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Extrapolation of Numerical Solutions for Elliptic Problems on Corner Domains

Hongci Huang

Department of Mathematics
Hong Kong Baptist University
Kowloon, Hong Kong

and

Weimin Han and Jinshi Zhou

Department of Mathematics
University of Iowa
Iowa City, Iowa 52242

ABSTRACT

We study the extrapolation method for the numerical solution of elliptic boundary value problems on corner domains. Most papers on the extrapolation method for solving differential equations require smoothness assumptions on exact solutions, which do not hold for boundary value problems on corner domains. We conjecture that asymptotic error expansions exist as long as there is enough smoothness on the input data except the domain. This is confirmed by our theoretical result on asymptotic error expansions for model problems on a rectangle. Numerical examples suggest that similar results hold for problems on more general corner domains, although it seems very difficult to prove them.

1. INTRODUCTION

The Richardson extrapolation technique is an efficient approach to increase the accuracy of numerical solutions of mathematical problems. The success of the extrapolation technique relies on the existence of asymptotic error expansions. The survey paper [1] summarizes many important results, available up to 1971, on the application of extrapolation in numerical

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integration and numerical ODEs. The monograph [2] presents a systematic treatment of the application of the extrapolation technique on the finite difference method for solving ordinary and partial differential equations, as well as on methods for solving singular linear systems and integral equations. The extrapolation technique is also useful in accelerating the convergence of rootfinding methods, cf. [3, 4].

In the context of the finite element method, a first step was made in [5] on using the extrapolation technique for convergence acceleration and a posteriori error estimates. There, asymptotic error expansions are proved under assumptions that the triangulation is uniform and the solution is smooth. Numerical results indicate the existence of asymptotic error expansions for numerical solutions of problems on re-entrant corner domains. In [6], asymptotic error expansions for finite element solutions over arbitrary initial triangulations are proved, under assumptions that exact solutions are sufficiently smooth and mesh refinement follows certain patterns. Subsequent work in this direction focused on decreasing the smoothness requirement on exact solutions. In most papers on this topic, however, asymptotic error expansions are proved under the assumption of extra degree smoothness on exact solution than actual problems allow. A typical result is the following (cf. [7]). Assume the solution domain Ω is a polygon which can be triangulated by line segments parallel to three fixed directions. Assume the solution of the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

is smooth: $u \in C^{4,\epsilon}(\Omega)$, for some $\epsilon \in (0,1]$ (for the definition of $C^{4,\epsilon}(\Omega)$, see Section 2). Then the linear finite element solution u^h admits the expansion

$$u^h(x) = u(x) + e(x,u)h^2 + O(h^4)$$

at every nodal point x, where h is the meshsize. We notice that the smoothness of the solution of (1.1) depends on the largest internal angle of the polygon Ω, and in general, we do not have the smoothness $u \in C^{4,\epsilon}(\Omega)$, no matter how smooth are the input data f and g. Indeed, let $\omega \in (0, 2\pi]$ be the largest internal angle of Ω, and denote $\alpha = \pi/\omega \geqslant 1/2$. Then for smooth functions f and g, the solution $u \in H^{\alpha+1-\delta}(\Omega)$ for any small $\delta > 0$, and in general, $u \notin H^{\alpha+1}(\Omega)$ (cf. [8]).

The dilemma on the smoothness requirement was circumvented in [9]. The main result of that paper is presented in the following. Consider solving a Dirichlet problem for Poisson's equation by linear finite elements. Assume Ω is a polygon, and ω_j, $1 \leqslant j \leqslant n$, are its internal angles. Under assumptions on the smoothness of the given data and certain local uniform condition on mesh refinement, at every nodal point x, there is an asymptotic error expansion

$$u^h(x) = u(x) + \sum_{k=1}^{n} e_k(x, u)\, h_k^{2\pi/\omega_k} + O(h^2|\log h|), \qquad (1.2)$$

where, h_k, $1 \leqslant k \leqslant n$, are local meshsize parameters, $h = \max_{1 \leqslant k \leqslant n} h_k$.

