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Error analysis of the reproducing kernel particle method ☆

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Abstract

Interest in meshfree (or meshless) methods has grown rapidly in recent years in solving boundary value problems (BVPs) arising in mechanics, especially in dealing with difficult problems involving large deformation, moving discontinuities, etc. In this paper, we provide a theoretical analysis of the reproducing kernel particle method (RKPM), which belongs to the family of meshfree methods. One goal of the paper is to set up a framework for error estimates of RKPM. We introduce the concept of a regular family of particle distributions and derive optimal order error estimates for RKP interpolants on a regular family of particle distributions. The interpolation error estimates can be used to yield error estimates for RKP solutions of BVPs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The finite element method has been the dominant numerical method in computational mechanics for several decades. Recently, a new family of numerical methods has attracted much interest in the community of computational mechanics. This new family of numerical methods shares a common feature that no mesh is needed and shape functions are constructed from sets of particles. These methods are designed to handle more effectively problems with large deformations, moving discontinuities and other difficult problems, and are hailed as numerical methods of the next generation (cf. Preface of [15]).

Currently there is no single universal name for the family of methods, with *meshless methods* or *meshfree methods* as possible choices. For example, *Meshless methods* is the title of a special issue of the journal *Computer Methods in Applied Mechanics and Engineering* [15] in 1996. Recently, however, the name *meshfree methods* becomes more popular. Various methods belong to this family, including smooth particle hydrodynamics (SPH) methods [19,21,22], diffuse element method (DEM) [23], element free Galerkin (EFG) method [2,3], reproducing kernel particle method (RKPM) [5,16,17], moving least-square reproducing kernel method [14,18], *h*-*p*-Clouds [8,9], partition of unity finite element method [1,20].

In this paper, we provide a theoretical analysis of the RKPM, which belongs to the family of meshfree methods. One goal of the paper is to set up a framework for error estimates of RKPM. We introduce the concept of a regular family of particle distributions and derive error estimates for RKP interpolants on a regular family of particle distributions. The interpolation error estimates are used to yield optimal order error estimates for RKP solutions of Neumann boundary value problems. Since the RKP shape functions

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do not have the Kronecker delta property, the treatment of Dirichlet boundary value conditions is more difficult than in the finite element method. We will show how to derive optimal order error estimates for RKP solutions of Dirichlet boundary value problems (BVPs) in one dimension. Several approaches are proposed in the literature to treat Dirichlet boundary value conditions (cf. [6,7,11,24]) numerically. We discuss error estimates in this paper under sufficient smoothness assumption on the functions being approximated. Error analysis of the method for singular problems will be given in a forthcoming paper [12].

The paper is organized as follows. In Section 2, we introduce some notations to be used later. For convenience of mathematics readers, we provide a precise introduction of RKPM in Section 3, emphasizing the ideas behind the development of the method. In Section 4, we derive optimal order error estimates for RKP interpolants. These results are comparable to those in the theory of the finite element method. In Section 5, we discuss error estimates for RKP solutions of boundary value problems. Numerical results presented in the last section demonstrate convergence orders of RKPM, confirming the theoretical error estimates.

2. Notations

Throughout the paper, we use the following notations. The letter d is a positive integer and is used for the spatial dimension. We denote $\Omega \subset \mathbb{R}^d$ to be a nonempty, open bounded set with a Lipschitz continuous boundary. In the one-dimensional case, $d = 1$, we choose $\Omega = (0, L)$ for some $L > 0$. A generic point in $\mathbb{R}^d$ is denoted by $\boldsymbol{x} = (x_1, \ldots, x_d)^{\mathrm{T}}$, or $\boldsymbol{y} = (y_1, \ldots, y_d)^{\mathrm{T}}$ or $\boldsymbol{z} = (z_1, \ldots, z_d)^{\mathrm{T}}$. We use Euclidean norm to measure the vector length:

$$\|\boldsymbol{x}\| = \left(\sum_{i=1}^{d} |x_i|^2 \right)^{1/2}.$$

It is convenient to use the multi-index notation for partial derivatives. A multi-index is an ordered collection of d nonnegative integers, $\alpha = (\alpha_1, \ldots, \alpha_d)$. The quantity $|\alpha| = \sum_{i=1}^{d} \alpha_i$ is said to be the length of α. For $\boldsymbol{z} = (z_1, \ldots, z_d)^{\mathrm{T}} \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$, we write $\alpha! = \alpha_1! \cdots \alpha_d!$ and $\boldsymbol{z}^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$. For any α with $|\alpha| \leqslant m$,

$$D^\alpha v(\boldsymbol{x}) = \frac{\partial^{|\alpha|} v(\boldsymbol{x})}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$$

is the αth order partial derivative. As usual, $D^0 v(\boldsymbol{x}) \equiv v(\boldsymbol{x})$.

For $\boldsymbol{x}_0 \in \mathbb{R}^d$ and $r > 0$, we use

$$B_r(\boldsymbol{x}_0) = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x} - \boldsymbol{x}_0\| \leqslant r\}$$

to denote the (closed) ball centered at $\boldsymbol{x}_0$ with radius r. In particular, when $\boldsymbol{x}_0 = \boldsymbol{0}$, the ball is denoted as B_r.

Let p be a nonnegative integer. We use the notation $\mathscr{P}_p = \mathscr{P}_p(\Omega)$ for the space of the polynomials of degree less than or equal to p on Ω. The dimension of the polynomial space is

$$N_p \equiv \dim \mathscr{P}_p = \binom{p+d}{d} = \frac{(p+d)!}{p!d!}.$$

Given $\xi_\alpha \in \mathbb{R}$ for all α with $|\alpha| \leqslant p$, we arrange them to a vector $\boldsymbol{\xi} \in \mathbb{R}^{N_p}$ in the lexical order: $\alpha = (0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1), (2, 0, \ldots, 0), (1, 1, 0, \ldots, 1), \ldots, (0, \ldots, 0, p)$.

3. Reproducing kernel particle approximation

We first introduce the concept of reproducing kernel approximation at the continuous level, which helps in understanding derivation of the RKP approximation.

3.1. Reproducing kernel approximation

Let $\delta(\cdot)$ denote the Dirac delta function. For $u \in C(\Omega)$, we write formally

$$u(\boldsymbol{x}) = \int_\Omega \delta(\boldsymbol{x}-\boldsymbol{y})\, u(\boldsymbol{y})\, \mathrm{d}y \quad \forall \boldsymbol{x} \in \Omega.$$

For practical use, the kernel function $\delta(\boldsymbol{x}-\boldsymbol{y})$ is approximated by finite-valued functions. A standard choice is $\varepsilon^{-d}\Phi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})$ with $\varepsilon > 0$ small and

$$\Phi_\varepsilon(\boldsymbol{z}) \equiv \Phi\left(\frac{\boldsymbol{z}}{\varepsilon}\right).$$

The function Φ, called a *generating function* or a *window function*, has the properties:

$$\begin{cases} \Phi \text{ is continuous,} \\ \operatorname{supp}\Phi = B_1, \\ \Phi(\boldsymbol{x}) > 0 \text{ for } \|\boldsymbol{x}\| < 1, \\ \int_{B_1} \Phi(\boldsymbol{x})\, \mathrm{d}x = 1. \end{cases}$$

