{1/2} + \|\boldsymbol{f}\|_{\boldsymbol{V}^*}}{\min\{\epsilon_0 - m, \mu_1^{-1}\}},$$

i.e., $\|\boldsymbol{E}_h\|_{\mathbf{curl},\Omega}$ is uniformly bounded with respect to h. □

We now establish Céa's inequality for the numerical solution.

THEOREM 5.3 Assume (3.3), $m < \epsilon_0$ and $\boldsymbol{V}_h \subset \boldsymbol{V}$. Let $\boldsymbol{E} \in \boldsymbol{V}$ and $\boldsymbol{E}_h \in \boldsymbol{V}_h$ be the solutions of problems (3.17) and (5.1), respectively. Then Céa's inequality holds:

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl},\Omega} \le c \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \left(\|\boldsymbol{E} - \boldsymbol{v}_h\|_{\mathbf{curl},\Omega} + \|\boldsymbol{E} - \boldsymbol{v}_h\|^{1/2}_{0,\Omega}\right). \tag{5.4}$$

Proof. Let $\boldsymbol{v}_h \in \boldsymbol{V}_h$ be arbitrary. Write

$$a(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{E} - \boldsymbol{E}_h) = a(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{E} - \boldsymbol{v}_h) + a(\boldsymbol{E}, \boldsymbol{v}_h - \boldsymbol{E}_h) + a(\boldsymbol{E}_h, \boldsymbol{E}_h - \boldsymbol{v}_h).$$

From (3.17) with $\boldsymbol{v} = \boldsymbol{E}_h - \boldsymbol{v}_h$,

$$a(\boldsymbol{E}, \boldsymbol{v}_h - \boldsymbol{E}_h) \le \int_\Omega \psi^0(\boldsymbol{E}; \boldsymbol{E}_h - \boldsymbol{v}_h)\,\mathrm{d}x + \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{E}_h\rangle.$$

From (5.1) with $\boldsymbol{v}_h$ replaced by $\boldsymbol{v}_h - \boldsymbol{E}_h$,

$$a(\boldsymbol{E}_h, \boldsymbol{E}_h - \boldsymbol{v}_h) \le \int_\Omega \psi^0(\boldsymbol{E}_h; \boldsymbol{v}_h - \boldsymbol{E}_h)\,\mathrm{d}x + \langle \boldsymbol{f}, \boldsymbol{E}_h - \boldsymbol{v}_h\rangle.$$

Hence,

$$a(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{E} - \boldsymbol{E}_h) \le I_1 + I_2, \tag{5.5}$$

where

$$I_1 = a(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{E} - \boldsymbol{v}_h), \tag{5.6}$$

$$I_2 = \int_\Omega \left[\psi^0(\boldsymbol{E}; \boldsymbol{E}_h - \boldsymbol{v}_h) + \psi^0(\boldsymbol{E}_h; \boldsymbol{v}_h - \boldsymbol{E}_h)\right]\mathrm{d}x. \tag{5.7}$$

Note that

$$a(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{E} - \boldsymbol{v}_h) \le a(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{E} - \boldsymbol{E}_h)^{1/2} a(\boldsymbol{E} - \boldsymbol{v}_h, \boldsymbol{E} - \boldsymbol{v}_h)^{1/2}.$$

By the modified Cauchy inequality with an arbitrarily small $\varepsilon > 0$,

$$a\,b \le \varepsilon a^2 + C\,b^2, \quad C = 1/(4\,\varepsilon) \quad \forall\, a, b \in \mathbb{R},$$

we have

$$I_1 \le \varepsilon\, a(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{E} - \boldsymbol{E}_h) + C\, a(\boldsymbol{E} - \boldsymbol{v}_h, \boldsymbol{E} - \boldsymbol{v}_h). \tag{5.8}$$

By the subadditivity (2.9) for the generalized directional derivative,

$$\psi^0(\boldsymbol{E}; \boldsymbol{E}_h - \boldsymbol{v}_h) \le \psi^0(\boldsymbol{E}; \boldsymbol{E}_h - \boldsymbol{E}) + \psi^0(\boldsymbol{E}; \boldsymbol{E} - \boldsymbol{v}_h),$$
$$\psi^0(\boldsymbol{E}_h; \boldsymbol{v}_h - \boldsymbol{E}_h) \le \psi^0(\boldsymbol{E}_h; \boldsymbol{E} - \boldsymbol{E}_h) + \psi^0(\boldsymbol{E}_h; \boldsymbol{v}_h - \boldsymbol{E}).$$

By (3.3)(d),

$$\psi^0(\boldsymbol{E}; \boldsymbol{E}_h - \boldsymbol{E}) + \psi^0(\boldsymbol{E}_h; \boldsymbol{E} - \boldsymbol{E}_h) \le m\,|\boldsymbol{E} - \boldsymbol{E}_h|^2.$$

By (3.4),

$$\psi^0(\boldsymbol{E}; \boldsymbol{E} - \boldsymbol{v}_h) \le \left(c_0 + c_1|\boldsymbol{E}|\right)|\boldsymbol{E} - \boldsymbol{v}_h|,$$
$$\psi^0(\boldsymbol{E}_h; \boldsymbol{v}_h - \boldsymbol{E}) \le \left(c_0 + c_1|\boldsymbol{E}_h|\right)|\boldsymbol{E} - \boldsymbol{v}_h|.$$

Thus,

$$I_2 \le \int_\Omega m\,|\boldsymbol{E} - \boldsymbol{E}_h|^2\,\mathrm{d}x + \int_\Omega \left(2c_0 + c_1|\boldsymbol{E}| + c_1|\boldsymbol{E}_h|\right)|\boldsymbol{E} - \boldsymbol{v}_h|\,\mathrm{d}x. \tag{5.9}$$

Combine (5.5), (5.8) and (5.9) to get

$$\int_\Omega \left[\left(\epsilon_0 - m - \varepsilon\right)|\boldsymbol{E} - \boldsymbol{E}_h|^2 + \left(\mu_1^{-1} - \varepsilon\right)|\mathbf{curl}(\boldsymbol{E} - \boldsymbol{E}_h)|^2\right]\mathrm{d}x$$
$$\le c\,\|\boldsymbol{E} - \boldsymbol{v}_h\|^2_{\mathbf{curl},\Omega} + \int_\Omega \left(2c_0 + c_1|\boldsymbol{E}| + c_1|\boldsymbol{E}_h|\right)|\boldsymbol{E} - \boldsymbol{v}_h|\,\mathrm{d}x. \tag{5.10}$$