Numerical experiments suggest that the extrapolation technique can be successfully applied without extra smoothness assumptions on exact solutions. Furthermore, in asymptotic error expansions, higher order terms exist, depending on the smoothness of the given data other than the domain. The main theoretical result of the paper is asymptotic error expansions of numerical solutions of Poisson's equation problem (1.1) on rectangles. We only assume the smoothness of f and g, and do not require any smoothness assumptions on exact solutions (which exhibit singularities of the form $r^2 \ln r$ around corners). Error expansions of numerical solutions contain as many terms as is allowed by the smoothness of the given f and g. The result is proved only for the model problem (1.1) on a rectangle. However, the importance of the result is that it shows what one might expect on the form of asymptotic error expansions and when the asymptotic error expansions might be true, for numerical solutions of more general problems with more general meshes. In the next section, we present some preliminary results to be used later. The main result is proved in Section 3. In the last section, we give some numerical examples to show that similar asymptotic error expansions are likely to exist for problems on more general re-entrant corner domains.

2. PRELIMINARIES

Our model problem is (1.1), with Ω a rectangle whose sides are parallel to the coordinate axes. Without loss of generality, we may assume Ω to be the unit square, $\Omega = (0, 1)^2$. We denote its four sides by e_i, $1 \leqslant i \leqslant 4$, i.e., $\partial\Omega = \cup_{i=1}^{4} \bar{e}_i$. We need a result from [10] on the error expansion for the five-point finite difference approximation of a harmonic function. To state the result, let N be a positive integer, $h = 1/N$, $\Omega_h = \{(ih, jh) \mid 1 \leqslant i, j \leqslant N - 1\}$, $\partial\Omega_h = \{(ih, jh) \in \partial\Omega \mid 0 \leqslant i, j \leqslant N\}$, $\overline{\Omega}_h = \Omega_h \cup \partial\Omega_h$, and $(e_i)_h = e_i \cap \partial\Omega_h$, $1 \leqslant i \leqslant 4$.

THEOREM 2.1. ([10]) *Consider the problem*

$$\begin{cases} -\Delta U = 0 & \text{in } \Omega, \\ U = \phi & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where ϕ is continuous on each of the four sides of the square. Define a finite difference solution U^h by

$$\begin{cases} -\Delta_h U^h = 0 & \text{in } \Omega_h, \\ U^h = \phi & \text{on } \partial\Omega_h, \end{cases} \tag{2.2}$$

where Δ_h is the five-point finite difference operator. Assume

$$\int_0^1 \phi(x,0) \sin n\pi x \, dx - h \sum_{i=1}^{N-1} \phi\left(\frac{i}{N}, 0\right) \sin \frac{n\pi i}{N}$$

$$= \sum_{i=1}^{t} d_i(n) n^{\rho_i} h^{\tau_i} + O(h^{\tau_{t+1}}), \quad \forall n \geqslant 1 \tag{2.3}$$

and similar relations on the other three sides e_2, e_3, and e_4. Here, $\rho_i \geqslant 0$, $0 = \tau_0 < \tau_1 < \cdots$, t a positive integer, and $d_i(n)$ bounded functions of $n \geqslant 1$. Then, at each mesh point $x \in \Omega_h$,

$$U(x) - U^h(x) = \sum_{i=1}^{r} s_i(x) h^{\sigma_i} + O(h^{\sigma_{r+1}}), \tag{2.4}$$

where, $s_i(x)$ are bounded for $x \in \Omega$, $\{\sigma_i\} = \{2j + \tau_k \mid j, k \geqslant 0, j + k > 0, 2j + \tau_k \leqslant \tau_{t+1}\}$ and $\sigma_{i+1} > \sigma_i$ $(i \geqslant 1)$.

We also need the following discrete maximum principle.

LEMMA 2.2. ([2]) *The solution of the finite difference problem*

$$\begin{cases} -\Delta_h \eta^h = \xi & \text{in } \Omega_h, \\ \eta^h = 0 & \text{on } \partial\Omega_h \end{cases}$$

satisfies

$$\max_{\overline{\Omega}_h} |\eta^h| \leqslant c \max_{\Omega_h} |\xi|.$$

For $I = (0,1)$, $l \geqslant 0$ an integer, $0 < \alpha \leqslant 1$, we denote the space

$$C^{l,\alpha}(I) = \left\{ v \in C(\bar{I}) \mid v^{(j)} \in C(\bar{I}), 1 \leqslant j \leqslant l, \right. \left. \sup_{x \neq y} \frac{|v^{(l)}(x) - v^{(l)}(y)|}{|x-y|^{\alpha}} < \infty \right\} \tag{2.5}$$

with the norm

$$\|v\|_{C^{l,\alpha}(I)} = \sum_{j=0}^{l} \sup_{x \in I} |v^{(j)}(x)| + \sup_{x \neq y} \frac{|v^{(l)}(x) - v^{(l)}(y)|}{|x-y|^{\alpha}}. \tag{2.6}$$

We will use $[x]$ to denote the integer part of x.