Correspondingly, properties of the function Φ_ε are

$$\begin{cases} \Phi_\varepsilon \text{ is continuous,} \\ \operatorname{supp}\Phi_\varepsilon = B_\varepsilon, \\ \Phi_\varepsilon(\boldsymbol{x}) > 0 \text{ for } \|\boldsymbol{x}\| < \varepsilon, \\ \int_{B_\varepsilon} \Phi_\varepsilon(\boldsymbol{x})\, \mathrm{d}x = \varepsilon^d. \end{cases}$$

There are infinitely many possible choices for the generating function. We first list some generating functions in one dimension. A popular choice in engineering computations is the cubic spline $\frac{1}{4}\Phi$ with

$$\Phi(z) = \begin{cases} \frac{2}{3} - 4|z|^2 + 4|z|^3, & 0 \leqslant |z| \leqslant \frac{1}{2}, \\ \frac{4}{3} - 4|z| + 4|z|^2 - \frac{4}{3}|z|^3, & \frac{1}{2} \leqslant |z| \leqslant 1, \\ 0, & |z| > 1. \end{cases}$$

This function has the smoothness C^2. Another popular choice is

$$\Phi(z) = \begin{cases} c_0 \mathrm{e}^{1/(z^2-1)}, & |z| < 1, \\ 0, & |z| \geqslant 1, \end{cases}$$

where $c_0 > 0$ is chosen such that

$$\int_{-1}^{1} \Phi(z)\, \mathrm{d}z = 1.$$

This function is infinitely smooth. One family of generating functions is given by the formula

$$\Phi_l(z) = \begin{cases} c_l(1-z^2)^l, & |z| \leqslant 1, \\ 0, & |z| > 1, \end{cases}$$

where $c_l = (2l+1)!/(2^{2l+1}l!^2)$. We observe that $\Phi_l \in C^{l-1}$.

Any one-dimensional generating function $\Phi(z)$ can be used to create a d-dimensional generating function either in the form $\Phi(\|\boldsymbol{z}\|)$ or by a tensor product $\prod_{i=1}^{d} \Phi(z_i)$. For definiteness in this paper, we consider only the first form. All the theoretical analysis presented in this paper goes through when a d-dimensional generating function is constructed as a tensor product of lower-dimensional generating functions.

We can then use

$$u_\varepsilon(\boldsymbol{x}) = \int_\Omega \varepsilon^{-d}\Phi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\, u(\boldsymbol{y})\, \mathrm{d}y, \quad \boldsymbol{x} \in \Omega, \tag{3.1}$$

to approximate the function $u(\boldsymbol{x})$. Denote

$$\Omega_\varepsilon = \{\boldsymbol{x} \in \Omega : \mathrm{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon\}.$$

We have the following standard result concerning the convergence of u_ε to u as $\varepsilon \to 0$ (see, e.g. [10]).

Theorem 3.1. *Assume $\Phi \in C^m(\mathbb{R}^d)$. If u is locally integrable, then $u_\varepsilon \in C^m(\Omega_\varepsilon)$. If $u \in C^k(\Omega)$ for some $k \leqslant m$, then for any α with $|\alpha| \leqslant k$,*

$$D^\alpha u_\varepsilon \to D^\alpha u \quad \text{as } \varepsilon \to 0+$$

uniformly on compact subsets of Ω.

In the literature on function spaces, $\varepsilon^{-d}\Phi_\varepsilon(\cdot)$ is called a *mollifier*. In the development of RKPM, however, it is more convenient to single out the factor ε^{-d} in the mollifier as we will see later. The convolution of the mollifier with a function u, defined in (3.1), is used to generate a sequence of smooth functions approximating the function u.

The pointwise convergence stated in Theorem 3.1 is the basis for the success of SPH methods for solving differential equations over the whole spatial space.

Note that the convergence of (3.1) is restricted to the interior of the domain. Near the boundary, u_ε does not represent an approximation to u. To have convergence also near the boundary, Liu et al. [17] modified the approximation formula (3.1) by introducing a correction term to the kernel function:

$$u_\varepsilon(\boldsymbol{x}) = \int_\Omega \varepsilon^{-d}\Phi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\, C(\boldsymbol{x}; \boldsymbol{x}-\boldsymbol{y})\, u(\boldsymbol{y})\, \mathrm{d}y, \quad \boldsymbol{x} \in \Omega, \tag{3.2}$$

where the correction function is

$$C(\boldsymbol{x}; \boldsymbol{z}) = \sum_{|\alpha| \leqslant p} \boldsymbol{z}^\alpha b_\alpha(\boldsymbol{x}), \quad p \geqslant 0 \text{ integer}.$$

With the introduction of the correction function in the kernel of (3.2), it is convenient for us to abandon the condition

$$\int_{B_1} \Phi(\boldsymbol{x})\, \mathrm{d}x = 1,$$

since any scaling factor in the function $\Phi_\varepsilon(\cdot)$ can be absorbed into the correction function. The coefficient functions $\{b_\alpha(\boldsymbol{x})\}_{|\alpha| \leqslant p}$ are chosen in such a way that formula (3.2) reproduces polynomials of degree less than or equal to p,

$$u(\boldsymbol{x}) = \int_\Omega \varepsilon^{-d}\Phi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\, C(\boldsymbol{x}; \boldsymbol{x}-\boldsymbol{y})\, u(\boldsymbol{y})\, \mathrm{d}y \quad \forall u \in \mathscr{P}_p, \tag{3.3}$$

i.e.

$$u(\boldsymbol{x}) = \sum_{|\alpha| \leqslant p} b_\alpha(\boldsymbol{x}) \int_\Omega \varepsilon^{-d}\Phi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\, (\boldsymbol{x}-\boldsymbol{y})^\alpha u(\boldsymbol{y})\, \mathrm{d}y \quad \forall u \in \mathscr{P}_p.$$

This introduces a system of N_p equations for the N_p coefficient functions $\{b_\alpha(\boldsymbol{x})\}_{|\alpha| \leqslant p}$. Taking $u(\boldsymbol{y}) = 1$ in (3.3), we obtain

$$\sum_{|\alpha| \leqslant p} b_\alpha(\boldsymbol{x}) \int_\Omega \varepsilon^{-d}\Phi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\, (\boldsymbol{x}-\boldsymbol{y})^\alpha\, \mathrm{d}y = 1.$$

Taking $u(\boldsymbol{y}) = (\boldsymbol{x}-\boldsymbol{y})^\beta$ with $0 < |\beta| \leqslant p$, we have

$$\sum_{|\alpha|\leqslant p} b_\alpha(\boldsymbol{x}) \int_\Omega \varepsilon^{-d} \Phi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\,(\boldsymbol{x}-\boldsymbol{y})^{\alpha+\beta}\,\mathrm{d}y = 0.$$

Combining these relations, we get the polynomial reproducing conditions

$$\sum_{|\alpha|\leqslant p} \tilde{m}_{\alpha+\beta}(\boldsymbol{x})\, b_\alpha(\boldsymbol{x}) = \delta_{|\beta|,0}, \quad |\beta| \leqslant p. \tag{3.4}$$

Here

$$\tilde{m}_\alpha(\boldsymbol{x}) = \int_\Omega \varepsilon^{-d} \Phi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\,(\boldsymbol{x}-\boldsymbol{y})^\alpha\,\mathrm{d}y$$

are the *moment functions*.

Proposition 3.2. *The system* (3.4) *has a unique solution* $\{b_\alpha(\boldsymbol{x})\}_{|\alpha|\leqslant p}$.