By the Cauchy–Schwarz inequality and the uniform boundedness of $\|\boldsymbol{E}_h\|_{0,\Omega}$,

$$\int_\Omega \left(2c_0 + c_1|\boldsymbol{E}| + c_1|\boldsymbol{E}_h|\right)|\boldsymbol{E} - \boldsymbol{v}_h|\,\mathrm{d}x \le c\left(1 + \|\boldsymbol{E}\|_{0,\Omega} + \|\boldsymbol{E}_h\|_{0,\Omega}\right)\|\boldsymbol{E} - \boldsymbol{v}_h\|_{0,\Omega}$$
$$\le c\,\|\boldsymbol{E} - \boldsymbol{v}_h\|_{0,\Omega}.$$

By taking $\varepsilon > 0$ sufficiently small, e.g., $\varepsilon = \min\{\epsilon_0 - m, \mu_1^{-1}\}/2$, we derive from (5.10) that

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|^2_{\mathbf{curl},\Omega} \le c\left(\|\boldsymbol{E} - \boldsymbol{v}_h\|^2_{\mathbf{curl},\Omega} + \|\boldsymbol{E} - \boldsymbol{v}_h\|_{0,\Omega}\right).$$

Since $\boldsymbol{v}_h \in \boldsymbol{V}_h$ is arbitrary we obtain Céa's inequality (5.4). □

REMARK 5.4 The Galerkin method (5.1) provides an approximation of problem (3.17), which originates from (3.15). In terms of the data in the original form (3.15) of the problem, Céa's inequality (5.4) reads

$$\begin{aligned}&\|\boldsymbol{E}-\boldsymbol{E}_h\|_{0,\Omega}+k\,\|\mathbf{curl}(\boldsymbol{E}-\boldsymbol{E}_h)\|_{0,\Omega}\\&\quad\le c\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\Big[\|\boldsymbol{E}-\boldsymbol{v}_h\|_{0,\Omega}+k^{1/2}\|\boldsymbol{E}-\boldsymbol{v}_h\|_{0,\Omega}^{1/2}+k\,\|\mathbf{curl}(\boldsymbol{E}-\boldsymbol{v}_h)\|_{0,\Omega}\Big],\end{aligned}$$

where the constant c does not depend on the time-step size k.

Céa's inequality (5.4) is the starting point for convergence analysis and error estimation.

COROLLARY 5.5 Assume (3.3), $m<\epsilon_0$, $\boldsymbol{V}_h\subset\boldsymbol{V}$ and $\{\boldsymbol{V}_h\}_{h>0}$ approximates $\boldsymbol{V}$. Let $\boldsymbol{E}_h\in\boldsymbol{V}_h$ be the numerical solution defined by (5.1). Then we have convergence:

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\mathbf{curl},\Omega}\to 0\quad\text{as } h\to 0. \tag{5.11}$$

Proof. Since $\{\boldsymbol{V}_h\}_{h>0}$ approximates $\boldsymbol{V}$ we can find a sequence $\{\tilde{\boldsymbol{v}}_h\}_{h>0}$ such that

$$\|\boldsymbol{E}-\tilde{\boldsymbol{v}}_h\|_{\mathbf{curl},\Omega}\to 0\quad\text{as } h\to 0.$$

By Céa's inequality (5.4),

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\mathbf{curl},\Omega}\le c\left(\|\boldsymbol{E}-\tilde{\boldsymbol{v}}_h\|_{\mathbf{curl},\Omega}+\|\boldsymbol{E}-\tilde{\boldsymbol{v}}_h\|_{0,\Omega}^{1/2}\right).$$

Hence, the convergence (5.11) is valid. □

6. Linear edge finite element approximation

In this section we consider the particular case where $\boldsymbol{V}_h$ is constructed from the linear edge finite element. For simplicity we assume Ω is a bounded polyhedral domain. Let $\{\mathcal{T}_h\}_h$ be a shape-regular family of decompositions of $\overline{\Omega}$ into tetrahedral elements. For a generic element $K\in\mathcal{T}_h$ denote $h_K=\operatorname{diam}(K)$ and $h=\max\{h_K : K\in\mathcal{T}_h\}$. Since the solution of the HVI (3.17) is not expected to have high regularity, we use low-degree elements. Let P_1 be the space of polynomials of degree ≤ 1, and use $\boldsymbol{P}_1=(P_1)^3$ for the corresponding vector-valued polynomial space. We choose the space of lowest-order edge elements of Nédélec's second family (Nédélec, 1986):

$$\boldsymbol{V}_h=\left\{\boldsymbol{v}_h\in\boldsymbol{V} : \boldsymbol{v}_h|_K\in\boldsymbol{P}_1(K)\;\forall K\in\mathcal{T}_h\right\}.$$

In the error estimation we will need the quasi-interpolation operator $\mathcal{J}_{h0}^c : \boldsymbol{L}^1(\Omega)\to\boldsymbol{V}_h$ introduced in Ern & Guermond (2016, Definition 6.4), where the reader can find the details. Using Ern & Guermond (2016, Theorem 6.5) together with Ern & Guermond (2017, Corollary 5.3) we have the following approximation result.

LEMMA 6.1 Let $\{\mathcal{T}_h\}_h$ be a shape-regular family of decompositions of $\overline{\Omega}$ into tetrahedral elements. Let $\boldsymbol{v} \in \boldsymbol{V} \cap \boldsymbol{H}^s(\Omega)$, with $s \in [0, 2]$. Then there exists a constant $c > 0$ such that

$$\|\boldsymbol{v} - \mathcal{J}^c_{h0}\boldsymbol{v}\|_{0,\Omega} \le ch^s|\boldsymbol{v}|_{s,\Omega}. \tag{6.1}$$

Moreover, let $\boldsymbol{v} \in \boldsymbol{V} \cap \boldsymbol{H}^s(\Omega)$ and $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^s(\Omega)$, with $s \in (0, 2]$. Then there exists a constant $c > 0$ such that

$$\|\boldsymbol{v} - \mathcal{J}^c_{h0}\boldsymbol{v}\|_{\mathbf{curl},\Omega} \le ch^s(|\boldsymbol{v}|_{s,\Omega} + |\mathbf{curl}\,\boldsymbol{v}|_{s,\Omega}). \tag{6.2}$$

As an application of Corollary 5.5 we have convergence of the linear edge finite element solutions.

