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NUMERICAL ANALYSIS OF A NONLINEAR EVOLUTIONARY SYSTEM WITH APPLICATIONS IN VISCOPLASTICITY*

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Abstract. We consider numerical approximations of a class of abstract nonlinear evolutionary systems arising in the study of quasi-static frictional contact problems for elastic-viscoplastic materials. Both semidiscrete and fully discrete schemes are analyzed. Strong convergence of both approximations is established under minimal solution regularity. The results are applied to two particular frictional contact problems for viscoplastic bodies, where the finite element method is employed to discretize the spatial domain. Under additional regularity assumptions on the exact solution, some error estimates are derived.

Key words. evolutionary variational inequality, frictional contact problem, viscoplasticity, numerical schemes, finite element method, convergence analysis, error estimation

AMS subject classifications. 65M12, 65M15, 65M60, 65N12, 65N15, 65N60, 74D10, 74S05, 74S20

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1. Introduction.

The aim of this paper is to provide numerical analysis of some problems arising in frictional contact between an elastic-viscoplastic body and a rigid foundation. Situations of frictional contact abound in industry and everyday life. Contacts of the braking pads with the wheel, the tire with the road, and the piston with the skirt are just a few simple examples. Because of the importance of the process of frictional contact, a considerable effort has been made in its modeling and numerical simulations. Indeed, the engineering literature concerning this topic is extensive. Most of it, however, is dedicated to simple geometries, specific settings, and mostly to numerical simulations.

In the applied mathematics literature, the study of general models for dynamic or quasi-static contact process involving elastic-viscoplastic materials is very recent. Rate-type viscoplastic constitutive laws of the form

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + G(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \tag{1.1}$$

are used in the literature to describe mechanical responses of such materials as rubber, various metals, rocks, pastes, etc. In (1.1), $\boldsymbol{\sigma}$ denotes the stress tensor, $\boldsymbol{u}$ the displacement field, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the linearized strain tensor, and $\mathcal{E}$ and G are material constitutive functions. The function $\mathcal{E}$ is assumed to be linear while G is in general nonlinear. Here and throughout the paper, a dot above a quantity represents its derivative with respect to the time variable t, and double dots denote the second-order derivative. Concrete examples, experimental background, and mechanical interpretations of such models may be found in [4] and references therein. Functional and numerical methods are discussed in [13] for initial and boundary value problems involving (1.1) with the usual displacement and traction boundary conditions.

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Existence of weak solutions to quasi-static frictional contact problems for materials modeled by a general rate-type constitutive law of the form (1.1) were established in [1, 2, 17]. In the abstract form, the frictional contact problems are formulated as a nonlinear evolution equation (the abstract version of (1.1)) coupled with an evolutionary variational inequality resulted from the equilibrium equation and the boundary conditions. The variational analysis of this abstract problem was done in [1].

In this paper, we consider the numerical analysis of frictional contact problems for elastic-viscoplastic materials of the type (1.1). The literature is abundant on numerical treatment of variational inequality; see, for instance, the monographs [7, 8, 12, 14]. Of particular relevance to this paper are the works on numerical analysis of variational inequalities arising in plasticity; cf. [9, 11, 10].

The paper is organized as follows. In section 2 we present the abstract problem, state the assumptions on the data, and recall the existence and uniqueness result proved in [1]. In section 3, a semidiscrete scheme and a fully discrete scheme for the approximation of the abstract problem are analyzed and some error estimates are presented. In deriving the error estimates, we will need to apply Gronwall's inequality, which is recalled here for convenience. Suppose $f, g \in C[a,b]$ and g is nondecreasing, $c_0 > 0$ is a constant, then

$$f(t) \le g(t) + c_0 \int_a^t f(s)\,ds,\ \ t \in [a,b] \Longrightarrow f(t) \le e^{c_0(t-a)} g(t),\ \ t \in [a,b]. \tag{1.2}$$

Convergence analysis for both approximations is done in section 4 under the solution regularity condition established in the proof of well-posedness of the problem. Finally, in section 5 we apply the results to the study of numerical approximations of two concrete examples of quasi-static frictional contact problems in rate-type viscoplasticity.

2. The abstract problem.

Let H be a real Hilbert space, V a closed subspace of H. We denote by $(\cdot,\cdot)_H$ the inner product of H and by $\|\cdot\|_H$ the associated norm. Let us remark that V itself is a real Hilbert space endowed with the inner product of H; for this reason we shall sometimes use the notation $(u,v)_V$, $\|u\|_V$ instead of $(u,v)_H$, $\|u\|_H$, if $u, v \in V$. Let $A : H \to H$ be a linear operator, $B : [0,T]\times H\times H \to H$ a possibly nonlinear operator, and $\varphi : H \to (-\infty, +\infty]$. Let $[0,T]$ be the time interval of interest.

We consider an abstract problem

$$\dot{y}(t) = A\dot{x}(t) + B(t, x(t), y(t)) \quad \text{almost everywhere (a.e.) } t \in (0,T), \tag{2.1}$$

$$y(t) + \partial\varphi(\dot{x}(t)) \ni f(t) \quad \text{a.e. } t \in (0,T), \tag{2.2}$$

$$x(0) = x_0, \quad y(0) = y_0. \tag{2.3}$$

Here the unknowns are the functions $x : [0,T] \to V$ and $y : [0,T] \to H$, while $x_0 \in V$, $y_0 \in H$ and $f : [0,T] \to V$ are given data. The symbol $\partial\varphi$ represents the subdifferential of the function φ, and the relation (2.2) is understood in the sense that for a.e. $t \in (0,T)$, $f(t) - y(t)$ is a subgradient of φ at $\dot{x}(t)$. We denote by $D(\varphi)$ the effective domain of φ defined by

$$D(\varphi) = \{\ x \in H \mid \varphi(x) < +\infty\ \}.$$

We assume in what follows $D(\varphi) = V$.

An equivalent formulation of the problem (2.1)–(2.3) is derived.

PROBLEM P. Find functions $x : [0,T] \to V$ and $y : [0,T] \to H$ such that

$$x(0) = x_0, \ y(0) = y_0, \tag{2.4}$$

and for a.e. $t \in (0,T)$,

$$\dot{y}(t) = A\dot{x}(t) + B(t, x(t), y(t)), \tag{2.5}$$

$$(y(t), w - \dot{x}(t))_H + \varphi(w) - \varphi(\dot{x}(t)) \geq (f(t), w - \dot{x}(t))_V \quad \forall\, w \in V. \tag{2.6}$$

In the study of Problem P we make the following assumptions:

$$\left\{ \begin{array}{l} A : H \to H \text{ is linear, continuous, positive definite, and symmetric, i.e.,} \\ \text{(a) there exists } c_0 > 0 \text{ such that } \ (Ax, x)_H \geq c_0 \|x\|_H^2 \quad \forall\, x \in H; \\ \text{(b) } (Ax, y)_H = (x, Ay)_H \quad \forall\, x,\, y \in H. \end{array} \right. \tag{2.7}$$

$$\left\{ \begin{array}{l} B : [0,T] \times H \times H \to H \text{ has the properties that} \\ \text{(a) there exists an } L > 0 \text{ such that} \\ \quad \|B(t, x_1, y_1) - B(t, x_2, y_2)\|_H \leq L\,(\|x_1 - x_2\|_H + \|y_1 - y_2\|_H) \\ \qquad \forall\, t \in [0,T] \ \forall\, x_1, x_2, y_1, y_2 \in H; \\ \text{(b) the mapping } t \mapsto B(t,x,y) \text{ is measurable } \forall\, x, y \in H; \\ \text{(c) the mapping } t \mapsto B(t,0,0) \in L^\infty(0,T;H). \end{array} \right. \tag{2.8}$$

$$\varphi : V \to \mathbb{R}_+ \text{ is a continuous seminorm.} \tag{2.9}$$

$$f \in W^{1,\infty}(0,T;V). \tag{2.10}$$

$$x_0 \in V, \ y_0 \in H. \tag{2.11}$$

$$y_0 + \partial\varphi(0) \ni f(0). \tag{2.12}$$

We see from the condition (2.7) that the quantity $(Ax, y)_H$ defines an inner product on H, and the corresponding induced norm

$$\|x\|_A = \sqrt{(Ax, x)_H}, \quad x \in H$$

is equivalent to the norm $\|x\|_H$. We also remark that the assumption (2.9) implies that φ is Lipschitz continuous on V.

Everywhere in the paper we use the standard notation for L^p, $W^{m,p}$, H^m, and C^m spaces, $1 \leq p \leq \infty$, $m \in \mathbb{N}$. Moreover, if X and Y are real Hilbert spaces, we denote in what follows by $X \times Y$ the product space endowed with the canonical inner product.

The well-posedness of Problem P has been investigated in [1] where the following result can be found.

THEOREM 2.1. *Under the assumptions* (2.7)–(2.12)*, Problem* P *has a unique solution* $x \in W^{1,\infty}(0,T;V)$*,* $y \in W^{1,\infty}(0,T;H)$*.*

The proof of Theorem 2.1 is carried out in several steps. It is based on time discretization method, standard arguments of elliptic variational inequalities, and a fixed point property. Because of the Sobolev embedding $W^{1,\infty}(0,T;X) \hookrightarrow C([0,T];X)$ for any Banach space X, the solution from Theorem 2.1 is (or more precisely, can be made) continuous: $x \in C([0,T];V)$, $y \in C([0,T];H)$.

In the next two sections, we assume (2.7)–(2.12) are satisfied.

3. Numerical analysis of the abstract problem.

In this section, we present and analyze approximation schemes for solving Problem P. We will give some preliminary results that will be applied to some concrete examples later.

3.1. A semidiscrete approximation. Let $h \in (0,1]$ be an index and $\{H^h\}$ a family of finite dimensional subspaces of H. Set $V^h = V \cap H^h$, which is nonempty since $0 \in V^h$. Let $\mathcal{P}_{H^h} : H \to H^h$ be the orthogonal projection defined through the relation

$$(\mathcal{P}_{H^h} q, q^h)_H = (q, q^h)_H \quad \forall\, q \in H,\ q^h \in H^h. \tag{3.1}$$

Then $\mathcal{P}_{H^h}$ is linear and we have

$$\|\mathcal{P}_{H^h} q\|_H \le \|q\|_H \quad \forall\, q \in H. \tag{3.2}$$

This property will be used on various occasions.

Now a semidiscrete approximation of Problem P follows.

Problem P^h. Find the functions $x^h : [0,T] \to V^h$ and $y^h : [0,T] \to H^h$ such that

$$x^h(0) = x_0^h,\ y^h(0) = y_0^h, \tag{3.3}$$

and for a.e. $t \in (0,T)$,

$$\dot{y}^h(t) = \mathcal{P}_{H^h} A\dot{x}^h(t) + \mathcal{P}_{H^h} B(t, x^h(t), y^h(t)), \tag{3.4}$$

$$(y^h(t), w^h - \dot{x}^h(t))_H + \varphi(w^h) - \varphi(\dot{x}^h(t)) \ge (f(t), w^h - \dot{x}^h(t))_V \quad \forall\, w^h \in V^h. \tag{3.5}$$

Here, $x_0^h = \mathcal{P}_{V^h} x_0 \in V^h$, $y_0^h = \mathcal{P}_{H^h} y_0 \in H^h$ are orthogonal projections of x_0 and y_0 to V^h and H^h, respectively. The definition of $\mathcal{P}_{V^h} : V \to V^h$ is similar to that of $\mathcal{P}_{H^h}$. From the definition of y_0^h, we see that the discrete analog of (2.12) is valid:

$$(y_0^h, w^h)_H + \varphi(w^h) \ge (f(0), w^h)_V \quad \forall\, w^h \in V^h.$$

This relation is needed to verify the existence of a solution of Problem P^h.

We observe that the projection operator $\mathcal{P}_{H^h}$ is introduced to ensure that the relation (3.4) is well defined on the space H^h.

Using the arguments in [1], it can be shown that Problem P^h has a unique solution $x^h \in W^{1,\infty}(0,T;V^h)$ and $y^h \in W^{1,\infty}(0,T;H^h)$. We have $x^h \in C([0,T];V^h)$ and $y^h \in C([0,T];H^h)$. Our main purpose here is to derive estimates for the errors $x - x^h$ and $y - y^h$.

To this end, let $t \in [0,T]$. We first integrate (2.5) and (3.4) and use the initial conditions (2.4) and (3.3) to obtain

$$y(t) = Ax(t) + \int_0^t B(s, x(s), y(s))\, ds + y_0 - Ax_0, \tag{3.6}$$

$$y^h(t) = \mathcal{P}_{H^h} Ax^h(t) + \mathcal{P}_{H^h} \int_0^t B(s, x^h(s), y^h(s))\, ds + y_0^h - \mathcal{P}_{H^h} Ax_0^h. \tag{3.7}$$

Then we subtract (3.7) from (3.6) to get

$$\begin{aligned} y(t) - y^h(t) = {} & \mathcal{P}_{H^h} A(x(t) - x^h(t)) \\ & + \mathcal{P}_{H^h} \int_0^t \Big[B(s, x(s), y(s)) - B(s, x^h(s), y^h(s)) \Big]\, ds \\ & + y_0 - y_0^h - \mathcal{P}_{H^h} A(x_0 - x_0^h) + (I_H - \mathcal{P}_{H^h})(y(t) - y_0), \end{aligned} \tag{3.8}$$

where $I_H : H \to H$ is the identity operator.

