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Stabilized low-order mixed finite element methods for a Navier-Stokes hemivariational inequality

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Abstract

In this paper, pressure projection stabilized low-order mixed finite element methods are studied to solve a Navier-Stokes hemivariational inequality for a boundary value problem of the Navier-Stokes equations involving a non-smooth non-monotone boundary condition. A new abstract mixed hemivariational inequality is introduced for the purpose of analyzing stabilized mixed finite element methods to solve the Navier-Stokes hemivariational inequality using velocity-pressure pairs without the discrete inf-sup condition. The well-posedness of the abstract problem is established through considerations of a related saddle-point formulation and fixed-point arguments. Then the results on the abstract problem are applied to the study of the Navier-Stokes hemivariational inequality and its stabilized mixed finite element approximations. Optimal order error estimates are derived for finite element solutions of the pressure projection stabilized lowest-order conforming pair and lowest equal order pair under appropriate solution regularity assumptions. Numerical results are reported on the performance of the pressure projection stabilized mixed finite element methods for solving the Navier-Stokes hemivariational inequality.

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1 Introduction

Hemivariational inequalities are a family of non-smooth problems arising in science and engineering applications that involve non-smooth, non-monotone and set-valued relations among physical quantities. The framework of hemivariational inequalities is more general than that of variational inequalities. The notion of hemivariational inequalities was introduced by Panagiotopoulos four decades ago [34]. Since then, modeling, analysis, numerical solution and applications of hemivariational inequalities have attracted increasing attention from the research community, and the number of publications on hemivariational inequalities grows substantially these years. As representative recent comprehensive references in the area, [37] focuses on well-posedness analysis of hemivariational inequalities, and [21] provides a survey of numerical analysis of hemivariational inequalities.

A large number of publications on hemivariational inequalities is devoted to application problems in solid mechanics. Meanwhile, hemivariational inequalities are also studied for applications in fluid mechanics, especially boundary or initial-boundary value problems associated with the Stokes equations or the Navier-Stokes equations involving non-smooth non-monotone slip or leak boundary conditions of friction type. Consideration of viscous incompressible fluid flows with non-smooth yet monotone slip or leak boundary conditions of friction type started in early 1990s [14, 15]. Weak formulations of such problems are variational inequalities and they are studied in many papers, e.g., [24–26] on Stokes variational inequalities, and [9, 27, 28, 35] on Navier-Stokes variational inequalities, just to mention a few. When the slip or leak boundary conditions are allowed to express more general non-smooth non-monotone relations, the corresponding mathematical problems are Stokes hemivariational inequalities or Navier-Stokes hemivariational inequalities. Some references on well-posedness analysis of such hemivariational inequalities are [13, 31]. Mixed finite element methods using velocity-pressure pairs satisfying the discrete inf-sup condition are analyzed in [12] for Stokes hemivariational inequalities and in [19] for Navier-Stokes hemivariational inequalities. Although the lowest-order velocity-pressure pairs of finite element spaces do not satisfy the discrete inf-sup condition, they are attractive for simulations of incompressible flow problems due to, e.g., a local mass conservation property by the lowest-order conforming pair of continuous piecewise linear, bilinear or trilinear velocities and piecewise constant pressures, simple and uniform data structures for lowest equal order pair of continuous piecewise linear, bilinear or trilinear elements for both velocities and pressures, and favorable size and bandwidth properties of discrete systems for these element pairs [5, 36]. For solving variational inequalities or hemivariational inequalities, low-order element pairs are even more preferred since the smoothness of the true solutions of the inequalities is quite limited and moreover, for inequality problems, even if the true solution is assumed to be smooth, it is not possible to have an optimal order error bound for high-order element solutions due to the inequality feature of the problems. In the literature, several stabilization techniques have been introduced to stabilize low-order element pairs that do not satisfy the discrete inf-sup condition. For instance, consistent stabilized methods [2, 6], local and global stabilized methods [23], pressure projection based stabilization methods [5, 29], and local pressure gradient projection stabilized methods [3, 4].

In this paper, following [5], we consider pressure projection stabilized mixed finite element methods for solving the Navier-Stokes hemivariational inequality. We start with the well-posedness analysis of a new abstract hemivariational inequality that is particularly suitable for the study of stabilized mixed finite element methods to solve Navier-Stokes hemivariational inequalities as well as Stokes hemivariational inequalities. The well-posedness of the abstract problem is established through considerations of a related saddle-point formulation and fixed-point arguments. This approach is more accessible to applied mathematicians and engineers. In comparison, a local pressure projection stabilized mixed method for a Stokes hemivariational inequality is studied in [30] using continuous piecewise linear approximations for both the velocity and pressure, and there, proving the existence of a stabilized mixed solution needs the rather complicated Knaster-Kuratowski-Mazurkiewicz principle. We comment that in addition, results on the new abstract hemivariational inequality can also be applied to study Navier-Stokes hemivariational inequalities and Stokes hemivariational inequalities, as well as their mixed finite element solutions using velocity-pressure pairs with the discrete inf-sup condition. For the stabilized mixed methods for the Navier-Stokes hemivariational inequality, we then derive optimal order error estimates of the velocity and pressure for both the lowest-order conforming pairs and the lowest equal order pairs, under appropriate solution regularity conditions.

Description of hemivariational inequalities requires the notions of the generalized directional derivative and generalized subdifferential in the sense of Clarke for a locally Lipschitz continuous function [8]. Let $\Psi : V \to \mathbb{R}$ be a locally Lipschitz continuous functional defined on a real Banach space V. Then its generalized (Clarke) directional derivative at $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u; v) := \limsup_{w \to u,\, \lambda \downarrow 0} \frac{\Psi(w + \lambda\, v) - \Psi(w)}{\lambda},$$

whereas the generalized subdifferential of Ψ at $u \in V$ is

$$\partial\Psi(u) := \left\{\xi \in V^* \mid \Psi^0(u; v) \geq \langle \xi, v\rangle \;\forall\, v \in V\right\}.$$

Given the generalized subdifferential, the generalized directional derivative can be determined by

$$\Psi^0(u; v) = \max\{\langle \xi, v\rangle \mid \xi \in \partial\Psi(u)\} \quad \forall\, u, v \in V.$$

If the locally Lipschitz continuous function $\Psi : V \to \mathbb{R}$ is also convex, then the subdifferential $\partial\Psi(u)$ at any $u \in V$ in the sense of Clarke coincides with the convex subdifferential $\partial\Psi(u)$. Hence, the notion of the Clarke subdifferential can be viewed as a generalization of that of the convex subdifferential. Two basic properties are

$$\partial(\lambda\,\Psi)(u) = \lambda\,\partial\Psi(u) \quad \forall\,\lambda \in \mathbb{R},\ \forall\,u \in V, \tag{1.1}$$

$$\Psi^0(u; v_1 + v_2) \le \Psi^0(u; v_1) + \Psi^0(u; v_2) \quad \forall\,\lambda \in \mathbb{R},\ \forall\,u, v_1, v_2 \in V. \tag{1.2}$$

For locally Lipschitz functions $\Psi_1, \Psi_2 \colon V \to \mathbb{R}$, the inclusion

$$\partial(\Psi_1 + \Psi_2)(u) \subset \partial\Psi_1(u) + \partial\Psi_2(u) \quad \forall\,u \in V \tag{1.3}$$

holds, which is equivalent to the inequality

$$(\Psi_1 + \Psi_2)^0(u; v) \le \Psi_1^0(u; v) + \Psi_2^0(u; v) \quad \forall\,u, v \in V. \tag{1.4}$$

Detailed discussions of the generalized directional derivative and the generalized subdifferential for locally Lipschitz continuous functionals, including their properties, can be found in several references, e.g. [8, 32].

The rest of the paper is organized as follows. In Sect. 2, we introduce the new abstract mixed hemivariational inequality and study its well-posedness. This abstract framework is especially suitable for the study of stabilization of mixed finite element methods for solving the Navier-Stokes hemivariational inequality considered in this paper. In Sect. 3, we apply the theoretical results in the study of the Navier-Stokes hemivariational inequality. In Sect. 4, we introduce pressure projection stabilized mixed finite element methods to solve the Navier-Stokes hemivariational inequality, and derive optimal order error estimates for the pressure projection stabilized mixed finite element solutions of both the lowest-order conforming pairs and the lowest equal order pairs under appropriate solution regularity assumptions. In Sect. 5, we present simulation results on a numerical example to illustrate the performance of the numerical methods, paying particular attention on the numerical convergence orders.

2 A new abstract mixed hemivariational inequality

Let V and Q be two real Hilbert spaces. Their dual spaces are denoted by V^* and Q^*. The symbol $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V^* and V, or between Q^* and Q; it should be clear from the context which duality pairing is meant by $\langle\cdot,\cdot\rangle$. Let there be given $a \colon V \times V \to \mathbb{R}$, $d \colon V \times V \times V \to \mathbb{R}$, $b \colon V \times Q \to \mathbb{R}$, $S \colon Q \times Q \to \mathbb{R}$, and $f \in V^*$. Denote by Δ the spatial domain of the problem, or a sub-domain, or the boundary or part of the boundary of the domain. Let ψ be a real-valued function defined on $\Delta \times \mathbb{R}^m$ for some positive integer m, and let γ_ψ be a linear continuous operator from V to $L^2(\Delta; \mathbb{R}^m)$. For applications in mechanics, the operator γ_ψ is either the normal trace operator and then $m = 1$, or the tangential component trace operator and then m is the dimension of the spatial domain.

Let us introduce assumptions on the data.

$\underline{H(a)}$ $a \colon V \times V \to \mathbb{R}$ is a bounded and V-elliptic bilinear form.

We will use $M_a > 0$ for the boundedness constant and $m_a > 0$ for the V-ellipticity constant:

$$|a(u,v)| \le M_a \|u\|_V \|v\|_V \quad \forall\, u, v \in V, \tag{2.1}$$

$$a(v,v) \ge m_a \|v\|_V^2 \quad \forall\, v \in V. \tag{2.2}$$

From the bilinear form $a(\cdot,\cdot)$, we can define an operator $A\colon V \to V^*$ by

$$\langle Au, v\rangle = a(u,v), \quad u, v \in V. \tag{2.3}$$

The properties (2.1) and (2.2) can be equivalently expressed in terms of A:

$$\|A\| \le M_a, \tag{2.4}$$

$$\langle Av, v\rangle \ge m_a \|v\|_V^2 \quad \forall\, v \in V. \tag{2.5}$$

$\underline{H(d)}$ $d\colon V \times V \times V \to \mathbb{R}$ is a bounded trilinear form such that

$$d(u,v,v) = 0 \quad \forall\, u, v \in V. \tag{2.6}$$

We will use $c_d > 0$ for the boundedness constant:

$$|d(u,v,w)| \le c_d \|u\|_V \|v\|_V \|w\|_V \quad \forall\, u, v, w \in V. \tag{2.7}$$

$\underline{H(\psi)}$ $\gamma_\psi \in \mathcal{L}(V; L^2(\Delta;\mathbb{R}^m))$; $\psi\colon \Delta \times \mathbb{R}^m \to \mathbb{R}$; $\psi(\cdot,z)$ is measurable on Δ for all $z \in \mathbb{R}^m$; for a function $z_0 \in L^2(\Delta;\mathbb{R}^m)$, $\psi(\cdot, z_0(\cdot)) \in L^1(\Delta)$; $\psi(\boldsymbol{x},\cdot)$ is locally Lipschitz continuous on $\mathbb{R}^m$ for a.e. $\boldsymbol{x} \in \Delta$; and for non-negative constants c_1, c_1 and α_ψ,

$$|\partial\psi(z)| \le c_0 + c_1 |z|_{\mathbb{R}^m} \quad \forall\, z \in \mathbb{R}^m, \text{ a.e. on } \Delta, \tag{2.8}$$

$$\psi^0(z_1; z_2 - z_1) + \psi^0(z_2; z_1 - z_2) \le \alpha_\psi |z_1 - z_2|_{\mathbb{R}^m}^2 \quad \forall\, z_1, z_2 \in \mathbb{R}^m, \text{ a.e. on } \Delta. \tag{2.9}$$

Note that to simplify the notation, we usually write $\psi(z)$ by suppressing its first argument. The relation (2.8) is a short-hand notation for

$$|\eta| \le c_0 + c_1 |z|_{\mathbb{R}^m} \quad \forall\, z \in \mathbb{R}^m,\ \eta \in \partial\psi(z), \text{ a.e. on } \Delta,$$

and it is equivalent to

$$\left|\psi^0(z_1; z_2)\right| \le \left(c_0 + c_1 |z_1|_{\mathbb{R}^m}\right) |z_2|_{\mathbb{R}^m} \quad \forall\, z_1, z_2 \in \mathbb{R}^m, \text{ a.e. on } \Delta.$$