LEMMA 2.3. *If $f \in C^{l,\alpha}(I)$, $l \geqslant 2$, $h = 1/N$, then*

$$\frac{h}{2}\left[f(0) + 2\sum_{i=1}^{N-1} f(ih) + f(1) \right] - \int_0^1 f(x)\,dx$$
$$= \sum_{j=1}^{[l/2]} \frac{B_{2j}}{(2j)!} h^{2j}\left[f^{(2j-1)}(1) - f^{(2j-1)}(0) \right] + O(h^{l+\alpha}), \tag{2.7}$$

where, $\{B_{2j}\}$ are Bernoulli numbers.

PROOF. When l is odd, $l = 2k+1$, we have ([11]),

$$\frac{h}{2}\left[f(0) + 2\sum_{i=1}^{N-1} f(ih) + f(1) \right] - \int_0^1 f(x)\,dx$$
$$= \sum_{j=1}^{[l/2]} \frac{B_{2j}}{(2j)!} h^{2j}\left[f^{(2j-1)}(1) - f^{(2j-1)}(0) \right] + R_k, \tag{2.8}$$

where

$$R_k = h^{2k+1} \int_0^1 (-1)^{k-1} \sum_{j=1}^{\infty} 2 \frac{\sin 2\pi jNx}{(2\pi j)^{2k+1}} f^{(2k+1)}(x)\,dx.$$

Note that

$$R_k = h^{2k+1}(-1)^{k-1} \sum_{j=1}^{\infty} \frac{2}{(2\pi j)^{2k+1}} \sum_{i=0}^{N-1} \int_0^h \sin(2\pi jNt) f^{(2k+1)}(t+ih)\,dt$$

$$= h^{2k+1}(-1)^{k-1} \sum_{j=1}^{\infty} \frac{2}{(2\pi j)^{2k+1}} \sum_{i=0}^{N-1} \int_0^h \sin(2\pi jNt)\big(f^{(2k+1)}(t+ih) - f^{(2k+1)}(ih)\big)\,dt.$$

Thus,

$$|R_k| \leqslant h^{2k+1} \sum_{j=1}^{\infty} \frac{2}{(2\pi j)^{2k+1}} \sum_{i=0}^{N-1} \int_0^h ct^{\alpha}\,dt = \tilde{c}h^{2k+1+\alpha}.$$

When $l = 2k$ is even, we have the relation (2.8) with

$$R_k = h^{2k} \int_0^1 (-1)^{k-1} \sum_{j=1}^{\infty} 2\frac{\cos 2\pi jNx}{(2\pi j)^{2k}} f^{(2k)}(x)\,dx.$$

We can prove similarly that $|R_k| \leqslant ch^{2k+\alpha}$. ■

The following extension result can be found in [12].

LEMMA 2.4. *Assume D is an open set in R^n, $l \geqslant 0$ is an integer, and $\alpha \in (0,1]$. Then, there exists a bounded mapping E from $C^{l,\alpha}(D)$ to $C^{l,\alpha}(R^n)$, such that $Ef|_{\overline{D}} = f$, for any $f \in C^{l,\alpha}(D)$.*

In Lemma 2.4, the space $C^{l,\alpha}(D)$ is defined similarly as $C^{l,\alpha}(I)$ in (2.5), i.e, $C^{l,\alpha}(D)$ contains all the functions v whose derivatives up to order l are continuous on $\overline{D}$, and the derivatives $D^l v$ of order l satisfy the Hölder condition with exponent α,

$$\sup_{x \neq y} \frac{|D^l v(x) - D^l v(y)|}{|x-y|^{\alpha}} < \infty.$$

3. ASYMPTOTIC ERROR EXPANSIONS

We consider numerical solutions of the problem (1.1) on the unit square Ω. Assume, for some integer $l \geqslant 2$, and $\alpha \in (0, 1]$,

$$f \in C^{l,\alpha}(\Omega), \quad g \in \cap_{i=1}^{4} C^{l,\alpha}(e_i). \tag{3.1}$$