Proof. Consider the quadratic form

$$\begin{aligned} Q(\boldsymbol{x};\boldsymbol{\xi}) &= \sum_{|\alpha|,|\beta|\leqslant p} \xi_\beta \tilde{m}_{\alpha+\beta}(\boldsymbol{x})\,\xi_\alpha \\ &= \int_\Omega \varepsilon^{-d}\Phi_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) \left[\sum_{|\alpha|\leqslant p} \xi_\alpha (\boldsymbol{x}-\boldsymbol{y})^\alpha\right]^2 \mathrm{d}y \end{aligned}$$

for $\boldsymbol{\xi} = (\xi_\alpha) \in \mathbb{R}^{N_p}$. We always have $Q(\boldsymbol{x};\boldsymbol{\xi}) \geqslant 0$. Suppose for some $\boldsymbol{x} \in \Omega$, $Q(\boldsymbol{x};\boldsymbol{\xi}) = 0$. Since $\Phi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})$ is positive for $|\boldsymbol{x}-\boldsymbol{y}| < \varepsilon$, there is a region D containing the origin with meas $(D) > 0$ such that

$$\sum_{|\alpha|\leqslant p} \xi_\alpha \boldsymbol{z}^\alpha = 0 \quad \forall \boldsymbol{z} \in D.$$

Then

$$\xi_\alpha = 0 \quad \forall \alpha : |\alpha| \leqslant p.$$

Therefore, the coefficient matrix of (3.4) is positive definite and the system (3.4) has a unique solution. □

Proposition 3.3. *For* $\boldsymbol{x} \in \Omega_\varepsilon$, $\tilde{m}_\alpha(\boldsymbol{x})$ *is constant. Hence the coefficient functions* $\{b_\alpha(\boldsymbol{x})\}_{|\alpha|\leqslant p}$ *are constant on* Ω_ε.

Proof. Recall that $\operatorname{supp}\Phi_\varepsilon = B_\varepsilon$ is the ball with radius ε centered at the origin. By a change of variables, we have

$$\tilde{m}_\alpha(\boldsymbol{x}) = (-1)^d \varepsilon^{-d} \int_{B_\varepsilon} \Phi_\varepsilon(\boldsymbol{z})\,\boldsymbol{z}^\alpha\,\mathrm{d}z, \quad \boldsymbol{x} \in \Omega_\varepsilon,$$

which is constant. □

One major role played by the correction function is to make the kernel approximation also valid near and on the boundary.

3.2. Reproducing kernel particle approximation

For practical computations, integrals are replaced by summations. For example, let $\{\boldsymbol{x}_i\}_{i=1}^I$ be a set of points, called *particles*, of the domain $\overline{\Omega}$. Then we can introduce a numerical integration formula

$$\int_\Omega f(\boldsymbol{y})\,\mathrm{d}y \approx \sum_{i=1}^I f(\boldsymbol{x}_i)\,w_i,$$

where $w_i > 0$ are suitable constants. With this numerical integration, we introduce a discrete kernel approximation

$$u_\varepsilon(\boldsymbol{x}) = \sum_{i=1}^{I} \varepsilon^{-d} \Phi_\varepsilon(\boldsymbol{x} - \boldsymbol{x}_i)\, C(\boldsymbol{x}; \boldsymbol{x} - \boldsymbol{x}_i)\, w_i u(\boldsymbol{x}_i). \tag{3.5}$$

The question is then how to choose the particles $\{\boldsymbol{x}_i\}_{i=1}^{I}$ and the weights $\{w_i\}_{i=1}^{I}$ suitably so that the polynomial reproducing property (3.3) is carried over to the discrete level, i.e. whether the relation

$$u(\boldsymbol{x}) = \sum_{i=1}^{I} \varepsilon^{-d} \Phi_\varepsilon(\boldsymbol{x} - \boldsymbol{x}_i)\, C(\boldsymbol{x}; \boldsymbol{x} - \boldsymbol{x}_i)\, w_i u(\boldsymbol{x}_i) \quad \forall u \in \mathscr{P}_p$$

holds?

A more plausible approach is achieved by specifying the polynomial reproducing property directly on the discrete kernel approximation, without recourse to the integral form. Let

$$\Psi_i(\boldsymbol{x}) = \varepsilon^{-d} \Phi_\varepsilon(\boldsymbol{x} - \boldsymbol{x}_i)\, C(\boldsymbol{x}; \boldsymbol{x} - \boldsymbol{x}_i)\, w_i.$$

Then formula (3.5) can be rewritten as

$$u_\varepsilon(\boldsymbol{x}) = \sum_{i=1}^{I} \Psi_i(\boldsymbol{x})\, u(\boldsymbol{x}_i). \tag{3.6}$$

By absorbing the weights w_i and the factor ε^{-d} into the coefficients $\{b_\alpha(\boldsymbol{x})\}_{|\alpha| \leqslant p}$ and recalling the form for the correction function $C(\boldsymbol{x}; \boldsymbol{z})$, we write

$$\Psi_i(\boldsymbol{x}) = \Phi_\varepsilon(\boldsymbol{x} - \boldsymbol{x}_i) \sum_{|\alpha| \leqslant p} (\boldsymbol{x} - \boldsymbol{x}_i)^\alpha b_\alpha(\boldsymbol{x}).$$

To have greater flexibility, for each particle $\boldsymbol{x}_i$, we can allow the support radius ε to be dependent on i. Let us replace ε by $r_i > 0$ and write

$$\Psi_i(\boldsymbol{x}) = \Phi_{r_i}(\boldsymbol{x} - \boldsymbol{x}_i) \sum_{|\alpha| \leqslant p} (\boldsymbol{x} - \boldsymbol{x}_i)^\alpha b_\alpha(\boldsymbol{x}), \quad 1 \leqslant i \leqslant I, \tag{3.7}$$

where

$$\Phi_{r_i}(\boldsymbol{x} - \boldsymbol{x}_i) = \Phi\left(\frac{\boldsymbol{x} - \boldsymbol{x}_i}{r_i}\right).$$

Since the domain Ω is assumed to be Lipschitz continuous, it is locally on one side of the boundary. In case the particle $\boldsymbol{x}_i$ lies on or close to the boundary so that $B_{r_i}(\boldsymbol{x}_i) \cap \partial\Omega \neq \emptyset$, we redefine the function value $\Phi_{r_i}(\boldsymbol{x} - \boldsymbol{x}_i)$ to be zero outside that side of Ω on which the particle $\boldsymbol{x}_i$ lies. This is implicitly assumed throughout the paper.

