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Convergence analysis of numerical solutions for optimal control of variational–hemivariational inequalities[☆]

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ABSTRACT

Variational–hemivariational inequalities and hemivariational inequalities form a powerful mathematical tool in modeling and studying problems in science and engineering where non-smooth, non-monotone and multi-valued relations among different physical quantities are present. In this paper, we consider the numerical solution of optimal control problems for variational–hemivariational inequalities or hemivariational inequalities, and prove the convergence of numerical solutions under rather general assumptions.

Optimal control is a research area with many important applications in science, engineering and technology. Optimal control problems for differential equations and variational inequalities have been the subject of many publications, see e.g. the comprehensive Refs. [1–3], and [4–6]. Recently, optimal control problems for variational–hemivariational inequalities have been studied in [7] where existence of optimal pairs is proved and necessary optimality conditions of first order are derived, and in [8,9] where existence of optimal pairs is proved and application to contact problems is illustrated.

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For applications, numerical methods are needed to solve optimal control problems for variational–hemivariational inequalities. In this regard, we note an early reference [10]; we also note some solution techniques in solving hemivariational inequalities considered in [11,12]. In this paper, we consider the numerical solution of such optimal control problems and prove convergence of numerical solutions under rather general assumptions.

Preliminaries. We will need two real reflexive Banach spaces for the optimal control problem: V for the state variable and Q for the control variable. We denote by $\|\cdot\|$ the norm in V, write V^* for its dual space and $\langle\cdot,\cdot\rangle$ for the duality pairing between V^* and V. Strong convergence is indicated by the symbol $\to$, whereas weak convergence is indicated by the symbol $\rightharpoonup$. We will use the notions of the generalized directional derivative and generalized subdifferential in the sense of Clarke (cf. [13]). Let $\psi\colon V\to\mathbb{R}$ be locally Lipschitz continuous. Then the generalized directional derivative of ϕ at $u\in V$ in the direction $v\in V$ is defined by

$$\psi^0(u;v):=\limsup_{w\to u,\,\lambda\downarrow 0}\frac{\psi(w+\lambda v)-\psi(w)}{\lambda}.$$

The generalized subdifferential of ψ at u is a subset of V^* given by

$$\partial\psi(u):=\left\{\xi\in V^*\mid \psi^0(u;v)\ge\langle\xi,v\rangle\ \forall\, v\in V\right\}.$$

We record the following properties:

$$\psi^0(u;v)=\max\left\{\langle\xi,v\rangle\mid \xi\in\partial\psi(u)\right\}, \tag{1}$$

$$u_n\to u \text{ and } v_n\to v \text{ in } V \quad\Longrightarrow\quad \limsup_{n\to\infty}\psi^0(u_n;v_n)\le\psi^0(u;v). \tag{2}$$

Let $\phi\colon V\to\mathbb{R}\cup\{+\infty\}$ be proper, convex and l.s.c. By a classical result in convex analysis, ϕ is bounded below by the summation of a continuous linear functional on V and a constant (cf. [14, Lemma 11.3.5] or [15, Prop. 5.2.25]). Therefore, there exist two constants $\overline{c}_0,\overline{c}_1$ such that

$$\phi(v)\ge\overline{c}_0+\overline{c}_1\|v\|_V\quad\forall\, v\in V. \tag{3}$$

Hypotheses. In studying the optimal control problem, we use the following hypotheses.

$\underline{(H_{Q_{ad}})}$ Q is a reflexive Banach space, and Q_{ad} is a nonempty, closed and convex set in Q.

$\underline{(H_K)}$ V is a reflexive Banach space, and K is a nonempty, closed and convex set in V.