Denote

$$e_0 = \|x_0 - x_0^h\|_V + \|y_0 - y_0^h\|_H. \tag{3.9}$$

Then we get the following inequality from (3.8), using the assumptions (2.7), (2.8), and the property (3.2):

$$\begin{aligned} \|y(t) - y^h(t)\|_H \le{}& c\,\|x(t) - x^h(t)\|_V + \|(I_H - \mathcal{P}_{H^h})(y(t) - y_0)\|_H \\ &+ c\int_0^t \left(\|y(s) - y^h(s)\|_H + \|x(s) - x^h(s)\|_A\right) ds + c\,e_0, \end{aligned} \tag{3.10}$$

where c is a positive constant which depends on operators A and B. Everywhere in this paper, except in section 5, the symbol c will represent a strictly positive constant which may change its value from place to place, and may depend on A, B, T, and φ, but not on the time or the input data.

Now plugging (3.6) in (2.6) with $w = \dot{x}^h(t)$, we have

$$\begin{aligned} (Ax(t), \dot{x}(t) - \dot{x}^h(t))_H \le{}& (f(t), \dot{x}(t) - \dot{x}^h(t))_V + \varphi(\dot{x}^h(t)) - \varphi(\dot{x}(t)) \\ &+ \left(\int_0^t B(s, x(s), y(s))\,ds, \dot{x}^h(t) - \dot{x}(t)\right)_H \\ &+ (y_0 - Ax_0, \dot{x}^h(t) - \dot{x}(t))_H. \end{aligned} \tag{3.11}$$

Let $w^h \in L^2(0,T;V^h)$ be arbitrary. We plug (3.7) in (3.5) with $w^h = w^h(t)$ to obtain

$$\begin{aligned} -(Ax^h(t), \dot{x}(t) - \dot{x}^h(t))_H \le{}& (f(t), \dot{x}^h(t) - w^h(t))_V + \varphi(w^h(t)) - \varphi(\dot{x}^h(t)) \\ &+ \left(\int_0^t B(s, x^h(s), y^h(s))\,ds, w^h(t) - \dot{x}^h(t)\right)_H \\ &+ (y_0^h - Ax_0^h, w^h(t) - \dot{x}^h(t))_H \\ &+ (Ax^h(t), w^h(t) - \dot{x}(t))_H. \end{aligned} \tag{3.12}$$

The relations (3.11) and (3.12) hold for a.e. t. Adding these relations we have

$$\begin{aligned} (A(x(t) - x^h(t)), \dot{x}(t) - \dot{x}^h(t))_H \le{}& (A(x(t) - x^h(t)), \dot{x}(t) - w^h(t))_H \\ &+ R(t; \dot{x}(t), w^h(t)) + (D(t), \dot{x}^h(t) - w^h(t))_H \end{aligned}$$

for a.e. $t \in (0,T)$, where

$$\begin{aligned} R(t; \dot{x}(t), w^h(t)) ={}& (y(t), w^h - \dot{x}(t))_H + \varphi(w^h(t)) - \varphi(\dot{x}(t)) \\ &- (f(t), w^h(t) - \dot{x}(t))_V \end{aligned} \tag{3.13}$$

and

$$D(t) = \int_0^t (B(s, x(s), y(s)) - B(s, x^h(s), y^h(s)))\,ds + y_0 - y_0^h - A(x_0 - x_0^h). \tag{3.14}$$

Using the assumption (2.7), we then have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|x(t) - x^h(t)\|_A^2 \le{}& \frac{1}{2}\|x(t) - x^h(t)\|_A^2 + \frac{1}{2}\|\dot{x}(t) - w^h(t)\|_A^2 + R(t; \dot{x}(t), w^h(t)) \\ &+ (D(t), \dot{x}^h(t) - \dot{x}(t))_H + (D(t), \dot{x}(t) - w^h(t))_H. \end{aligned}$$

Integrate the above inequality from 0 to t to obtain

$$\begin{aligned}\|x(t)-x^h(t)\|_A^2 \le \|x_0-x_0^h\|_A^2 &+ c\int_0^t \left(|R(s;\dot{x}(s),w^h(s))| + \|D(s)\|_H^2\right) ds\\ &+ c\int_0^t \left(\|x(s)-x^h(s)\|_A^2 + \|\dot{x}(s)-w^h(s)\|_A^2\right) ds\\ &+ \int_0^t (D(s),\dot{x}^h(s)-\dot{x}(s))_H\, ds,\end{aligned}$$

which holds for all $t\in[0,T]$ owing to the continuity of $x(t)$ and $x^h(t)$. For the last term above, we perform an integration by parts,

$$\begin{aligned}&\int_0^t (D(s),\dot{x}^h(s)-\dot{x}(s))_H\, ds\\ &\quad = (D(t),x^h(t)-x(t))_H - (D(0),x_0^h-x_0)_H\\ &\qquad - \int_0^t (B(s,x(s),y(s))-B(s,x^h(s),y^h(s)),x^h(s)-x(s))_H\, ds.\end{aligned}$$

Using (2.8), we have the estimate

$$\begin{aligned}\int_0^t (D(s),\dot{x}^h(s)-\dot{x}(s))_H\, ds \le c\,\|D(t)\|_H^2 + \frac{1}{2}\|x(t)-x^h(t)\|_A^2 + c\,e_0^2&\\ + c\int_0^t (\|x(s)-x^h(s)\|_A^2 + \|y(s)-y^h(s)\|_H^2)\, ds.&\end{aligned}$$

From the definition (3.14) and the assumptions (2.7) and (2.8), we have

$$\|D(t)\|_H \le c\int_0^t (\|x(s)-x^h(s)\|_A + \|y(s)-y^h(s)\|_H)\, ds + c\,e_0.$$

Combine the last several relations,

$$\begin{aligned}\|x(t)-x^h(t)\|_A^2 \le c\,e_0^2 &+ c\int_0^t \left(\|\dot{x}(s)-w^h(s)\|_A^2 + |R(s;\dot{x}(s),w^h(s))|\right) ds\\ &+ c\int_0^t \left(\|x(s)-x^h(s)\|_A^2 + \|y(s)-y^h(s)\|_H^2\right) ds.\end{aligned} \tag{3.15}$$

This inequality, together with (3.10), implies

$$\begin{aligned}&\|x(t)-x^h(t)\|_A^2 + \|y(t)-y^h(t)\|_H^2\\ &\quad \le c\,e_0^2 + c\,\|(I_H-\mathcal{P}_{H^h})(y(t)-y_0)\|_H^2\\ &\qquad + c\int_0^t \left(\|\dot{x}(s)-w^h(s)\|_A^2 + |R(s;\dot{x}(s),w^h(s))|\right) ds\\ &\qquad + c\int_0^t \left(\|x(s)-x^h(s)\|_A^2 + \|y(s)-y^h(s)\|_H^2\right) ds.\end{aligned} \tag{3.16}$$

Applying the Gronwall inequality (1.2), we get

$$\sup_{t\in[0,T]}\Big(\|x(t)-x^h(t)\|_A^2+\|y(t)-y^h(t)\|_H^2\Big)$$
$$\le c\,e_0^2+c\sup_{t\in[0,T]}\|(I_H-\mathcal{P}_{H^h})(y(t)-y_0)\|_H^2$$
$$+c\int_0^T\big(\|\dot x(t)-w^h(t)\|_A^2+|R(t;\dot x(t),w^h(t))|\big)\,dt.$$

Summarizing, we have shown the following result.

THEOREM 3.1. *Let* $(x,y)\in W^{1,\infty}(0,T;V\times H)$ *and* $(x^h,y^h)\in W^{1,\infty}(0,T;V^h\times H^h)$ *be the solutions of Problem* P *and* P^h*, respectively. Then we have the error estimate*

(3.17)
$$\|x-x^h\|_{L^\infty(0,T;V)}+\|y-y^h\|_{L^\infty(0,T;H)}$$
$$\le c\,(\|x_0-x_0^h\|_V+\|y_0-y_0^h\|_H)+c\,\|(I_H-\mathcal{P}_{H^h})(y-y_0)\|_{L^\infty(0,T;H)}$$
$$+c\inf_{w^h\in L^2(0,T;V^h)}\left\{\|\dot x-w^h\|_{L^2(0,T;V)}+\left(\int_0^T|R(s;\dot x(s),w^h(s))|\,ds\right)^{1/2}\right\},$$

where $R(\cdot;\cdot,\cdot)$ *is defined in* (3.13).

The inequality (3.17) is the basis of convergence and error analysis for the semidiscrete solutions. Concrete order error estimates will be established when Theorem 3.1 is applied in section 5 to some examples arising in mechanics.

3.2. A fully discrete approximation. In addition to the finite dimensional spaces V^h and H^h introduced in the previous subsection, we divide the time interval $[0,T]$ into N equal parts and denote the step-size by $k=T/N$, the nodal points by $t_n=nk$, $n=0,1,\dots,N$, and the subintervals $I_n=[t_{n-1},t_n]$, $n=1,\dots,N$. The arguments and results of this subsection can be easily extended to the case of nonuniform partition of the time interval. For a continuous function $w(t)$ with values in H or V, we use the notation $w_n\equiv w(t_n)$. For a sequence $\{w_n\}_{n=0}^N$, we denote $\delta w_n=(w_n-w_{n-1})/k$, $n=1,\dots,N$.

Then a fully discrete approximation based on a forward Euler scheme which we will analyze is the following.

PROBLEM P^{hk}. Find $x^{hk}=\{x_n^{hk}\}_{n=0}^N\subset V^h$ and $y^{hk}=\{y_n^{hk}\}_{n=0}^N\subset H^h$ such that

$$x_0^{hk}=x_0^h,\quad y_0^{hk}=y_0^h, \tag{3.18}$$

and for $n=1,\dots,N$,

$$\delta y_n^{hk}=\mathcal{P}_{H^h}A\delta x_n^{hk}+\mathcal{P}_{H^h}B(t_{n-1},x_{n-1}^{hk},y_{n-1}^{hk}), \tag{3.19}$$

$$(y_n^{hk},w^h-\delta x_n^{hk})_H+\varphi(w^h)-\varphi(\delta x_n^{hk})\ge(f_n,w^h-\delta x_n^{hk})_V\quad\forall\,w^h\in V^h. \tag{3.20}$$

Here again, $x_0^h=\mathcal{P}_{V^h}x_0\in V^h$, $y_0^h=\mathcal{P}_{H^h}y_0\in H^h$ are orthogonal projections of x_0 and y_0 to V^h and H^h, respectively.

We first inductively show the unique solvability of Problem P^{hk}. Let $x_{n-1}^{hk}\in V^h$ and $y_{n-1}^{hk}\in H^h$ be given. We rewrite (3.19) as

$$y_n^{hk}=k\,\mathcal{P}_{H^h}A\delta x_n^{hk}+k\,\mathcal{P}_{H^h}B(t_{n-1},x_{n-1}^{hk},y_{n-1}^{hk})+y_{n-1}^{hk} \tag{3.21}$$

and use it in (3.20) to yield

$$k(A\delta x_n^{hk}, w^h - \delta x_n^{hk})_H + \varphi(w^h) - \varphi(\delta x_n^{hk}) \geq (f_n - k\, B(t_{n-1}, x_{n-1}^{hk}, y_{n-1}^{hk}) - y_{n-1}^{hk}, w^h - \delta x_n^{hk})_H \quad \forall\, w^h \in V^h.$$

This is a discrete elliptic variational inequality of the second kind for the variable δx_n^{hk}. By a standard result (cf. [7]), this inequality has a unique solution $\delta x_n^{hk} \in V^h$ from which we can determine x_n^{hk}. Then y_n^{hk} can be obtained from (3.21). Therefore Problem P^{hk} has a unique solution.

For convergence analysis and error estimation, we need the Lipschitz continuity of the function $B(t; x, y)$ with respect to t, i.e., instead of (2.8), we assume

$$\|B(t_1; x_1, y_1) - B(t_2; x_2, y_2)\|_H \leq L\,(|t_1 - t_2| + \|x_1 - x_2\|_H + \|y_1 - y_2\|_H) \quad \forall\, t_1, t_2 \in [0, T] \quad \forall\, x_1, y_1, x_2, y_2 \in H. \tag{3.22}$$

If B does not depend on t, then the assumptions (2.8) and (3.22) are the same.