The inequality (2.9) is equivalent to [37, p. 124]

$$\langle \eta_1 - \eta_2, z_1 - z_2\rangle \ge -\alpha_\psi |z_1 - z_2|_{\mathbb{R}^m}^2 \quad \forall\, z_i \in \mathbb{R}^m,\ \eta_i \in \partial\psi(z_i),\ i = 1, 2, \text{ a.e. on } \Delta.$$

Let I_Δ stand for the integration operator over Δ. Denote by $c_\Delta > 0$ the smallest constant in the inequality

$$I_\Delta(|\gamma_\psi v|^2_{\mathbb{R}^m}) \le c_\Delta^2 \|v\|_V^2 \quad \forall\, v \in V. \tag{2.10}$$

From the assumptions (2.8), (2.9), we can deduce that

$$\psi^0(z; -z) \le c_0 |z|_{\mathbb{R}^m} + \alpha_\psi |z|^2_{\mathbb{R}^m} \quad \forall\, z \in \mathbb{R}^m, \text{ a.e. on } \Delta. \tag{2.11}$$

Then

$$I_\Delta(\psi^0(\gamma_\psi v; -\gamma_\psi v)) \le c_0 c_\Delta |\Delta|^{1/2} \|v\|_V + \alpha_\psi c_\Delta^2 \|v\|_V^2 \quad \forall\, v \in V, \tag{2.12}$$

where $|\Delta| := I_\Delta(1)$ is the measure of Δ. Define the functional

$$\Psi(v) = I_\Delta(\psi(\gamma_\psi v)), \quad v \in V. \tag{2.13}$$

Then under the assumption $H(\psi)$, similar to the results and arguments in [32, Section 3.3], it can be shown that $\Psi(\cdot)$ is well-defined and locally Lipschitz on V, and

$$\Psi^0(u; v) \le I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v)) \quad \forall\, u, v \in V. \tag{2.14}$$

$\underline{H(f)}$ $f \in V^*$.
$\underline{H(S)}$ $S\colon Q \times Q \to \mathbb{R}$ is bilinear, bounded, symmetric, and non-negative:

$$S(q, q) \ge 0 \quad \forall\, q \in Q. \tag{2.15}$$

It is easy to derive the following two inequalities from (2.15):

$$S(q, r)^2 \le S(q, q)\, S(r, r) \quad \forall\, q, r \in Q, \tag{2.16}$$

$$|S(q, r)| \le \frac{1}{2} S(q, q) + \frac{1}{2} S(r, r) \quad \forall\, q, r \in Q. \tag{2.17}$$

Introduce an extended bilinear form

$$\tilde{b}((u, p); (v, q)) := a(u, v) - b(v, p) + b(u, q) + S(p, q) \quad \forall\, (u, p), (v, q) \in V \times Q. \tag{2.18}$$

$\underline{H(b)}$ $b : V \times Q \to \mathbb{R}$ is bilinear, bounded, and there exists a constant $\alpha_{\tilde{b}} > 0$ such that

$$\sup_{(v,q) \in V \times Q} \frac{\tilde{b}((u, p); (v, q))}{\|v\|_V + \|q\|_Q} \ge \alpha_{\tilde{b}} \left(\|u\|_V + \|p\|_Q \right) \quad \forall\, (u, p) \in V \times Q. \tag{2.19}$$

We will use $M_b > 0$ for the boundedness constant of $b(\cdot, \cdot)$:

$$|b(v, q)| \le M_b \|v\|_V \|q\|_Q \quad \forall\, v \in V,\ q \in Q. \tag{2.20}$$

We can introduce an operator $B \in \mathcal{L}(V; Q^*)$ and its dual $B^* \in \mathcal{L}(Q; V^*)$ by

$$\langle Bv, q\rangle = \langle B^* q, v\rangle = b(v, q) \quad \forall\, (v, q) \in V \times Q,$$

and we have $\|B\| \le M_b$.

The abstract mixed hemivariational inequality is the following.

Problem 2.1 Find $(u, p) \in V \times Q$ such that

$$a(u, v) + d(u, u, v) - b(v, p) + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v)) \ge \langle f, v\rangle \quad \forall\, v \in V, \tag{2.21}$$
$$b(u, q) + S(p, q) = 0 \quad \forall\, q \in Q. \tag{2.22}$$

In the well-posedness analysis of Problem 2.1, we need to assume

$$\alpha_\psi c_\Delta^2 < m_a. \tag{2.23}$$

Then, it is convenient to introduce a constant

$$M_f = \frac{c_0 c_\Delta |\Delta|^{1/2} + \|f\|_{V^*}}{m_a - \alpha_\psi c_\Delta^2} \tag{2.24}$$

and a subset of V:

$$K_f = \left\{ v \in V \mid \|v\|_V \le M_f \right\}. \tag{2.25}$$

We will further assume

$$\alpha_\psi c_\Delta^2 + c_d M_f < m_a. \tag{2.26}$$

Conditions such as (2.23) and (2.26) are known as smallness conditions in the literature [32].

To prepare for the well-posedness analysis of Problem 2.1, we first present a boundedness result.

Lemma 2.1 *Assume $H(a)$, $H(d)$, $H(\psi)$, $H(f)$, $H(S)$, $H(b)$, and (2.23). If Problem 2.1 has a solution $(u, p) \in V \times Q$, then $u \in K_f$.*

Proof We take $v = -u$ in (2.21) to obtain

$$a(u, u) \le -d(u, u, u) + b(u, p) + I_\Delta(\psi^0(\gamma_\psi u; -\gamma_\psi u)) + \langle f, v\rangle. \tag{2.27}$$

By (2.2), $a(u, u) \ge m_a \|u\|_V^2$. By (2.6), $d(u, u, u) = 0$. By (2.22) and (2.15), $b(u, p) = -S(p, p) \le 0$. The term $I_\Delta(\psi^0(\gamma_\psi u; -\gamma_\psi u))$ is bounded with (2.12). In addition, $\langle f, v\rangle \le \|f\|_{V^*} \|u\|_V$. Hence, from (2.27), we know that

$$\left(m_a - \alpha_\psi c_\Delta^2\right) \|u\|_V^2 \le c_0 c_\Delta |\Delta|^{1/2} \|u\|_V + \|f\|_{V^*} \|u\|_V,$$

from which, we derive that $\|u\| \le M_f$. □

Denote by $\mathcal{J}\colon V^* \to V$ the Riesz mapping from the Riesz representation theorem:

$$\langle g, v\rangle = (\mathcal{J} g, v) \quad \forall\, g \in V^*,\ v \in V.$$

We have the next result which is easy to prove (e.g., [1, p. 337]).

Lemma 2.2 *Under the assumption $H(a)$, we have the bound*

$$\|\mathcal{I} - \theta\, \mathcal{J} A\| \le (1 - 2\,\theta\, m_a + \theta^2 M_a^2)^{1/2} \quad \forall\, \theta \ge 0, \tag{2.28}$$

where $\mathcal{I}$ is the identity operator.

Let $\tilde{u} \in K_f$ be arbitrary and fixed. We introduce an auxiliary problem for Problem 2.1.

Problem 2.2 Find $(u, p) \in V \times Q$ such that

$$a(u, v) + d(\tilde{u}, u, v) - b(v, p) + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v)) \ge \langle f, v\rangle \quad \forall\, v \in V, \tag{2.29}$$

$$b(u, q) + S(p, q) = 0 \quad \forall\, q \in Q. \tag{2.30}$$

Theorem 2.1 *Assume $H(a)$, $H(d)$, $H(\psi)$, $H(f)$, $H(S)$, $H(b)$, (2.23) and (2.26). Then Problem 2.2 has a solution $(u, p) \in V \times Q$, u being unique and $u \in K_f$.*

Proof Let $\theta > 0$ be sufficiently small. For any $w \in V$, consider the problem of finding $(u, p) \in V \times Q$ such that

$$(u, v)_V - \theta\, b(v, p) + \theta\, I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v)) \ge \ell_\theta(w; v) + \theta\, \langle f, v\rangle \quad \forall\, v \in V, \tag{2.31}$$

$$b(u, q) + S(p, q) = 0 \quad \forall\, q \in Q. \tag{2.32}$$

where

$$\ell_\theta(w; v) = (w, v)_V - \theta\, (a(w, v) + d(\tilde{u}, w, v)). \tag{2.33}$$

Introduce a Lagrangian functional

$$L_\theta(v, q) = \frac{1}{2}\, \|v\|_V^2 - \theta\, b(v, q) - \frac{\theta}{2}\, S(q, q) + \theta\, \Psi(v) - \ell_\theta(w; v) - \theta\, \langle f, v\rangle,\ (v, q) \in V \times Q. \tag{2.34}$$

This functional has the following four properties.

(P_1) For any fixed $q \in Q$, $L_\theta(\cdot, q)$ is locally Lipschitz continuous and strongly convex on V.

(P_2) For any fixed $v \in V$, $L_\theta(v, \cdot)$ is continuous and concave on Q.

(P_3) $\lim_{\|v\|_V \to \infty} L_\theta(v, 0) = \infty$.

(P_4) $\lim_{\|q\|_Q \to \infty} \inf_{v \in V} L_\theta(v, q) = -\infty$.

The strong convexity in (P_1) is proved as follows. Denote by $\partial L_\theta(v, q)$ the generalized sub-differential of L_θ with respect to v. For $v_1, v_2 \in V$ and $v_1^* \in \partial L_\theta(v_1, q)$,

$v_2^* \in \partial L_\theta(v_2, q)$, similar to the proof of [18, Proposition 3.4], we write

$$v_i^* = \mathcal{J}^{-1} v_i - \theta\, B^* q + \theta\, \eta_i - \ell_\theta(w; \cdot) - \theta\, f,\ \eta_i \in \partial \Psi(v_i),\ i = 1, 2,$$

where $\mathcal{J}^{-1} v_i$ is the sub-differential of the term $\|v_i\|_V^2/2 = \langle \mathcal{J}^{-1} v_i, v_i \rangle/2$, $B^* q$ is the sub-differential of $b(v_i, q) = \langle B^* q, v_i \rangle$ with respect to v_i, and note that $\ell_\theta(w; v_i)$ and $\langle f, v_i \rangle$ are linear with respect to v_i. By using (2.9) and (2.14), we have

$$\langle v_1^* - v_2^*, v_1 - v_2 \rangle \geq \|v_1 - v_2\|_V^2 - \theta\, I_\Delta(\alpha_\psi |\gamma_\psi v_1 - \gamma_\psi v_2|_{\mathbb{R}^m}^2) \geq \left(1 - \theta\, \alpha_\psi c_\Delta^2\right) \|v_1 - v_2\|_V^2. \tag{2.35}$$

For $\theta > 0$ sufficiently small, $1 - \theta\, \alpha_\psi c_\Delta^2 > 0$ and $L_\theta(\cdot, q)$ is strongly convex [11, Proposition 3.1].