$\underline{(H_A)}$ $A\colon V\to V^*$ is bounded, continuous, and strongly monotone with a constant $m_A>0$:

$$\langle Av_1-Av_2,v_1-v_2\rangle\ge m_A\|v_1-v_2\|^2\quad\forall\, v_1,v_2\in V.$$

$\underline{(H_\Phi)}$ $\Phi\colon V\times V\to\mathbb{R}$ is such that $\Phi(u,\cdot)\colon K\to\mathbb{R}$ is convex and continuous for all $u\in V$, and for some constant $\alpha_\Phi>0$,

$$\Phi(u_1,v_2)-\Phi(u_1,v_1)+\Phi(u_2,v_1)-\Phi(u_2,v_2)\le\alpha_\Phi\|u_1-u_2\|\,\|v_1-v_2\|\quad\forall\, u_1,u_2,v_1,v_2\in V.$$

$\underline{(H_\Psi)}$ $\Psi\colon V\to\mathbb{R}$ is locally Lipschitz continuous, and for some constants c_0,c_1 and $\alpha_\Psi>0$,

$$\|\partial\Psi(v)\|_{V^*}\le c_0+c_1\|v\|\quad\forall\, v\in V, \tag{4}$$

$$\Psi^0(v_1;v_2-v_1)+\Psi^0(v_2;v_1-v_2)\le\alpha_\Psi\|v_1-v_2\|^2\quad\forall\, v_1,v_2\in V. \tag{5}$$

$\underline{(H_f)}$ $f\in V^*$.

$\underline{(H_B)}$ $B\in\mathcal{L}(Q,V^*)$ is compact.

$\underline{(H_s)}$ $\alpha_\Phi+\alpha_\Psi<m_A$.

We comment on some of the hypotheses. The admissible set of the control variable is Q_{ad}. In [16], the operator A is assumed to be pseudomonotone and strongly monotone, whereas $\Phi\colon V \to \mathbb{R}\cup\{+\infty\}$ is assumed to be proper, convex and l.s.c. We note that (H_A) is easy to verify and it implies that the operator A is pseudomonotone and strongly monotone. For applications, it is sufficient to consider nonsmooth convex functionals of the form $\Phi + I_K$ where $\Phi\colon V \to \mathbb{R}$ is convex and l.s.c., and I_K is the indicator function of K. Since a l.s.c. convex functional $\Phi\colon V \to \mathbb{R}$ on V is continuous [17], we state the hypothesis on Φ in the form (H_Φ). The other hypotheses are commonly used in the literature.

A variational–hemivariational inequality for the state variable. Under the stated hypotheses, for any $p \in Q$, the variational–hemivariational inequality

$$\begin{aligned} u \in K, \quad & \langle Au, v-u\rangle + \Phi(u,v) - \Phi(u,u) + \Psi^0(u; v-u) \\ & \geq \langle f, v-u\rangle + \langle Bp, v-u\rangle \quad \forall\, v \in K \end{aligned} \tag{6}$$

has a unique solution (cf. [16,18]). We will write $u = S(p)$ for this solution, where the operator $S\colon Q \to K$. The variable p will play the role of control, whereas $u = S(p)$ is the state variable. We note the following weak-strong continuity property of the operator S.

Proposition 1. *The operator $S\colon Q \to K$ is sequentially weakly-strongly continuous; in other words, if $p_n \rightharpoonup p$ in Q, then $u_n = S(p_n) \to u = S(p)$ in V.*

Proof. By definition,

$$\begin{aligned} u_n \in K, \quad & \langle Au_n, v-u_n\rangle + \Phi(u_n,v) - \Phi(u_n,u_n) + \Psi^0(u_n; v-u_n) \\ & \geq \langle f, v-u_n\rangle + \langle Bp_n, v-u_n\rangle \quad \forall\, v \in K. \end{aligned} \tag{7}$$

Fix an arbitrary $w \in K$ and let $v = w$ in (7) to get

$$\langle Au_n, u_n - w\rangle \leq \Phi(u_n,w) - \Phi(u_n,u_n) + \Psi^0(u_n; w-u_n) - \langle f, w-u_n\rangle - \langle Bp_n, w-u_n\rangle. \tag{8}$$

By (H_A),

$$m_A\|u_n - w\|^2 \leq \langle Au_n, u_n - w\rangle - \langle Aw, u_n - w\rangle.$$

Apply (8),

$$m_A\|u_n - w\|^2 \leq \Phi(u_n,w) - \Phi(u_n,u_n) + \Psi^0(u_n; w-u_n) - \langle Aw - f - Bp_n, u_n - w\rangle. \tag{9}$$