In the next section, we demonstrate the convergence of the fully discrete solution under the proved solution regularity $x \in W^{1,\infty}(0, T; V)$ and $y \in W^{1,\infty}(0, T; H)$. Our goal here is to derive estimates for the errors $\{x_n - x_n^{hk}\}_{n=1}^N$ and $\{y_n - y_n^{hk}\}_{n=1}^N$. For this purpose, we assume additionally $x \in C^1([0, T]; V)$ so that (2.5) and (2.6) hold for all $t \in [0, T]$. We apply (3.21) recursively to get

$$y_n^{hk} = \mathcal{P}_{H^h} A x_n^{hk} + \sum_{j=1}^{n} k \mathcal{P}_{H^h} B(t_{j-1}, x_{j-1}^{hk}, y_{j-1}^{hk}) + y_0^{hk} - \mathcal{P}_{H^h} A x_0^{hk}. \tag{3.23}$$

Then subtracting (3.23) from (3.6) at $t = t_n$, we obtain

$$\begin{aligned} y_n - y_n^{hk} = {} & (I_H - \mathcal{P}_{H^h})(y_n - y_0) + \mathcal{P}_{H^h} A(x_n - x_n^{hk}) \\ & + \mathcal{P}_{H^h} \left[\int_0^{t_n} B(s, x(s), y(s))\, ds - \sum_{j=1}^{n} k B(t_{j-1}, x_{j-1}^{hk}, y_{j-1}^{hk}) \right] \\ & + y_0 - y_0^{hk} - \mathcal{P}_{H^h} A(x_0 - x_0^{hk}). \end{aligned} \tag{3.24}$$

We denote

$$e_n = \|x_n - x_n^{hk}\|_V + \|y_n - y_n^{hk}\|_H, \quad n = 0, \ldots, N$$

for the numerical solution errors.

We first present a preparatory result.

Lemma 3.2. *There exists a constant $c > 0$ such that for $n = 1, \ldots, N$,*

$$\left\| \int_{t_{n-1}}^{t_n} B(s, x(s), y(s))\, ds - k B(t_{n-1}, x_{n-1}^{hk}, y_{n-1}^{hk}) \right\|_H \leq c\, k e_{n-1} + c\, k^2 \left(1 + \|\dot{x}\|_{L^\infty(0,T;V)} + \|\dot{y}\|_{L^\infty(0,T;H)}\right). \tag{3.25}$$

Therefore,

$$\left\| \int_0^{t_n} B(s, x(s), y(s))\, ds - \sum_{j=1}^{n} k B(t_{j-1}, x_{j-1}^{hk}, y_{j-1}^{hk}) \right\|_H \leq c \sum_{j=1}^{n} k\, e_{j-1} + c\, k \left(1 + \|\dot{x}\|_{L^\infty(0,T;V)} + \|\dot{y}\|_{L^\infty(0,T;H)}\right). \tag{3.26}$$

Proof. We write

$$\begin{aligned}\int_{t_{n-1}}^{t_n} B(s,x(s),y(s))\,ds - kB(t_{n-1},x_{n-1}^{hk},y_{n-1}^{hk})\\ = \int_{t_{n-1}}^{t_n} \left[B(s,x(s),y(s)) - B(t_{n-1},x_{n-1},y_{n-1})\right] ds\\ + k\left[B(t_{n-1},x_{n-1},y_{n-1}) - B(t_{n-1},x_{n-1}^{hk},y_{n-1}^{hk})\right].\end{aligned}$$

Using the assumption (3.22), we have

$$\|B(s,x(s),y(s)) - B(t_{n-1},x_{n-1},y_{n-1})\|_H \le c\,k\left(1 + \|\dot{x}\|_{L^\infty(0,T;V)} + \|\dot{y}\|_{L^\infty(0,T;H)}\right)$$

and

$$\|B(t_{n-1},x_{n-1},y_{n-1}) - B(t_{n-1},x_{n-1}^{hk},y_{n-1}^{hk})\|_H \le c\,ke_{n-1}.$$

Thus (3.25) holds. The inequality (3.26) follows from (3.25). ∎

We obtain the following inequality from (3.24) by the use of Lemma 3.2, the assumptions (2.7), and the property (3.2):

$$\begin{aligned}\|y_n - y_n^{hk}\|_H \le (I_H - \mathcal{P}_{H^h})(y_n - y_0)\|_H + c\,\|x_n - x_n^{hk}\|_H + c\,e_0\\ + c\sum_{j=1}^{n} k\,e_{j-1} + c\,k\,(1 + \|\dot{x}\|_{L^\infty(0,T;V)} + \|\dot{y}\|_{L^\infty(0,T;H)}).\end{aligned} \tag{3.27}$$

Now let us bound $r_n \equiv x_n - x_n^{hk}$, $n = 1, \ldots, N$. For this, we plug (3.6) into (2.6) at $t = t_n$ with $w = \delta x_n^{hk}$ and get

$$\begin{aligned}(Ax_n, \dot{x}_n - \delta x_n^{hk})_H \le (f_n, \dot{x}_n - \delta x_n^{hk})_V + \varphi(\delta x_n^{hk}) - \varphi(\dot{x}_n)\\ + \left(\int_0^{t_n} B(s,x(s),y(s))\,ds, \delta x_n^{hk} - \dot{x}_n\right)_H\\ + (y_0 - Ax_0, \delta x_n^{hk} - \dot{x}_n)_H.\end{aligned} \tag{3.28}$$

Similarly, plugging (3.23) into (3.20) with an arbitrary $w^h = w_n^h \in V^h$ yields

$$\begin{aligned}-(Ax_n^{hk}, w_n^h - \delta x_n^{hk})_H \le (f_n, \delta x_n^{hk} - w_n^h)_V + \varphi(w_n^h) - \varphi(\delta x_n^{hk})\\ + \left(\sum_{j=1}^{n} kB(t_{j-1},x_{j-1}^{hk},y_{j-1}^{hk}), w_n^h - \delta x_n^{hk}\right)_H\\ + (y_0^{hk} - Ax_0^{hk}, w_n^h - \delta x_n^{hk})_H.\end{aligned} \tag{3.29}$$

Let us consider the quantity

$$A_n \equiv (Ar_n, \delta r_n)_H = (A(x_n - x_n^{hk}), \delta x_n - \delta x_n^{hk})_H.$$

The following lower bound for A_n can be obtained by using the assumptions (2.7):

$$A_n \ge \frac{1}{2k}(\|r_n\|_A^2 - \|r_{n-1}\|_A^2). \tag{3.30}$$

For any $w_n^h \in V^h$, we write

$$A_n = (Ax_n, \delta x_n - \dot{x}_n)_H + (Ax_n, \dot{x}_n - \delta x_n^{hk})_H - (Ax_n^{hk}, \delta x_n - w_n^h)_H - (Ax_n^{hk}, w_n^h - \delta x_n^{hk})_H.$$

We then use (3.28) and (3.29) to bound the second and fourth terms,

$$\begin{aligned} A_n &\le (Ar_n, \delta x_n - w_n^h)_H + R_n(\dot{x}_n, w_n^h) + I_1(n) + I_2(n) + I_3(n) \\ &\le \frac{1}{2}\|r_n\|_A^2 + \frac{1}{2}\|\delta x_n - w_n^h\|_A^2 + R_n(\dot{x}_n, w_n^h) + I_1(n) + I_2(n) + I_3(n), \end{aligned}$$

where

$$R_n(\dot{x}_n, w_n^h) = (y_n, w_n^h - \dot{x}_n)_H + \varphi(w_n^h) - \varphi(\dot{x}_n) - (f_n, w_n^h - \dot{x}_n)_V, \tag{3.31}$$

and

$$\begin{aligned} I_1(n) &= \left(\int_0^{t_n} B(s, x(s), y(s))\, ds - \sum_{j=1}^n kB(t_{j-1}, x_{j-1}^{hk}, y_{j-1}^{hk}), \delta x_n^{hk} - \delta x_n\right)_H, \\ I_2(n) &= (y_0 - y_0^{hk} - A(x_0 - x_0^{hk}), \delta x_n^{hk} - \delta x_n)_H, \\ I_3(n) &= \left(\int_0^{t_n} B(s, x(s), y(s))\, ds - \sum_{j=1}^n kB(t_{j-1}, x_{j-1}^{hk}, y_{j-1}^{hk}) \right. \\ &\qquad \left. + \, y_0 - y_0^{hk} - A(x_0 - x_0^{hk}), \delta x_n - w_n^h\right)_H. \end{aligned}$$

Combining the lower and upper bounds for A_n, we have

$$\begin{aligned} \|r_n\|_A^2 - \|r_{n-1}\|_A^2 \le k\,\|r_n\|_A^2 + k\,\|\delta x_n - w_n^h\|_A^2 \\ + 2\,k\,(R_n(\dot{x}_n, w_n^h) + I_1(n) + I_2(n) + I_3(n)). \end{aligned} \tag{3.32}$$

In the inequality (3.32), we change the index n to j, and sum over j from 1 to n:

$$\begin{aligned} \|r_n\|_A^2 \le \|r_0\|_A^2 + \sum_{j=1}^n k\,\|r_j\|_A^2 + \sum_{j=1}^n k\,\|\delta x_j - w_j^h\|_A^2 \\ + 2\sum_{j=1}^n k\left[R_j(\dot{x}_j, w_j^h) + I_1(j) + I_2(j) + I_3(j)\right]. \end{aligned} \tag{3.33}$$

We now write

$$\begin{aligned} \sum_{j=1}^n kI_1(j) &= \sum_{j=1}^n \left(\int_0^{t_j} B(s, x(s), y(s))\, ds - \sum_{i=1}^j k\,B(t_{i-1}, x_{i-1}^{hk}, y_{i-1}^{hk}), r_{j-1} - r_j\right)_H \\ &= \left(\int_0^{t_1} B(s, x(s), y(s))\, ds - kB(t_0, x_0^{hk}, y_0^{hk}), r_0\right)_H \\ &\quad - \left(\int_0^{t_n} B(s, x(s), y(s))\, ds - \sum_{i=1}^n kB(t_{i-1}, x_{i-1}^{hk}, y_{i-1}^{hk}), r_n\right)_H \\ &\quad + \sum_{j=1}^{n-1} \left(\int_{t_j}^{t_{j+1}} B(s, x(s), y(s))\, ds - kB(t_j, x_j^{hk}, y_j^{hk}), r_j\right)_H. \end{aligned}$$

Using the assumption (2.7) and the estimates (3.26) and (3.25), we obtain

$$\sum_{j=1}^{n} kI_1(j) \le c\,e_0^2 + c\sum_{j=1}^{n-1} k\,(\|r_j\|_V^2 + \|y_j - y_j^{hk}\|_H^2) + \frac{1}{16}\|r_n\|_A^2 + c\,k^2(1 + \|\dot{x}\|_{L^\infty(0,T;V)} + \|\dot{y}\|_{L^\infty(0,T;H)})^2. \tag{3.34}$$

Similarly, we write

$$\sum_{j=1}^{n} kI_2(j) = \left(y_0 - y_0^{hk} - A(x_0 - x_0^{hk}), \sum_{j=1}^{n}(r_{j-1} - r_j)\right)_H = (y_0 - y_0^{hk} - A(x_0 - x_0^{hk}), r_0 - r_n)_H$$

and use the assumption (2.7) to obtain

$$\sum_{j=1}^{n} kI_2(j) \le c\,e_0^2 + \frac{1}{16}\|r_n\|_A^2. \tag{3.35}$$

Finally we use the assumption (2.8) and the estimates (3.26) and (3.25) to find

$$\sum_{j=1}^{n} kI_3(j) \le c\,e_0^2 + c\sum_{j=1}^{n-1} k\,(\|r_j\|_V^2 + \|y_j - y_j^{hk}\|_H^2) + c\,k^2(1 + \|\dot{x}\|_{L^\infty(0,T;V)} + \|\dot{y}\|_{L^\infty(0,T;H)})^2 + c\sum_{j=1}^{n} k\,\|\delta x_j - w_j^h\|_V^2. \tag{3.36}$$

We now combine the estimates (3.27), (3.33), and (3.34)–(3.36) to obtain

$$\|x_n - x_n^{hk}\|_V^2 + \|y_n - y_n^{hk}\|_H^2 \le c\,J_n + c\sum_{j=1}^{n-1} k(\|x_j - x_j^{hk}\|_V^2 + \|y_j - y_j^{hk}\|_H^2), \tag{3.37}$$

where

$$J_n = e_0^2 + k^2(1 + \|\dot{x}\|_{L^\infty(0,T;V)} + \|\dot{y}\|_{L^\infty(0,T;H)})^2 + \|(I_H - \mathcal{P}_{H^h})(y_n - y_0)\|_H^2 + \sum_{j=1}^{n} k\,\|\delta x_j - w_j^h\|_V^2 + \sum_{j=1}^{n} k\,|R_j(\dot{x}_j, w_j^h)|.$$

An error estimate can be derived based on (3.37).