Property (P_4) is proved as follows. For each fixed $q \in Q$, we know from (P_1) that the mapping $v \mapsto L_\theta(v, q)$ is locally Lipschitz and strongly convex on V. Thus, $L_\theta(\cdot, q)$ has a unique minimizer u_q on V. Then,

$$0 \in \partial L_\theta(u_q, q). \tag{2.36}$$

By making use of Lemma 2.4 and Proposition 3.3 in [20], we have

$$L_\theta(0, q) - L_\theta(u_q, q) \geq \frac{1 - \theta\, \alpha_\psi c_\Delta^2}{2} \|u_q\|_V^2.$$

As is commented in [20], Lemma 2.4 there follows from a combination of Proposition 3.1 and Theorem 3.4 in [11], whereas Proposition 3.3 there follows from the second paragraph of the proof of Proposition 3.4 in [18]. Hence,

$$L_\theta(u_q, q) \leq \theta\, I_\Delta(\psi(0)) - \frac{\theta}{2} S(q, q) - \frac{1 - \theta\, \alpha_\psi c_\Delta^2}{2} \|u_q\|_V^2. \tag{2.37}$$

Apply (2.19) with $u = 0$ and p replaced by q,

$$\alpha_{\tilde{b}} \|q\|_Q \leq \sup_{\|v\|_V + \|r\|_Q = 1} [-b(v, q) + S(q, r)]. \tag{2.38}$$

Now from (2.36) and (2.14),

$$-b(v, q) \leq -\theta^{-1}(u_q, v)_V + I_\Delta(\psi^0(\gamma_\psi u_q; -\gamma_\psi v)) + \theta^{-1} \ell_\theta(w; v) + \langle f, v \rangle \quad \forall\, v \in V. \tag{2.39}$$

By (2.8), the Cauchy-Schwarz inequality, and (2.10),

$$\left|\psi^0(\gamma_\psi u_q; -\gamma_\psi v)\right| \leq \left(c_0 + c_1 |\gamma_\psi u_q|_{\mathbb{R}^m}\right) |\gamma_\psi v|_{\mathbb{R}^m},$$
$$\left|I_\Delta(\psi^0(\gamma_\psi u_q; -\gamma_\psi v))\right| \leq \left(c_0 c_\Delta |\Delta|^{1/2} + c_1 c_\Delta^2 \|u_q\|_V\right) \|v\|_V.$$

Hence, from (2.39),

$$-\,b(v,q) \le M_b(u_q)\,\|v\|_V \quad \forall\, v \in V \tag{2.40}$$

where

$$M_b(u_q)=\big(\theta^{-1}+c_1c_\Delta^2\big)\,\|u_q\|_V+c_0c_\Delta|\Delta|^{1/2}+\big(\theta^{-1}+M_a+c_d\|\tilde u\|_V\big)\,\|w\|_V+\|f\|_{V^*}. \tag{2.41}$$

Use (2.40) and (2.17) in (2.38),

$$\alpha_{\tilde b}\|q\|_Q \le \frac{1}{2}\,S(q,q) + \sup_{\|v\|_V+\|r\|_Q=1}\left[M_b(u_q)\,\|v\| + \frac{1}{2}\,S(r,r)\right].$$

Then, for some constant c_2 depending on the boundedness constant of $S(\cdot,\cdot)$, we have the inequality

$$\alpha_{\tilde b}\|q\|_Q \le \frac{1}{2}\,S(q,q) + M_b(u_q) + c_2 \quad \forall\, q \in Q. \tag{2.42}$$

By the definition (2.41) for $M_b(u_q)$, we deduce from (2.42) the implication

$$\|q\|_Q \to \infty \quad \Longrightarrow \quad S(q,q) + \|u_q\|_V \to \infty. \tag{2.43}$$

By (2.37), we conclude that $L_\theta(u_q,q) \to -\infty$ as $\|q\|_Q \to \infty$.

Properties (P_2) and (P_3) are obvious, and so is the local Lipschitz continuity of $L_\theta(\cdot,q)$ in (P_1). With the properties (P_1)–(P_4) established, we are in a position to apply [10, Chapter VI, Proposition 2.4] and conclude that L_θ has a saddle-point $(u,p) \in V \times Q$:

$$L_\theta(u,q) \le L_\theta(u,p) \le L_\theta(v,p) \quad \forall\, v \in V,\ q \in Q. \tag{2.44}$$

Note that the first inequality in (2.44) is equivalent to

$$-b(u,q) - \frac{1}{2}\,S(q,q) \le -b(u,p) - \frac{1}{2}\,S(p,p) \quad \forall\, q \in Q,$$

which is equivalent to (2.32). The second inequality in (2.44) implies $0 \in \partial L_\theta(u,p)$, or

$$(u,v)_V - \theta\, b(v,p) + \theta\, \Psi^0(u;v) - \ell_\theta(w;v) - \theta\,\langle f,v\rangle \ge 0 \quad \forall\, v \in V.$$

Since $\Psi^0(u;v) \le I_\Delta(\psi^0(\gamma_\psi u;\gamma_\psi v))$, we see that (2.31) is satisfied. In conclusion, we have shown that the saddle-point of L_θ is a solution of the problem (2.31)–(2.32).

Now, let (u_1,p_1) and (u_2,p_2) both satisfy (2.31)–(2.32): for $i = 1,2$,

$$(u_i,v)_V - \theta\, b(v,p_i) + \theta\, I_\Delta(\psi^0(\gamma_\psi u_i;\gamma_\psi v)) \ge \ell_\theta(w;v) + \theta\,\langle f,v\rangle \quad \forall\, v \in V, \tag{2.45}$$

$$b(u_i,q) + S(p_i,q) = 0 \quad \forall\, q \in Q. \tag{2.46}$$

Take $v = u_2 - u_1$ in (2.45) for $i = 1$, $v = u_1 - u_2$ in (2.45) for $i = 2$, and add the two inequalities,

$$\|u_1 - u_2\|_V^2 \le \theta\, b(u_1 - u_2, p_1 - p_2) + \theta\, I_\Delta(\psi^0(\gamma_\psi u_1; \gamma_\psi u_2 - \gamma_\psi u_1) + \psi^0(\gamma_\psi u_2; \gamma_\psi u_1 - \gamma_\psi u_2)). \tag{2.47}$$

From (2.46) for $i = 1, 2$, we can deduce that

$$b(u_1 - u_2, p_1 - p_2) = -S(p_1 - p_2, p_1 - p_2) \le 0.$$

Then from (2.47), (2.9), and (2.10),

$$\|u_1 - u_2\|_V^2 \le \theta\, I_\Delta(\alpha_\psi |\gamma_\psi(u_1 - u_2)|_{\mathbb{R}^m}^2) \le \theta\, \alpha_\psi c_\Delta^2 \|u_1 - u_2\|_V^2.$$

For $\theta > 0$ sufficiently small, the above inequality implies $u_1 = u_2$.

Therefore, for $\theta > 0$ sufficiently small, we can define an operator $P_\theta : V \to V$ by $P_\theta(w) = u$ where u is the first component of a solution of the problem (2.31)–(2.32). Let's prove that P_θ is contractive. For this purpose, for $i = 1, 2$, let $w_i \in V$ and denote $u_i = P_\theta(w_i)$. Then for some element $p_i \in Q$, we have

$$(u_i, v)_V - \theta\, b(v, p_i) + \theta\, I_\Delta(\psi^0(\gamma_\psi u_i; \gamma_\psi v)) \ge \ell_\theta(w_i; v) + \theta\, \langle f, v\rangle \quad \forall\, v \in V, \tag{2.48}$$

$$b(u_i, q) + S(p_i, q) = 0 \quad \forall\, q \in Q. \tag{2.49}$$

Take $v = u_2 - u_1$ in (2.48) for $i = 1$, take $v = u_1 - u_2$ in (2.48) for $i = 2$, and add to obtain

$$\begin{aligned}\|u_1 - u_2\|_V^2 \le{}& \theta\, b(u_1 - u_2, p_1 - p_2) + \theta\, I_\Delta(\psi^0(\gamma_\psi u_1; \gamma_\psi u_2 - \gamma_\psi u_1) + \psi^0(\gamma_\psi u_2; \gamma_\psi u_1 - \gamma_\psi u_2)) \\ &+ \ell_\theta(w_1 - w_2; u_1 - u_2).\end{aligned} \tag{2.50}$$

By (2.49) for $i = 1, 2$,

$$b(u_1 - u_2, p_1 - p_2) = -S(p_1 - p_2, p_1 - p_2) \le 0.$$

Write

$$\begin{aligned}\ell_\theta(w_1 - w_2; u_1 - u_2) &= (w_1 - w_2, u_1 - u_2)_V \\ &\qquad - \theta\,(a(w_1 - w_2, u_1 - u_2) + d(\tilde{u}, w_1 - w_2, u_1 - u_2)) \\ &= ((I - \theta\, \mathcal{J}A)(w_1 - w_2), u_1 - u_2) - \theta\, d(\tilde{u}, w_1 - w_2, u_1 - u_2) \\ &\le \left(\|I - \theta\, \mathcal{J}A\| + \theta\, c_d M_f\right) \|w_1 - w_2\|_V \|u_1 - u_2\|_V.\end{aligned}$$

We apply the bound (2.28) for $\|I - \theta\, \mathcal{J}A\|$. Then from (2.50) and

$$I_\Delta(\psi^0(\gamma_\psi u_1; \gamma_\psi u_2 - \gamma_\psi u_1) + \psi^0(\gamma_\psi u_2; \gamma_\psi u_1 - \gamma_\psi u_2)) \le \alpha_\psi c_\Delta^2 \|u_1 - u_2\|_V^2,$$

we find that

$$\|u_1 - u_2\|_V \le \kappa_\theta \|w_1 - w_2\|_V, \qquad \kappa_\theta := \frac{(1 - 2\theta\, m_a + \theta^2 M_a^2)^{1/2} + \theta\, c_d M_f}{1 - \theta\, \alpha_\psi c_\Delta^2}.$$

For $\theta > 0$ sufficiently small, due to (2.26),

$$\kappa_\theta = 1 - \theta \left(m_a - (\alpha_\psi c_\Delta^2 + c_d M_f) \right) + O(\theta^2) < 1,$$

and consequently, $P_\theta \colon V \to V$ is a contraction. By the Banach fixed-point theorem, P_θ has a unique fixed-point $u \in K_f$. It is easy to see that for the fixed-point u, there is an element $p \in Q$ such that (u, p) is a solution of Problem 2.2. Similar to Lemma 2.1, we can show that $u \in K_f$. □

Finally, we consider the well-posedness of Problem 2.1.

Theorem 2.2 *Assume $H(a)$, $H(d)$, $H(\psi)$, $H(f)$, $H(S)$, $H(b)$, (2.23) and (2.26). Then Problem 2.1 has a solution $(u, p) \in V \times Q$, u being unique and $u \in K_f$. Moreover, u depends Lipschitz continuously on f.*

Proof By Theorem 2.1, we can define an operator $P \colon K_f \to K_f$ such that for $\tilde{u} \in K_f$, $u = P(\tilde{u})$ is the unique first component of a solution to Problem 2.2. For $i = 1, 2$, let $\tilde{u}_i \in K_f$ and let $u_i = P(\tilde{u}_i)$. Then for some element $p_i \in Q$,

$$a(u_i, v) + d(\tilde{u}_i, u_i, v) - b(v, p_i) + I_\Delta(\psi^0(\gamma_\psi u_i; \gamma_\psi v)) \ge \langle f, v \rangle \quad \forall\, v \in V, \tag{2.51}$$

$$b(u_i, q) + S(p_i, q) = 0 \quad \forall\, q \in Q. \tag{2.52}$$

We take $v = u_2 - u_1$ in (2.51) for $i = 1$, take $v = u_1 - u_2$ in (2.51) for $i = 2$, and add to obtain

$$\begin{aligned} a(u_1 - u_2, u_1 - u_2) \le\ & d(\tilde{u}_1, u_1, u_2 - u_1) + d(\tilde{u}_2, u_2, u_1 - u_2) \\ & + b(u_1 - u_2, p_1 - p_2) + I_\Delta(\psi^0(\gamma_\psi u_1; \gamma_\psi u_2 - \gamma_\psi u_1) \\ & + \psi^0(\gamma_\psi u_2; \gamma_\psi u_1 - \gamma_\psi u_2)). \end{aligned} \tag{2.53}$$

Recalling (2.6), we can write

$$\begin{aligned} d(\tilde{u}_1, u_1, u_2 - u_1) + d(\tilde{u}_2, u_2, u_1 - u_2) &= d(\tilde{u}_1, u_2, u_2 - u_1) + d(\tilde{u}_2, u_2, u_1 - u_2) \\ &= d(\tilde{u}_1 - \tilde{u}_2, u_2, u_2 - u_1) \end{aligned}$$

and then bound:

$$d(\tilde{u}_1, u_1, u_2 - u_1) + d(\tilde{u}_2, u_2, u_1 - u_2) \le c_d M_f \|\tilde{u}_1 - \tilde{u}_2\|_V \|u_1 - u_2\|_V.$$

By (2.52) for $i = 1, 2$,

$$b(u_1 - u_2, p_1 - p_2) = -S(p_1 - p_2, p_1 - p_2) \le 0.$$

So we derive from (2.53) the inequality

$$m_a \|u_1 - u_2\|_V^2 \le c_d M_f \|\tilde{u}_1 - \tilde{u}_2\|_V \|u_1 - u_2\|_V + \alpha_\psi c_\Delta^2 \|u_1 - u_2\|_V^2.$$

Thus,

$$\|u_1 - u_2\|_V \le \kappa \, \|\tilde{u}_1 - \tilde{u}_2\|_V, \quad \kappa := \frac{c_d M_f}{m_a - \alpha_\psi c_\Delta^2}.$$

By the assumption (2.26), $\kappa < 1$. Hence, $P\colon K_f \to K_f$ is a contraction, admitting a unique fixed-point $u \in K_f$. Together with some element $p \in Q$, (u, p) solves Problem 2.1.