By (H_Φ) and (H_Ψ),

$$\begin{aligned} \Phi(u_n,w) - \Phi(u_n,u_n) &\leq \Phi(w,w) - \Phi(w,u_n) + \alpha_\Phi\|u_n - w\|^2, \\ \Psi^0(u_n; w-u_n) &\leq -\Psi^0(w; u_n - w) + \alpha_\Psi\|u_n - w\|^2. \end{aligned}$$

Thus, from (9),

$$(m_A - \alpha_\Phi - \alpha_\Psi)\,\|u_n - w\|^2 \leq \Phi(w,w) - \Phi(w,u_n) - \Psi^0(w; u_n - w) - \langle Aw - f - Bp_n, u_n - w\rangle. \tag{10}$$

By (3), for some constants $\overline{c}_0$ and $\overline{c}_1$ depending on w such that

$$\Phi(w,u_n) \geq \overline{c}_0 + \overline{c}_1\|u_n\|,$$

and by (H_Ψ),

$$-\Psi^0(w; u_n - w) \leq (c_0 + c_1\|w\|)\,\|u_n - w\|.$$

Also,

$$-\langle Aw - f - Bp_n, u_n - w\rangle \le \|Aw - f - Bp_n\|_{V^*}\|u_n - w\|.$$

Use these bounds in (10), for a constant c depending on w,

$$(m_A - \alpha_\Phi - \alpha_\Psi)\,\|u_n - w\|^2 \le c\,(1 + \|u_n - w\|) + \|Aw - f - Bp_n\|_{V^*}\|u_n - w\|.$$

By the smallness condition (H_s), $m_A - \alpha_\Phi - \alpha_\Psi > 0$, and we derive from the above inequality the boundedness of the sequence $\{\|u_n - w\|\}$ and then also the boundedness of the sequence $\{\|u_n\|\}$.

Since V is reflexive, there exist some element $\tilde{u} \in V$ and a subsequence of $\{u_n\}$, still denoted as $\{u_n\}$, such that $u_n \rightharpoonup \tilde{u}$ in V. Moreover, since K is weakly closed due to (H_K), $\tilde{u} \in K$. Next we prove the strong convergence $u_n \to \tilde{u}$ in V. Similar to (10), we have

$$(m_A - \alpha_\Phi - \alpha_\Psi)\,\|u_n - \tilde{u}\|^2 \le \Phi(\tilde{u},\tilde{u}) - \Phi(\tilde{u},u_n) - \Psi^0(\tilde{u}; u_n - \tilde{u}) - \langle A\tilde{u} - f - Bp_n, u_n - \tilde{u}\rangle. \tag{11}$$

Since Φ is weakly sequentially l.s.c. with respect to its second argument due to (H_Φ),

$$\limsup\,[-\Phi(\tilde{u},u_n)] = -\liminf \Phi(\tilde{u},u_n) \le -\Phi(\tilde{u},\tilde{u}). \tag{12}$$

Now for any $\xi_{\tilde{u}} \in \partial\Psi(\tilde{u})$,

$$-\Psi^0(\tilde{u}; u_n - \tilde{u}) \le -\langle \xi_{\tilde{u}}, u_n - \tilde{u}\rangle \to 0.$$

Thus,

$$\limsup\left[-\Psi^0(\tilde{u}; u_n - \tilde{u})\right] \le 0.$$

Moreover, by (H_B), for some subsequence again denoted by $\{p_n\}$, $Bp_n \to Bp$ in V^*. Hence, take the upper limit of both sides of (11),

$$(m_A - \alpha_\Phi - \alpha_\Psi)\limsup\|u_n - \tilde{u}\|^2 \le 0.$$

Therefore, $u_n \to \tilde{u}$ in V.

Finally, we show that $\tilde{u} = u$ is the solution of the problem (6). Note that by (H_Φ),

$$\Phi(u_n,v) - \Phi(u_n,u_n) \le \Phi(\tilde{u},v) - \Phi(\tilde{u},u_n) + \alpha_\Phi\|u_n - \tilde{u}\|\,\|v - u_n\| \quad \forall\, v \in K,$$

and we can deduce from (7) that

$$\begin{aligned}&\langle Au_n, v - u_n\rangle + \Phi(\tilde{u},v) - \Phi(\tilde{u},u_n) + \alpha_\Phi\|u_n - \tilde{u}\|\,\|v - u_n\| + \Psi^0(u_n; v - u_n)\\ &\quad \ge \langle f, v - u_n\rangle + \langle Bp_n, v - u_n\rangle \quad \forall\, v \in K.\end{aligned} \tag{13}$$