Imposing the polynomial reproducing conditions on formula (3.6),

$$u(\boldsymbol{x}) = \sum_{i=1}^{I} \Psi_i(\boldsymbol{x})\, u(\boldsymbol{x}_i) \quad \forall u \in \mathscr{P}_p, \tag{3.8}$$

we have, similar to (3.4), that

$$\sum_{|\alpha| \leqslant p} m_{\alpha+\beta}(\boldsymbol{x})\, b_\alpha(\boldsymbol{x}) = \delta_{|\beta|,0}, \quad |\beta| \leqslant p, \tag{3.9}$$

where

$$m_\alpha(\boldsymbol{x}) = \sum_{i=1}^{I} \Phi_{r_i}(\boldsymbol{x} - \boldsymbol{x}_i)\, (\boldsymbol{x} - \boldsymbol{x}_i)^\alpha, \quad |\alpha| \leqslant p, \tag{3.10}$$

are the *discrete moment functions*. Conditions (3.9) can be seen as a consistency condition for the shape functions $\{\Psi_i(\boldsymbol{x})\}$:

$$\sum_{i=1}^{I} \Psi_i(\boldsymbol{x})\,(\boldsymbol{x}-\boldsymbol{x}_i)^{\beta} = \delta_{|\beta|,0}, \quad |\beta| \leqslant p. \tag{3.11}$$

Denote the discrete moment matrix from (3.9) by $M(\boldsymbol{x})$. Then

$$M(\boldsymbol{x}) = \sum_{i=1}^{I} \Phi_{r_i}(\boldsymbol{x}-\boldsymbol{x}_i)\,\boldsymbol{h}(\boldsymbol{x}-\boldsymbol{x}_i)\,\boldsymbol{h}(\boldsymbol{x}-\boldsymbol{x}_i)^{\mathrm{T}}, \tag{3.12}$$

where

$$\boldsymbol{h}(\boldsymbol{z}) = (\boldsymbol{z}^{\alpha})_{|\alpha|\leqslant p} \in \mathbb{R}^{N_p}.$$

Obviously, the matrix $M(\boldsymbol{x})$ is symmetric and positive semi-definite.

Since $M(\boldsymbol{x})$ is a summation of rank-one matrices, for $M(\boldsymbol{x})$ to be nonsingular, a necessary condition is: for any $\boldsymbol{x}$, there are at least

$$N_p = \dim \mathscr{P}_p = \binom{p+d}{d}$$

nonzero terms in the summation (3.12).

Definition 3.4. A point $\boldsymbol{x} \in \overline{\Omega}$ is said to be covered by m shape functions if there are m indices $i_1, \ldots, i_m$ such that

$$|\boldsymbol{x}-\boldsymbol{x}_{i_j}| < r_{i_j}, \quad j = 1, \ldots, m.$$

Proposition 3.5. *For any $\boldsymbol{x} \in \Omega$, a necessary condition for $M(\boldsymbol{x})$ to be invertible is that $\boldsymbol{x}$ is covered by at least $N_p = \dim \mathscr{P}_p$ shape functions.*

Let us try to find a sufficient condition for nonsingularity of $M(\boldsymbol{x})$. For any vector $\boldsymbol{\xi} = (\xi_\alpha) \in \mathbb{R}^{N_p}$, consider the quadratic form

$$\boldsymbol{\xi}^{\mathrm{T}} M(\boldsymbol{x})\,\boldsymbol{\xi} = \sum_{i=1}^{I} \Phi_{r_i}(\boldsymbol{x}-\boldsymbol{x}_i)\,|\boldsymbol{h}(\boldsymbol{x}-\boldsymbol{x}_i)^{\mathrm{T}}\boldsymbol{\xi}|^2.$$

Assuming $\boldsymbol{\xi}^{\mathrm{T}} M(\boldsymbol{x})\,\boldsymbol{\xi} = 0$, then

$$\Phi_{r_i}(\boldsymbol{x}-\boldsymbol{x}_i) > 0 \;\Rightarrow\; \boldsymbol{h}(\boldsymbol{x}-\boldsymbol{x}_i)^{\mathrm{T}}\boldsymbol{\xi} = 0.$$

Suppose the necessary condition of Proposition 3.5 is satisfied and let $i_1, \ldots, i_{N_p}$ be indices with $\Phi_{r_i}(\boldsymbol{x}-\boldsymbol{x}_i) > 0$. Then

$$\begin{pmatrix} \boldsymbol{h}(\boldsymbol{x}-\boldsymbol{x}_{i_1})^{\mathrm{T}} \\ \vdots \\ \boldsymbol{h}(\boldsymbol{x}-\boldsymbol{x}_{i_{N_p}})^{\mathrm{T}} \end{pmatrix} \boldsymbol{\xi} = \boldsymbol{0}. \tag{3.13}$$

A sufficient condition for invertibility of $M(\boldsymbol{x})$ follows if the coefficient matrix in (3.13) is nonsingular.

In the one-dimensional case, $\Omega = (0, L)$, we introduce particles $x_1, \ldots, x_I \in \overline{\Omega}$. We have $N_p \equiv \dim \mathscr{P}_p = p+1$. The moment matrix is

$$M(x) = \sum_{i=1}^{I} \Phi\left(\frac{x-x_i}{r_i}\right) \boldsymbol{h}(x-x_i)\,\boldsymbol{h}(x-x_i)^{\mathrm{T}},$$

where

$$\boldsymbol{h}(z) = (1, z, \ldots, z^p)^{\mathrm{T}}.$$

Assume the necessary condition of Proposition 3.5 is satisfied and let $i_0, \ldots, i_p$ be indices with $\Phi((x - x_{i_j})/r_{i_j}) > 0$. Then the system (3.13) is

$$\begin{pmatrix} 1 & x - x_{i_0} & \cdots & (x - x_{i_0})^p \\ 1 & x - x_{i_1} & \cdots & (x - x_{i_1})^p \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x - x_{i_p} & \cdots & (x - x_{i_p})^p \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}.$$

The determinant of the coefficient matrix is equal to

$$\prod_{0 \leqslant l < k \leqslant p} [(x - x_{i_k}) - (x - x_{i_l})] = \prod_{0 \leqslant l < k \leqslant p} (x_{i_l} - x_{i_k}),$$

which is nonzero since the particles are distinct. So we have the following result.

Theorem 3.6. *In the one-dimensional case, the discrete moment matrix $M(x)$ is invertible if and only if x is covered by at least $p + 1$ shape functions.*

For RKPM to work well, we need conditions stronger than the nonsingularity of the discrete moment matrix. The notion of an (r, p)-regular family of particle distributions to be introduced and discussed in the following section is one such condition. We will verify the (r, p)-regularity for some important situations, leading immediately to some sufficient conditions for the nonsingularity of the discrete moment matrix.

3.3. Properties of the shape functions

We list below some properties of the shape functions $\{\Psi_i\}$. We assume $M(\boldsymbol{x})$ is nonsingular so that we can solve (3.9) uniquely and define the shape functions $\{\Psi_i\}_{i=1}^{I}$ from (3.7).

Property 3.7. *The shape functions have compact supports: $\operatorname{supp} \Psi_i = \operatorname{supp} \Phi_{r_i}$.*

This property follows immediately from formula (3.7).

Property 3.8. *The shape functions $\{\Psi_i\}_{i=1}^{I}$ form a partition of unity.*

Indeed, using (3.8) with $u(\boldsymbol{x}) \equiv 1$, we obtain

$$\sum_{i=1}^{I} \Psi_i(\boldsymbol{x}) = 1.$$

Property 3.9. *If $\Phi \in C^k$, then $\Psi_i \in C^k$, $i = 1, \ldots, I$.*

This follows from the observation

$$\Phi \in C^k \quad \Rightarrow \quad b_\alpha \in C^k \ \ \forall \alpha : |\alpha| \leqslant p,$$

using (3.9) and (3.10). So unlike the finite element method, in RKPM it is easy to construct shape functions of any degree of smoothness. Thus the solution of higher-order differential equations does not present any special difficulty in the construction of conforming RKP shape functions. This is a common feature shared by other meshfree methods.

It is shown in [18] that the consistency condition (3.11) of the shape functions leads to consistency relations of their derivatives.