THEOREM 6.2 Assume (3.3) and $m < \epsilon_0$. Let $\boldsymbol{E}_h \in \boldsymbol{V}_h$ be the linear edge finite element solution defined by (5.1). Then we have convergence:

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl},\Omega} \to 0 \quad \text{as } h \to 0. \tag{6.3}$$

Proof. Let $\varepsilon > 0$ be an arbitrary but fixed number. Since $\boldsymbol{V}$ is the closure of $\boldsymbol{C}_0^\infty(\Omega)$ with respect to the $\boldsymbol{H}(\mathbf{curl}, \Omega)$ norm there exists $\tilde{\boldsymbol{E}} \in \boldsymbol{C}_0^\infty(\Omega)$ such that

$$\|\boldsymbol{E} - \tilde{\boldsymbol{E}}\|_{\mathbf{curl},\Omega} \le \frac{\varepsilon}{2}.$$

Applying Lemma 6.1 we know that there exists $h_\varepsilon > 0$ such that for $h \le h_\varepsilon$,

$$\|\tilde{\boldsymbol{E}} - \mathcal{J}^c_{h0}\tilde{\boldsymbol{E}}\|_{\mathbf{curl},\Omega} \le \frac{\varepsilon}{2}.$$

Hence, for $h \le h_\varepsilon$,

$$\|\boldsymbol{E} - \mathcal{J}^c_{h0}\tilde{\boldsymbol{E}}\|_{\mathbf{curl},\Omega} \le \|\boldsymbol{E} - \tilde{\boldsymbol{E}}\|_{\mathbf{curl},\Omega} + \|\tilde{\boldsymbol{E}} - \mathcal{J}^c_{h0}\tilde{\boldsymbol{E}}\|_{\mathbf{curl},\Omega} \le \varepsilon.$$

By Céa's inequality (5.4),

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl},\Omega} \le c\left(\|\boldsymbol{E} - \mathcal{J}^c_{h0}\tilde{\boldsymbol{E}}\|_{\mathbf{curl},\Omega} + \|\boldsymbol{E} - \mathcal{J}^c_{h0}\tilde{\boldsymbol{E}}\|^{1/2}_{0,\Omega}\right) \le c\left(\varepsilon + \varepsilon^{1/2}\right).$$

Since $\varepsilon > 0$ is arbitrary we conclude the convergence (6.3). □

THEOREM 6.3 Assume (3.3) and $m < \epsilon_0$. Let $\boldsymbol{E}_h \in \boldsymbol{V}_h$ be the linear edge finite element solution defined by (5.1). Assume $\boldsymbol{E} \in \boldsymbol{H}^s(\Omega)$ and $\mathbf{curl}\,\boldsymbol{E} \in \boldsymbol{H}^s(\Omega)$, with $s \in (0, 1]$. Then

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl},\Omega} \le c\,h^{s/2}\left(|\boldsymbol{E}|_{s,\Omega} + |\mathbf{curl}\,\boldsymbol{E}|_{s,\Omega} + |\boldsymbol{E}|^{1/2}_{s,\Omega}\right). \tag{6.4}$$

Moreover, assume $\boldsymbol{E} \in \boldsymbol{H}^2(\Omega)$; we have an optimal-order error estimate

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl},\Omega} \le c\,h\left(|\boldsymbol{E}|_{1,\Omega} + |\mathbf{curl}\,\boldsymbol{E}|_{1,\Omega} + |\boldsymbol{E}|^{1/2}_{2,\Omega}\right). \tag{6.5}$$

Proof. By Céa's inequality (5.4),

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\mathbf{curl},\Omega} \le c\left(\|\boldsymbol{E}-\mathcal{J}_{h0}^c\boldsymbol{E}\|_{\mathbf{curl},\Omega} + \|\boldsymbol{E}-\mathcal{J}_{h0}^c\boldsymbol{E}\|_{0,\Omega}^{1/2}\right).$$

Then the error bounds (6.4) and (6.5) follow from an application of the interpolation error bounds (6.1)–(6.2). □

7. Fully discrete scheme

In this section we consider a fully discrete scheme of the problem defined by (3.1a)–(3.1e) and (3.2) through discretizing the semidiscrete problem (3.15) with a mixed Galerkin approximation in space based on Nédélec's edge elements for the electric field $\boldsymbol{E}$ and piecewise constant elements for the magnetic induction $\boldsymbol{B}$. We assume the continuous initial data $(\boldsymbol{E}_0, \boldsymbol{B}_0)$ satisfies the following compatibility system:

$$\begin{aligned}&\int_\Omega \left(\tilde{\epsilon}\boldsymbol{E}_0\cdot\boldsymbol{v} + \tilde{\mu}^{-1}\boldsymbol{B}_0\cdot\boldsymbol{w}\right)\mathrm{d}x + \int_\Omega\left(\tilde{\mu}^{-1}\mathbf{curl}\,\boldsymbol{E}_0\cdot\boldsymbol{w} - \tilde{\mu}^{-1}\boldsymbol{B}_0\cdot\mathbf{curl}\,\boldsymbol{v}\right)\mathrm{d}x\\ &\quad + \int_\Omega \psi^0(\boldsymbol{E}_0;\boldsymbol{v})\,\mathrm{d}x \ge \int_\Omega \boldsymbol{l}(0)\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall\,(\boldsymbol{v},\boldsymbol{w})\in\boldsymbol{V}\times\boldsymbol{L}^2(\Omega).\end{aligned} \tag{7.1}$$

In addition to the partition of the time interval $[0,T]$ in Section 3 and the finite element space $\boldsymbol{V}_h$ introduced in Section 6 we let

$$\boldsymbol{W}_h = \left\{\boldsymbol{w}_h\in\boldsymbol{L}^2(\Omega) : \boldsymbol{w}_h|_K\in\boldsymbol{P}_0(K)\ \forall\,K\in\mathcal{T}_h\right\}.$$

Note that $\mathbf{curl}\,\boldsymbol{V}_h\subset\boldsymbol{W}_h$.