Assume the numerical scheme for solving (1.1) can be written in the form

$$\begin{cases} -\Delta_h u^h = \sum_{j=0}^{[l/2]} h^{2j} F_{l-2j,\alpha}^{(0)} + h^{l+\alpha} R_{l,\alpha}^{(0)} & \text{in } \Omega_h, \\ u^h = \sum_{j=0}^{[l/2]} h^{2j} G_{l-2j,\alpha} + h^{l+\alpha} S_{l,\alpha} & \text{on } \partial\Omega_h. \end{cases} \tag{3.2}$$

In (3.2), $F_{l,\alpha}^{(0)} = f$, $G_{l,\alpha} = g$ are the given data. For $j \geqslant 1$, $F_{l-2j,\alpha}^{(0)} \in C^{l-2j,\alpha}(\Omega)$, $G_{l-2j,\alpha} \in \cap_{i=1}^{4} C^{l-2j,\alpha}(e_i)$. Usually, $F_{l-2j,\alpha}^{(0)}$ is some linear combination of $(2j)$-th derivatives of f, and on each side, e_i, $1 \leqslant i \leqslant 4$, $G_{l-2j,\alpha}$ is some linear combination of $(2j)$-th derivatives of g. The remainder coefficients $R_{l,\alpha}^{(0)}$ and $S_{l,\alpha}$ satisfy $R_{l,\alpha}^{(0)} \in C(\overline{\Omega})$ and $S_{l,\alpha} \in \cap_{i=1}^{4} C(\bar{e}_i)$.

Let us write

$$u^h = u_0^h + h^2 v_0^h, \tag{3.3}$$

where u_0^h and v_0^h are defined by the problems

$$\begin{cases} -\Delta_h u_0^h = f & \text{in } \Omega_h, \\ u_0^h = g & \text{on } \partial\Omega_h, \end{cases} \tag{3.4}$$

and

$$\begin{cases} -\Delta_h v_0^h = \sum_{j=0}^{[l/2]-1} h^{2j} F_{l-2-2j,\alpha}^{(0)} + h^{l-2+\alpha} R_{l,\alpha}^{(0)} & \text{in } \Omega_h, \\ v_0^h = \sum_{j=0}^{[l/2]-1} h^{2j} G_{l-2-2j,\alpha} + h^{l-2+\alpha} S_{l,\alpha} & \text{on } \partial\Omega_h. \end{cases} \tag{3.5}$$

We consider the problem (3.4) first. Let C be a circular region containing $\overline{\Omega}$. We extend $f \in C^{l,\alpha}(\Omega)$ to $\tilde{f} \in C^{l,\alpha}(C)$ (cf. Lemma 2.4). The problem

$$\begin{cases} -\Delta \tilde{u} = \tilde{f} & \text{in } C, \\ \tilde{u} = 0 & \text{on } \partial C \end{cases} \tag{3.6}$$

has a unique solution $\tilde{u} \in C^{l+2,\alpha}(\Omega)$ (cf. [13]). Denote $\overline{u} = u - \tilde{u}$. Then

$$\begin{cases} -\Delta \overline{u} = 0 & \text{in } \Omega, \\ \overline{u} = g - \tilde{u} & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Let $\tilde{u}^h$ and $\overline{u}^h$ be the five-point finite difference solutions of $\tilde{u}$ and $\overline{u}$ on Ω, respectively. Using Theorem 2.1 and Lemma 2.3, we get the expansion

$$\overline{u}^h - \overline{u} = \sum_{i=1}^{[l/2]} s_i^{(0)} h^{2i} + O(h^{l+\alpha}) \quad \text{in } \Omega_h.$$

Hence,

$$u_0^h - u = (\tilde{u}^h - \tilde{u}) + \sum_{i=1}^{[l/2]} s_i^{(0)} h^{2i} + O(h^{l+\alpha}) \quad \text{in } \Omega_h. \tag{3.8}$$

Writing

$$\tilde{u}^h = \tilde{u} + h^2 w^h \tag{3.9}$$

and using the following relation

$$\begin{cases} -\Delta_h \tilde{u}^h = f & \text{in } \Omega_h, \\ \tilde{u}^h = \tilde{u} & \text{on } \partial\Omega_h, \end{cases}$$

we find that w^h satisfies

$$\begin{cases} -\Delta_h w^h = \displaystyle\sum_{i=0}^{[l/2]-1} h^{2i} L_{l-2-2i,\alpha}^{(0)} + h^{l-2+\alpha} \tilde{R}_{l,\alpha}^{(0)} & \text{in } \Omega_h, \\ w^h = 0 \quad \text{on } \partial\Omega_h, \end{cases} \tag{3.10}$$

where, $L_{l-2-2i,\alpha}^{(0)}$ is some linear combination of $(2i+4)$-th derivatives of $\tilde{u}$.