Property 3.10. *Assume $\Phi \in C^k$. Then*

$$\sum_{i=1}^{I} D^\alpha \Psi_i(\boldsymbol{x})(\boldsymbol{x}-\boldsymbol{x}_i)^\beta = (-1)^{|\alpha|}\beta!\,\delta_{\alpha\beta} \quad \forall |\alpha| \leqslant k,\ |\beta| \leqslant p. \tag{3.14}$$

Here $\delta_{\alpha\beta}$ equals 1 if $\beta = \alpha$, and is zero otherwise.

The question whether the shape functions $\{\Psi_i\}_{i=1}^I$ are independent seems difficult to answer in general. Let us consider the one-dimensional case with $p = 0$. Then the system (3.9) reduces to a single equation

$$m_0(x)\,b_0(x) = 1$$

with

$$m_0(x) = \sum_{i=1}^{I} \Phi_{r_i}(x-x_i).$$

Hence

$$b_0(x) = \frac{1}{\sum_{i=1}^{I} \Phi_{r_i}(x-x_i)}$$

and the "basis" functions Ψ_i from (3.7) are

$$\Psi_i(x) = \Phi_{r_i}(x-x_i)\,b_0(x) = \frac{\Phi_{r_i}(x-x_i)}{\sum_{j=1}^{I} \Phi_{r_j}(x-x_j)} = \frac{\Phi((x-x_i)/r_i)}{\sum_{j=1}^{I} \Phi\big((x-x_j)/r_j\big)}. \tag{3.15}$$

To see if the set (3.15) is linearly independent, suppose there are constants $c_1, \ldots, c_I$ such that

$$\sum_{i=1}^{I} c_i \Phi\left(\frac{x-x_i}{r_i}\right) \equiv 0 \quad \text{in } \Omega.$$

In the case $p = 0$, each x must be covered by at least one shape function. Suppose for each i, $\operatorname{supp}\Psi_i$ contains a nontrivial portion that does not intersect the supports of the remaining shape functions, then $\{\Psi_i\}$ as given in (3.15) are independent.

Unlike basis functions in the finite element method, the shape functions $\{\Psi_i\}$ do not enjoy the Kronecker delta property, i.e. we do not have $\Psi_i(\boldsymbol{x}_j) = 0$ for $j \neq i$. This lack of the Kronecker delta property is the source of difficulty in the implementation of Dirichlet boundary conditions and in error analysis of RKPM for solving Dirichlet BVPs.

4. Interpolation error estimates

We will derive error estimates for the case of quasiuniform support sizes, i.e. there exist two constants $c_1, c_2 \in (0, \infty)$ such that

$$c_1 \leqslant \frac{r_i}{r_j} \leqslant c_2 \quad \forall i, j.$$

For such particle distributions, there exists a parameter $r > 0$ such that

$$\tilde{c}_1 \leqslant \frac{r_i}{r} \leqslant \tilde{c}_2 \quad \forall i.$$

The more general case of arbitrary support sizes will be studied in a forthcoming paper.

4.1. Regularity of particle distributions

Rewrite the system (3.9) in the form

$$M_0(\boldsymbol{x})\tilde{\boldsymbol{b}}(\boldsymbol{x}) = \boldsymbol{e}_1 \equiv \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{N_p},$$

where

$$\tilde{\boldsymbol{b}}(\boldsymbol{x}) = (r^{|\alpha|} b_\alpha(\boldsymbol{x}))_{|\alpha| \leqslant p}$$

and

$$M_0(\boldsymbol{x}) = \sum_{i=1}^{I} \Phi\left(\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r_i}\right) \boldsymbol{h}\left(\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r}\right) \boldsymbol{h}\left(\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r}\right)^{\mathrm{T}}$$

is the *scaled discrete moment matrix*.

Definition 4.1. A family of particle distributions $\{\{\boldsymbol{x}_i\}_{i=1}^{I}\}$ is said to be *(r,p)-regular* (or we simply say the particle distributions are (r,p)-regular) if there is a constant L_0 such that

$$\max_{\boldsymbol{x}\in\overline{\Omega}} \|M_0(\boldsymbol{x})^{-1}\|_2 \leqslant L_0$$

for all the particle distributions in the family.

Since on a finite dimensional space all norms are equivalent, the spectral norm $\|\cdot\|_2$ in the definition can be replaced by any other matrix norm. We observe that the essential point is to have $M_0(\boldsymbol{x})^{-1}$ uniformly bounded, or equivalently, the vectors $\{\boldsymbol{h}((\boldsymbol{x}-\boldsymbol{x}_i)/r)\}$, for which $\Phi((\boldsymbol{x}-\boldsymbol{x}_i)/r) \geqslant c_0 > 0$, are "uniformly" linearly independent.

It is easy to deduce the following result from Definition 4.1.

Proposition 4.2. *A family of particle distributions is (r,p)-regular if it is $(r,p+1)$-regular, but not conversely.*

Let us verify the (r,p)-regularity for two important situations.

First we consider the case of one-dimension. Then $\tilde{\boldsymbol{b}}(x) = (b_0(x), r b_1(x), \ldots, r^p b_p(x))^{\mathrm{T}}$ is the solution of the system

$$M_0(x)\,\tilde{\boldsymbol{b}}(x) = \boldsymbol{e}_1 \equiv \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{p+1},$$

where

$$M_0(x) = \sum_{i=1}^{I} \Phi\left(\frac{x-x_i}{r_i}\right) \boldsymbol{h}\left(\frac{x-x_i}{r}\right) \boldsymbol{h}\left(\frac{x-x_i}{r}\right)^{\mathrm{T}}$$

with

$$\boldsymbol{h}(z) = (1, z, \ldots, z^p)^{\mathrm{T}}.$$

Theorem 4.3. *Assume there exist two constants* $c_0 > 0$, $\sigma_0 > 0$ *such that for any* $x \in [0, L]$, *there are* $i_0 < i_1 < \cdots < i_p$ *with*

$$\min_{0 \leqslant j \leqslant p} \Phi\left(\frac{x - x_{i_j}}{r_{i_j}}\right) \geqslant c_0 > 0, \tag{4.1}$$

and

$$\min_{j \neq k} \left| \frac{x_{i_j} - x_{i_k}}{r} \right| \geqslant \sigma_0 > 0. \tag{4.2}$$

Then the family of particle distributions $\{\{x_i\}_{i=1}^I\}$ *is* (r, p)*-regular, i.e. there exists a constant* $L(c_0, \sigma_0)$ *such that*

$$\max_{0 \leqslant x \leqslant L} \|M_0(x)^{-1}\|_2 \leqslant L(c_0, \sigma_0). \tag{4.3}$$

Before proving the theorem, let us make some remarks. The first condition, (4.1), is a strengthened version of the necessary condition that any point must be covered by $p + 1$ shape functions (cf. Proposition 3.5 or Theorem 3.6). Condition (4.2) can be equivalently written as

$$\min_{0 \leqslant j \leqslant p-1} \frac{x_{i_{j+1}} - x_{i_j}}{r} \geqslant \sigma_0 > 0.$$

A geometrical interpretation of condition (4.2) is that in any local region, at least $p + 1$ particles do not coalesce as the refinement goes (i.e. as $r \to 0$).