Note that $\alpha_\Phi\|u_n - \tilde{u}\|\,\|v - u_n\| \to 0$ for fixed v, and due to the u.s.c. property (2),

$$\limsup \Psi^0(u_n; v - u_n) \le \Psi^0(\tilde{u}; v - \tilde{u}).$$

We can take the upper limit of (13) to obtain

$$\langle A\tilde{u}, v - \tilde{u}\rangle + \Phi(\tilde{u},v) - \Phi(\tilde{u},\tilde{u}) + \Psi^0(\tilde{u}; v - \tilde{u}) \ge \langle f, v - \tilde{u}\rangle + \langle Bp, v - \tilde{u}\rangle.$$

Since $v \in K$ is arbitrary and since the solution of the problem (6) is unique, we conclude that $\tilde{u} = u$ is the solution of the problem (6). ■

An optimal control problem. Consider the following optimal control problem.

Problem 2.

$$\inf\left\{J(u,p) \mid p \in Q_{ad},\ u = S(p)\right\}. \tag{14}$$

Regarding the real-valued objective functional $J(u,p)$, we introduce the following hypothesis:
$\underline{(H_J)}$ $u_n \to u$ in V and $p_n \rightharpoonup p$ in Q imply $J(u,p) \le \liminf J(u_n,p_n)$. And, if Q_{ad} is unbounded, $\inf_{u\in K} J(u,p) \to \infty$ as $\|p\|_Q \to \infty$.

Theorem 3. *Under the stated hypotheses, Problem 2 has a solution.*

Proof. Denote $m = \inf\{J(u,p) \mid p \in Q_{ad},\, u = S(p)\}$. By (H_J), $m \in \mathbb{R}$. Let $\{(u_n,p_n)\}$ be a minimizing sequence for (14), i.e., $J(u_n,p_n) \to m$. Here and below, $u_n = S(p_n)$. By (H_J), $\{p_n\}$ is bounded in Q. So for a subsequence of $\{p_n\}$, still denoted as $\{p_n\}$, we have $p_n \rightharpoonup p$ in Q for some element $p \in Q$. By $(H_{Q_{ad}})$, $p \in Q_{ad}$. By Proposition 1, $u_n \to u = S(p)$ in V. Then by (H_J), $J(u,p) \le \liminf_{n\to\infty} J(u_n,p_n) = m$. Thus, (u,p) is a solution of Problem 2. ■

Numerical approximation of optimal control problem. Since it is not realistic to have an exact solution of Problem 2, one has to use a numerical method to solve the problem. Let V^h and Q^h be finite dimensional subspaces of V and Q, and let $K^h \subset V^h$ and $Q^h_{ad} \subset Q^h$, where $h > 0$ is a discretization parameter, e.g., h can be the mesh-size of a finite element partition of the spatial domain Ω over which the spaces V and Q are defined. Consider the following discrete optimal control problem.

Problem 4.

$$\inf\left\{J(u^h,p^h) \mid p^h \in Q^h_{ad},\, u^h = S^h(p^h)\right\}. \tag{15}$$

In Problem 4, $u^h = S^h(p^h)$ stands for a numerical solution of (6) with p replaced by p^h:

$$\begin{aligned} u^h \in K^h, \quad & \langle Au^h, v^h - u^h\rangle + \Phi(u^h,v^h) - \Phi(u^h,u^h) + \Psi^0(u^h; v^h - u^h) \\ & \ge \langle f, v^h - u^h\rangle + \langle Bp^h, v^h - u^h\rangle \quad \forall\, v^h \in K^h. \end{aligned} \tag{16}$$

Similar to Problem 2, it can be shown that Problem 4 has a solution (u^h,p^h).

For convergence analysis, we will additionally assume
$\underline{(H_h)}$ $K^h \xrightarrow{M} K$ and $Q^h_{ad} \xrightarrow{M} Q_{ad}$ as $h \to 0$.