Then the fully discrete scheme is to find $\{(\boldsymbol{E}_h^n,\boldsymbol{B}_h^n)\}_{n=1}^N\subset\boldsymbol{V}_h\times\boldsymbol{W}_h$ such that

$$(\boldsymbol{E}_h^0,\boldsymbol{B}_h^0) = (\boldsymbol{E}_{0h},\boldsymbol{B}_{0h}) \tag{7.2}$$

and for $1\le n\le N$,

$$\begin{aligned}&\int_\Omega\left(\tilde{\epsilon}\frac{\boldsymbol{E}_h^n-\boldsymbol{E}_h^{n-1}}{k}\cdot\boldsymbol{v}_h + \tilde{\mu}^{-1}\frac{\boldsymbol{B}_h^n-\boldsymbol{B}_h^{n-1}}{k}\cdot\boldsymbol{w}_h\right)\mathrm{d}x\\ &\quad + \int_\Omega\left(\tilde{\mu}^{-1}\mathbf{curl}\,\boldsymbol{E}_h^n\cdot\boldsymbol{w}_h - \tilde{\mu}^{-1}\boldsymbol{B}_h^n\cdot\mathbf{curl}\,\boldsymbol{v}_h\right)\mathrm{d}x\\ &\quad + \int_\Omega\psi^0(\boldsymbol{E}_h^n;\boldsymbol{v}_h)\,\mathrm{d}x \ge \int_\Omega \boldsymbol{l}^n\cdot\boldsymbol{v}_h\,\mathrm{d}x \quad \forall\,(\boldsymbol{v}_h,\boldsymbol{w}_h)\in\boldsymbol{V}_h\times\boldsymbol{W}_h,\end{aligned} \tag{7.3}$$

where $(\boldsymbol{E}_{0h}, \boldsymbol{B}_{0h}) \in \boldsymbol{V}_h \times \boldsymbol{W}_h$ is the finite element approximation of $(\boldsymbol{E}_0, \boldsymbol{B}_0)$, which satisfies

$$\int_\Omega \left(\tilde{\epsilon}\boldsymbol{E}_{0h} \cdot \boldsymbol{v}_h + \tilde{\mu}^{-1}\boldsymbol{B}_{0h} \cdot \boldsymbol{w}_h\right) \mathrm{d}x + \int_\Omega \left(\tilde{\mu}^{-1}\mathbf{curl}\, \boldsymbol{E}_{0h} \cdot \boldsymbol{w}_h - \tilde{\mu}^{-1}\boldsymbol{B}_{0h} \cdot \mathbf{curl}\, \boldsymbol{v}_h\right) \mathrm{d}x$$
$$+ \int_\Omega \psi^0(\boldsymbol{E}_{0h}; \boldsymbol{v}_h)\, \mathrm{d}x \geq \int_\Omega \boldsymbol{l}(0) \cdot \boldsymbol{v}_h\, \mathrm{d}x \quad \forall\, (\boldsymbol{v}_h, \boldsymbol{w}_h) \in \boldsymbol{V}_h \times \boldsymbol{W}_h. \tag{7.4}$$

Let $\boldsymbol{v}_h = \mathbf{0}$ in (7.4) to get

$$\int_\Omega \tilde{\mu}^{-1}(\boldsymbol{B}_{0h} + \mathbf{curl}\, \boldsymbol{E}_{0h}) \cdot \boldsymbol{w}_h\, \mathrm{d}x = 0 \quad \forall\, \boldsymbol{w}_h \in \boldsymbol{W}_h.$$

Since $\mathbf{curl}\, \boldsymbol{V}_h \subset \boldsymbol{W}_h$ we have

$$\boldsymbol{B}_{0h} = -\mathbf{curl}\, \boldsymbol{E}_{0h}. \tag{7.5}$$

Thus, $\boldsymbol{B}_{0h} \in \mathbf{curl}\, \boldsymbol{V}_h$ holds for all $h > 0$. Next we choose $\boldsymbol{w}_h = \mathbf{0}$ in (7.4) to get

$$\int_\Omega \left(\tilde{\epsilon}\boldsymbol{E}_{0h} \cdot \boldsymbol{v}_h + \tilde{\mu}^{-1}\mathbf{curl}\, \boldsymbol{E}_{0h} \cdot \mathbf{curl}\, \boldsymbol{v}_h\right) \mathrm{d}x + \int_\Omega \psi^0(\boldsymbol{E}_{0h}; \boldsymbol{v}_h)\, \mathrm{d}x$$
$$\geq \int_\Omega \boldsymbol{l}(0) \cdot \boldsymbol{v}_h\, \mathrm{d}x \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{7.6}$$

Similar to (7.5), by taking $\boldsymbol{v}_h = \mathbf{0}$ we can derive from (7.3) that

$$\boldsymbol{B}_h^n = \boldsymbol{B}_h^{n-1} - k\, \mathbf{curl}\, \boldsymbol{E}_h^n. \tag{7.7}$$

Thanks to (7.5) and (7.7), $\boldsymbol{B}_h^n \in \mathbf{curl}\, \boldsymbol{V}_h$ holds for all $h > 0$ and $1 \leq n \leq N$. Using (7.7) and taking $\boldsymbol{w}_h = \mathbf{0}$ in (7.3), we obtain

$$a(\boldsymbol{E}_h^n, \boldsymbol{v}_h) + \int_\Omega \psi^0(\boldsymbol{E}_h^n; \boldsymbol{v}_h)\, \mathrm{d}x \geq \langle \boldsymbol{f}^n, \boldsymbol{v}_h\rangle \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{7.8}$$

where

$$\langle \boldsymbol{f}^n, \boldsymbol{v}_h\rangle = \int_\Omega \left(\boldsymbol{l}^n \cdot \boldsymbol{v}_h + k^{-1}\tilde{\epsilon}\, \boldsymbol{E}_h^{n-1} \cdot \boldsymbol{v}_h + \tilde{\mu}^{-1}\boldsymbol{B}_h^{n-1} \cdot \mathbf{curl}\, \boldsymbol{v}_h\right) dx.$$