Theorem 3.3. *Let $(x, y) \in W^{1,\infty}(0,T;V) \times W^{1,\infty}(0,T;H)$ and $\{(x_n^{hk}, y_n^{hk})\}_{n=1}^N$ be the solutions of Problems* P *and* P^{hk}*, respectively. Assume $x \in C^1([0,T];V)$. Then we have the error estimate*

$$\max_{1\le n\le N}\left(\|x_n - x_n^{hk}\|_V + \|y_n - y_n^{hk}\|_H\right) \le c\,(\|x_0 - x_0^{hk}\|_V + \|y_0 - y_0^{hk}\|_H) + c\,k\,(1 + \|\dot{x}\|_{L^\infty(0,T;V)} + \|\dot{y}\|_{L^\infty(0,T;H)}) + c\max_{1\le n\le N}\|(I_H - \mathcal{P}_{H^h})(y_n - y_0)\|_H + c\left[\sum_{j=1}^{N} k\inf_{w_j^h\in V^h}\left(\|\delta x_j - w_j^h\|_V^2 + |R_j(\dot{x}_j, w_j^h)|\right)\right]^{1/2}. \tag{3.38}$$

Proof. Denote $E_n = \sum_{j=1}^n k\, e_n^2$. Then (3.37) can be rewritten as

$$e_n^2 \le c\, J_n + c\, E_{n-1}. \tag{3.39}$$

Now

$$E_n - E_{n-1} = k\, e_n^2 \le c\, k\, J_n + c\, k\, E_{n-1}.$$

Hence we have

$$E_n - (1 + c\, k) E_{n-1} \le c\, k\, J_n,$$

or equivalently,

$$\frac{E_n}{(1+c\,k)^n} - \frac{E_{n-1}}{(1+c\,k)^{n-1}} \le \frac{c\,k\,J_n}{(1+c\,k)^n}.$$

By an inductive argument, we get

$$E_n \le c\, k \sum_{i=1}^n (1+c\,k)^{n-i} J_i \le ((1+c\,k)^n - 1) \max_{1\le i\le n} J_i \le (e^{c\,T} - 1) \max_{1\le i\le n} J_i.$$

Using (3.39), we have

$$e_n^2 \le c \max_{1\le i\le n} J_i,$$

which implies (3.38). ∎

The estimate (3.38) will be used for error analysis of the fully discrete solutions provided the exact solution possesses certain regularity.

4. Convergence analysis. In this section, we analyze the convergence of the semidiscrete and fully discrete solutions for the Problem P under the basic regularity condition $(x, y) \in W^{1,\infty}(0, T; V \times H)$, available from Theorem 2.1. First we make the following additional assumptions on the function spaces H, V and the finite dimensional spaces H^h and V^h.

Assumption H1. There exist a subspace $V_0 \subset V$ which is dense in V and a function $\alpha(h) \ge 0$ such that

$$\lim_{h\to 0^+} \alpha(h) = 0,$$

and

$$\inf_{w^h \in V^h} \|w - w^h\|_V = \|w - \mathcal{P}_{V^h} w\|_V \le \alpha(h) \|w\|_{V_0} \quad \forall\, w \in V_0.$$

Assumption H2. There exist a subspace $H_0 \subset H$ which is dense in H and a function $\beta(h) \ge 0$ such that

$$\lim_{h\to 0^+} \beta(h) = 0,$$

and

$$\inf_{z^h \in H^h} \|z - z^h\|_H = \|z - \mathcal{P}_{H^h} z\|_H \le \beta(h) \|z\|_{H_0} \quad \forall\, z \in H_0.$$

These two hypotheses will be verified for the application problems discussed in the next section.

We will need the following result, which can be found in [18].

LEMMA 4.1. *Assume that X is a Banach space, $X_0 \subset X$ is dense in X. Then $H^1(0,T;X_0)$ is dense in $H^1(0,T;X)$.*

Now we are ready to study the convergence of the semidiscrete solution for Problem P, based on the estimate (3.17).

THEOREM 4.2. *Let $(x,y) \in W^{1,\infty}(0,T;V\times H)$ be the solution of Problem* P *and $(x^h,y^h) \in W^{1,\infty}(0,T;V^h\times H^h)$ the solution of corresponding semidiscrete Problem* P^h. *Then under Assumptions* H1 *and* H2 *we have convergence:*

$$\|x-x^h\|_{L^\infty(0,T;V)} + \|y-y^h\|_{L^\infty(0,T;H)} \to 0 \quad \text{as } h\to 0. \tag{4.1}$$

Proof. From Assumptions H1 and H2, we know

$$\|x_0-x_0^h\|_V \to 0, \quad \|y_0-y_0^h\|_H \to 0 \quad \text{as } h\to 0. \tag{4.2}$$

Since φ is Lipschitz continuous on V, using the definition (3.13), we have

$$|R(t;\dot{x}(t),w^h(t))| \le (\|y(t)\|_H + \|f(t)\|_V + c)\,\|w^h(t)-\dot{x}(t)\|_V.$$

Hence, the estimate (3.17) can be rewritten as

$$\begin{aligned}\|x^h-x\|_{L^\infty(0,T;V)} &+ \|y^h-y\|_{L^\infty(0,T;H)}\\ &\le c\,e_0 + c\,\|(I_H-\mathcal{P}_{H^h})(y-y_0)\|_{L^\infty(0,T;H)}\\ &\quad + c \inf_{w^h\in L^2(0,T;V^h)} \|\dot{x}-w^h\|^{1/2}_{L^2(0,T;V)}.\end{aligned} \tag{4.3}$$

Using Lemma 4.1 and Assumption H1, we know that $H^1(0,T;V_0)$ is dense in $H^1(0,T;V)$. So for any $\varepsilon\in(0,1)$, there exists $\tilde{x}\in H^1(0,T;V_0)$ such that

$$\|x-\tilde{x}\|_{H^1(0,T;V)} \le \varepsilon, \tag{4.4}$$

which can be combined with Assumption H1 again to yield

$$\begin{aligned}&\inf_{w^h\in L^2(0,T;V^h)} \|\dot{x}-w^h\|^{1/2}_{L^2(0,T;V)}\\ &\quad\le \|\dot{x}-\dot{\tilde{x}}\|^{1/2}_{L^2(0,T;V)} + \inf_{w^h\in L^2(0,T;V^h)} \|\dot{\tilde{x}}-w^h\|^{1/2}_{L^2(0,T;V)}\\ &\quad\le \sqrt{\varepsilon} + \sqrt{\alpha(h)}\,\|\tilde{x}\|^{1/2}_{H^1(0,T;V_0)}.\end{aligned} \tag{4.5}$$

Similarly, from Lemma 4.1 and Assumption H2, we know $H^1(0,T;H_0)$ is dense in $H^1(0,T;H)$. So there exists $\tilde{y}\in H^1(0,T;H_0)$ such that

$$\|y-y_0-\tilde{y}\|_{H^1(0,T;H)} \le \varepsilon.$$

Now we are ready to bound the first term on the right-hand side of (4.3):

$$\begin{aligned}\|(I_H-\mathcal{P}_{H^h})(y-y_0)\|_{L^\infty(0,T;H)} &\le c\,\|(I_H-\mathcal{P}_{H^h})(y-y_0)\|_{H^1(0,T;H)}\\ &\le c\,\|y-y_0-\tilde{y}\|_{H^1(0,T;H)}\\ &\quad + \|(I_H-\mathcal{P}_{H^h})\tilde{y}\|_{H^1(0,T;H)}\\ &\le c\,\varepsilon + \beta(h)\,\|\tilde{y}\|_{H^1(0,T;H_0)}.\end{aligned} \tag{4.6}$$

Above we used the embedding result $H^1(0,T;H) \hookrightarrow L^\infty(0,T;H)$ (cf. [18]). The convergence result (4.1) then follows from (4.2)–(4.6). ∎

We now turn to a convergence analysis of the fully discrete scheme. Notice that we cannot use the estimate (3.38), because under the basic regularity condition $(x,y) \in W^{1,\infty}(0,T;V\times H)$, the pointwise values $\dot{x}_j$ and $\dot{y}_j$ are not well defined. Here we follow the approach developed in [11] for a convergence analysis. For this purpose we will need another density result, which can also be found in [18].

LEMMA 4.3. *The space $C^\infty([0,T];V)$ is dense in $H^1(0,T;V)$; that is, given $w \in H^1(0,T;V)$, for any $\varepsilon > 0$ there exists $\bar{w} \in C^\infty([0,T];V)$ such that*

$$\|w - \bar{w}\|_{H^1(0,T;V)} \le \varepsilon.$$

Let us consider the quantity $A_n = (Ar_n, \delta r_n)_H$. We have the lower bound (3.30) for A_n. To obtain an upper bound we begin with

$$A_n = (Ax_n, \delta x_n - \delta x_n^{hk})_H - (Ax_n^{hk}, \delta x_n - w_n^h)_H - (Ax_n^{hk}, w_n^h - \delta x_n^{hk})_H,$$

where $w_n^h \in V^h$ is arbitrary. Using (3.29) to bound the last term, we obtain

$$\begin{aligned} A_n \le\ & (Ax_n, \delta x_n - \delta x_n^{hk})_H - (Ax_n^{hk}, \delta x_n - w_n^h)_H \\ &+ (f_n, \delta x_n^{hk} - w_n^h)_V + \varphi(w_n^h) - \varphi(\delta x_n^{hk}) \\ &+ \left(\sum_{j=1}^n kB(t_{j-1}, x_{j-1}^{hk}, y_{j-1}^{hk}), w_n^h - \delta x_n^{hk} \right)_H \\ &+ (y_0^{hk} - Ax_0^{hk}, w_n^h - \delta x_n^{hk})_H. \end{aligned} \tag{4.7}$$

Now integrate (2.6) with $w = \delta x_n^{hk}$ from $t = t_{n-1}$ to t_n and use (3.6) to obtain

$$\begin{aligned} 0 \le\ & \frac{1}{k}\int_{I_n} (Ax(t), \delta x_n^{hk} - \dot{x}(t))_H dt + \varphi(\delta x_n^{hk}) - \frac{1}{k}\int_{I_n} \varphi(\dot{x}(t))dt \\ &+ \frac{1}{k}\int_{I_n} \left(\int_0^t B(s, x(s), y(s))ds, \delta x_n^{hk} - \dot{x}(t) \right)_H dt \\ &+ (y_0 - Ax_0, \delta x_n^{hk} - \delta x_n)_H + \frac{1}{k}\int_{I_n} (f(t), \dot{x}(t) - \delta x_n^{hk})_V dt. \end{aligned} \tag{4.8}$$

Then we add (4.8) and (4.7) to obtain

$$A_n \le R_1 + R_2 + R_3 + R_4 + R_5, \tag{4.9}$$

where

$$R_1 = (Ax_n, \delta x_n - \delta x_n^{hk})_H - (Ax_n^{hk}, \delta x_n - w_n^h)_H + \frac{1}{k}\int_{I_n} (Ax(t), \delta x_n^{hk} - \dot{x}(t))_H dt,$$

$$R_2 = \varphi(w_n^h) - \frac{1}{k}\int_{I_n} \varphi(\dot{x}(t))dt,$$

$$R_3 = (f_n, \delta x_n^{hk} - w_n^h)_V + \frac{1}{k}\int_{I_n} (f(t), \dot{x}(t) - \delta x_n^{hk})_V dt,$$

$$R_4 = (y_0^{hk} - Ax_0^{hk}, w_n^h - \delta x_n^{hk})_H + (y_0 - Ax_0, \delta x_n^{hk} - \delta x_n)_H,$$

$$\begin{aligned} R_5 =\ & \left(\sum_{j=1}^n kB(t_{j-1}, x_{j-1}^{hk}, y_{j-1}^{hk}), w_n^h - \delta x_n^{hk} \right)_H \\ &+ \frac{1}{k}\int_{I_n} \left(\int_0^t B(s, x(s), y(s))ds, \delta x_n^{hk} - \dot{x}(t) \right)_H dt. \end{aligned}$$