Since $\tilde{u} \in C^{l+2,\alpha}(\Omega)$, we have $L^{(0)}_{l-2-2i,\alpha} \in C^{l-2-2i,\alpha}(\Omega)$ and $R^{(0)}_{l,\alpha} \in C(\overline{\Omega})$.

Combining (3.3), (3.8), (3.9), and (3.10), we then have

$$u^h - u = h^2 u_1^h + \sum_{i=1}^{[l/2]} s_i^{(0)} h^{2i} + O(h^{l+\alpha}), \tag{3.11}$$

where u_1^h solves

$$\begin{cases} -\Delta_h u_1^h = \displaystyle\sum_{j=0}^{[l/2]-1} h^{2j} F^{(1)}_{l-2-2j,\alpha} + h^{l-2+\alpha} R^{(1)}_{l-2,\alpha} & \text{in } \Omega_h, \\ u_1^h = \displaystyle\sum_{j=0}^{[l/2]-1} h^{2j} G_{l-2-2j,\alpha} + h^{l-2+\alpha} S_{l,\alpha} & \text{on } \partial\Omega_h, \end{cases} \tag{3.12}$$

with $F^{(1)}_{l-2-2j,\alpha} = F^{(0)}_{l-2-2j,\alpha} + L^{(0)}_{l-2-2j,\alpha} \in C^{l-2-2j,\alpha}(\Omega)$, $R^{(1)}_{l-2,\alpha} = R^{(0)}_{l,\alpha} + \tilde{R}^{(0)}_{l,\alpha} \in C(\overline{\Omega})$.

Applying the same technique, a mathematical induction leads to the following result.

THEOREM 3.1. *Under the smoothness assumption* (3.1) *on the input data, we have the asymptotic error expansion*

$$u^h = u + \sum_{i=1}^{[l/2]} s_i h^{2i} + O(h^{l+\alpha}) \quad \text{in } \Omega_h, \tag{3.13}$$

where, $s_i(x)$ are finite for $x \in \Omega$.

The error expansion (3.13) holds as long as the assumption (3.1) is true. We observe that the boundary value g is allowed to have jumps at the corners.

An immediate consequence of Theorem 3.1 is

COROLLARY 3.2. *For the five-point finite difference solution u^h, we have the error expansion* (3.13).

Now consider uniform triangulations of the types shown in Figure 1.

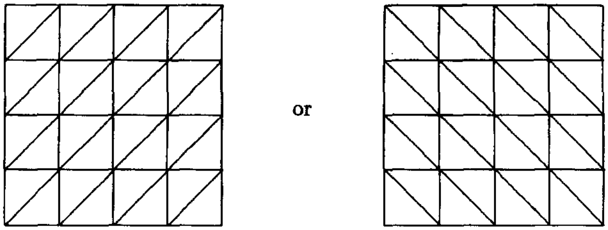

FIG. 1. Uniform triangulations of the square.

Denote h the meshsize. After quite a bit of manipulation, it can be shown that the linear finite element solution u^h solves a finite difference scheme of the form

$$\begin{cases} -\Delta_h u^h = \sum_{j=0}^{[l/2]} h^{2j} F_{l-2j,\alpha}^{(0)} + h^{l+\alpha} R_{l,\alpha}^{(0)} & \text{in } \Omega_h, \\ u^h = g \quad \text{on } \partial\Omega_h, \end{cases} \tag{3.14}$$

where, $F_{l,\alpha}^{(0)} = f$, $F_{l-2j,\alpha}^{(0)} \in C^{l-2j,\alpha}(\Omega)$, $j \geqslant 1$. Hence, from Theorem 3.1, we obtain

COROLLARY 3.3. *Assume* (3.1). *For linear finite element solutions over the above shown uniform meshes, we have*

$$u^h(x) = u(x) + \sum_{i=1}^{[l/2]} s_i(x) h^{2i} + O(h^{l+\alpha}) \text{ at any nodal point } x. \tag{3.15}$$

In practice, actual computation of the load vector involves numerical integrations. As long as the employed numerical quadrature has certain symmetry property to allow an evenpower term expansion for components of the load vector as in (3.14), we have the asymptotic error expansion (3.15).