As a further remark, assume equal support size $r_1 = \cdots = r_I \equiv r$ and consider the situation where Φ is increasing on $[-1, 0]$ and decreasing on $[0, 1]$, and is symmetric with respect to 0, as is the case in actual computations. If for any x, we can find $i_{-1} < i_0 < \cdots < i_{p+1}$ such that

$$|x - x_{i_j}| \leqslant r, \quad -1 \leqslant j \leqslant p + 1$$

with

$$\min_{-1 \leqslant j \leqslant p} \frac{x_{i_{j+1}} - x_{i_j}}{r} \geqslant \sigma_0 > 0,$$

then (4.1) is automatically satisfied with

$$c_0 \geqslant \Phi(1 - \sigma_0).$$

As one more remark, a set of particles is regularly distributed if it contains a regularly distributed subset of particles. Thus in a regular distribution, we allow aggregation of particles in any local area where the exact solution is expected to be rough and more particles are needed to approximate the solution more accurately (however, see Hypothesis (H) introduced in Subsection 4.3).

Proof of Theorem 4.3. Under the given assumption, the symmetric matrix $M_0(x)$ is positive definite and is thus invertible. For a symmetric, positive semidefinite matrix $A \in \mathbb{R}^{(p+1)\times(p+1)}$, we arrange its eigenvalues in increasing order:

$$(0 \leqslant)\, \lambda_0(A) \leqslant \cdots \leqslant \lambda_p(A).$$

Recall that

$$\|A\|_2 = \max_{0 \leqslant k \leqslant p} \lambda_k(A) = \lambda_p(A).$$

When A is symmetric and positive definite, the eigenvalues of A^{-1} are

$$(0 <)\, \lambda_p(A)^{-1} \leqslant \cdots \leqslant \lambda_0(A)^{-1},$$

and we have

$$\|A^{-1}\|_2 = \lambda_0(A)^{-1}.$$

Also recall that if A and B are both symmetric and $A - B$ is positive semidefinite, then

$$\lambda_k(A) \geqslant \lambda_k(B), \quad k = 0, 1, \ldots, p.$$

Using these results, we have

$$\|M_0(x)^{-1}\|_2 = \lambda_0(M_0(x))^{-1}.$$

We will use the notation

$$z_i = \frac{x - x_i}{r}$$

and denote the matrix

$$H(z_{i_0}, \ldots, z_{i_p}) = (\boldsymbol{h}(z_{i_0}), \ldots, \boldsymbol{h}(z_{i_p})) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_{i_0} & z_{i_1} & \cdots & z_{i_p} \\ \vdots & \vdots & \ddots & \vdots \\ z_{i_0}^p & z_{i_1}^p & \cdots & z_{i_p}^p \end{pmatrix}.$$

Now

$$M_0(x) - c_0 \sum_{j=0}^{p} \boldsymbol{h}(z_{i_j}) \boldsymbol{h}(z_{i_j})^{\mathrm{T}} = \sum_{j=0}^{p} (\Phi(z_{i_j}) - c_0) \boldsymbol{h}(z_{i_j}) \boldsymbol{h}(z_{i_j})^{\mathrm{T}} + \sum_{i \neq i_j, 0 \leqslant j \leqslant p} \Phi(z_i) \boldsymbol{h}(z_i) \boldsymbol{h}(z_i)^{\mathrm{T}}$$

is positive semidefinite. Then

$$\lambda_0(M_0(x)) \geqslant \lambda_0\left(c_0 \sum_{j=0}^{p} \boldsymbol{h}(z_{i_j}) \boldsymbol{h}(z_{i_j})^{\mathrm{T}} \right) = c_0 \lambda_0(H(z_{i_0}, \ldots, z_{i_p}) H(z_{i_0}, \ldots, z_{i_p})^{\mathrm{T}}).$$

So

$$\left\| M_0(x)^{-1} \right\|_2 \leqslant c_0^{-1} \left\| \left(H(z_{i_0}, \ldots, z_{i_p}) H(z_{i_0}, \ldots, z_{i_p})^{\mathrm{T}} \right)^{-1} \right\|_2 = c_0^{-1} \left\| H(z_{i_0}, \ldots, z_{i_p})^{-1} \right\|_2^2.$$

With condition (4.2), which is rewritten as

$$\min_{j \neq k} |z_{i_j} - z_{i_k}| \geqslant \sigma_0 > 0,$$

the formula

$$\det H(z_{i_0}, \ldots, z_{i_p}) = \prod_{0 \leqslant l < k \leqslant p} (z_{i_k} - z_{i_j})$$

and the Cramer formula for an inverse matrix, we conclude that the entries of the matrix $H(z_{i_0}, \ldots, z_{i_p})^{-1}$ are uniformly bounded in x. Therefore, $\|H(z_{i_0}, \ldots, z_{i_p})^{-1}\|_2$ is uniformly bounded and (4.3) holds. □

Next we consider the case $p = 1$ in d dimensions. This is the case in many engineering calculations. Let

$$\boldsymbol{z}_i = \frac{\boldsymbol{x} - \boldsymbol{x}_i}{r} = (z_{i,1}, \ldots, z_{i,d})^{\mathrm{T}},$$

and

$$\boldsymbol{h}(\boldsymbol{z}_i) = (1, z_{i,1}, \ldots, z_{i,d})^{\mathrm{T}}.$$

Let us evaluate the determinant of

$$H(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_{d+1}) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_{1,1} & z_{2,1} & \cdots & z_{d+1,1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{1,d} & z_{2,d} & \cdots & z_{d+1,d} \end{pmatrix}.$$

After some elementary manipulations, we have

$$\det H(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_{d+1}) = \frac{(-1)^d}{r^d} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_{1,1} & x_{2,1} & \cdots & x_{d+1,1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,d} & x_{2,d} & \cdots & x_{d+1,d} \end{vmatrix}.$$

So the determinant is nonzero if and only if the points $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{d+1}$ are the vertices of a nondegenerate d-simplex, or equivalently, the $(d+1)$ points $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{d+1}$ are not contained in a hyperplane in $\mathbb{R}^d$.

After translating (negatively) by $\boldsymbol{x}$ and scaling by r, we obtain a d-simplex with the vertices $\boldsymbol{z}_1,\ldots,\boldsymbol{z}_{d+1}$. The value $\det H(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_{d+1})$ is proportional to the volume of the scaled d-simplex. Thus the regularity requirement (cf. the proof of Theorem 4.3) is that

$$|\det H(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_{d+1})| \geqslant \overline{c}_0 > 0,$$

or

$$\left| \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_{1,1} & x_{2,1} & \cdots & x_{d+1,1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,d} & x_{2,d} & \cdots & x_{d+1,d} \end{vmatrix} \right| \geqslant \tilde{c}_0 r^d.$$

That is, the d-simplex with the vertices $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{d+1}$ has a volume larger than $\tilde{c}_0 r^d$ for some $\tilde{c}_0 > 0$. This condition is satisfied if the largest sphere inscribed in the d-simplex has a diameter bounded below by some constant times r.

Summarizing, we have shown the following result.

Theorem 4.4. *A family of particle distributions* $\{\{\boldsymbol{x}_i\}_{i=1}^I\}$ *in* $\mathbb{R}^d$ *is* $(r,1)$*-regular if there exist two constants* $c_0, \tilde{c}_0 > 0$ *such that for any* $\boldsymbol{x} \in \overline{\Omega}$, *there are* $d+1$ *particles* $\boldsymbol{x}_{i_0},\ldots,\boldsymbol{x}_{i_d}$ *satisfying*

$$\min_{0\leqslant j\leqslant d} \Phi\left(\frac{\boldsymbol{x}-\boldsymbol{x}_{i_j}}{r}\right) \geqslant c_0 > 0,$$

and the d-simplex with the vertices $\boldsymbol{x}_{i_0},\ldots,\boldsymbol{x}_{i_d}$ *has a volume larger than* $\tilde{c}_0 r^d$.