We need to find appropriate bounds for R_i, $1 \le i \le 5$. First let us estimate R_1. Using the assumption (2.7) and the properties of inner product, we have

$$
\begin{aligned}
R_1 &= \frac{1}{k}\left(Ax_n - \frac{1}{k}\int_{I_n} Ax(t)\,dt, r_n - r_{n-1}\right)_H \\
&\quad + \frac{1}{k}\int_{I_n} (Ax(t), \delta x_n - \dot{x}(t))_H dt \\
&\quad + (Ar_n, \delta x_n - w_n^h)_H - (Ax_n, \delta x_n - w_n^h)_H \\
&\le \frac{1}{k^2}\left(A\int_{I_n} (t - t_{n-1})\dot{x}(t)\,dt, r_n - r_{n-1}\right)_H \\
&\quad + \frac{1}{k}\int_{I_n} (Ax(t), \delta x_n - \dot{x}(t))_H dt \\
&\quad + \frac{1}{2}\|r_n\|_A^2 + \frac{1}{2}\|\delta x_n - w_n^h\|_A^2 - (Ax_n, \delta x_n - w_n^h)_H \\
&\le c\,\|\dot{x}\|_{L^\infty(I_n;V)}(\|r_n\|_V + \|r_{n-1}\|_V) \\
&\quad + \frac{c}{k}\|x\|_{L^\infty(I_n;V)}\int_{I_n}\|\delta x_n - \dot{x}(t)\|_V dt \\
&\quad + \frac{1}{2}\|r_n\|_A^2 + c\,\|\delta x_n - w_n^h\|_V^2 + c\,\|x_n\|_V\|\delta x_n - w_n^h\|_V.
\end{aligned} \tag{4.10}
$$

Using the Lipschitz continuity of φ on V, we find a bound for R_2,

$$
|R_2| = \frac{1}{k}\left|\int_{I_n} (\varphi(w_n^h) - \varphi(\dot{x}(t))\,dt\right| \le \frac{c}{k}\int_{I_n}\|w_n^h - \dot{x}(t)\|_V dt. \tag{4.11}
$$

For R_3, we have

$$
\begin{aligned}
R_3 &= \frac{1}{k^2}\left(\int_{I_n}(t - t_{n-1})\dot{f}(t)dt, r_{n-1} - r_n\right)_H + (f_n, \delta x_n - w_n^h)_H \\
&\quad - \frac{1}{k}\int_{I_n}(f(t), \delta x_n - \dot{x}(t))_H dt \\
&\le \|\dot{f}\|_{L^\infty(I_n;V)}(\|r_n\|_V + \|r_{n-1}\|_V) + \|f_n\|_V\|\delta x_n - w_n^h\|_V \\
&\quad + \frac{1}{k}\|f\|_{L^\infty(I_n;V)}\int_{I_n}\|\delta x_n - \dot{x}(t)\|_V dt.
\end{aligned} \tag{4.12}
$$

For R_4, we have

$$
\begin{aligned}
R_4 &= \frac{1}{k}(y_0 - y_0^{hk} - A(x_0 - x_0^{hk}), r_{n-1} - r_n)_H - (y_0 - Ax_0, \delta x_n - w_n^h)_H \\
&\quad + (y_0 - y_0^{hk} - A(x_0 - x_0^{hk}), \delta x_n - w_n^h)_H \\
&\le \frac{1}{k}(y_0 - y_0^{hk} - A(x_0 - x_0^{hk}), r_{n-1} - r_n)_H \\
&\quad + c\,(e_0 + \|y_0\|_H + \|x_0\|_V)\|\delta x_n - w_n^h\|_V.
\end{aligned} \tag{4.13}
$$

Express R_5 as

$$
\begin{aligned}
R_5 = {} & \left(\sum_{j=1}^{n} kB(t_{j-1}, x_{j-1}^{hk}, y_{j-1}^{hk}) - \int_0^{t_n} B(s, x(s), y(s))\, ds, w_n^h - \delta x_n^{hk} \right)_H \\
& + \left(\int_0^{t_n} B(s, x(s), y(s))\, ds, w_n^h - \delta x_n^{hk} \right)_H \\
& + \frac{1}{k} \int_{I_n} \left(\int_0^t B(s, x(s), y(s))\, ds, (\delta x_n^{hk} - \delta x_n) + (\delta x_n - \dot{x}(t)) \right)_H dt.
\end{aligned} \tag{4.14}
$$

Integrating by parts, we obtain

$$
\begin{aligned}
& \frac{1}{k} \int_{I_n} \left(\int_0^t B(s, x(s), y(s))\, ds, \delta x_n - \dot{x}(t) \right)_H dt \\
& \quad = \frac{1}{k} \int_{I_n} (B(t, x(t), y(t)), \Pi x(t))_H dt,
\end{aligned} \tag{4.15}
$$

where

$$
\Pi x(t) = x(t) - \frac{t - t_{n-1}}{k}\, x_n - \frac{t_n - t}{k}\, x_{n-1}, \quad t_{n-1} \le t \le t_n. \tag{4.16}
$$

By an elementary manipulation, we have

$$
\begin{aligned}
& \frac{1}{k} \int_{I_n} \left(\int_0^t B(s, x(s), y(s))\, ds, \delta x_n^{hk} - \delta x_n \right)_H dt \\
& \quad = \left(\int_0^{t_n} B(s, x(s), y(s))\, ds, \delta x_n^{hk} - \delta x_n \right)_H \\
& \qquad - \frac{1}{k} \left(\int_{I_n} (s - t_{n-1}) B(s, x(s), y(s))\, ds, \delta x_n^{hk} - \delta x_n \right)_H .
\end{aligned} \tag{4.17}
$$

Using (4.15) and (4.17), we can rewrite R_5 as

$$
\begin{aligned}
R_5 = {} & \left(\sum_{j=1}^{n} kB(t_{j-1}, x_{j-1}^{hk}, y_{j-1}^{hk}) - \int_0^{t_n} B(s, x(s), y(s))\, ds, w_n^h - \delta x_n \right)_H \\
& + \frac{1}{k} \left(\sum_{j=1}^{n} k\, B(t_{j-1}, x_{j-1}^{hk}, y_{j-1}^{hk}) - \int_0^{t_n} B(s, x(s), y(s))\, ds, r_n - r_{n-1} \right)_H \\
& - \frac{1}{k^2} \left(\int_{I_n} (s - t_{n-1}) B(s, x(s), y(s))\, ds, r_{n-1} - r_n \right)_H \\
& + \frac{1}{k} \int_{I_n} (B(t, x(t), y(t)), \Pi x(t))_H\, dt + \left(\int_0^{t_n} B(s, x(s), y(s))\, ds, w_n^h - \delta x_n \right)_H .
\end{aligned}
$$

We then apply the estimates (3.26) and (3.25),

(4.18)

$$
\begin{aligned}
R_5 \le{}& c\sum_{j=1}^{n} k(\|x_{j-1}-x^{hk}_{j-1}\|^2_V + \|y_{j-1}-y^{hk}_{j-1}\|^2_V)\\
&+ c\,k^2(1+\|\dot{x}\|^2_{L^\infty(0,t_n;V)} + \|\dot{y}\|^2_{L^\infty(0,t_n;H)})\\
&+ \frac{1}{2}\|w^h_n-\delta x_n\|^2_V + \frac{1}{k}\|B(\cdot,x,y)\|_{L^\infty(I_n;H)}\int_{I_n}\|\Pi x(t)\|_H dt\\
&+ \frac{1}{k}\left(\sum_{j=1}^{n} kB(t_{j-1},x^{hk}_{j-1},y^{hk}_{j-1}) - \int_0^{t_n} B(s,x(s),y(s))ds, r_n-r_{n-1}\right)_H\\
&+ \|B(\cdot,x,y)\|_{L^\infty(I_n;H)}(\|r_n\|_V + \|r_{n-1}\|_V + t_n\|w^h_n-\delta x_n\|_V).
\end{aligned}
$$

Combining the relations (3.30), (4.9)–(4.13), and (4.18), we find that

(4.19)

$$
\begin{aligned}
&\|r_n\|^2_A - \|r_{n-1}\|^2_A\\
&\quad\le k\,\|r_n\|^2_A + c\,k\,(\|\dot{x}\|_{L^\infty(I_n;V)} + \|\dot{f}\|_{L^\infty(I_n;V)}\\
&\qquad+ \|B(\cdot,x,y)\|_{L^\infty(I_n;V)})\,(\|r_n\|_V + \|r_{n-1}\|_V)\\
&\qquad+ c\,(\|x\|_{L^\infty(I_n;V)} + \|f\|_{L^\infty(I_n;V)})\int_{I_n}\|\delta x_n - \dot{x}(t)\|_V dt\\
&\qquad+ c\,k\,\|\delta x_n - w^h_n\|^2_V + c\,(\|x_n\|_V + \|f_n\|_V + e_0 + \|y_0\|_H + \|x_0\|_V\\
&\qquad+ \|B(\cdot,x,y)\|_{L^\infty(I_n;V)})\|\delta x_n - w^h_n\|_V\\
&\qquad+ c\int_{I_n}\|w^h_n - \dot{x}(t)\|_V dt + 2\,(y_0 - y^{hk}_0 - A(x_0 - x^{hk}_0), r_{n-1}-r_n)_H\\
&\qquad+ 2\,\left(\sum_{j=1}^{n} k\,B(t_{j-1},x^{hk}_{j-1},y^{hk}_{j-1}) - \int_0^{t_n} B(s,x(s),y(s))\,ds, r_n-r_{n-1}\right)_H\\
&\qquad+ c\sum_{j=1}^{n} k^2(\|x_{j-1}-x^{hk}_{j-1}\|^2_V + \|y_{j-1}-y^{hk}_{j-1}\|^2_V)\\
&\qquad+ c\,\|B(\cdot,x,y)\|_{L^\infty(I_n;H)}\int_{I_n}\|\Pi x(t)\|_H dt\\
&\qquad+ c\,k^3(1+\|\dot{x}\|^2_{L^\infty(0,t_n;V)} + \|\dot{y}\|^2_{L^\infty(0,t_n;V)}).
\end{aligned}
$$

By a series of manipulations similar to those leading to (3.37), we obtain from (4.19) that

$$
\begin{aligned}
(4.20)\qquad &\|x_n - x_n^{hk}\|_V^2 + \|y_n - y_n^{hk}\|_H^2 \\
&\quad \le c \sum_{j=1}^{n-1} k(\|x_j - x_j^{hk}\|_V^2 + \|y_j - y_j^{hk}\|_V^2) + c\, e_0^2 \\
&\quad + c\, k\,(1 + \|x\|^2_{W^{1,\infty}(0,T;V)} + \|f\|^2_{W^{1,\infty}(0,T;V)}) \\
&\quad + c\, k\,(\|y_0\|_H^2 + \|x_0\|_V^2 + \|B(\cdot,x,y)\|^2_{L^\infty(0,T;V)}) \\
&\quad + c\,(\|x\|_{L^\infty(0,T;V)} + \|f\|_{L^\infty(0,T;V)}) \sum_{j=1}^{n} \int_{I_j} \|\delta x_j - \dot{x}(t)\|_V\, dt \\
&\quad + c \sum_{j=1}^{n} k\, \|\delta x_j - w_j^h\|_V^2 + c \sum_{j=1}^{n} \int_{I_j} \|w_j^h - \dot{x}(t)\|_V\, dt \\
&\quad + c\, \|B(\cdot,x,y)\|_{L^\infty(0,t_n;H)} \sum_{j=1}^{n} \int_{I_j} \|\Pi x(t)\|_H\, dt \\
&\quad + \|(I_H - \mathcal{P}_{H^h})(y_n - y_0)\|_H^2.
\end{aligned}
$$

We can then apply the technique of the proof of Theorem 3.3 to find the estimate

$$
\begin{aligned}
(4.21)\quad &\max_{1\le n\le N}(\|x_n - x_n^{hk}\|_V^2 + \|y_n - y_n^{hk}\|_V^2) \\
&\le c\Bigg(e_0^2 + \max_{1\le n\le N} \|(I_H - \mathcal{P}_{H^h})(y_n - y_0)\|_H^2 + k\,(1 + \|x\|^2_{W^{1,\infty}(0,T;V)} \\
&\quad + \|f\|^2_{W^{1,\infty}(0,T;V)} \|y_0\|_H^2 + \|x_0\|_V^2 + \|B(\cdot,x,y)\|^2_{L^\infty(0,T;V)}) \\
&\quad + (\|x\|_{L^\infty(0,T;V)} + \|f\|_{L^\infty(0,T;V)}) \sum_{j=1}^{N} \int_{I_j} \|\delta x_j - \dot{x}(t)\|_V\, dt \\
&\quad + \sum_{j=1}^{N} k\, \|\delta x_j - w_j^h\|_V^2 + \sum_{j=1}^{N} \int_{I_j} \|w_j^h - \dot{x}(t)\|_V\, dt \\
&\quad + \|B(\cdot,x,y)\|_{L^\infty(0,T;H)} \sum_{j=1}^{N} \int_{I_j} \|\Pi x(t)\|_H\, dt\Bigg).
\end{aligned}
$$

We now show a convergence result using (4.21).