Finally, with $i = 1, 2$, for $f_i \in V^*$, let $(u_i, p_i) \in V \times Q$ solve Problem 2.1:

$$a(u_i, v) + d(u_i, u_i, v) - b(v, p_i) + I_\Delta(\psi^0(\gamma_\psi u_i; \gamma_\psi v)) \ge \langle f_i, v\rangle \quad \forall\, v \in V, \tag{2.54}$$

$$b(u_i, q) + S(p_i, q) = 0 \quad \forall\, q \in Q. \tag{2.55}$$

Take $v = u_2 - u_1$ in (2.54) for $i = 1$, take $v = u_1 - u_2$ in (2.55) for $i = 2$, and add to obtain

$$\begin{aligned} a(u_1 - u_2, u_1 - u_2) &\le d(u_1, u_1, u_2 - u_1) + d(u_2, u_2, u_1 - u_2) + b(u_1 - u_2, p_1 - p_2) \\ &\quad + I_\Delta(\psi^0(\gamma_\psi u_1; \gamma_\psi u_2 - \gamma_\psi u_1) + \psi^0(\gamma_\psi u_2; \gamma_\psi u_1 - \gamma_\psi u_2)) + \langle f_1 - f_2, u_1 - u_2\rangle. \end{aligned}$$

The terms on the right-hand side of the above inequality can be bounded as previously and we deduce that

$$m_a \|u_1 - u_2\|_V^2 \le c_d M_f \|u_1 - u_2\|_V^2 + \alpha_\psi c_\Delta^2 \|u_1 - u_2\|_V^2 + \|f_1 - f_2\|_{V^*} \|u_1 - u_2\|_V.$$

Then,

$$\|u_1 - u_2\|_V \le \frac{1}{m_a - (\alpha_\psi c_\Delta^2 + c_d M_f)} \, \|f_1 - f_2\|_{V^*}.$$

Thus, u depends Lipschitz continuously on f. □

We comment that Theorem 2.2 is rather general. It can be applied to studies of Navier-Stokes hemivariational inequalities by taking $S \equiv 0$ in Problem 2.1, of Stokes hemivariational inequalities by taking $S \equiv 0$ and $d \equiv 0$ in Problem 2.1. We can also apply Theorem 2.2 in the finite-dimensional settings to study mixed finite element methods for solving the hemivariational inequalities using velocity-pressure pairs with the discrete inf-sup condition by taking $S \equiv 0$ or stabilized mixed finite element methods using velocity-pressure pairs without the discrete inf-sup condition. Since the generalized subdifferential reduces to the (convex) subdifferential when the locally Lipschitz function is convex, a slight variation of Theorem 2.2 can be applied to studied of Stokes or Navier-Stokes variational inequalities and their mixed finite element approximations.

3 A Navier-Stokes Hemivariational Inequality

In this section we consider a hemivariational inequality for the Navier-Stokes equations. Let $\Omega \subset \mathbb{R}^d$ ($d \le 3$ in applications) be an open bounded connected set with a Lipschitz boundary. The boundary is split into two parts: $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_S}$ with $|\Gamma_D| > 0$, $|\Gamma_S| > 0$, and $\Gamma_D \cap \Gamma_S = \emptyset$. We will impose a Dirichlet boundary condition on Γ_D and a no-leak slip boundary condition of friction type on Γ_S. Since Ω is a Lipschitz domain, the unit outward normal $\boldsymbol{\nu}$ exists a.e. on $\partial\Omega$. For a vector-valued function $\boldsymbol{u}$ defined on the boundary, the normal component and the tangential component are $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, respectively. With the velocity field $\boldsymbol{u}$ and the pressure p, we define the strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$ and the stress tensor $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})$, where $\boldsymbol{I}$ is the identity matrix, $\mu > 0$ is the viscosity coefficient. The normal component and the tangential component of $\boldsymbol{\sigma}$ are $\sigma_\nu = \boldsymbol{\nu} \cdot \boldsymbol{\sigma}\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. The identities

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_\nu v_\nu + \boldsymbol{u}_\tau \cdot \boldsymbol{v}_\tau, \quad (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{v} = \sigma_\nu v_\nu + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau$$

are useful in derivation of the hemivariational inequality of the following boundary value problem of the Navier-Stokes equations

$$-\operatorname{div}(2\mu\, \boldsymbol{\varepsilon}(\boldsymbol{u})) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{3.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{3.2}$$

supplemented by the boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D, \tag{3.3}$$

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_S. \tag{3.4}$$

Here, $\boldsymbol{f}$ and ψ are given functions, $\psi : \Gamma_S \times \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz continuous with respect to its second argument. To simplify the notation, we write $\psi(\boldsymbol{u}_\tau)$ for $\psi(\boldsymbol{x}, \boldsymbol{u}_\tau)$, and $\partial\psi$ is the subdifferential of ψ in the sense of Clarke with respect to its second argument. The condition (3.4) is a no-leak slip boundary condition of friction type. The first part $u_\nu = 0$ reflects the fact that the fluid can not pass through Γ_S outside the domain. The second part represents a friction condition, relating the frictional force $\boldsymbol{\sigma}_\tau$ with the tangential velocity $\boldsymbol{u}_\tau$.

For the study of the problem (3.1)–(3.4), we introduce function spaces

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D,\ v_\nu = 0 \text{ on } \Gamma_S\}, \tag{3.5}$$

$$Q = L_0^2(\Omega) = \left\{q \in L^2(\Omega) \mid I_\Omega(q) = 0\right\} \tag{3.6}$$

for the velocity and pressure variables. As a consequence of Korn's inequality [33, p. 79],

$$\boldsymbol{V} \ni \boldsymbol{v} \mapsto \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega} := \left(\int_\Omega \sum_{i,j=1}^{d} |\varepsilon_{ij}(\boldsymbol{v})|^2\, d\boldsymbol{x}\right)^{\frac{1}{2}}$$

defines a norm which is equivalent to the standard $\boldsymbol{H}^1(\Omega)$-norm on $\boldsymbol{V}$. We use $\|\cdot\|_{\boldsymbol{V}} = \|\boldsymbol{\varepsilon}(\cdot)\|_{0,\Omega}$ for the norm on $\boldsymbol{V}$ and use the standard $L^2(\Omega)$ norm for Q. Then we introduce forms

$$a(\boldsymbol{u}, \boldsymbol{v}) = 2\mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, d\boldsymbol{x} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}, \tag{3.7}$$

$$b(\boldsymbol{v}, q) = \int_\Omega q\, \operatorname{div}\boldsymbol{v}\, d\boldsymbol{x} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\ q \in Q, \tag{3.8}$$

$$d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\, d\boldsymbol{x} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}. \tag{3.9}$$

Note that the bilinear form $a(\cdot, \cdot)$ is continuous and coercive on $\boldsymbol{V} \times \boldsymbol{V}$:

$$|a(\boldsymbol{u}, \boldsymbol{v})| \le 2\mu \|\boldsymbol{u}\|_{\boldsymbol{V}} \|\boldsymbol{v}\|_{\boldsymbol{V}}, \quad a(\boldsymbol{v}, \boldsymbol{v}) = 2\mu \|\boldsymbol{v}\|_{\boldsymbol{V}}^2 \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}. \tag{3.10}$$

The bilinear form $b(\cdot, \cdot)$ is bounded on $\boldsymbol{V} \times Q$:

$$|b(\boldsymbol{v}, q)| \le c\, \|\boldsymbol{v}\|_{\boldsymbol{V}} \|q\|_Q \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\ q \in Q. \tag{3.11}$$

The trilinear form $d(\cdot, \cdot, \cdot)$ is continuous on $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V}$:

$$|d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le c_d \|\boldsymbol{u}\|_{\boldsymbol{V}} \|\boldsymbol{v}\|_{\boldsymbol{V}} \|\boldsymbol{w}\|_{\boldsymbol{V}} \quad \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}. \tag{3.12}$$

Moreover,

$$d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}. \tag{3.13}$$

Concerning the super-potential ψ, we assume the following properties:
$\underline{H(\psi)}$. $\psi : \Gamma_S \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $\psi(\cdot, \boldsymbol{z})$ is measurable on Γ_S for all $\boldsymbol{z} \in \mathbb{R}^d$ and $\psi(\cdot, \boldsymbol{0}) \in L^1(\Gamma_S)$;
(ii) $\psi(\boldsymbol{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in \Gamma_S$;
(iii) $|\boldsymbol{\eta}|_{\mathbb{R}^d} \le c_0 + c_1 |\boldsymbol{z}|_{\mathbb{R}^d}\ \forall\, \boldsymbol{z} \in \mathbb{R}^d$, $\boldsymbol{\eta} \in \partial\psi(\boldsymbol{x}, \boldsymbol{z})$ a.e. $\boldsymbol{x} \in \Gamma_S$ with $c_0, c_1 \ge 0$;
(iv) $\psi^0(\boldsymbol{x}, \boldsymbol{z}_1; \boldsymbol{z}_2 - \boldsymbol{z}_1) + \psi^0(\boldsymbol{x}, \boldsymbol{z}_2; \boldsymbol{z}_1 - \boldsymbol{z}_2) \le \alpha_\psi |\boldsymbol{z}_1 - \boldsymbol{z}_2|_{\mathbb{R}^d}^2\ \forall\, \boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^d$ a.e. $\boldsymbol{x} \in \Gamma_S$ with $\alpha_\psi \ge 0$.

We assume $\boldsymbol{f} \in \boldsymbol{V}^*$. By a standard procedure, we can derive the following Navier-Stokes hemivariational inequality for the problem (3.1)–(3.4).

Problem 3.1 Find $\boldsymbol{u} \in \boldsymbol{V}$ and $p \in Q$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) + I_{\Gamma_S}(\psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)) \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{3.14}$$

$$b(\boldsymbol{u}, q) = 0 \quad \forall\, q \in Q. \tag{3.15}$$

Problem 3.1 is a special case of the abstract Problem 2.1 with the spaces and forms defined by (3.5)–(3.9), $\Delta = \Gamma_S$, $\gamma_\psi \boldsymbol{v} = \boldsymbol{v}_\tau$ for $\boldsymbol{v} \in \boldsymbol{V}$, $m = d$, and $S \equiv 0$. For the constant c_Δ occurring in (2.10), we have $c_\Delta = \lambda_0^{-1/2}$, $\lambda_0 > 0$ being the smallest eigenvalue of the eigenvalue problem:

$$\boldsymbol{u} \in \boldsymbol{v}, \quad \int_\Omega \sum_{i,j=1}^{d} \varepsilon_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\boldsymbol{v})\, d\boldsymbol{x} = \lambda \int_{\Gamma_S} \boldsymbol{u}_\tau \cdot \boldsymbol{v}_\tau\, ds \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}.$$

We verify that $H(a)$ is valid with $A \in \mathcal{L}(\boldsymbol{V}; \boldsymbol{V}^*)$, $M_a = 2\mu$, $m_a = 2\mu$; $H(\psi)$ is valid by assumption; $H(f)$ is valid since $\boldsymbol{f} \in \boldsymbol{V}^*$; $H(S)$ is trivial since $S \equiv 0$; $H(b)$ is valid and (2.19) follows from the standard inf-sup condition [16, 38]

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}_0} \frac{b(\boldsymbol{v}, p)}{\|\boldsymbol{v}\|_{\boldsymbol{V}}} \geq \alpha_b \|p\|_Q \quad \forall\, p \in Q, \tag{3.16}$$

where $\boldsymbol{V}_0 = \boldsymbol{H}_0^1(\Omega)$ is a subspace of $\boldsymbol{V}$. From (3.16), it can be shown that there exists a constant $\alpha_{\tilde{b}} > 0$ such that

$$\sup_{(\boldsymbol{v},q) \in \boldsymbol{V} \times Q} \frac{a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) + b(\boldsymbol{u}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{V}} + \|q\|_Q} \geq \alpha_{\tilde{b}} \left(\|\boldsymbol{u}\|_{\boldsymbol{V}} + \|p\|_Q \right) \quad \forall\, (\boldsymbol{u}, p) \in \boldsymbol{V} \times Q. \tag{3.17}$$

Let

$$M_{\boldsymbol{f}} := \frac{c_0 \lambda_0^{-1/2} |\Gamma_S|^{1/2} + \|\boldsymbol{f}\|_{\boldsymbol{V}^*}}{2\mu - \alpha_\psi \lambda_0^{-1}}. \tag{3.18}$$

Theorem 3.1 *Assume* $H(\psi)$ *and*

$$\alpha_\psi \lambda_0^{-1} < 2\mu, \quad \alpha_\psi \lambda_0^{-1} + c_d M_{\boldsymbol{f}} < 2\mu. \tag{3.19}$$

Then, Problem 3.1 has a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$, $\|\boldsymbol{u}\|_{\boldsymbol{V}} \leq M_{\boldsymbol{f}}$, *and* $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ *depends Lipschitz continuously on* $\boldsymbol{f} \in \boldsymbol{V}^*$.