Here the symbol $\xrightarrow{M}$ stands for set convergence in the sense of Mosco [19]. Recall that $K^h \xrightarrow{M} K$ means that (i) for an arbitrarily fixed $v \in K$, there exists $v^h \in K^h$ for each h such that $v^h \to v$ in V as $h \to 0$; (ii) if $v^h \rightharpoonup v$ in V and $v^h \in K^h$, then $v \in K$. The meaning for $Q^h_{ad} \xrightarrow{M} Q_{ad}$ is similar. An assumption of the form $\underline{(H_h)}$ is usually employed in convergence analysis for numerical solutions under minimal solution regularity, cf. [20] for variational inequalities and [18] for hemivariational inequalities.

We first note a convergence result for the numerical solution of the hemivariational inequality (6), which can be proved by slightly modifying the proof of Theorem 4 in [21].

Proposition 5. *If $p^h \rightharpoonup p$ in Q, then $u^h \to u$ in V as $h \to 0$, where $u^h = S^h(p^h) \in K^h$ is the solution of (16) and $u = S(p) \in K$ is the solution of (6).*

Now in addition to $\underline{(H_J)}$, we further assume
$\underline{(H_J')}$ $u_n \to u$ in V and $p_n \to p$ in Q imply $J(u_n,p_n) \to J(u,p)$.

Theorem 6. *Let all the hypotheses hold. For each $h > 0$, let (u^h,p^h) be a solution of Problem 4. Then for each sequence $\{(u^h,p^h)\}$, there are a subsequence, still denoted as $\{(u^h,p^h)\}$, and an element $p \in Q_{ad}$ such that with $u = S(p)$, we have*

$$u^h \to u \text{ in } V, \quad p^h \rightharpoonup p \text{ in } Q, \tag{17}$$

and (u,p) is a solution of Problem 2.

Proof. Fix an arbitrary element $p_0 \in Q_{ad}$. By (H_h), there is a sequence $\{\tilde{p}^h\}$, $\tilde{p}^h \in Q_{ad}^h$ such that $\tilde{p}^h \to p_0$ in Q. With $\tilde{u}^h = S^h(\tilde{p}^h)$ and $u_0 = S(p_0)$, we have from (H_J'), $J(\tilde{u}^h, \tilde{p}^h) \to J(u_0, p_0)$, where we have used the fact that $\tilde{u}^h \to u_0$ in V by an application of Proposition 5.

Since $J(u^h, p^h) \le J(\tilde{u}^h, \tilde{p}^h)$, we see that $\{J(u^h, p^h)\}$ is bounded from above. By (H_J), $\{p^h\}$ is bounded in Q. So for a subsequence, still denoted by $\{p^h\}$, we have an element $p \in Q$ such that $p^h \rightharpoonup p$ in Q. By (H_h), we know $p \in Q_{ad}$. Applying Proposition 5, we have $u^h \to u = S(p)$ in V and so by (H_J),

$$J(u, p) \le \liminf J(u^h, p^h). \tag{18}$$

We need to prove that (u, p) is a solution of Problem 2. For this purpose, let $(\overline{u}, \overline{p})$ be a solution of Problem 2. By (H_h), we can choose $\overline{p}^h \in Q_{ad}^h$ with $\overline{p}^h \to \overline{p}$ in Q. Then by Proposition 5, $\overline{u}^h = S^h(\overline{p}^h) \to \overline{u} = S(\overline{p})$ in V. By (H_J'), $J(\overline{u}^h, \overline{p}^h) \to J(\overline{u}, \overline{p})$. Since (u^h, p^h) is a solution of Problem 4, we have $J(u^h, p^h) \le J(\overline{u}^h, \overline{p}^h)$. Take the lower limit in the above inequality as $h \to 0$ and note (18) to obtain the relation $J(u, p) \le J(\overline{u}, \overline{p})$. Since $(\overline{u}, \overline{p})$ is a solution of Problem 2, we find from the above inequality that (u, p) is a solution of Problem 2. ■

We comment that if $\Phi \equiv 0$, then (6) represents an ordinary hemivariational inequality, whereas if $\Psi \equiv 0$, then (6) is reduced to a variational inequality. The convergence result of numerical solutions, Theorem 6, corresponds to that for the optimal control problem of a hemivariational inequality and the optimal control problem of a variational inequality.

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