THEOREM 4.4. *Let* $(x, y) \in W^{1,\infty}(0,T; V \times H)$ *be the solution of Problem* P *and* $\{(x_n^{hk}, y_n^{hk})\}_{n=1}^N \subset V^h \times H^h$ *be the solution of the corresponding fully discrete Problem* P^{hk}. *Then under Assumptions* H1 *and* H2 *we have*

$$
(4.22)\qquad \max_{1\le n\le N}(\|x_n - x_n^{hk}\|_V^2 + \|y_n - y_n^{hk}\|_V^2) \to 0 \quad \text{as } h, k \to 0.
$$

Proof. We still have (4.2). By Lemma 4.3, for any $\varepsilon > 0$, there exists $\bar{x} \in C^\infty([0,T]; V)$ such that

$$
\|x - \bar{x}\|_{H^1(0,T;V)} \le \varepsilon.
$$

Since

$$\begin{aligned}\int_{I_j} \|\delta x_j - \dot{x}(t)\|_V dt &= \int_{I_j} \left\| \frac{1}{k} \int_{I_j} (\dot{x}(s) - \dot{x}(t))\, ds \right\|_V dt \\ &\le \frac{1}{k} \int_{I_j} \int_{I_j} \big[\|\dot{x}(s) - \dot{\tilde{x}}(s)\|_V + \|\dot{x}(t) - \dot{\tilde{x}}(t)\|_V \\ &\qquad + \|\dot{\tilde{x}}(s) - \dot{\tilde{x}}(t)\|_V \big]\, ds\, dt \\ &\le 2 \int_{I_j} \|\dot{x}(t) - \dot{\tilde{x}}(t)\|_V dt + k \int_{I_j} \|\ddot{\tilde{x}}(t)\|_V dt,\end{aligned}$$

we have

$$\sum_{j=1}^N \int_{I_j} \|\delta x_j - \dot{x}(t)\|_V dt \le c\,\varepsilon + k\, \|\ddot{\tilde{x}}(t)\|_{L^1(0,T;V)}. \tag{4.23}$$

From

$$\|w_j^h - \dot{x}(t)\|_V \le \|w_j^h - \delta x_j\|_V + \|\delta x_j - \dot{x}(t)\|_V,$$

we see that

$$\sum_{j=1}^N \int_{I_j} \|w_j^h - \dot{x}(t)\|_V dt \le k \sum_{j=1}^N \|w_j^h - \delta x_j\|_V + \sum_{j=1}^N \int_{I_j} \|\delta x_j - \dot{x}(t)\|_V dt. \tag{4.24}$$

It suffices to estimate the term $k \sum_{j=1}^N \|w_j^h - \delta x_j\|_V$. Noticing that

$$w_j^h - \delta x_j = w_j^h - \delta \tilde{x}_j + \frac{1}{k} \int_{I_j} (\dot{\tilde{x}}(t) - \dot{x}(t)) dt,$$

we have

$$\begin{aligned}k \sum_{j=1}^N \|w_j^h - \delta x_j\|_V &\le k \sum_{j=1}^N \|w_j^h - \delta \tilde{x}_j\|_V + \|\dot{x} - \dot{\tilde{x}}\|_{L^1(0,T;V)} \\ &\le k \sum_{j=1}^N \|w_j^h - \delta \tilde{x}_j\|_V + c\,\varepsilon.\end{aligned} \tag{4.25}$$

Here $\tilde{x}$ is the function used in the proof of Theorem 4.2 such that (4.4) holds.

It remains to bound the last term in (4.21). From the definition (4.16), we have

$$\Pi x(t) = \int_{t_{n-1}}^{t} \frac{t_n - t}{k} \dot{x}(s) ds - \int_t^{t_n} \frac{t - t_{n-1}}{k} \dot{x}(s) ds, \quad t_{n-1} \le t \le t_n.$$

Therefore,

$$\sum_{j=1}^N \int_{I_j} \|\Pi x(t)\|_H dt \le \sum_{j=1}^N k \int_{I_j} \|\dot{x}(s)\|_V ds = k\, \|\dot{x}\|_{L^1(0,T;V)}. \tag{4.26}$$

The bounds (4.23)–(4.26) are now used in the estimate (4.21). Noticing the arbitrariness of $w_j^h \in V^h$, $1 \le j \le N$, and recalling the estimate (4.6), we get

$$\max_{1\le n\le N} (\|x_n - x_n^{hk}\|_V^2 + \|y_n - y_n^{hk}\|_V^2) \le c\,(e_0^2 + \varepsilon + \beta(h) + k + D_{hk}(\tilde{x})), \tag{4.27}$$

where

$$D_{hk}(\tilde{x}) = k \sum_{j=1}^{N} \inf_{w_j^h \in V^h} \|\delta \tilde{x}_j - w_j^h\|_V$$

and the constant c depends on $x, y, \tilde{y}, f, B(t, x(t), y(t))$ but is independent of ε, h, k. By Assumption H1, we see that

$$\inf_{w_j^h \in V^h} \|\delta \tilde{x}_j - w_j^h\|_V \le \alpha(h) \|\delta \tilde{x}_j\|_{V_0} \le \frac{\alpha(h)}{k} \int_{I_j} \|\dot{\tilde{x}}(t)\|_{V_0} dt,$$

and thus

$$D_{hk}(\tilde{x}) \le \alpha(h)\, \|\dot{\tilde{x}}\|_{L^1(0,T;V_0)}. \tag{4.28}$$

Then the convergence result (4.22) follows directly from (4.2), (4.27), and (4.28). ∎

We remark here that the estimate (3.38) is used for deriving optimal order error estimates provided the exact solution has higher degree regularity, while the estimate (4.21) is suitable for convergence analysis under the basic solution regularity condition.

5. Applications to contact problems in rate-type viscoplasticity. The aim of this section is to apply the results obtained in sections 3 and 4 to the numerical analysis of two nonlinear quasi-static frictional contact problems for viscoplastic materials. In this section, the symbol c may depend on the exact solution, but it is independent of the discretization parameters h and k.

Let us consider a viscoplastic body whose material particles occupy an open, bounded, connected domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3). The boundary of Ω, being assumed Lipschitz continuous, is partitioned into three disjoint measurable parts Γ_1, Γ_2, and Γ_3, with meas$\,(\Gamma_1) > 0$. Since the boundary is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu}$ exists a.e. on the boundary. Let $[0, T]$ be a time interval of interest. Displacement, surface traction, and contact conditions will be specified on Γ_1, Γ_2, and Γ_3, respectively. We assume that the body is fixed on Γ_1, a body force of density $\boldsymbol{b}$ acts in Ω, and a surface traction of density $\boldsymbol{F}$ acts on Γ_2. Both $\boldsymbol{b}$ and $\boldsymbol{F}$ can be time dependent but we assume a slow variation of these functions in time so that the inertia term in the equations of motion may be neglected. We choose (1.1) as the constitutive relation for the viscoplastic material, in which $\mathcal{E}$ is a fourth-order tensor and G is a given constitutive function, possibly nonlinear. The unknowns of the problem are the displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$ and the stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$, $\mathbb{S}^d$ being the set of second-order symmetric tensors on $\mathbb{R}^d$. We have the following equations and initial-boundary value conditions:

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + G(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in} \quad \Omega \times (0, T), \tag{5.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \quad \text{in} \quad \Omega \times (0, T), \tag{5.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1 \times (0, T), \tag{5.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{F} \quad \text{on} \quad \Gamma_2 \times (0, T), \tag{5.4}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{5.5}$$

Here $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ are the given initial data. These relations will be supplemented by a contact condition on Γ_3.

We define the inner product and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i,\quad |\boldsymbol{v}|=(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}\quad \forall\,\boldsymbol{u}=(u_i),\boldsymbol{v}=(v_i)\in\mathbb{R}^d,$$
$$\boldsymbol{\xi}\cdot\boldsymbol{\eta}=\xi_{ij}\eta_{ij},\quad |\boldsymbol{\xi}|=(\boldsymbol{\xi}\cdot\boldsymbol{\xi})^{1/2}\quad \forall\,\boldsymbol{\xi}=(\xi_{ij}),\boldsymbol{\eta}=(\eta_{ij})\in\mathbb{S}^d.$$

For every vector field $\boldsymbol{v}$, we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and the tangential components of $\boldsymbol{v}$ on the boundary given by

$$v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu},\quad \boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$$

and let $\boldsymbol{\varepsilon}(\boldsymbol{v})$ denote the tensor field defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{v})=(\varepsilon_{ij}(\boldsymbol{v})),\quad \varepsilon_{ij}(\boldsymbol{v})=\frac{1}{2}(v_{i,j}+v_{j,i}),$$

where the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

We denote in what follows by H the real Hilbert space defined by

$$H=\{\boldsymbol{\tau}=(\tau_{ij})\mid \tau_{ij}=\tau_{ji}\in L^2(\Omega),\quad 1\le i,j\le d\}$$

with the inner product

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_H=\int_\Omega \sigma_{ij}\tau_{ij}dx,\quad \boldsymbol{\sigma},\boldsymbol{\tau}\in H.$$

We assume in what follows that $\mathcal{E}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d$ and $G:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\to\mathbb{S}^d$ satisfy the following assumptions.

$$\left\{\begin{array}{l}\text{(a) } \mathcal{E}_{ijkl}\in L^\infty(\Omega),\ 1\le i,j,k,l\le d.\\ \text{(b) } \mathcal{E}\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\boldsymbol{\sigma}\cdot\mathcal{E}\boldsymbol{\tau}\ \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d \text{ a.e. in } \Omega.\\ \text{(c) There exists an } \alpha_0>0 \text{ such that}\\ \qquad \mathcal{E}\boldsymbol{\tau}\cdot\boldsymbol{\tau}\ge\alpha_0|\boldsymbol{\tau}|^2\ \ \forall\,\boldsymbol{\tau}\in\mathbb{S}^d \text{ a.e. in } \Omega.\end{array}\right.$$

$$\left\{\begin{array}{l}\text{(a) There exists an } \mathcal{L}>0 \text{ such that } \forall\,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d \text{ a.e. in } \Omega,\\ \qquad |G(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1)-G(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)|\le\mathcal{L}\,(|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2|+|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|).\\ \text{(b) For any } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathbb{S}^d,\ \boldsymbol{x}\mapsto G(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \text{ is measurable.}\\ \text{(c) The mapping } \boldsymbol{x}\mapsto G(\boldsymbol{x},\mathbf{0},\mathbf{0})\in H.\end{array}\right.$$

These assumptions will be used to verify (2.7) and (2.8) in the context of mechanical applications later.

For the input data, we assume that

$$\boldsymbol{b}\in W^{1,\infty}(0,T;(L^2(\Omega))^d),\quad \boldsymbol{F}\in W^{1,\infty}(0,T;(L^2(\Gamma_2))^d).$$

Finally, we suppose that the viscoplastic body is in contact with a rigid foundation on $\Gamma_3\times(0,T)$. This contact involves friction. In what follows we consider two different friction laws which lead us to the following examples.

5.1. Contact with simplified Coulomb's friction law. Consider a contact condition modeled by a simplified version of Coulomb's law of dry friction (see, e.g., [6, 16]), i.e.,

$$\text{(5.6)}\quad \begin{cases} \sigma_\nu = S, \quad |\boldsymbol{\sigma}_\tau| \le \mu\,|\sigma_\nu| \\ |\boldsymbol{\sigma}_\tau| < \mu\,|\sigma_\nu| \Rightarrow \dot{\boldsymbol{u}}_\tau = \boldsymbol{0} \\ |\boldsymbol{\sigma}_\tau| = \mu\,|\sigma_\nu| \Rightarrow \text{ there exists}\,\lambda \ge 0\,\text{such that}\,\boldsymbol{\sigma}_\tau = -\lambda\dot{\boldsymbol{u}}_\tau \end{cases} \quad \text{on } \Gamma_3 \times (0,T).$$

Here σ_ν denotes the normal stress on the contact boundary, $\boldsymbol{\sigma}_\tau$ is the tangential force on the contact boundary, $\dot{\boldsymbol{u}}_\tau$ denotes the tangential velocity, $S \in L^\infty(\Gamma_3)$ is a given function, and $\mu \ge 0$ is the coefficient of friction.

Let

$$U = \{\boldsymbol{v} \in (H^1(\Omega))^d \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\},$$
$$j : U \to \mathbb{R}_+,\ j(\boldsymbol{v}) = \mu \int_{\Gamma_3} |S|\,|\boldsymbol{v}_\tau|\,ds,$$
$$L : [0,T] \times U \to \mathbb{R},\ L(t,\boldsymbol{v}) = \int_\Omega \boldsymbol{b}(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{F}(t)\cdot\boldsymbol{v}\,ds + \int_{\Gamma_3} S\,v_\nu\,ds,$$

and let $\boldsymbol{u}_0 \in U$, $\boldsymbol{\sigma}_0 \in H$ denote initial data such that

$$(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}))_H + j(\boldsymbol{v}) \ge L(0,\boldsymbol{v}) \quad \forall\,\boldsymbol{v} \in U.$$

The weak formulation of the mechanical problem (5.1)–(5.5) and (5.6) is (see, e.g., [1]) the following.