Proof We can apply Theorem 2.2 for the special case $S \equiv 0$ to conclude that Problem 3.1 has a solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$, $\boldsymbol{u}$ is unique and depends Lipschitz continuously on $\boldsymbol{f}$, and $\|\boldsymbol{u}\|_{\boldsymbol{V}} \leq M_{\boldsymbol{f}}$.

Uniqueness of p is proved by applying the inf-sup condition (3.16). Suppose both $(\boldsymbol{u}, p_1)$ and $(\boldsymbol{u}, p_2)$ are solutions of Problem 3.1. Then from (3.14),

$$a(\boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p_1) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0,$$
$$a(\boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p_2) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0.$$

Subtract the two equalities,

$$b(\boldsymbol{v}, p_1 - p_2) = 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0.$$

Apply the inf-sup condition (3.16),

$$\alpha_b \|p_1 - p_2\|_Q \le \sup_{\boldsymbol{v}\in \boldsymbol{V}_0} \frac{b(\boldsymbol{v}, p_1 - p_2)}{\|\boldsymbol{v}\|_V} = 0.$$

We conclude that $p_1 = p_2$.

To prove the Lipschitz continuous dependence of p on $\boldsymbol{f}$, let $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \boldsymbol{v}^*$ and let $(\boldsymbol{u}_1, p_1), (\boldsymbol{u}_2, p_2) \in \boldsymbol{V} \times Q$ be the corresponding solutions of Problem 3.1. Then, from (3.14),

$$a(\boldsymbol{u}_1, \boldsymbol{v}) + d(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{v}) - b(\boldsymbol{v}, p_1) = \langle \boldsymbol{f}_1, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0,$$
$$a(\boldsymbol{u}_2, \boldsymbol{v}) + d(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{v}) - b(\boldsymbol{v}, p_2) = \langle \boldsymbol{f}_2, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0.$$

Subtract the two equalities to obtain

$$b(\boldsymbol{v}, p_1 - p_2) = a(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{v}) + d(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{v}) - d(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{v}) - \langle \boldsymbol{f}_1 - \boldsymbol{f}_2, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0. \tag{3.20}$$

Since $\boldsymbol{v} \in \boldsymbol{V}_0$,

$$\begin{aligned} d(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{v}) - d(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{v}) &= -d(\boldsymbol{u}_1, \boldsymbol{v}, \boldsymbol{u}_1) + d(\boldsymbol{u}_2, \boldsymbol{v}, \boldsymbol{u}_2) \\ &= d(\boldsymbol{u}_2 - \boldsymbol{u}_1, \boldsymbol{v}, \boldsymbol{u}_2) + d(\boldsymbol{u}_1, \boldsymbol{v}, \boldsymbol{u}_2 - \boldsymbol{u}_1). \end{aligned}$$

Thus, by the Sobolev embedding $\boldsymbol{v} \hookrightarrow \boldsymbol{L}^4(\Omega)$ and boundedness of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in $\boldsymbol{V}$,

$$\begin{aligned} |d(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{v}) - d(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{v})| &\le c\left(\|\boldsymbol{u}_1\|_{L^4(\Omega)} + \|\boldsymbol{u}_2\|_{L^4(\Omega)}\right)\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^4(\Omega)}\|\boldsymbol{v}\|_V \\ &\le c\,\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V\|\boldsymbol{v}\|_V. \end{aligned}$$

Then we derive from (3.20) that

$$b(\boldsymbol{v}, p_1 - p_2) \le c\left(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{V^*}\right)\|\boldsymbol{v}\|_V.$$

This inequality and the inf-sup condition (3.16) together imply

$$\|p_1 - p_2\|_Q \le c\left(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{V^*}\right).$$

Since $\boldsymbol{u}$ depends Lipschitz continuously on $\boldsymbol{f}$, we deduce from the above inequality that p also depends Lipschitz continuously on $\boldsymbol{f}$. □

We comment that solution existence and uniqueness can also be proved by applying the theory of pseudomonotone operators (cf. [19]). Nevertheless, the approach presented here is more accessible and more general, and it can be naturally applied in the study of stabilized mixed finite element methods to solve Navier-Stokes hemivariational inequalities.

4 Stabilized mixed finite element method for the Navier-Stokes hemivariational inequality

We now consider stabilized mixed finite element methods for solving Problem 3.1. For simplicity, we assume Ω is a polygonal/polyhedral domain in this section. We express $\overline{\Gamma_S}$ as the union of closed flat components with disjoint interior: $\overline{\Gamma_S} = \cup_{l=1}^{l_S} \Gamma_{S,l}$. Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\overline{\Omega}$ into triangles or quadrilaterals in two-dimensions, or tetrahedrons or hexahedrons in three-dimensions. We assume the partitions are compatible with the boundary splitting $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_S}$ in the sense that each face of any element on the boundary lies entirely in $\overline{\Gamma_D}$ or $\overline{\Gamma_S}$. For a generic element $K \in \mathcal{T}^h$, denote by $h_K = \text{diam}(K)$ the diameter of K. The mesh-size of $\mathcal{T}^h$ is $h = \max\{h_K \mid K \in \mathcal{T}^h\}$. For a triangular or tetrahedral element K, let $P_k(K)$ be the space of the polynomials of degree $\le k$ on K. Corresponding to the partition $\mathcal{T}^h$ into triangular or tetrahedral elements, we introduce the finite element spaces

$$\boldsymbol{V}^h = \{\boldsymbol{v}^h \in C^0(\overline{\Omega}) \mid \boldsymbol{v}^h|_K \in P_1(K)^d \ \forall K \in \mathcal{T}^h\} \cap \boldsymbol{V}, \tag{4.1}$$

$$Q_1^h = \{q^h \in C^0(\overline{\Omega}) \mid q^h|_K \in P_1(K) \ \forall K \in \mathcal{T}^h\}, \tag{4.2}$$

$$Q_0^h = \{q^h \in L^2(\Omega) \mid q^h|_K \in P_0(K) \ \forall K \in \mathcal{T}^h\}. \tag{4.3}$$

For quadrilateral or hexahedral partitions, let $\hat{K}$ be a reference element, and let $Q_k(\hat{K})$ be the space of polynomials of individual degrees $\le k$ on $\hat{K}$. For a quadrilateral or hexahedral element K, denote by $F_K \in Q_1(\hat{K})$ the one-to-one mapping from $\hat{K}$ to K. We then introduce the corresponding finite element spaces

$$\boldsymbol{V}^h = \{\boldsymbol{v}^h \in C^0(\overline{\Omega}) \mid \boldsymbol{v}^h|_K \circ F_K \in Q_1(\hat{K})^d \ \forall K \in \mathcal{T}^h\} \cap \boldsymbol{V}, \tag{4.4}$$

$$Q_1^h = \{q^h \in C^0(\overline{\Omega}) \mid q^h|_K \circ F_K \in Q_1(\hat{K}) \ \forall K \in \mathcal{T}^h\}, \tag{4.5}$$

and Q_0^h as in (4.3). In the following, we let $Q^h = Q_1^h \cap Q$ for Q_1^h defined by (4.2) or (4.5), or $Q^h = Q_0^h \cap Q$ for Q_0^h defined by (4.3).

The stabilizing term $S^h(p, q)$ is defined by

$$S^h(p, q) = I_\Omega((p - \Pi^h p)(q - \Pi^h q)) \quad \forall\, p, q \in Q, \tag{4.6}$$

where the projection operator Π^h has the following property.

$\underline{H(\Pi^h)}$ $\Pi^h \in \mathcal{L}(L^2(\Omega); L^2(\Omega))$; $\Pi^h = \Pi_0^h\colon L^2(\Omega) \to Q_0^h$ for $Q^h = Q_1^h \cap Q$ with Q_1^h defined by (4.2) or (4.5), and $\Pi^h = \Pi_1^h\colon L^2(\Omega) \to Q_1^h$ for $Q^h = Q_0^h \cap Q$ with Q_0^h defined by (4.3).

Introduce the extended bilinear form for $(\boldsymbol{u}^h, p^h), (\boldsymbol{v}^h, q^h) \in \boldsymbol{V}^h \times Q^h$:

$$\tilde{b}^h((\boldsymbol{u}^h, p^h); (\boldsymbol{v}^h, q^h)) = a(\boldsymbol{u}^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p^h) + b(\boldsymbol{u}^h, q^h) + S^h(p^h, q^h). \tag{4.7}$$

The following inf-sup condition result for the bilinear form (4.7) is shown in [30]; see [5] in the case $\boldsymbol{V}^h$ is replaced by $\boldsymbol{V}_0^h = \boldsymbol{V}^h \cap \boldsymbol{H}_0^1(\Omega)$.

Lemma 4.1 *There exists a positive constant c independent of h such that*

$$\sup_{(\boldsymbol{v}^h, q^h) \in \boldsymbol{V}^h \times Q^h} \frac{\tilde{b}^h((\boldsymbol{u}^h, p^h); (\boldsymbol{v}^h, q^h))}{\|\boldsymbol{v}^h\|_V + \|q^h\|_Q} \ge c \left(\|\boldsymbol{u}^h\|_V + \|p^h\|_Q\right) \forall (\boldsymbol{u}^h, p^h) \in \boldsymbol{V}^h \times Q^h, \tag{4.8}$$

$$\sup_{(\boldsymbol{v}^h, q^h) \in \boldsymbol{V}_0^h \times Q^h} \frac{\tilde{b}^h((\boldsymbol{0}, p^h); (\boldsymbol{v}^h, q^h))}{\|\boldsymbol{v}^h\|_V + \|q^h\|_Q} \ge c \, \|p^h\|_Q \quad \forall \, p^h \in Q^h. \tag{4.9}$$

We now consider the stabilized mixed finite element method for Problem 3.1.

Problem 4.1 Find $(\boldsymbol{u}^h, p^h) \in \boldsymbol{V}^h \times Q^h$ such that

$$a(\boldsymbol{u}^h, \boldsymbol{v}^h) + d(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p^h) + I_{\Gamma_S}(\psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h)) \ge \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall \, \boldsymbol{v}^h \in \boldsymbol{V}^h, \tag{4.10}$$

$$b(\boldsymbol{u}^h, q^h) + S^h(p^h, q^h) = 0 \quad \forall q^h \in Q^h. \tag{4.11}$$

Theorem 4.1 *Assume $H(\psi)$, (3.19), and $H(\Pi^h)$. Then, there is a unique solution $(\boldsymbol{u}^h, p^h) \in \boldsymbol{V}^h \times Q^h$ to Problem 4.1, $\|\boldsymbol{u}^h\|_V \le M_{\boldsymbol{f}}$, and $(\boldsymbol{u}^h, p^h) \in \boldsymbol{V}^h \times Q^h$ depends Lipschitz continuously on $\boldsymbol{f} \in \boldsymbol{V}^*$.*

Proof Similar to the proof of Theorem 3.1, we can apply Theorem 2.2 in the finite-dimensional setting and conclude that under the stated conditions, Problem 4.1 has a solution $(\boldsymbol{u}^h, p^h) \in \boldsymbol{V}^h \times Q^h$, $\boldsymbol{u}^h$ is unique, $\|\boldsymbol{u}^h\|_V \le M_{\boldsymbol{f}}$ with $M_{\boldsymbol{f}}$ defined by (3.18), and $\boldsymbol{u}^h$ depends Lipschitz continuously on $\boldsymbol{f} \in \boldsymbol{V}^*$. To show the uniqueness of p^h, let $(\boldsymbol{u}^h, p_1^h), (\boldsymbol{u}^h, p_2^h) \in \boldsymbol{V}^h \times Q^h$ be solutions of Problem 4.1. Then,

$$b(\boldsymbol{u}^h, q^h) + S^h(p_1^h, q^h) = 0 \quad \forall q^h \in Q^h,$$
$$b(\boldsymbol{u}^h, q^h) + S^h(p_2^h, q^h) = 0 \quad \forall q^h \in Q^h.$$

Thus,

$$S^h(p_1^h - p_2^h, q^h) = 0 \quad \forall q^h \in Q^h.$$

In particular, $S^h(p_1^h - p_2^h, p_1^h - p_2^h) = 0$ and hence,

$$p_1^h - p_2^h = \Pi^h(p_1^h - p_2^h). \tag{4.12}$$

In the case $Q^h = Q_1^h \cap Q$, we have $\Pi^h = \Pi_0^h\colon L^2(\Omega) \to Q_0^h$, and in the case $Q^h = Q_0^h \cap Q$, we have $\Pi^h = \Pi_1^h\colon L^2(\Omega) \to Q_1^h$. In either case, $p_1^h - p_2^h$ is a continuous piecewise constant function with a vanishing integral over Ω. Therefore, $p_1^h = p_2^h$.