Notice that the number of terms in the error expansion (3.13) is determined by the smoothness of the input data. Although we proved the result only for the special case of a square, numerical experiments show that for problems on general corner domains, we still have asymptotic error expansions which contain as many terms as the smoothness of the input data allow.

We remark that since the solution of the model problem has singularities of the form $r^2 \ln r$ near each corner, the optimal maximum norm error estimate for linear finite element solutions is $O(h^2|\ln h|)$. On the other hand, at any fixed nodal point, the asymptotic error expansion (3.15) shows that the pointwise error is $O(h^2)$.

4. NUMERICAL EXAMPLES

We recall a standard extrapolation procedure based on an asymptotic error expansion. Assume T_0 is a desired quantity, T_h is its numerical approximation depending on a discretization parameter h. Assume we have the relation

$$T_h = T_0 + c_1 h^{\alpha_1} + c_2 h^{\alpha_2} + \cdots + c_n h^{\alpha_n} + \cdots, \tag{4.1}$$

where, $\{\alpha_i\}$ is a sequence of increasing positive numbers. Assume we have a sequence of numerical approximations $T_h, T_{2^{-1}h}, \ldots, T_{2^{-n}h}$. Denote $T^{(0)}_{2^{-j}h} \equiv T_{2^{-j}h}$, $0 \leqslant j \leqslant n$. For $i = 1, \ldots, n$, we compute

$$T^{(i)}_{2^{-j}h} = \frac{2^{\alpha_i} T^{(i-1)}_{2^{-(j+1)}h} - T^{(i-1)}_{2^{-j}h}}{2^{\alpha_i} - 1}, \quad 0 \leqslant j \leqslant n - i. \tag{4.2}$$

Then we have

$$T^{(n)}_h = T_0 + O(h^{\alpha_{n+1}}). \tag{4.3}$$

EXAMPLE 4.1. In the first example, we consider the problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega = (0,1)^2, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

We have a formula for the exact solution,

$$u(x, y) = \frac{16}{\pi^4} \sum_{i,j=0}^{\infty} \frac{\sin[(2i+1)\pi x]\sin[(2j+1)\pi y]}{(2i+1)(2j+1)\left[(2i+1)^2 + (2j+1)^2\right]}.$$

We divide the unit interval $[0, 1]$ into N equal parts, and set $h = 1/N$.

Then we solve the problem by using linear finite elements on the meshes shown in Figure 1. We have the asymptotic error expansion (3.15), where l can be an arbitrarily large positive integer. So in the extrapolation algorithm (4.2), we use $\alpha_i = 2i$. We then compute solution errors at various nodal points. The numerical results completely agree with the theory. For instance, at the point (0.25, 0.75), we have Table 1, where we denote $u^{h,k}$ the numerical solution after k steps of extrapolation.

We see that extrapolating from the three solutions corresponding to $h = 1/4, 1/8$ and $1/16$, we get a numerical solution with error $.9781 \times 10^{-6}$, which is smaller than the numerical solution error ($.2505 \times 10^{-5}$) corresponding to $h = 1/128$. At other nodal points, we have the similar improvement on the numerical solution accuracy by using extrapolation.

EXAMPLE 4.2. Next, we consider a crack domain problem. The domain is shown in Figure 2.

When we impose Dirichlet boundary conditions along the two folds of the crack OA, the leading singularity of the solution of a Poisson equation is of the type $r^{1/2}\sin(\theta/2)$, where (r, θ) is the local polar coordinate system at the crack tip O (cf. [8]). We solve the following problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases} \tag{4.5}$$

such that the exact solution $u = r^{1/2}\sin(\theta/2)$. Because the problem is symmetric with respect to the central horizontal line, we only need to solve a half problem, and we use linear finite elements on uniform meshes sketched in Figure 1. From (1.2), we see that the leading term in the error expansion is $c_1 h$. Numerical experiments show that at any nodal point, we have the asymptotic error expansion

$$u^h = u + c_1 h + c_2 h^2 + c_3 h^3 + \cdots .$$

TABLE 1

	$u - u^h$	$u - u^{h,1}$	$u - u^{h,2}$
$h = 1/4$	$.2317 \times 10^{-2}$	$.5791 \times 10^{-4}$	$.9781 \times 10^{-6}$
$h = 1/8$	$.6228 \times 10^{-3}$	$.4536 \times 10^{-5}$	
$h = 1/16$	$.1591 \times 10^{-3}$		