Under the assumption (3.3) and the smallness condition $m < \tilde{\epsilon}_0$, problem (7.6) has a unique solution $\boldsymbol{E}_{0h} \in \boldsymbol{V}_h$. In view of (7.5), problem (7.4) has a unique solution $(\boldsymbol{E}_{0h}, \boldsymbol{B}_{0h}) \in \boldsymbol{V}_h \times \mathbf{curl}\boldsymbol{V}_h$. We then apply Theorem 6.2 and deduce that $(\boldsymbol{E}_{0h}, \boldsymbol{B}_{0h}) \in \boldsymbol{V}_h \times \mathbf{curl}\boldsymbol{V}_h$ converges strongly to $(\boldsymbol{E}_0, \boldsymbol{B}_0)$ as $h \to 0$. Moreover, the condition $k\, m < \tilde{\epsilon}_0$ is always satisfied if the step size k is sufficiently small; then problem (7.8) has a unique solution $\boldsymbol{E}_h^n \in \boldsymbol{V}_h$ (cf. the proof of Theorem 4.3). Inserting the solution $\boldsymbol{E}_h^n$ of problem (7.8) into (7.7), we finally obtain a unique solution $\{(\boldsymbol{E}_h^n, \boldsymbol{B}_h^n)\}_{n=1}^N \subset \boldsymbol{V}_h \times \mathbf{curl}\, \boldsymbol{V}_h$ of problem (7.3).

REMARK 7.1 The idea of using the compatibility condition (7.1) was employed in Winckler & Yousept (2019) in the case of a hyperbolic Maxwell variational inequality. However, the compatibility condition leads to an undesired restriction since the given initial data $(\boldsymbol{E}_0, \boldsymbol{B}_0)$ may not necessarily satisfy (7.1). The issue with the restrictive compatibility assumption on the initial data was circumvented in the recent work Yousept (2021) for a hyperbolic Maxwell quasi-variational inequality through the introduction of an auxiliary initial current density and appropriate initial difference quotients (not enforced to be the initial discrete values). This new idea of Yousept (2021) may be adapted and extended to the hemivariational inequalities considered in this paper, a topic for future work.

In stability estimates for the fully discrete solution to the inequality problem (7.3) we will need a discrete Gronwall inequality (Han & Sofonea, 2002).

LEMMA 7.2 For a fixed T and a positive integer N let $k = T/N$. Assume $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of non-negative numbers satisfying

$$e_n \le c g_n + ck \sum_{i=1}^{n-1} e_i, \quad n = 1, \ldots, N,$$

for a constant $c > 0$, independent of T and k. Then, for a possibly different constant $c > 0$, independent of T and k,

$$\max_{1 \le n \le N} e_n \le c \max_{1 \le n \le N} g_n.$$

THEOREM 7.3 Assume $m < \tilde{\epsilon}_0$ and $\boldsymbol{l} \in H^1(0,T;\boldsymbol{L}^2(\Omega))$. Then there is a constant $C > 0$ such that

$$\begin{aligned} &\max_{1\le n\le N} \|\delta \boldsymbol{E}_h^n\|_{0,\Omega}^2 + \max_{1\le n\le N} \|\delta \boldsymbol{B}_h^n\|_{0,\Omega}^2 \\ &\quad + \sum_{n=1}^N \|\delta \boldsymbol{E}_h^n - \delta \boldsymbol{E}_h^{n-1}\|_{0,\Omega}^2 + \sum_{n=1}^N \|\delta \boldsymbol{B}_h^n - \delta \boldsymbol{B}_h^{n-1}\|_{0,\Omega}^2 \le C \end{aligned} \tag{7.9}$$

and

$$\max_{1\le n\le N} \|\mathbf{curl}\, \boldsymbol{E}_h^n\|_{0,\Omega}^2 \le C, \tag{7.10}$$

where $\delta \boldsymbol{E}_h^0 = \boldsymbol{E}_{0h}$, $\delta \boldsymbol{B}_h^0 = \boldsymbol{B}_{0h}$ and, for $n = 1, \ldots, N$,

$$\delta \boldsymbol{E}_h^n = \frac{\boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1}}{k} \quad \text{and} \quad \delta \boldsymbol{B}_h^n = \frac{\boldsymbol{B}_h^n - \boldsymbol{B}_h^{n-1}}{k}.$$

Proof. We comment that with the definitions $\delta \boldsymbol{E}_h^0 = \boldsymbol{E}_{0h}$ and $\delta \boldsymbol{B}_h^0 = \boldsymbol{B}_{0h}$, we can extend (7.3) to include the case $n = 0$ since this is exactly (7.4). With an argument similar to that in Lemma 5.2 we can derive that $\boldsymbol{E}_{0h}$ and $\boldsymbol{B}_{0h}$ are uniformly bounded in h.

Take $(\boldsymbol{v}_h, \boldsymbol{w}_h) = (\boldsymbol{E}_h^{n-1} - \boldsymbol{E}_h^n, \boldsymbol{B}_h^{n-1} - \boldsymbol{B}_h^n)$ in (7.3) with index n, $(\boldsymbol{v}_h, \boldsymbol{w}_h) = (\boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1}, \boldsymbol{B}_h^n - \boldsymbol{B}_h^{n-1})$ in (7.3) with index $(n-1)$ and add the two resulting inequalities to get

$$\begin{aligned}
\int_\Omega \left[\tilde{\epsilon}(\delta\boldsymbol{E}_h^n - \delta\boldsymbol{E}_h^{n-1}) \cdot \delta\boldsymbol{E}_h^n + \tilde{\mu}^{-1}(\delta\boldsymbol{B}_h^n - \delta\boldsymbol{B}_h^{n-1}) \cdot \delta\boldsymbol{B}_h^n\right] \mathrm{d}x \\
\le \frac{1}{k}\int_\Omega \left[\psi^0(\boldsymbol{E}_h^n; \boldsymbol{E}_h^{n-1} - \boldsymbol{E}_h^n) + \psi^0(\boldsymbol{E}_h^{n-1}; \boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1})\right] \mathrm{d}x \\
+ \int_\Omega (\boldsymbol{l}^n - \boldsymbol{l}^{n-1}) \cdot \delta\boldsymbol{E}_h^n \, \mathrm{d}x. \qquad (7.11)
\end{aligned}$$