For the Lipschitz continuous dependence of p^h on $\boldsymbol{f}$, let $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \boldsymbol{V}^*$ and let $(\boldsymbol{u}_1^h, p_1^h), (\boldsymbol{u}_2^h, p_2^h) \in \boldsymbol{V}^h \times Q^h$ be the corresponding solutions of Problem 4.1. Then we can derive from the defining relations for the solutions that

$$\begin{aligned} & a(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, \boldsymbol{v}^h) + d(\boldsymbol{u}_1^h, \boldsymbol{u}_1^h, \boldsymbol{v}^h) - d(\boldsymbol{u}_2^h, \boldsymbol{u}_2^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p_1^h - p_2^h) \\ & \quad = \langle \boldsymbol{f}_1 - \boldsymbol{f}_2, \boldsymbol{v}^h \rangle \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}_0^h, \end{aligned} \tag{4.13}$$

$$b(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, q^h) + S^h(p_1^h - p_2^h, q^h) = 0 \quad \forall\, q^h \in Q^h. \tag{4.14}$$

From (4.9),

$$\|p_1^h - p_2^h\|_Q \le c \sup_{(\boldsymbol{v}^h, q^h) \in \boldsymbol{V}_0^h \times Q^h} \frac{-b(\boldsymbol{v}^h, p_1^h - p_2^h) + S^h(p_1^h - p_2^h, q^h)}{\|\boldsymbol{v}^h\|_V + \|q^h\|_Q}.$$

By making use of (4.13) and (4.14), we can rewrite the numerator of the fraction in the above inequality as

$$\langle \boldsymbol{f}_1 - \boldsymbol{f}_2, \boldsymbol{v}^h \rangle - a(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, \boldsymbol{v}^h) - \left[d\big(\boldsymbol{u}_1^h, \boldsymbol{u}_1^h, \boldsymbol{v}^h\big) - d\big(\boldsymbol{u}_2^h, \boldsymbol{u}_2^h, \boldsymbol{v}^h\big) \right] - b\big(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, q^h\big),$$

which can be bounded by

$$c \left(\|\boldsymbol{u}_1^h - \boldsymbol{u}_2^h\|_V + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{V^*} \right) \|\boldsymbol{v}^h\|_V + c\, \|\boldsymbol{u}_1^h - \boldsymbol{u}_2^h\|_V \|q^h\|_Q$$

as in the last part of the proof of Theorem 3.1. Therefore,

$$\|p_1^h - p_2^h\|_Q \le c \left(\|\boldsymbol{u}_1^h - \boldsymbol{u}_2^h\|_V + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{V^*} \right),$$

and p^h depends Lipschitz continuously on $\boldsymbol{f}$. □

Now we turn to an error analysis. Define a bilinear form

$$\tilde{b}((\boldsymbol{u}, p); (\boldsymbol{v}, q)) = a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) + b(\boldsymbol{u}, q), \quad (\boldsymbol{u}, p), (\boldsymbol{v}, q) \in \boldsymbol{V} \times Q. \tag{4.15}$$

Then (3.14)–(3.15) are equivalent to

$$\tilde{b}((\boldsymbol{u}, p); (\boldsymbol{v}, q)) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + I_{\Gamma_S}(\psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)) \ge \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, (\boldsymbol{v}, q) \in \boldsymbol{V} \times Q, \tag{4.16}$$

whereas (4.10)–(4.11) are equivalent to

$$\tilde{b}^h((\boldsymbol{u}^h, p^h); (\boldsymbol{v}^h, q^h)) + d(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) + I_{\Gamma_S}(\psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h)) \geq \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \forall (\boldsymbol{v}^h, q^h) \in \boldsymbol{V}^h \times Q^h. \tag{4.17}$$

For $(\boldsymbol{v}^h, q^h) \in \boldsymbol{V}^h \times Q^h$ arbitrary, we write

$$\begin{aligned} 2\,\mu\, \|\boldsymbol{u}^h - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 &\leq \tilde{b}^h((\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)) \\ &= \tilde{b}^h((\boldsymbol{u}^h, p^h); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)) \\ &\quad + \tilde{b}^h((-\boldsymbol{v}^h, -q^h); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)) \\ &= \tilde{b}^h((\boldsymbol{u}^h, p^h); (\boldsymbol{u}^h \boldsymbol{v}^h, p^h - q^h)) - \tilde{b}((\boldsymbol{u}, p); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)) \\ &\quad + \tilde{b}((\boldsymbol{u} - \boldsymbol{v}^h, p - q^h); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)) - S^h(q^h, p^h - q^h). \end{aligned} \tag{4.18}$$

From (4.17),

$$\begin{aligned} \tilde{b}^h((\boldsymbol{u}^h, p^h); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)) \leq d(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h) \\ + I_{\Gamma_S}(\psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h)) - \langle \boldsymbol{f}, \boldsymbol{v}^h - \boldsymbol{u}^h \rangle. \end{aligned}$$

From (4.16),

$$\begin{aligned} -\tilde{b}((\boldsymbol{u}, p); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)) \leq d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}^h - \boldsymbol{v}^h) \\ + I_{\Gamma_S}(\psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{v}_\tau^h)) - \langle \boldsymbol{f}, \boldsymbol{u}^h - \boldsymbol{v}^h \rangle. \end{aligned}$$

Use these two relations in (4.18) to obtain

$$2\,\mu\, \|\boldsymbol{u}^h - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 \leq I_1 + I_2 + I_3 + I_4, \tag{4.19}$$

where

$$\begin{aligned} I_1 &= d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}^h - \boldsymbol{v}^h) + d(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h), \\ I_2 &= I_{\Gamma_S}(\psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{v}_\tau^h) + \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h)), \\ I_3 &= \tilde{b}((\boldsymbol{u} - \boldsymbol{v}^h, p - q^h); (\boldsymbol{u}^h - \boldsymbol{v}^h, p^h - q^h)), \\ I_4 &= -S^h(q^h, p^h - q^h). \end{aligned}$$

Let us bound each of these four terms. Write

$$\begin{aligned} I_1 &= d(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h) + d(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h) \\ &= d(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{v}^h) + d(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h), \end{aligned}$$

where we used the equality $d(\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{u}^h, \boldsymbol{u}^h-\boldsymbol{u})=0$. Then

$$\begin{aligned} I_1 &\le c_d \|\boldsymbol{u}\|_V \|\boldsymbol{u}-\boldsymbol{u}^h\|_V \|\boldsymbol{u}-\boldsymbol{v}^h\|_V + c_d \|\boldsymbol{u}-\boldsymbol{u}^h\|_V \|\boldsymbol{u}^h\|_V \|\boldsymbol{u}^h-\boldsymbol{v}^h\|_V \\ &\le c_d M_{\boldsymbol{f}} \|\boldsymbol{u}-\boldsymbol{u}^h\|_V \|\boldsymbol{u}-\boldsymbol{v}^h\|_V + c_d M_{\boldsymbol{f}} \|\boldsymbol{u}-\boldsymbol{u}^h\|_V \|\boldsymbol{u}^h-\boldsymbol{v}^h\|_V. \end{aligned}$$

We then use the triangle inequality

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_V \le \|\boldsymbol{u}-\boldsymbol{v}^h\|_V + \|\boldsymbol{u}^h-\boldsymbol{v}^h\|_V \tag{4.20}$$

and the modified Cauchy-Schwarz inequality

$$a\,b \le \epsilon\, a^2 + (4\,\epsilon)^{-1} b^2 \quad \forall\, a, b \in \mathbb{R},\ \forall\, \epsilon > 0, \tag{4.21}$$

and find that for some ϵ-dependent constant $c > 0$,

$$I_1 \le \left(c_d M_{\boldsymbol{f}} + 2\,\epsilon\right) \|\boldsymbol{u}^h-\boldsymbol{v}^h\|_V^2 + c\,\|\boldsymbol{u}-\boldsymbol{v}^h\|_V^2.$$

To bound the term I_2, we first notice that by (1.2),

$$\begin{aligned} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h-\boldsymbol{v}_\tau^h) &\le \psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h-\boldsymbol{u}_\tau) + \psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h), \\ \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h-\boldsymbol{u}_\tau^h) &\le \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{u}_\tau-\boldsymbol{u}_\tau^h) + \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h-\boldsymbol{u}_\tau). \end{aligned}$$

By $H(\psi)$ (iv),

$$\psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h-\boldsymbol{u}_\tau) + \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{u}_\tau-\boldsymbol{u}_\tau^h) \le \alpha_\psi |\boldsymbol{u}_\tau-\boldsymbol{u}_\tau^h|_{\mathbb{R}^d}^2.$$

By $H(\psi)$ (iii),

$$\begin{aligned} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h) &\le \left(c_0 + c_1|\boldsymbol{u}_\tau|_{\mathbb{R}^d}\right) |\boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h|_{\mathbb{R}^d}, \\ \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h-\boldsymbol{u}_\tau) &\le \left(c_0 + c_1|\boldsymbol{u}_\tau^h|_{\mathbb{R}^d}\right) |\boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h|_{\mathbb{R}^d}. \end{aligned}$$

Hence, using the Cauchy-Schwarz inequality and the bounds $\|\boldsymbol{u}\|_V \le M_{\boldsymbol{f}}$, $\|\boldsymbol{u}^h\|_V \le M_{\boldsymbol{f}}$,

$$\begin{aligned} I_2 &\le \alpha_\psi \|\boldsymbol{u}_\tau-\boldsymbol{u}_\tau^h\|_{L^2(\Gamma_S)}^2 + I_{\Gamma_S}((2\,c_0 + c_1|\boldsymbol{u}_\tau|_{\mathbb{R}^d} + c_1|\boldsymbol{u}_\tau^h|_{\mathbb{R}^d})|\boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h|_{\mathbb{R}^d}) \\ &\le \alpha_\psi \|\boldsymbol{u}_\tau-\boldsymbol{u}_\tau^h\|_{L^2(\Gamma_S)}^2 + c\,\|\boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h\|_{L^2(\Gamma_S)} \\ &\le \alpha_\psi \lambda_0^{-1} \|\boldsymbol{u}-\boldsymbol{u}^h\|_V^2 + c\,\|\boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h\|_{L^2(\Gamma_S)}. \end{aligned}$$

Applying the triangle inequality (4.20) and the modified Cauchy-Schwarz inequality (4.21) to the first term on the right side, we obtain

$$I_2 \le \left(\alpha_\psi \lambda_0^{-1} + \epsilon\right) \|\boldsymbol{u}^h-\boldsymbol{v}^h\|_V^2 + c\left(\|\boldsymbol{u}-\boldsymbol{v}^h\|_V^2 + \|\boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h\|_{L^2(\Gamma_S)}\right)$$

for some ϵ-dependent constant $c > 0$. For

$$I_3 = a(\boldsymbol{u} - \boldsymbol{v}^h, \boldsymbol{u}^h - \boldsymbol{v}^h) - b(\boldsymbol{u}^h - \boldsymbol{v}^h, p - q^h) + b(\boldsymbol{u} - \boldsymbol{v}^h, p^h - q^h),$$

we use the boundedness of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, and apply the modified Cauchy-Schwarz inequality (4.21) to the term

$$a(\boldsymbol{u} - \boldsymbol{v}^h, \boldsymbol{u}^h - \boldsymbol{v}^h) \leq 2\,\mu\, \|\boldsymbol{u}^h - \boldsymbol{v}^h\|_{\boldsymbol{V}} \|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}}.$$

As a result, for some ϵ-dependent constant $c > 0$,

$$I_3 \leq \epsilon\, \|\boldsymbol{u}^h - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 + c \left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 + \|p - q^h\|_Q^2 + \|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}} \|p^h - q^h\|_Q \right).$$

Write

$$I_4 = S^h(p - q^h, p^h - q^h) - S^h(p, p^h - q^h)$$

and bound the term as follows:

$$I_4 \leq \left(\|p - q^h\|_Q + \|p - \Pi^h p\|_Q \right) \|p^h - q^h\|_Q.$$

Summarizing, from (4.19) and the above inequalities,

$$\begin{aligned}
&\left(2\,\mu - \alpha_\psi \lambda_0^{-1} - c_d M_{\boldsymbol{f}} - 4\,\epsilon\right) \|\boldsymbol{u}^h - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 \\
&\quad \leq c \left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 + \|p - q^h\|_Q^2 + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{\boldsymbol{L}^2(\Gamma_S)} \right) \\
&\qquad + c \left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}} + \|p - q^h\|_Q + \|p - \Pi^h p\|_Q \right) \|p^h - q^h\|_Q.
\end{aligned}$$