PROBLEM P_1. Find the displacement field $\boldsymbol{u} : [0,T] \to U$ and the stress field $\boldsymbol{\sigma} : [0,T] \to H$ such that

$$\text{(5.7)}\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0,$$

and for a.e. $t \in (0,T)$,

$$\text{(5.8)}\qquad \dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + G(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))),$$
$$\text{(5.9)}\qquad (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}(t)))_H + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \ge L(t, \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) \quad \forall\,\boldsymbol{v} \in U.$$

Let V be the subspace of H given by

$$V = \boldsymbol{\varepsilon}(U) = \{\boldsymbol{\varepsilon}(\boldsymbol{v}) \mid \boldsymbol{v} \in U\}.$$

Since meas$\,(\Gamma_1) > 0$, Korn's inequality holds (see [15]):

$$\|\boldsymbol{v}\|_U \le c\,\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_H \quad \forall\,\boldsymbol{v} \in U.$$

It follows that V is a closed subspace of H and that the deformation operator $\boldsymbol{\varepsilon} : U \to V$ is a linear, continuous invertible operator. We denote the inverse of $\boldsymbol{\varepsilon} : U \to V$ by $\boldsymbol{\varepsilon}^{-1} : V \to U$, which is a linear, continuous operator. Now variational Problem P_1

can be viewed as a special case of abstract Problem P, after we make the following identifications:

$$\left\{ \begin{array}{l} x \leftrightarrow \boldsymbol{\varepsilon}(\boldsymbol{u}),\ y \leftrightarrow \boldsymbol{\sigma},\ x_0 \leftrightarrow \boldsymbol{\varepsilon}(\boldsymbol{u}_0),\ y_0 \leftrightarrow \boldsymbol{\sigma}_0,\ A \leftrightarrow \mathcal{E},\ B \leftrightarrow G, \\ \varphi(\boldsymbol{w}) \leftrightarrow j(\boldsymbol{\varepsilon}^{-1}(\boldsymbol{w})),\ (f(t), \boldsymbol{w})_V \leftrightarrow L(t, \boldsymbol{\varepsilon}^{-1}(\boldsymbol{w})) \quad \text{for } \boldsymbol{w} \in V. \end{array} \right. \tag{5.10}$$

The conditions (2.7)–(2.12) can then be verified by using the assumptions made on the constitutive functions $\mathcal{E}$ and G as well as on the data $\boldsymbol{b}$, $\boldsymbol{F}$, $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$, S, and μ. Therefore, from Theorem 2.1 it follows that Problem P_1 has a unique solution $(\boldsymbol{u}, \boldsymbol{\sigma}) \in W^{1,\infty}(0, T; U \times H)$.

We now briefly specify how to construct the finite dimensional spaces V^h and H^h via the finite element method. Details can be found in [3]. For simplicity, we assume that Ω is a polygon or polyhedron and $\overline{\Gamma}_1 \cap \overline{\Gamma}_3 = \emptyset$. We have $\overline{\Gamma}_3 = \cup_{i=1}^{I} \overline{\Gamma}_{3,i}$ with each piece $\overline{\Gamma}_{3,i}$ represented by an affine function. Let $\mathcal{T}^h$ be a regular finite element partition of Ω in such a way that if a side of an element lies on the boundary, the side belongs entirely to one of the subsets $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_{3,i}$, $1 \le i \le I$. Let $U^h \subset U$ consist of linear elements, let us use piecewise constants for H^h, and recall that $V^h = H^h \cap V$. It can be shown that $V^h = \boldsymbol{\varepsilon}(U^h)$.

With the above specifications, let us show that Assumptions H1 and H2 are satisfied. We need a density result proved in [5].

LEMMA 5.1. *Let $\Omega \subset \mathbb{R}^d$, $d \ge 1$, be an open, bounded, Lipschitz domain, and let $\Gamma_1 \subset \partial\Omega$ be a relatively open set with a Lipschitz relative boundary. Then the space $\{v \in C^\infty(\overline{\Omega}) : v = 0$ in a neighborhood of $\Gamma_1\}$ is dense in $\{v \in H^1(\Omega) : v = 0$ a.e. on $\Gamma_1\}$.*

From this result, we see immediately that the space

$$\{\boldsymbol{v} \in [C^\infty(\overline{\Omega})]^d : \boldsymbol{v} = \boldsymbol{0} \text{ in a neighborhood of } \Gamma_1\},$$

and therefore $U_0 = U \cap [H^2(\Omega)]^d$ also, is dense in U. Let $V_0 = \boldsymbol{\varepsilon}(U_0)$ and $H_0 = H \cap (H^1(\Omega))^{d\times d}$. Then the spaces V_0 and H_0 are dense in V and H.

For any $\boldsymbol{w} \in V_0$, there exists $\bar{\boldsymbol{w}} \in U_0$ such that $\boldsymbol{w} = \boldsymbol{\varepsilon}(\bar{\boldsymbol{w}})$. Let $\Pi^h \bar{\boldsymbol{w}} \in U^h$ be the piecewise linear interpolant of $\bar{\boldsymbol{w}}$. Then from the standard finite element interpolation theory (cf. [3]), we have

$$\|\bar{\boldsymbol{w}} - \Pi^h \bar{\boldsymbol{w}}\|_{(H^1(\Omega))^d} \le c\, h\, \|\bar{\boldsymbol{w}}\|_{(H^2(\Omega))^d}.$$

By Korn's inequality, we get

$$\begin{aligned} \inf_{\boldsymbol{w}^h \in V^h} \|\boldsymbol{w} - \boldsymbol{w}^h\|_V &\le c \inf_{\boldsymbol{z}^h \in U^h} \|\bar{\boldsymbol{w}} - \boldsymbol{z}^h\|_{(H^1(\Omega))^d} \\ &\le c\, \|\bar{\boldsymbol{w}} - \Pi^h \bar{\boldsymbol{w}}\|_{(H^1(\Omega))^d} \\ &\le c\, h\, \|\bar{\boldsymbol{w}}\|_{(H^2(\Omega))^d} \\ &\le c\, h\, \|\boldsymbol{w}\|_{(H^1(\Omega))^d}. \end{aligned}$$

So Assumption H1 is satisfied and we may take $\alpha(h) = c\, h$. Assumption H2 can be verified similarly. Therefore by Theorems 4.2 and 4.4, both the semidiscrete and fully discrete solutions corresponding to Problem P_1 converge to the solution of Problem P_1 as h, and h and k go to zero.

In order to derive error estimates via (3.17) and (3.38), we need to make assumptions on the regularity of the exact solution. Let us specialize the estimate (3.17) for the semidiscrete approximation of Problem P_1 and obtain

(5.11)
$$\begin{aligned}\|\boldsymbol{u}^h-\boldsymbol{u}\|_{L^\infty(0,T;U)}&+\|\boldsymbol{\sigma}^h-\boldsymbol{\sigma}\|_{L^\infty(0,T;H)}\\ &\le c\,\|(I_H-\mathcal{P}_{H^h})(\boldsymbol{\sigma}-\boldsymbol{\sigma}_0)\|_{L^\infty(0,T;H)}\\ &\quad+c\,(\|\boldsymbol{u}_0^h-\boldsymbol{u}_0\|_U+\|\boldsymbol{\sigma}_0^h-\boldsymbol{\sigma}_0\|_H)\\ &\quad+c\inf_{\boldsymbol{z}^h\in L^2(0,T;U^h)}\left\{\|\dot{\boldsymbol{u}}-\boldsymbol{z}^h\|_{L^2(0,T;U)}+\left(\int_0^T|R(t;\dot{\boldsymbol{u}}(t),\boldsymbol{z}^h(t))|\,dt\right)^{1/2}\right\},\end{aligned}$$

where

$$R(t;\dot{\boldsymbol{u}}(t),\boldsymbol{z}^h(t))=(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{z}^h(t)-\dot{\boldsymbol{u}}(t)))_H+j(\boldsymbol{z}^h(t))-j(\dot{\boldsymbol{u}}(t))-L(t,\boldsymbol{z}^h(t)-\dot{\boldsymbol{u}}(t)).$$

Under the assumption

$$\boldsymbol{\sigma\nu}\in C([0,T];(L^2(\Gamma))^d),$$

we can follow a standard argument (e.g., [14]) to get the relations

$$\begin{aligned}\mathrm{Div}\boldsymbol{\sigma}(t)+\boldsymbol{b}(t)=\boldsymbol{0}\quad&\text{a.e. in }\Omega,\\ \boldsymbol{\sigma}(t)\boldsymbol{\nu}=\boldsymbol{F}(t)\quad&\text{a.e. on }\Gamma_2,\\ \sigma_\nu(t)=S\quad&\text{a.e. on }\Gamma_3\end{aligned}$$

for all $t\in[0,T]$. We then have

$$\begin{aligned}R(t;\dot{\boldsymbol{u}}(t),\boldsymbol{z}^h(t))&=\int_{\Gamma_3}\boldsymbol{\sigma}_\tau(t)\cdot(\boldsymbol{z}^h(t)_\tau-\dot{\boldsymbol{u}}_\tau(t))\,ds\\ &\quad+\mu\int_{\Gamma_3}(|\dot{\boldsymbol{u}}_\tau(t)|-|\boldsymbol{z}_\tau^h(t)|)\,ds.\end{aligned}\tag{5.12}$$

The error analysis of the semidiscrete solution is given by the following result.

THEOREM 5.2. *Let* $(\boldsymbol{u},\boldsymbol{\sigma})\in W^{1,\infty}(0,T;U\times H)$ *be the solution of Problem* P_1 *and* $(\boldsymbol{u}^h,\boldsymbol{\sigma}^h)\in W^{1,\infty}(0,T;U^h\times H^h)$ *be the corresponding semidiscrete solution. For the initial values, we choose* $\boldsymbol{u}_0^h\in U^h$ *to be the orthogonal projection of* $\boldsymbol{u}_0$ *into* U^h *with respect to the inner product* $(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_V$*, and* $\boldsymbol{\sigma}_0^h\in H^h$ *the orthogonal projection of* $\boldsymbol{\sigma}_0$ *into* H^h *with respect to the inner product of* H*. Assume*

$$\boldsymbol{\sigma\nu}\in C([0,T];(L^2(\Gamma))^d),\quad\boldsymbol{\sigma}\in W^{1,\infty}(0,T;H_0),\quad\boldsymbol{u}\in H^1(0,T;U_0),$$

and

$$\boldsymbol{\sigma}_0\in H^2(\Omega)^{d\times d},\quad\boldsymbol{u}_0\in H^2(\Omega)^d;$$

then we have the error estimate

$$\|\boldsymbol{u}^h-\boldsymbol{u}\|_{L^\infty(0,T;U)}+\|\boldsymbol{\sigma}^h-\boldsymbol{\sigma}\|_{L^\infty(0,T;H)}\le c\,h^{3/4}.\tag{5.13}$$

If we further assume

$$\boldsymbol{u}_\tau\in H^1(0,T;H^2(\Gamma_{3,i})),\quad 1\le i\le I,$$

then we have the optimal order error estimate

$$\|\boldsymbol{u}^h - \boldsymbol{u}\|_{L^\infty(0,T;U)} + \|\boldsymbol{\sigma}^h - \boldsymbol{\sigma}\|_{L^\infty(0,T;H)} \le c\,h. \tag{5.14}$$

Proof. First from the choice of the initial values $\boldsymbol{u}_0^h$ and $\boldsymbol{\sigma}_0^h$, we have

$$\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_U \le c\,h, \quad \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_H \le c\,h. \tag{5.15}$$

The condition $\boldsymbol{u} \in H^1(0,T;U_0)$ implies $\boldsymbol{u}_\tau \in H^1(0,T;H^{3/2}(\Gamma_{3,i}))$, $1 \le i \le I$. Let $\Pi^h \dot{\boldsymbol{u}}(t) \in U^h$ be the piecewise linear interpolant of $\dot{\boldsymbol{u}}(t)$ for a.e. $t \in (0,T)$. Then we have the estimates (cf. [3])

$$\|\dot{\boldsymbol{u}}(t) - \Pi^h\dot{\boldsymbol{u}}(t)\|_U \le c\,h\,\|\dot{\boldsymbol{u}}(t)\|_{(H^2(\Omega))^d},$$
$$\|\dot{\boldsymbol{u}}_\tau(t) - (\Pi^h\dot{\boldsymbol{u}}(t))_\tau\|_{L^2(\Gamma_{3,i})} \le c\,h^{3/2}\|\dot{\boldsymbol{u}}_\tau(t)\|_{H^{3/2}(\Gamma_{3,i})}.$$

Keep in mind the definition (3.13) and the transformation (5.10). From (5.12), we obtain

$$\begin{aligned} |R(t;\dot{\boldsymbol{u}}(t),\Pi^h\dot{\boldsymbol{u}}(t))| &\le (\|\boldsymbol{\sigma}_\tau(t)\|_{L^2(\Gamma_3)} + c\,\mu)\,\|\dot{\boldsymbol{u}}_\tau(t) - (\Pi^h\dot{\boldsymbol{u}}(t))_\tau\|_{L^2(\Gamma_3)} \\ &\le c\,(\|\boldsymbol{\sigma}(t)\|_{H_0} + \mu)\,h^{3/2}\sum_{i=1}^{I}\|\dot{\boldsymbol{u}}_\tau(t)\|_{H^{3/2}(\Gamma_{3,i})}. \end{aligned} \tag{5.16}$$

From the property of the projection, we have

$$\|(I_H - \mathcal{P}_{H^h})(\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_0)\|_H \le c\,h\,\|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_0\|_{H_0}. \tag{5.17}$$

Then the error estimate (5.13) follows from (5.11), (5.15), (5.16), and (5.17).