4.2. Bounds on the shape functions and their derivatives

Since

$$\tilde{\boldsymbol{b}}(\boldsymbol{x}) = M_0(\boldsymbol{x})^{-1}\boldsymbol{e}_1,$$

we have the following result.

Theorem 4.5. *Assume the distributions of the particles are* (r,p)*-regular. Then there is a constant* $c < \infty$ *such that*

$$\max_{\alpha:|\alpha|\leqslant p} r^{|\alpha|}\|b_\alpha\|_\infty \leqslant c.$$

Under the assumptions of Theorem 4.5, we see that $\|b_\alpha\|_\infty \leqslant cr^{-|\alpha|}$. So as $r \to 0$, only $b_{(0,\dots,0)}$ stays bounded, while for $\alpha \neq (0,\dots,0)$, $\|b_\alpha\|_\infty$ tends to infinity at the rate of $r^{-|\alpha|}$. The phenomenon is observed in actual computations. This gives us a warning of possible numerical instability of a code implementing RKPM with high degree p, and partially explains that most engineering computations with RKPM are done with a low degree p.

Recall that the shape functions are given by the formulas

$$\Psi_i(\boldsymbol{x}) = \Phi\left(\frac{\boldsymbol{x} - \boldsymbol{x}_i}{r_i}\right) \sum_{|\alpha| \leqslant p} \left(\frac{\boldsymbol{x} - \boldsymbol{x}_i}{r}\right)^\alpha r^{|\alpha|} b_\alpha(\boldsymbol{x}), \quad 1 \leqslant i \leqslant I.$$

We immediately have the following result.

Theorem 4.6. *Assume $\Phi \in C$ and the particle distributions are (r,p)-regular, then there is a constant $c < \infty$ such that*

$$\max_{1 \leqslant i \leqslant N} \|\Psi_i\|_\infty \leqslant c.$$

To bound the first derivatives of the shape functions, we need to assume the generating function Φ to be continuously differentiable in $\mathbb{R}^d$,

$$\|D\Phi\|_\infty \equiv \max_{\beta:|\beta|=1} \max_{\boldsymbol{x} \in B_1} \|D^\beta \Phi(\boldsymbol{x})\| < \infty.$$

For any multi-index β with $|\beta| = 1$, we differentiate the equation

$$M_0(\boldsymbol{x}) \tilde{\boldsymbol{b}}(\boldsymbol{x}) = \boldsymbol{e}_1$$

to obtain

$$M_0(\boldsymbol{x}) D^\beta \tilde{\boldsymbol{b}}(\boldsymbol{x}) = -D^\beta M_0(\boldsymbol{x}) \tilde{\boldsymbol{b}}(\boldsymbol{x}).$$

Easily,

$$\max_{\beta:|\beta|=1} \max_{\boldsymbol{x} \in \overline{\Omega}} \|D^\beta M_0(\boldsymbol{x})\|_2 \leqslant \frac{c}{r}.$$

Then under the assumption that the distribution of the particles is (r,p)-regular, we have

$$\max_{\beta:|\beta|=1} \max_{\alpha:|\alpha| \leqslant p} r^{|\alpha|} \|D^\beta b_\alpha\|_\infty \leqslant \frac{c}{r}.$$

In general, under the assumption that Φ is k-times continuously differentiable,

$$\max_{0 \leqslant l \leqslant k} \|D^l \Phi\|_\infty \equiv \max_{0 \leqslant l \leqslant k} \max_{\beta:|\beta|=l} \max_{\boldsymbol{x} \in B_1} \|D^\beta \Phi(\boldsymbol{x})\| < \infty,$$

it can be shown inductively that

$$\max_{\alpha:|\alpha| \leqslant p} \max_{\beta:|\beta| \leqslant k} r^{|\alpha|} \|D^\beta b_\alpha\|_\infty \leqslant \frac{c}{r^k}.$$

Then from the expressions

$$\Psi_i(\boldsymbol{x}) = \Phi\left(\frac{\boldsymbol{x} - \boldsymbol{x}_i}{r_i}\right) \sum_{|\alpha| \leqslant p} \left(\frac{\boldsymbol{x} - \boldsymbol{x}_i}{r}\right)^\alpha r^{|\alpha|} b_\alpha(\boldsymbol{x}),$$

we obtain the following result.

Theorem 4.7. *Assume the particle distributions are (r,p)-regular and the generating function Φ is k-times continuously differentiable. Then*

$$\max_{1\leqslant i\leqslant I}\max_{\beta:|\beta|=l}\|D^\beta\Psi_i\|_\infty\leqslant\frac{c}{r^l},\quad 0\leqslant l\leqslant k.$$

4.3. Error estimates for interpolants

Throughout this subsection, we assume $\Phi\in C^k$, $k\geqslant 1$.

We will need some results concerning polynomial approximations of Sobolev functions found in [4, Chapter 4]. For this purpose, we first introduce some concepts. Let B be a ball. Then a domain Ω_1 is said to be star-shaped with respect to B if for any $\boldsymbol{x}\in\Omega_1$, the closed convex hull of $\{\boldsymbol{x}\}\cup B$ is a subset of Ω_1. The *chunkiness parameter* of Ω_1 is defined to be $\mathrm{diam}(\Omega_1)/\rho_{\max}$, where

$$\rho_{\max}=\sup\{\rho:\Omega_1 \text{ is star-shaped with respect to a ball of radius } \rho\}.$$

Let $u\in W^{m+1,q}(\Omega)$, $m\geqslant 0$, $q\in[1,\infty]$. We assume $(m+1)q>d$ if $q>1$, or $m+1\geqslant d$ if $q=1$. Then by the Sobolev embedding theorem, $u\in C(\overline{\Omega})$, and it is meaningful to use pointwise values of $u(\boldsymbol{x})$. We define the RKP interpolant of $u(\boldsymbol{x})$ by the formula

$$u^I(\boldsymbol{x})=\sum_{i=1}^{I}u(\boldsymbol{x}_i)\,\Psi_i(\boldsymbol{x}),\quad \boldsymbol{x}\in\overline{\Omega}.$$

Note that in general, $u^I(\boldsymbol{x}_i)\neq u(\boldsymbol{x}_i)$, so u^I is an interpolant of u in a generalized sense.

We are interested in estimating the error $u-u^I$ in various Sobolev norms. Denote $p_1=\min\{m+1,p+1\}$. For error analysis, we assume that the family of particle distributions is (r,p)-regular and the following hypothesis is satisfied.

Hypothesis (H). There is a constant integer I_0 such that for any $\boldsymbol{x}\in\overline{\Omega}$, there are at most I_0 of $\boldsymbol{x}_i$ satisfying the relation $\|\boldsymbol{x}-\boldsymbol{x}_i\|<r_i$, i.e. each point in $\overline{\Omega}$ is covered by at most I_0 shape functions.

Hypothesis (H) is quite natural since otherwise as the number of shape functions covering a local area increases, the shape functions tend to be more and more linearly dependent in the local area.