Then take $\epsilon = \left(2\,\mu - \alpha_\psi \lambda_0^{-1} - c_d M_{\boldsymbol{f}}\right)/8$ to get

$$\begin{aligned}
\|\boldsymbol{u}^h - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 &\leq c \left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 + \|p - q^h\|_Q^2 + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{\boldsymbol{L}^2(\Gamma_S)} \right) \\
&\quad + c \left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}} + \|p - q^h\|_Q + \|p - \Pi^h p\|_Q \right) \|p^h - q^h\|_Q.
\end{aligned}$$

From this inequality and the triangle inequality (4.20), we obtain

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}}^2 &\leq c \left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}}^2 + \|p - q^h\|_Q^2 + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{\boldsymbol{L}^2(\Gamma_S)} \right) \\
&\quad + c \left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}} + \|p - q^h\|_Q + \|p - \Pi^h p\|_Q \right) \|p^h - q^h\|_Q.
\end{aligned} \tag{4.22}$$

By (4.9),

$$c\,\|p^h - q^h\|_Q \le \sup_{(\boldsymbol{v}^h, r^h)\in \boldsymbol{V}_0^h\times Q^h} \frac{-b(\boldsymbol{v}^h, p^h - q^h) + S^h(p^h - q^h, r^h)}{\|\boldsymbol{v}^h\|_{\boldsymbol{V}} + \|r^h\|_Q}. \tag{4.23}$$

Take $\boldsymbol{v} = \boldsymbol{v}^h \in \boldsymbol{V}_0^h$ arbitrary in (3.14) and (4.10):

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}^h) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p) &= \langle \boldsymbol{f}, \boldsymbol{v}^h\rangle,\\ a(\boldsymbol{u}^h, \boldsymbol{v}^h) + d(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p^h) &= \langle \boldsymbol{f}, \boldsymbol{v}^h\rangle. \end{aligned}$$

Then,

$$\begin{aligned} b(\boldsymbol{v}^h, p - p^h) &= a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}^h) - d(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h)\\ &= a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) + d(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}, \boldsymbol{v}^h) + d(\boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h). \end{aligned}$$

From (3.15) and (4.11), for any $q^h \in Q^h$,

$$\begin{aligned} &b(\boldsymbol{u}, q^h) = 0,\\ &b(\boldsymbol{u}^h, q^h) + S^h(p^h, q^h) = 0. \end{aligned}$$

Thus,

$$S^h(p^h, r^h) = -b(\boldsymbol{u}^h, r^h) = b(\boldsymbol{u} - \boldsymbol{u}^h, r^h) \quad \forall\, r^h \in Q^h.$$

So we can write

$$\begin{aligned} &-b(\boldsymbol{v}^h, p^h - q^h) + S^h(p^h - q^h, r^h)\\ &\quad = -b(\boldsymbol{v}^h, p^h - p) - b(\boldsymbol{v}^h, p - q^h) + b(\boldsymbol{u} - \boldsymbol{u}^h, r^h) - S^h(q^h, r^h)\\ &\quad = a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) + d(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}, \boldsymbol{v}^h) + d(\boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h)\\ &\qquad - b(\boldsymbol{v}^h, p - q^h) + b(\boldsymbol{u} - \boldsymbol{u}^h, r^h) + S^h(p - q^h, r^h) - S^h(p, r^h). \end{aligned}$$

Note that

$$\begin{aligned} d(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}, \boldsymbol{v}^h) + d(\boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) &\le c_d \left(\|\boldsymbol{u}\|_{\boldsymbol{V}} + \|\boldsymbol{u}^h\|_{\boldsymbol{V}}\right) \|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} \|\boldsymbol{v}^h\|_{\boldsymbol{V}}\\ &\le c\,\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} \|\boldsymbol{v}^h\|_{\boldsymbol{V}} \end{aligned}$$

since $\|\boldsymbol{u}\|_{\boldsymbol{V}}$ and $\|\boldsymbol{u}^h\|_{\boldsymbol{V}}$ are bounded by $M_{\boldsymbol{f}}$. Therefore,

$$\begin{aligned} &-b(\boldsymbol{v}^h, p^h - q^h) + S^h(p^h - q^h, r^h)\\ &\quad \le c\left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} + \|p - q^h\|_Q\right) \|\boldsymbol{v}^h\|_{\boldsymbol{V}}\\ &\qquad + c\left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} + \|p - q^h\|_Q + \|p - \Pi^h p\|_Q\right) \|r^h\|_Q. \end{aligned}$$

Use this inequality in (4.23) to obtain

$$\|p^h - q^h\|_Q \le c\left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} + \|p - q^h\|_Q + \|p - \Pi^h p\|_Q\right). \tag{4.24}$$

This inequality, combined with the triangle inequality $\|p - p^h\|_Q \le \|p - q^h\|_Q + \|p^h - q^h\|_Q$, leads to

$$\|p - p^h\|_Q \le c\left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} + \|p - q^h\|_Q + \|p - \Pi^h p\|_Q\right). \tag{4.25}$$

From (4.22) and (4.24),

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} \le c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}} + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|^{1/2}_{\boldsymbol{L}^2(\Gamma_S)} + \|p - q^h\|_Q + \|p - \Pi^h p\|_Q\right). \tag{4.26}$$

Then from (4.25),

$$\|p - p^h\|_Q \le c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}} + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|^{1/2}_{\boldsymbol{L}^2(\Gamma_S)} + \|p - q^h\|_Q + \|p - \Pi^h p\|_Q\right). \tag{4.27}$$

Based on (4.26)–(4.27), we can then derive optimal order error estimates for the pressure projection stabilized $\boldsymbol{P}_1$-P_1 and $\boldsymbol{P}_1$-P_0 mixed finite element solutions by an application of the standard finite element interpolation error bounds [7], under certain solution regularity assumptions. We summarize the results in the form of a theorem.

Theorem 4.2 *Assume $H(\psi)$, (3.19), and $H(\Pi^h)$. Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}^h, p^h)$ be the solutions of Problems 3.1 and 4.1. Then for any $(\boldsymbol{v}^h, q^h) \in \boldsymbol{V}^h \times Q^h$,*

$$\begin{aligned}&\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{V}} + \|p - p^h\|_Q \\ &\quad \le c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{\boldsymbol{V}} + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|^{1/2}_{\boldsymbol{L}^2(\Gamma_S)} + \|p - q^h\|_Q + \|p - \Pi^h p\|_Q\right).\end{aligned}$$

Assume the solution regularities $\boldsymbol{u} \in H^2(\Omega)^d$, $\boldsymbol{u}_\tau|_{\Gamma_{S,l}} \in H^2(\Gamma_{S,l})^d$, $1 \le l \le l_S$, $p \in H^1(\Omega)$, and assume the projection error bound

$$\|q - \Pi^h q\|_Q \le c\, h\, \|q\|_{H^1(\Omega)} \quad \forall\, q \in H^1(\Omega). \tag{4.28}$$

Then we have the error bound

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{V}} + \|p - p_h\|_Q \le c\, h. \tag{4.29}$$

Following [5, Section 6], we can choose Π_0^h as the piecewise L^2-projection operator: $\Pi_0^h|_K = \Pi_0^K$ for $K \in \mathcal{T}^h$ and

$$\Pi_0^K q = \frac{1}{|K|}\int_K q\, dx \quad \forall\, q \in L^2(K),$$

and we can choose $\Pi_1^h\colon L^2(\Omega) \to Q_1^h$ as a Clément-like interpolant. For both projection operators, the error bound (4.28) holds.

5 Numerical results

In the numerical example, the domain Ω is two-dimensional. We consider the superpotential function of the form

$$\psi(\boldsymbol{v}_\tau) = \int_0^{|\boldsymbol{v}_\tau|} g(t)\,dt, \quad g(t) = (a-b)e^{-\alpha t} + b,\ \alpha > 0,\ a > b > 0.$$

Then the boundary condition

$$-\boldsymbol{\sigma}_\tau \in \partial\psi(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_S$$

is equivalent to

$$|\boldsymbol{\sigma}_\tau| \le g(0) \text{ if } \boldsymbol{u}_\tau = \boldsymbol{0}, \quad \boldsymbol{\sigma}_\tau = -g(|\boldsymbol{u}_\tau|)\,\frac{\boldsymbol{u}_\tau}{|\boldsymbol{u}_\tau|} \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0}, \quad \text{on } \Gamma_S. \tag{5.1}$$

For the purpose of introducing Algorithm 1 below for solving Problem 4.1, let us reformulate Problem 3.1 and its finite element approximation. From (5.1), we know that $\boldsymbol{\sigma}_\tau \in \boldsymbol{L}^\infty(\Gamma_S)$. Introduce a Lagrangian multiplier $\boldsymbol{\lambda} = -\boldsymbol{\sigma}_\tau/g(|\boldsymbol{u}_\tau|)$, which belongs to the set

$$\boldsymbol{\Lambda} = \left\{\boldsymbol{\lambda} \in \boldsymbol{L}^\infty(\Gamma_S) \mid \|\boldsymbol{\lambda}\|_{\boldsymbol{L}^\infty(\Gamma_S)} \le 1\right\}.$$

Then another weak formulation of the problem (3.1)–(3.4) is the following.

Problem 5.1 Find $\boldsymbol{u} \in \boldsymbol{V}$, $p \in Q$, and $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that

$$\begin{cases} a(\boldsymbol{u},\boldsymbol{v}) + d(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) - b(\boldsymbol{v},p) + I_{\Gamma_S}(g(|\boldsymbol{u}_\tau|)\boldsymbol{\lambda}\cdot\boldsymbol{v}_\tau) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, & (5.2a)\\ b(\boldsymbol{u},q) = 0 \quad \forall\, q \in Q, & (5.2b)\\ \boldsymbol{\lambda}\cdot\boldsymbol{u}_\tau = |\boldsymbol{u}_\tau| \quad \text{a.e. on } \Gamma_S. & (5.2c)\end{cases}$$

Here, (5.2a) is derived from (3.1) together with the boundary conditions and the definition of $\boldsymbol{\lambda}$, whereas $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ and (5.2c) are equivalent to (5.1) for $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}_\tau$ related by $\boldsymbol{\lambda} = -\boldsymbol{\sigma}_\tau/g(|\boldsymbol{u}_\tau|)$. The stabilized mixed finite element method for Problem 5.1 is the following.

Problem 5.2 Find $\boldsymbol{u}^h \in \boldsymbol{V}^h$, $p^h \in Q^h$, and $\boldsymbol{\lambda}^h \in \boldsymbol{\Lambda}$ such that

$$\begin{cases} a(\boldsymbol{u}^h,\boldsymbol{v}^h) + d(\boldsymbol{u}^h;\boldsymbol{u}^h,\boldsymbol{v}^h) - b(\boldsymbol{v}^h,p^h) + I_{\Gamma_S}(g(|\boldsymbol{u}_\tau^h|)\boldsymbol{\lambda}^h\cdot\boldsymbol{v}_\tau^h) & \\ \quad = \langle \boldsymbol{f},\boldsymbol{v}^h\rangle \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h, & (5.3a)\\ b(\boldsymbol{u}^h,q^h) + S^h(p^h,q^h) = 0 \quad \forall\, q^h \in Q^h, & (5.3b)\\ \boldsymbol{\lambda}^h\cdot\boldsymbol{u}_\tau^h = |\boldsymbol{u}_\tau^h| \quad \text{a.e. on } \Gamma_S. & (5.3c)\end{cases}$$

Note that $\boldsymbol{\lambda}^h \in \boldsymbol{\Lambda}$ and (5.3c) together imply $g(|\boldsymbol{u}_\tau^h|)\boldsymbol{\lambda}^h \in \partial\psi(\boldsymbol{u}_\tau^h)$. Hence, we can eliminate $\boldsymbol{\lambda}^h$ and derive Problem 4.1 from Problem 5.2. In other words, $(\boldsymbol{u}^h, p^h) \in \boldsymbol{V}^h \times Q^h$ obtained from Problem 5.2 is the solution of Problem 4.1.

We use the following iterative algorithm to solve Problem 5.2 [17].

Algorithm 1

Step 1. Choose $\boldsymbol{u}_0^h \in \boldsymbol{V}^h$, $\boldsymbol{\lambda}_1^h \in \boldsymbol{L}^\infty(\Gamma_S)$ and $\rho > 0$, $n := 1$.