From (3.3)(d),

$$\begin{aligned}
\frac{1}{k}\int_\Omega \left[\psi^0(\boldsymbol{E}_h^n; \boldsymbol{E}_h^{n-1} - \boldsymbol{E}_h^n) + \psi^0(\boldsymbol{E}_h^{n-1}; \boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1})\right] \mathrm{d}x &\le \frac{1}{k}\int_\Omega m|\boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1}|^2 \, \mathrm{d}x \\
&= k\, m\, \|\delta\boldsymbol{E}_h^n\|_{0,\Omega}^2.
\end{aligned}$$

In addition, for any small $\varepsilon > 0$ to be determined later, there is a constant C inversely proportional to ε such that

$$\int_\Omega (\boldsymbol{l}^n - \boldsymbol{l}^{n-1}) \cdot \delta\boldsymbol{E}_h^n \, \mathrm{d}x \le k\,\varepsilon\, \|\delta\boldsymbol{E}_h^n\|_{0,\Omega}^2 + C\,k^{-1}\|\boldsymbol{l}^n - \boldsymbol{l}^{n-1}\|_{0,\Omega}^2.$$

Notice that

$$(\delta\boldsymbol{E}_h^n - \delta\boldsymbol{E}_h^{n-1}) \cdot \delta\boldsymbol{E}_h^n = \frac{1}{2}\left[|\delta\boldsymbol{E}_h^n|^2 - |\delta\boldsymbol{E}_h^{n-1}|^2 + |\delta\boldsymbol{E}_h^n - \delta\boldsymbol{E}_h^{n-1}|^2\right].$$

Then from (7.11) we can derive

$$\begin{aligned}
\int_\Omega \tilde{\epsilon}\left[|\delta\boldsymbol{E}_h^n|^2 - |\delta\boldsymbol{E}_h^{n-1}|^2 + |\delta\boldsymbol{E}_h^n - \delta\boldsymbol{E}_h^{n-1}|^2\right] \mathrm{d}x \\
+ \int_\Omega \tilde{\mu}^{-1}\left[|\delta\boldsymbol{B}_h^n|^2 - |\delta\boldsymbol{B}_h^{n-1}|^2 + |\delta\boldsymbol{B}_h^n - \delta\boldsymbol{B}_h^{n-1}|^2\right] \mathrm{d}x \\
\le 2(m+\varepsilon)k\|\delta\boldsymbol{E}_h^n\|_{0,\Omega}^2 + C\,k^{-1}\|\boldsymbol{l}^n - \boldsymbol{l}^{n-1}\|_{0,\Omega}^2. \qquad (7.12)
\end{aligned}$$

We replace n with i in (7.12) and sum over i from 1 to n:

$$\begin{aligned}
\int_\Omega \tilde{\epsilon}\left[|\delta\boldsymbol{E}_h^n|^2 - |\boldsymbol{E}_{0h}|^2 + \sum_{i=1}^n |\delta\boldsymbol{E}_h^i - \delta\boldsymbol{E}_h^{i-1}|^2\right] \mathrm{d}x \\
+ \int_\Omega \tilde{\mu}^{-1}\left[|\delta\boldsymbol{B}_h^n|^2 - |\boldsymbol{B}_{0h}|^2 + \sum_{i=1}^n |\delta\boldsymbol{B}_h^i - \delta\boldsymbol{B}_h^{i-1}|^2\right] \mathrm{d}x \\
\le 2(m+\varepsilon)k\sum_{i=1}^n \|\delta\boldsymbol{E}_h^i\|_{0,\Omega}^2 + C\,k^{-1}\sum_{i=1}^n \|\boldsymbol{l}^i - \boldsymbol{l}^{i-1}\|_{0,\Omega}^2, \qquad (7.13)
\end{aligned}$$

where we have used $(\delta\boldsymbol{E}_h^0, \delta\boldsymbol{B}_h^0) = (\boldsymbol{E}_{0h}, \boldsymbol{B}_{0h})$. Then we rewrite (7.13) as

$$\tilde{\epsilon}_0\|\delta\boldsymbol{E}_h^n\|_{0,\Omega}^2 + \tilde{\mu}_1^{-1}\|\delta\boldsymbol{B}_h^n\|_{0,\Omega}^2 + \tilde{\epsilon}_0\sum_{i=1}^n\|\delta\boldsymbol{E}_h^i - \delta\boldsymbol{E}_h^{i-1}\|_{0,\Omega}^2 + \tilde{\mu}_1^{-1}\sum_{i=1}^n\|\delta\boldsymbol{B}_h^i - \delta\boldsymbol{B}_h^{i-1}\|_{0,\Omega}^2$$
$$\le 2(m+\varepsilon)k\sum_{i=1}^n\|\delta\boldsymbol{E}_h^i\|_{0,\Omega}^2 + C\,k^{-1}\sum_{i=1}^n\|\boldsymbol{l}^i - \boldsymbol{l}^{i-1}\|_{0,\Omega}^2 + \tilde{\epsilon}_1\|\boldsymbol{E}_{0h}\|_{0,\Omega}^2 + \tilde{\mu}_0^{-1}\|\boldsymbol{B}_{0h}\|_{0,\Omega}^2. \quad (7.14)$$

From

$$\boldsymbol{l}^i - \boldsymbol{l}^{i-1} = \int_{t_{i-1}}^{t_i}\dot{\boldsymbol{l}}(t)\,\mathrm{d}t,$$

where $\dot{\boldsymbol{l}}$ denotes the time derivative of $\boldsymbol{l}$, we find

$$\|\boldsymbol{l}^i - \boldsymbol{l}^{i-1}\|_{0,\Omega}^2 \le k\int_{t_{i-1}}^{t_i}\|\dot{\boldsymbol{l}}(t)\|_{0,\Omega}^2\,\mathrm{d}t.$$

Hence,

$$k^{-1}\sum_{i=1}^n\|\boldsymbol{l}^i - \boldsymbol{l}^{i-1}\|_{0,\Omega}^2 \le \|\boldsymbol{l}\|_{H^1(0,T;\boldsymbol{L}^2(\Omega))}^2.$$