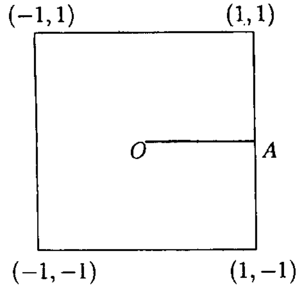

FIG. 2. A crack domain.

Indeed, at a typical point in the domain, say (0.5, 0.5), we have Table 2 for numerical results.

With $h = 1/128$, the linear finite element solution gives $(u - u^h)(0.5, 0.5) = .2257 \times 10^{-3}$ which is larger than the error of the extrapolated solution obtained from the three solutions corresponding to $h = 1/4$, $1/8$, and $1/16$.

EXAMPLE 4.3. We then solve the problem (4.5) over the L-shape domain shown in Figure 3.

We select the function g so that the solution of the boundary value problem is $u = r^{2/3}\sin(2\theta/3)$, which is the leading singularity associated with an L-shape corner (when a same type of boundary condition is specified around the corner). Once again, because of the symmetry of the problem with respect to the line $x + y = 0$, we only need solve a half problem. We use linear finite elements on uniform meshes similar to those in Figure 1. Numerical results show that at any nodal point, we have the asymptotic error expansion

$$u^h = u + c_1 h^{4/3} + c_2 h^2 + c_3 h^{8/3} + c_4 h^4 + \cdots ,$$

TABLE 2

	$u - u^h$	$u - u^{h,1}$	$u - u^{h,2}$	$u - u^{h,3}$
$h = 1/4$	$.6674 \times 10^{-2}$	$.3320 \times 10^{-3}$	$-.2470 \times 10^{-4}$	$-.6558 \times 10^{-6}$
$h = 1/8$	$.3503 \times 10^{-2}$	$.6446 \times 10^{-4}$	$-.3661 \times 10^{-5}$	
$h = 1/16$	$.1784 \times 10^{-2}$	$.1337 \times 10^{-4}$		
$h = 1/32$	$.8986 \times 10^{-3}$			

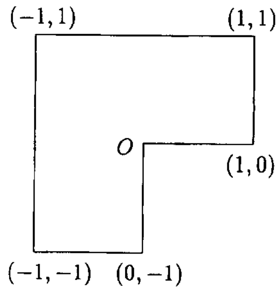

FIG. 3. An L-shape domain.

i.e., the error expansion contains all the terms of the form h^{2j} and $h^{4j/3}$, for any positive integer j. As in the last example, we list some numerical results at the nodal point (0.5, 0.5).

For the linear element solution on a mesh with $h = 1/128$, the error is $.3642 \times 10^{-4}$ at (0.5, 0.5). We observe dramatic improvement on the accuracy of numerical solutions using the extrapolation technique.

In the previous examples, we used uniform meshes for the numerical computations, which was possible due to the special shapes of the domains of the problems. In the case of a boundary value problem with a polygonal domain, it is no longer possible to use uniform meshes. What can be done for solving the problem is first to construct an initial mesh with certain degree of refinement in neighborhoods of corners, and then to uniformly refine the mesh subsequently. Numerical experiments in [5] still suggest the existence of asymptotic error expansions and, as is shown in the numerical examples presented here, in the error expansions the exponents of the meshsize are positive integers and positive integer multiples of numbers of the form $2\pi/\omega$, where ω is an internal angle of the polygonal domain.

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TABLE 3

	$u - u^h$	$u - u^{h,1}$	$u - u^{h,2}$	$u - u^{h,3}$
$h = 1/4$	$.3074 \times 10^{-2}$	$.1904 \times 10^{-3}$	$-.9224 \times 10^{-5}$	$-.5144 \times 10^{-6}$
$h = 1/8$	$.1335 \times 10^{-2}$	$.4067 \times 10^{-4}$	$-.1019 \times 10^{-5}$	
$h = 1/16$	$.5543 \times 10^{-3}$	$.9404 \times 10^{-5}$		
$h = 1/32$	$.2256 \times 10^{-3}$			

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