Under the additional assumption $\boldsymbol{u}_\tau \in H^1(0,T;H^2(\Gamma_{3,i}))$, $1 \le i \le I$, we have

$$\|\dot{\boldsymbol{u}}_\tau(t) - (\Pi^h\dot{\boldsymbol{u}}(t))_\tau\|_{L^2(\Gamma_{3,i})} \le c\,h^2\|\dot{\boldsymbol{u}}_\tau(t)\|_{H^2(\Gamma_{3,i})},$$

and then the estimate (5.14) follows. ∎

Now we turn to error analysis of the fully discrete solution.

THEOREM 5.3. *Let $(\boldsymbol{u},\boldsymbol{\sigma}) \in W^{1,\infty}(0,T;U\times H)$ be the solution of Problem* P_1 *and $\{(\boldsymbol{u}_n^{hk},\boldsymbol{\sigma}_n^{hk})\}_{n=1}^N$ be the corresponding fully discrete solution. For the initial values, we choose $\boldsymbol{u}_0^h \in U^h$ to be the orthogonal projection of $\boldsymbol{u}_0$ into U^h with respect to the inner product $(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_V$, and $\boldsymbol{\sigma}_0^h \in H^h$ the orthogonal projection of $\boldsymbol{\sigma}_0$ into H^h with respect to the inner product of H. Assume*

$$\boldsymbol{\sigma\nu} \in C([0,T];(L^2(\Gamma))^d), \quad \boldsymbol{\sigma} \in W^{1,\infty}(0,T;H_0), \quad \boldsymbol{u} \in C^1([0,T];U_0),$$

and

$$\boldsymbol{\sigma}_0 \in H^2(\Omega)^{d\times d}, \quad \boldsymbol{u}_0 \in H^2(\Omega)^d;$$

then we have the error estimate

$$\max_{1\le n\le N}\left(\|\boldsymbol{u}_n^{hk} - \boldsymbol{u}_n\|_U + \|\boldsymbol{\sigma}_n^{hk} - \boldsymbol{\sigma}_n\|_H\right) \le c\,(h^{3/4} + k + I^k(\dot{\boldsymbol{u}})^{1/2}), \tag{5.18}$$

where

$$I^k(\dot{\boldsymbol{u}}) = \sum_{j=1}^{N}\int_{I_j}\|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}_j\|_U^2 dt. \tag{5.19}$$

If we further assume

$$\boldsymbol{u}_\tau \in H^1(0,T;H^2(\Gamma_{3,i})), \quad 1 \le i \le I,$$

then

$$\max_{1\le n\le N} \left(\|\boldsymbol{u}_n^{hk} - \boldsymbol{u}_n\|_U + \|\boldsymbol{\sigma}_n^{hk} - \boldsymbol{\sigma}_n\|_H\right) \le c\,(h + k + I^k(\dot{\boldsymbol{u}})^{1/2}). \tag{5.20}$$

Proof. The condition $\boldsymbol{u} \in C^1([0,T];U_0)$ implies $\boldsymbol{u}_\tau \in C^1([0,T];H^{3/2}(\Gamma_{3,i}))$, $1 \le i \le I$. We specialize the estimate (3.38) for the case of the full discretization of Problem P_1:

$$\begin{aligned}
\max_{1\le n\le N} &\left(\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_U + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_H\right) \\
&\le c\,k\,(\|\dot{\boldsymbol{u}}\|_{L^\infty(0,T;U)} + \|\dot{\boldsymbol{\sigma}}\|_{L^\infty(0,T;H)}) \\
&\quad + c\left[\sum_{j=1}^N k \inf_{\boldsymbol{z}^h\in U^h} \left(\|\delta\boldsymbol{u}_j - \boldsymbol{z}^h\|_U^2 + |R_j(\dot{\boldsymbol{u}}_j, \boldsymbol{z}^h)|\right)\right]^{1/2} \\
&\quad + c \max_{1\le n\le N} \|(I_H - \mathcal{P}_{H^h})(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_0)\|_H \\
&\quad + c\left(\|\boldsymbol{u}_0 - \boldsymbol{u}_0^{hk}\|_U + \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^{hk}\|_H\right),
\end{aligned} \tag{5.21}$$

where

$$R_j(\dot{\boldsymbol{u}}_j, \boldsymbol{z}^h) = (\boldsymbol{\sigma}_j, \boldsymbol{\varepsilon}(\boldsymbol{z}^h - \dot{\boldsymbol{u}}_j))_H + j(\boldsymbol{z}^h) - j(\dot{\boldsymbol{u}}_j) - L(t_j, \boldsymbol{z}^h - \dot{\boldsymbol{u}}_j).$$

Again, we have the estimate (5.15) for the approximation of the initial values. For each j, let $\Pi^h\dot{\boldsymbol{u}}_j$ be the piecewise linear interpolant of $\dot{\boldsymbol{u}}_j$. Then

$$\begin{aligned}
\|\delta\boldsymbol{u}_j - \Pi^h\dot{\boldsymbol{u}}_j\|_U &\le c\,(\|\delta\boldsymbol{u}_j - \dot{\boldsymbol{u}}_j\|_U + \|\dot{\boldsymbol{u}}_j - \Pi^h\dot{\boldsymbol{u}}_j\|_U) \\
&\le \frac{c}{k}\int_{I_j} \|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}_j\|_U\,dt + c\,h\,\|\dot{\boldsymbol{u}}_j\|_{U_0}.
\end{aligned}$$

So

$$\sum_{j=1}^N k\|\delta\boldsymbol{u}_j - \Pi^h\dot{\boldsymbol{u}}_j\|_U^2 \le c\,I^k(\dot{\boldsymbol{u}}) + c\,h^2\|\dot{\boldsymbol{u}}\|^2_{L^\infty(0,T;U_0)}.$$

Similar to (5.16) we have

$$|R_j(\dot{\boldsymbol{u}}_j, \Pi^h\dot{\boldsymbol{u}}_j)| \le c\,h^{3/2}(\|\boldsymbol{\sigma}\|_{L^\infty(0,T;H_0)} + \mu) \sum_{i=1}^I \|\dot{\boldsymbol{u}}_\tau\|_{L^\infty(0,T;H^{3/2}(\Gamma_{3,i}))}.$$

The error estimate (5.18) then follows from (5.21) together with the bounds for the various terms on the right-hand side of (5.21). The error estimate (5.20) is derived using an improved bound for $|R_j(\dot{\boldsymbol{u}}_j, \Pi^h\dot{\boldsymbol{u}}_j)|$ under the additional solution regularity assumption. ∎

Both estimates (5.18) and (5.20) involve the quantity $I^k(\dot{\boldsymbol{u}})$ defined in (5.19). Under the physically unrealistic assumption $\boldsymbol{u} \in H^2(0,T;U)$, we have

$$I^k(\dot{\boldsymbol{u}}) \le c\,k^2\|\ddot{\boldsymbol{u}}\|^2_{L^2(0,T;U)},$$

and then the estimates (5.18) and (5.20) give the bounds $O(h^{3/4}+k)$ and $O(h+k)$, respectively.

We emphasize that the error estimates (5.13), (5.14), (5.18), and (5.20) are only sample results under the stated regularity conditions. If the regularity conditions are different, the error estimates need to be changed accordingly, but that follows easily from (5.11) and (5.21).

5.2. Bilateral contact with Tresca's friction law. We assume a bilateral contact modeled by Tresca's friction law (see, e.g., [2, 6]), i.e.,

$$\begin{cases} u_\nu = 0, \quad |\boldsymbol{\sigma}_\tau| \le g \\ |\boldsymbol{\sigma}_\tau| < g \Rightarrow \dot{\boldsymbol{u}}_\tau = \mathbf{0} \\ |\boldsymbol{\sigma}_\tau| = g \Rightarrow \text{ there exists}\, \lambda \ge 0\, \text{such that}\, \boldsymbol{\sigma}_\tau = -\lambda \dot{\boldsymbol{u}}_\tau \end{cases} \quad \text{on } \Gamma_3 \times (0,T). \tag{5.22}$$

Here u_ν represents the normal displacement, $\dot{\boldsymbol{u}}_\tau$ denotes the tangential velocity, $\boldsymbol{\sigma}_\tau$ is the tangential force on the contact boundary, and $g \ge 0$ is the friction bound, i.e., the magnitude of the limiting friction traction at which slip begins. For simplicity, we assume here that g is a constant but, with minor changes, the nonhomogenous case may also be considered. In (5.22) the strict inequality holds in the stick zone and the equality in the slip zone. The contact is assumed to be bilateral, i.e., there is no loss of the contact during the process.

Let

$$\begin{aligned} &U = \{\boldsymbol{v} \in (H^1(\Omega))^d \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1, v_\nu = 0 \text{ on } \Gamma_3\}, \\ &j : U \to \mathbb{R}_+,\ j(\boldsymbol{v}) = g \int_{\Gamma_3} |\boldsymbol{v}_\tau|\, ds, \\ &L : [0,T] \times U \to \mathbb{R},\ L(t,\boldsymbol{v}) = \int_\Omega \boldsymbol{b}(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{F}(t) \cdot \boldsymbol{v}\, ds \end{aligned}$$

and let $\boldsymbol{u}_0 \in U$, $\boldsymbol{\sigma}_0 \in H$ be given initial data such that

$$(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}))_H + j(\boldsymbol{v}) \ge L(0,\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in U.$$

In [2] the following weak formulation of the mechanical problem (5.1)–(5.5) and (5.22) was derived.

Problem P_2. Find the displacement field $\boldsymbol{u} : [0,T] \to U$ and the stress field $\boldsymbol{\sigma} : [0,T] \to H$ such that

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \tag{5.23}$$

and for a.e. $t \in (0,T)$,

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + G(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \tag{5.24}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}(t)))_H + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \ge L(t, \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) \quad \forall\, \boldsymbol{v} \in U. \tag{5.25}$$

Let V be the subspace of H given by

$$V = \boldsymbol{\varepsilon}(U) = \{\boldsymbol{\varepsilon}(\boldsymbol{v}) \mid \boldsymbol{v} \in U\}.$$

V is a closed subspace of H and that the deformation operator $\boldsymbol{\varepsilon} : U \to V$ is a linear, continuous invertible operator. We denote the inverse of $\boldsymbol{\varepsilon} : U \to V$ by $\boldsymbol{\varepsilon}^{-1} : V \to U$, which is a linear, continuous operator. Variational Problem P_2 can be viewed as a special case of abstract Problem P, after we make the identifications (5.10).

The rest of the discussion on Problem P_2 is similar to that for Problem P_1 of the previous subsection. In particular, the conditions (2.7)–(2.12) can be verified by using the assumptions made on the constitutive functions $\mathcal{E}$ and G as well as on the data $\boldsymbol{b}$, $\boldsymbol{F}$, $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$, and g; so it follows from Theorem 2.1 that Problem P_2 has a unique solution having the regularity $(\boldsymbol{u}, \boldsymbol{\sigma}) \in W^{1,\infty}(0,T;U \times H)$.

We make the same assumptions on the domain Ω and its finite element partition. The finite element space $U^h \subset U$ now differs from that in the previous subsection by requiring a vanishing normal component on $\Gamma_{3,i}$ for $i = 1, \ldots, I$. We still use piecewise constants for H^h and we have $V^h = \varepsilon(U^h) = H^h \cap V$.

To show the convergence of both semidiscrete and fully discrete solutions for Problem P_2, we need to verify Assumptions H1 and H2. Again, the crucial ingredient is a density result. By adapting the proof of Lemma 3.2 on page 141 from [12], we know that for $d = 2$, the space $U \cap [C^\infty(\overline{\Omega})]^2$ is dense in U. Using this result, then it can be shown that for $d = 2$, Assumptions H1 and H2 are satisfied and we have the convergence of the discrete solutions under the proved solution regularity $(\boldsymbol{u}, \boldsymbol{\sigma}) \in W^{1,\infty}(0,T;U \times H)$.

Finally, the error estimates presented in Theorems 5.2 and 5.3 also hold for numerical approximations of Problem P_2. We skip the detailed arguments for these results because they are very similar to those used in proving Theorems 5.2 and 5.3.

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