Together with the uniform boundedness of $\|\boldsymbol{E}_{0h}\|_{0,\Omega}$ and $\|\boldsymbol{B}_{0h}\|_{0,\Omega}$ we derive from (7.14) that

$$\|\delta\boldsymbol{E}_h^n\|_{0,\Omega}^2 \le C + 2\tilde{\epsilon}_0^{-1}(m+\varepsilon)k\sum_{i=1}^n\|\delta\boldsymbol{E}_h^i\|_{0,\Omega}^2. \quad (7.15)$$

Since $m < \tilde{\epsilon}_0$ we can choose $\varepsilon = (\tilde{\epsilon}_0 - m)/2 > 0$ and $k < 1/2$ to get

$$\|\delta\boldsymbol{E}_h^n\|_{0,\Omega}^2 \le C + Ck\sum_{i=1}^{n-1}\|\delta\boldsymbol{E}_h^i\|_{0,\Omega}^2. \quad (7.16)$$

Applying Lemma 7.2 we get

$$\|\delta\boldsymbol{E}_h^n\|_{0,\Omega}^2 \le C. \quad (7.17)$$

Since n is arbitrary applying the above estimate to (7.14) yields (7.9). Finally, (7.10) follows from (7.7) and (7.9). □

8. Numerical examples

In this section we report numerical results on linear edge finite element solutions of the $\boldsymbol{H}(\mathbf{curl})$-elliptic HVI (3.17), both in two and in three dimensions, and we pay particular attention to the numerical convergence orders. The discussion in previous sections was for three-dimensional problems.

Consideration of the numerical method for two-dimensional problems can be converted to that for three-dimensional problems through the following procedure: we extend vectors $\boldsymbol{u}(x,y) = (u_1(x,y), u_2(x,y))$ and $\boldsymbol{n} = (n_1, n_2)$ into three dimensions by $\boldsymbol{u}(x,y,z) = (u_1(x,y), u_2(x,y), 0)$ and $\boldsymbol{n} = (n_1, n_2, 0)$. Then

$$\mathbf{curl}\,\boldsymbol{u} = \left(0, 0, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right),$$
$$\boldsymbol{n} \times \boldsymbol{u} = (0, 0, n_1 u_2 - n_2 u_1).$$

For three-dimensional problems the spatial variable $\boldsymbol{x} = (x,y,z)$, and for two-dimensional problems, $\boldsymbol{x} = (x,y)$.

For the numerical examples we let $\Omega = (0,1)^d$ $(d = 2,3)$, and use uniform triangular partitions in two dimensions and tetrahedral partitions in three dimensions. The uniform partitions start with splitting the unit interval $[0,1]$ into $1/h$ equal-size subintervals and we use h as the mesh size in reporting the numerical results. The physical parameters ϵ and μ are both taken to be 1. The source function is chosen as

$$\boldsymbol{f}(x,y) = (\pi + 2\pi^3)\begin{bmatrix} -\cos(\pi x)\sin(\pi y) \\ \sin(\pi x)\cos(\pi y)\end{bmatrix}$$

for two dimensions, or

$$\boldsymbol{f}(x,y,z) = \frac{1+\pi^2}{\pi}\begin{bmatrix} \sin(\pi y) - \sin(\pi z) \\ \sin(\pi z) - \sin(\pi x) \\ \sin(\pi x) - \sin(\pi y)\end{bmatrix}$$

for three dimensions. For positive parameters $a > b$ and α we let

$$\omega(t) = (a-b)\,e^{-\alpha t} + b, \quad \psi(\boldsymbol{E}) = \int_0^{|\boldsymbol{E}|} \omega(t)\,\mathrm{d}t.$$

Then $\boldsymbol{J} \in \partial\psi(\boldsymbol{E})$ from (1.6) is equivalent to

$$|\boldsymbol{J}| \le \omega(0) \quad \text{if } \boldsymbol{E} = \boldsymbol{0}, \quad \boldsymbol{J} = \omega(|\boldsymbol{E}|)\frac{\boldsymbol{E}}{|\boldsymbol{E}|} \quad \text{if } \boldsymbol{E} \ne \boldsymbol{0}, \quad \text{in } \Omega. \tag{8.1}$$

It can be verified that for this choice of ψ, (3.3)(d) is satisfied with $m = \alpha(a-b)$. Since $\omega(t)$ is a decreasing function, ψ is nonconvex. Note that ψ is Lipschitz continuous and nonsmooth. We take $a = 0.004$, $b = 0.002$ and $\alpha = 100$ for the function ψ in the numerical tests.

Discrete problem (5.1) is solved by an algorithm based on a sequence of convex programming problems; cf. Barboteu *et al.* (2013). The main idea of the iterative algorithm is to update the value of the function ω at each iteration with the numerical solution found in the previous iteration.

TABLE 1 *Numerical errors of the linear edge finite element method in two dimensions*

h	$\|\boldsymbol{E}^* - \boldsymbol{E}_h\|_{0,\Omega}$	Order	$\|\boldsymbol{E}^* - \boldsymbol{E}_h\|_{\mathrm{curl},\Omega}$	Order
2^{-2}	2.4919e−01	–	2.4836	–
2^{-3}	5.3835e−02	2.2107	1.2142	1.0324
2^{-4}	1.2677e−02	2.0863	5.9858e−01	1.0204
2^{-5}	3.0879e−03	2.0375	2.9627e−01	1.0146
2^{-6}	7.5058e−04	2.0406	1.4625e−01	1.0185

TABLE 2 *Numerical errors of the linear edge finite element method in three dimensions*

h	$\|\boldsymbol{E}^* - \boldsymbol{E}_h\|_{0,\Omega}$	Order	$\|\boldsymbol{E}^* - \boldsymbol{E}_h\|_{\mathrm{curl},\Omega}$	Order
2^{-2}	1.0509e−02	–	1.3826e−01	–
2^{-3}	2.9857e−03	1.8155	7.4189e−02	0.8981
2^{-4}	7.6382e−04	1.9667	3.7808e−02	0.9725
2^{-5}	1.8604e−04	2.0376	1.9100e−02	0.9851