Step 2. With $\boldsymbol{\lambda}_n^h$ known, compute $(\boldsymbol{u}_n^h, p_n^h) \in \boldsymbol{V}^h \times Q^h$ from

$$\begin{cases} a(\boldsymbol{u}_n^h, \boldsymbol{v}^h) + d(\boldsymbol{u}_{n-1}^h; \boldsymbol{u}_{n-1}^h, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p_n^h) + I_{\Gamma_S}(g(|\boldsymbol{u}_{n-1,\tau}^h|)\,\boldsymbol{\lambda}_n^h \cdot \boldsymbol{v}_\tau^h) = \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall \boldsymbol{v}^h \in \boldsymbol{V}^h, \\ b(\boldsymbol{u}_n^h, q^h) + S^h(p_n^h, q^h) = 0 \quad \forall q^h \in Q^h. \end{cases} \tag{5.4}$$

Step 3. Set $n := n + 1$ and compute $\boldsymbol{\lambda}_{n+1}^h = P(\boldsymbol{\lambda}_n^h + \rho \boldsymbol{u}_{n,\tau}^h)$.

Step 4. Go to Steps 2 and 3 until $\|\boldsymbol{u}_n^h - \boldsymbol{u}_{n-1}^h\|_{\boldsymbol{V}} < \varepsilon_{tol}$.

In Step 3, P is the component-wise operator projecting each component to $[-1, 1]$. Algorithm 1 is known as the projection method, also called the Uzawa algorithm, cf. [17] for similar algorithms in solving elliptic variational inequalities. In our simulations, the trapezoidal rule is adopted for the term $I_{\Gamma_S}(\cdot)$, we choose $\boldsymbol{u}_0^h = \boldsymbol{0}$, $\boldsymbol{\lambda}_1^h = \boldsymbol{0}$, and $\varepsilon_{tol} = 10^{-6}$.

In the numerical experiment, we let $\Omega = (0, 1) \times (0, 1)$ be the unit square and we take the parameter $\mu = 1$. The impermeability and slip boundary conditions are imposed along the bottom of the domain $\Gamma_S = (0, 1) \times \{0\}$ and a homogeneous

Table 1 Errors of stabilized $\boldsymbol{P}_1$–P_1 FE solutions

h	$a = 0.255$	$b = 0.25$		$a = 0.85$	$b = 0.8$		$a = 5.01$	$b = 5.0$	
	$E_{L^2}^{\boldsymbol{u}}$	$E_{H^1}^{\boldsymbol{u}}$	$E_{L^2}^{p}$	$E_{L^2}^{\boldsymbol{u}}$	$E_{H^1}^{\boldsymbol{u}}$	$E_{L^2}^{p}$	$E_{L^2}^{\boldsymbol{u}}$	$E_{H^1}^{\boldsymbol{u}}$	$E_{L^2}^{p}$
2^{-3}	1.65e−02	1.30e−01	3.87e−01	1.64e−02	1.30e−01	4.01e−01	1.78e−02	2.46e−01	3.67e−01
2^{-4}	4.59e−03	4.42e−02	1.20e−01	4.60e−03	4.45e−02	1.22e−01	4.77e−03	1.12e−01	1.13e−01
2^{-5}	1.19e−03	1.44e−02	3.61e−02	1.19e−03	1.57e−02	3.80e−02	1.23e−03	5.26e−02	3.48e−02
2^{-6}	2.87e−04	4.63e−03	1.03e−02	2.89e−04	5.45e−03	1.12e−02	3.10e−04	2.55e−02	1.08e−02
Order	2.05	1.64	1.81	2.05	1.53	1.76	1.99	1.04	1.69

Table 2 Errors of stabilized $\boldsymbol{P}_1$-P_0 FE solutions

h	$a = 0.255$	$b = 0.25$		$a = 0.85$	$b = 0.8$		$a = 5.01$	$b = 5.0$	
	$E_{L^2}^{\boldsymbol{u}}$	$E_{H^1}^{\boldsymbol{u}}$	$E_{L^2}^{p}$	$E_{L^2}^{\boldsymbol{u}}$	$E_{H^1}^{\boldsymbol{u}}$	$E_{L^2}^{p}$	$E_{L^2}^{\boldsymbol{u}}$	$E_{H^1}^{\boldsymbol{u}}$	$E_{L^2}^{p}$
2^{-3}	6.33e−02	4.38e−01	1.37e+00	6.09e−02	4.75e−01	1.51e+00	6.21e−02	5.28e−01	1.36e+00
2^{-4}	2.43e−02	1.86e−01	5.66e−01	2.41e−02	2.05e−01	6.24e−01	2.46e−02	3.45e−01	5.57e−01
2^{-5}	7.23e−03	6.61e−02	2.15e−01	7.27e−03	7.26e−02	2.33e−01	7.51e−03	9.10e−02	1.94e−01
2^{-6}	1.87e−03	2.11e−02	7.87e−02	1.89e−03	2.29e−02	8.37e−02	2.05e−03	3.55e−02	6.29e−02
Order	1.95	1.65	1.45	1.94	1.66	1.48	1.87	1.36	1.62

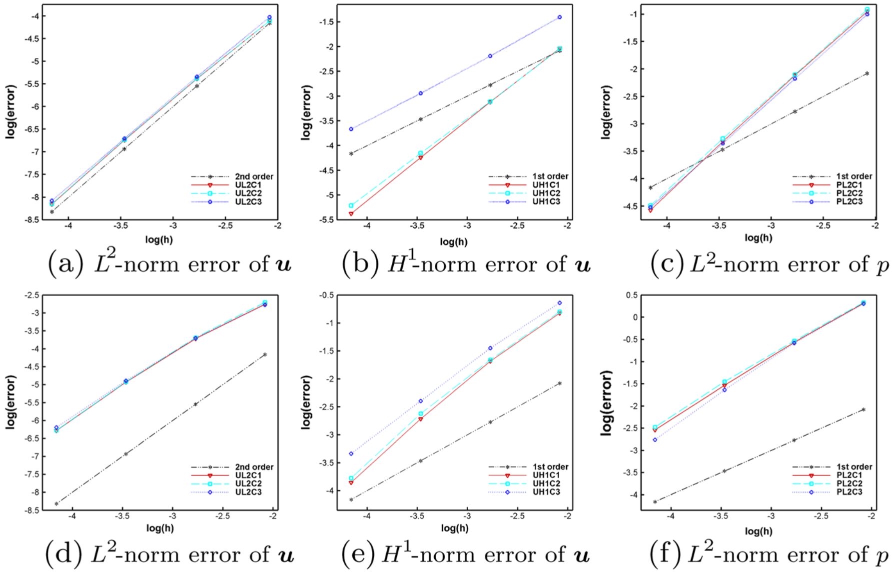

Fig. 1 Errors of velocity and pressure for stabilized $\boldsymbol{P}_1$-P_1 (**a**–**c**) and $\boldsymbol{P}_1$-P_0 (**d**–**f**) FE solutions

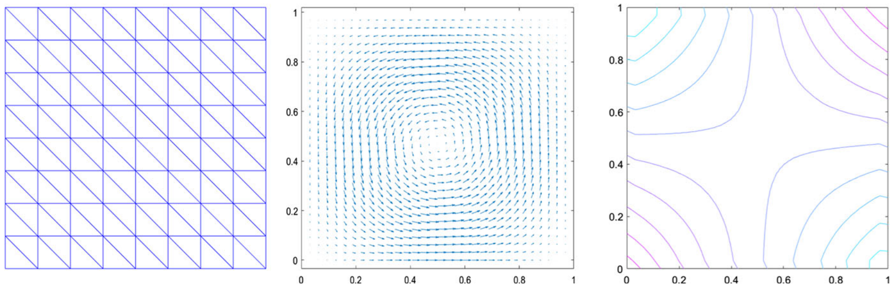

Fig. 2 Mesh (left), velocity field (middle), pressure isobars (right)

Dirichlet boundary condition along the remaining portion of the boundary. The source function is defined by $\boldsymbol{f}_0 = -\text{div}(2\boldsymbol{\varepsilon}(\boldsymbol{u}_0)) + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0 + \nabla p_0$ with

$$\boldsymbol{u}_0(x, y) = \begin{pmatrix} 20x^2(1-x)^2 y(1-y)(1-2y) \\ -20x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix}, \quad p_0(x, y) = 10(2x-1)(2y-1).$$

We use a sequence of uniform triangular meshes with the interval [0, 1] being split into h^{-1} equal sub-intervals for $h = 1/8, 1/16, \cdots$. The pressure stabilized $\boldsymbol{P}_1$-P_1 and $\boldsymbol{P}_1$-P_0 finite element pairs on triangular meshes are used to solve the problem. In computing the numerical solution errors, we use $\boldsymbol{u}^* = \boldsymbol{u}^{1/256}$ and $p^* = p^{1/256}$ as the reference solution. The numerical convergence order of the numerical solution is

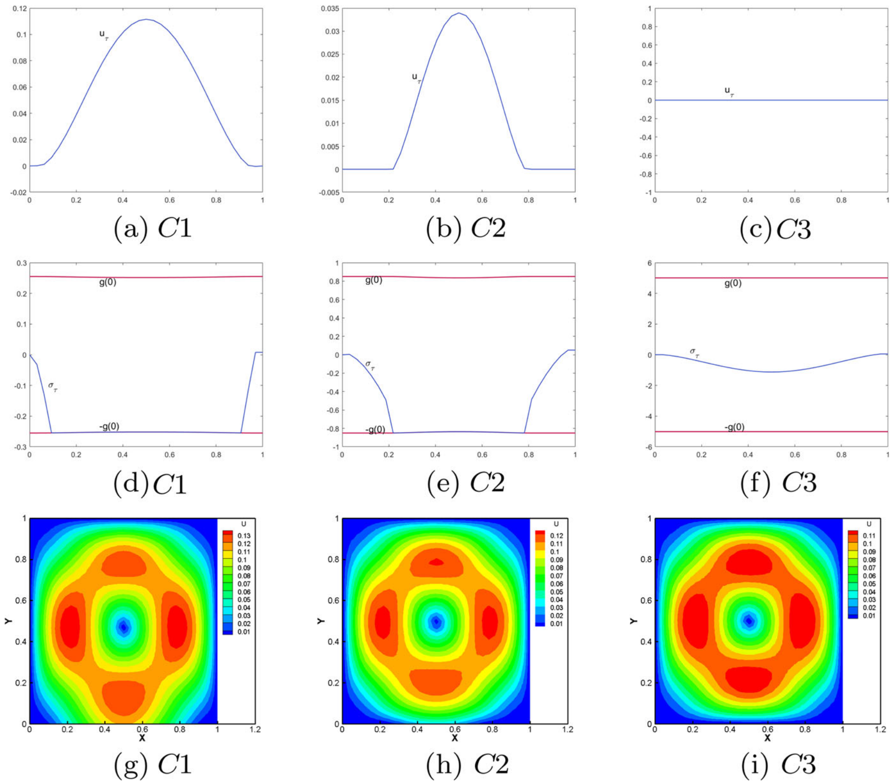

Fig. 3 Tangential components $\boldsymbol{u}_\tau$ and $\boldsymbol{\sigma}_\tau$, and $|\boldsymbol{u}|$ for different thresholds: $C1$ (left), $C2$ (middle), $C3$ (right) with stabilized $\boldsymbol{P}_1$-P_1 FE pair

computed by the log function in MATLAB as follows:

$$\text{Order} = \log(e^h/e^{h/2})/\log(2),$$

where $e^h = \|\boldsymbol{u}^h - \boldsymbol{u}^*\|_{L^2}$ for $E^{\boldsymbol{u}}_{L^2}(h)$, $e^h = |\boldsymbol{u}^h - \boldsymbol{u}^*|_{H^1}$ for $E^{\boldsymbol{u}}_{H^1}(h)$, and $e^h = \|p^h - p^*\|_{L^2}$ for $E^p_{L^2}(h)$.

We let $\alpha = 10$ and experiment on three sets of a and b: $(C1)$ $a = 0.255$, $b = 0.25$; $(C2)$ $a = 0.85$, $b = 0.8$; $(C3)$ $a = 5.01$, $b = 5.0$.

Errors of different finite element approximations are reported in Tables 1 and 2 and Figs. 1, 2 and 3. For the numerical results in Table 1, we used the iteration parameter $\rho = 100$; for the numerical results in Table 2, we used $\rho = 200$ and $\rho = 10$. For these choices of the value of ρ, the Uzawa algorithm converges in at most one or two dozens of steps. The results are consistent with the theoretical results derived in Sect. 4. The friction function $g(|\boldsymbol{u}_\tau|)$, the tangential component of velocity along the slip boundary Γ_S, the tangential component of stress tensor $\boldsymbol{\sigma}$ are drawn in Fig. 3, all with the mesh-size $h = 1/32$.

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Declarations

Conflict of interest The authors declared that they have no conflicts of interest to this work.

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