For initialization choose a small value ε used in a stopping criterion of the iteration and choose an initial guess $\boldsymbol{E}_h^0 \in \boldsymbol{V}_h$. Then for $\ell = 0, 1, \ldots$, find $\boldsymbol{E}_h^{\ell+1} \in \boldsymbol{V}_h$ as the solution of the following problem:

$$a(\boldsymbol{E}_h^{\ell+1}, \boldsymbol{v}_h) + \int_\Omega \boldsymbol{\lambda}_h^{\ell+1} \cdot \boldsymbol{v}_h \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{8.2}$$

with

$$\boldsymbol{\lambda}_h^{\ell+1} \in \omega(|\boldsymbol{E}_h^\ell|)\partial|\boldsymbol{E}_h^{\ell+1}| \tag{8.3}$$

until

$$\|\boldsymbol{E}_h^{\ell+1} - \boldsymbol{E}_h^\ell\|_{0,\Omega} \le \varepsilon \|\boldsymbol{E}_h^\ell\|_{0,\Omega} \quad \text{and} \quad \|\boldsymbol{\lambda}_h^{\ell+1} - \boldsymbol{\lambda}_h^\ell\|_{0,\Omega} \le \varepsilon \|\boldsymbol{\lambda}_h^\ell\|_{0,\Omega}.$$

The sequence $\{\boldsymbol{\lambda}_h^\ell\}$ can be viewed as a sequence approximating a Lagrange multiplier $\boldsymbol{\lambda}_h$.

In the numerical experiment we let $\varepsilon = 10^{-7}$ and determine the initial guess $\boldsymbol{E}_h^0 \in \boldsymbol{V}_h$ as the solution of the problem

$$a(\boldsymbol{E}_h^0, \boldsymbol{v}_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Because the true solution is not known, the linear edge finite element solution on a sufficiently refined mesh is used as the reference solution $\boldsymbol{E}^*$; here, we use the finite element solution with $h = 2^{-8}$ for two dimensions and that with $h = 2^{-7}$ for three dimensions. Then we compare the numerical solutions $\boldsymbol{E}_h$ on coarser meshes with $h = 2^{-n}$ for $2 \le n \le 6$ in two dimensions and for $2 \le n \le 5$ in three dimensions with $\boldsymbol{E}^*$. We compute the errors $\|\boldsymbol{E}^* - \boldsymbol{E}_h\|_{0,\Omega}$ and $\|\boldsymbol{E}^* - \boldsymbol{E}_h\|_{\mathbf{curl},\Omega}$. The numerical results for the linear edge finite element method are reported in Tables 1–2. We observe that the numerical convergence orders in the $\boldsymbol{H}(\mathbf{curl}, \Omega)$-norm are around 1, which matches the theoretical result in Theorem 6.3. The numerical solutions $\boldsymbol{E}_h$ on meshes with $h = 2^{-4}$ and $h = 2^{-5}$ in two dimensions, with $h = 2^{-3}$ and $h = 2^{-4}$ in three dimensions, are also shown in Figs 1 and 2.

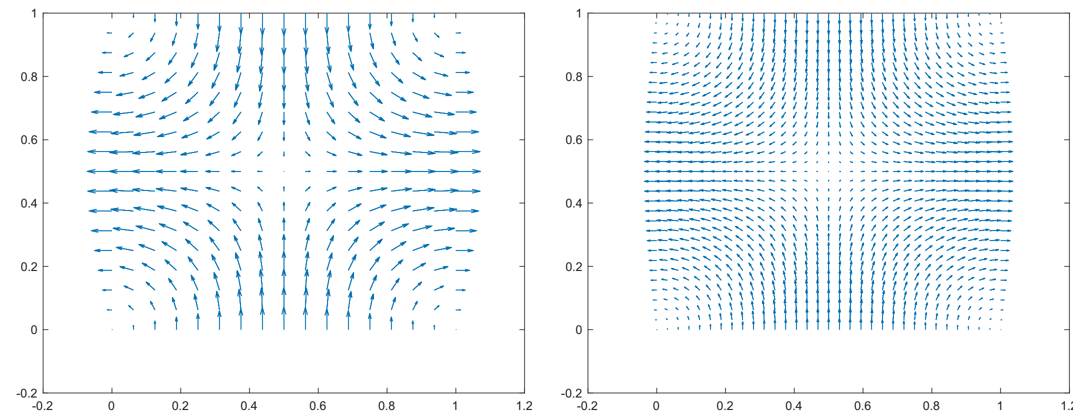

FIG. 1. Numerical solutions $\boldsymbol{E}_h$ on meshes with $h = 2^{-4}$ and $h = 2^{-5}$ in two dimensions.

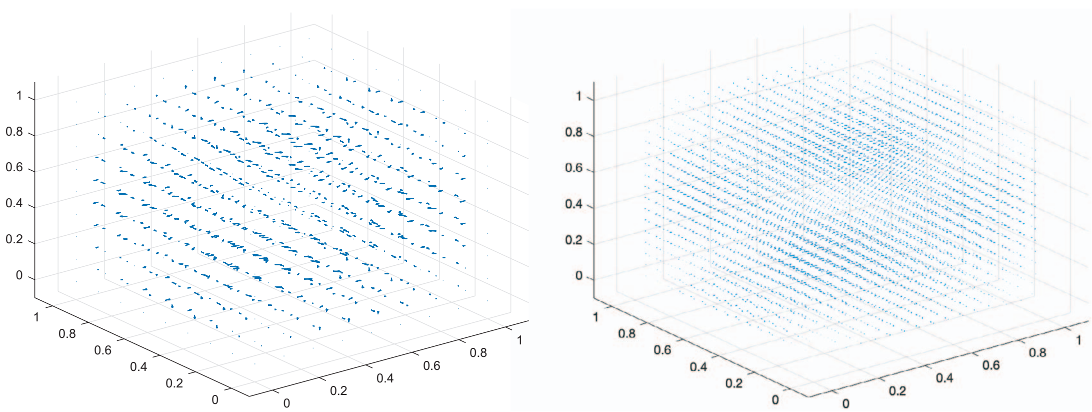

FIG. 2. Numerical solutions $\boldsymbol{E}_h$ on meshes with $h = 2^{-3}$ and $h = 2^{-4}$ in three dimensions.

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