To simplify the notation, we write $B_i\equiv B_{r_i}(\boldsymbol{x}_i)$, $1\leqslant i\leqslant I$. We first bound the error $u-u^I$ in Sobolev norms over $B_j\cap\overline{\Omega}$ for $j=1,\ldots,I$. Define

$$\Omega_j=\left\{\boldsymbol{x}:\|\boldsymbol{x}-\boldsymbol{x}_j\|<r_j+\max_{1\leqslant i\leqslant I}r_i\right\},$$

and let

$$S_j=\{i:\mathrm{dist}\ (\boldsymbol{x}_i,B_j)<r_i\}.$$

Then by Hypothesis (H), card (S_j), $1\leqslant j\leqslant I$, are uniformly bounded. If $\overline{\Omega}_j\subset\overline{\Omega}$, then $\overline{\Omega}_j\cap\overline{\Omega}=\overline{\Omega}_j$ is star-shaped with respect to $\tilde{B}_j\equiv B_j$, and the chunkiness parameter of $\overline{\Omega}_j\cap\overline{\Omega}$ is uniformly bounded. Now suppose $\overline{\Omega}_j\not\subset\overline{\Omega}$. Then since $\partial\Omega$ is Lipschitz continuous, if r is sufficiently small, we can choose a ball $\tilde{B}_j$ of radius $r_j/2$ with $\boldsymbol{x}_j$ on its boundary such that $\overline{\Omega}_j\cap\overline{\Omega}$ is star-shaped with respect to $\tilde{B}_j$. Again we see that the chunkiness parameter of $\overline{\Omega}_j\cap\overline{\Omega}$ is uniformly bounded.

Let $Q_j^{p_1}u$ be the Taylor polynomial of degree p_1-1 of u averaged over $\tilde{B}_j$ (cf. [4, Section 4.1]), and denote

$$R_j^{p_1}u(\boldsymbol{x})=u(\boldsymbol{x})-Q_j^{p_1}u(\boldsymbol{x}).$$

Then since the chunkiness parameters of $\overline{\Omega}_j \cap \overline{\Omega}$ are uniformly bounded when r is sufficiently small, from the results of [4, Section 4.3], we have the estimates:

$$\|R_j^{p_1} u\|_{W^{l,q}(\Omega_j\cap\Omega)} \leqslant c r^{p_1 - l} |u|_{W^{p_1,q}(\Omega_j\cap\Omega)}, \quad l = 0, \ldots, p_1, \tag{4.4}$$

$$\|R_j^{p_1} u\|_{L^\infty(\Omega_j\cap\Omega)} \leqslant c r^{p_1 - d/q} |u|_{W^{p_1,q}(\Omega_j\cap\Omega)}, \tag{4.5}$$

where the constant c depends only on p_1, d and q, and is independent of j.

Now for $\boldsymbol{x} \in B_j \cap \overline{\Omega}$, we write

$$u(\boldsymbol{x}) - u^I(\boldsymbol{x}) = Q_j^{p_1} u(\boldsymbol{x}) - \sum_{i=1}^{I} Q_j^{p_1} u(\boldsymbol{x}_i)\, \Psi_i(\boldsymbol{x}) + R_j^{p_1} u(\boldsymbol{x}) - \sum_{i\in S_j} R_j^{p_1} u(\boldsymbol{x}_i)\, \Psi_i(\boldsymbol{x}).$$

By the polynomial reproducing property (3.8),

$$\sum_{i=1}^{I} Q_j^{p_1} u(\boldsymbol{x}_i)\, \Psi_i(\boldsymbol{x}) = Q_j^{p_1} u(\boldsymbol{x}).$$

Thus

$$u(\boldsymbol{x}) - u^I(\boldsymbol{x}) = R_j^{p_1} u(\boldsymbol{x}) - \sum_{i\in S_j} R_j^{p_1} u(\boldsymbol{x}_i)\, \Psi_i(\boldsymbol{x}).$$

Note that $\boldsymbol{x}_i \in \overline{\Omega}_j \cap \overline{\Omega}$ for $i \in S_j$. So

$$\|u - u^I\|_{W^{l,q}(B_j\cap\Omega)} \leqslant \|R_j^{p_1} u\|_{W^{l,q}(B_j\cap\Omega)} + \|R_j^{p_1} u\|_{L^\infty(\Omega_j\cap\Omega)} \sum_{i\in S_j} \|\Psi_i\|_{W^{l,q}(B_j\cap\Omega)}.$$

Since card (S_j) is uniformly bounded, applying the estimates (4.4), (4.5) and Theorem 4.7, we have

$$\|u - u^I\|_{W^{l,q}(B_j\cap\Omega)} \leqslant c r^{p_1 - l} |u|_{W^{p_1,q}(\Omega_j\cap\Omega)}, \quad 0 \leqslant l \leqslant \min\{p_1, k\},\ 1 \leqslant j \leqslant I.$$

Therefore, recalling again Hypothesis (H),

$$\|u - u^I\|_{W^{l,q}(\Omega)} \leqslant c r^{p_1 - l} |u|_{W^{p_1,q}(\Omega)}, \quad 0 \leqslant l \leqslant \min\{p_1, k\}.$$

Summarizing, we have shown the following theorem.

Theorem 4.8. *Assume the particle distributions are* (r,p)*-regular,* $\Phi \in C^k$*, and Hypothesis* (H) *holds. Let* $m \geqslant 0$, $q \in [1,\infty]$ *with* $(m+1)q > d$ *if* $q > 1$*, or* $m + 1 \geqslant d$ *if* $q = 1$*. Then for any* $u \in W^{m+1,q}(\Omega)$*, we have the optimal order interpolation error estimates*

$$\|u - u^I\|_{W^{l,q}(\Omega)} \leqslant c r^{\min\{m+1,p+1\}-l} |u|_{W^{\min\{m+1,p+1\},q}(\Omega)} \quad \forall\, l \leqslant \min\{m+1, p+1, k\}. \tag{4.6}$$

Note that when $m \geqslant p$, the error estimate (4.6) reduces to

$$\|u - u^I\|_{W^{l,q}(\Omega)} \leqslant c r^{p+1-l} |u|_{W^{p+1,q}(\Omega)} \quad \forall\, l \leqslant \min\{p+1, k\}.$$

5. Reproducing kernel particle method and error analysis

The RKPM is a Galerkin method combined with the use of RKP spaces. To explain the method in a concrete problem setting, we take a linear elliptic boundary value problem as an example. It is equally fine to consider nonlinear elliptic BVPs if we wish. Since nonhomogeneous Dirichlet boundary conditions can be rendered homogeneous in a standard way (see [13, Chapter 6] or one of many texts on modern PDE), we will assume Dirichlet boundary conditions, if any, are homogeneous. The weak formulation is

$$u \in V: \quad a(u,v) = \ell(v) \quad \forall v \in V. \tag{5.1}$$

Here V is a Sobolev space. For Neumann BVPs, V is a complete Sobolev space without boundary condition constraints, e.g. $H^1(\Omega)$ for second-order problems, and $H^2(\Omega)$ for fourth-order problems, Ω being the spatial domain of the differential equation. Otherwise, V is a subspace of a complete Sobolev space (e.g. $H_0^1(\Omega)$). The bilinear form $a(\cdot,\cdot)$ is an elliptic bilinear form on V, and ℓ is a linear continuous form on V. By the Lax–Milgram lemma, the variational problem (5.1) has a unique solution $u \in V$.

On $\overline{\Omega}$, introduce a set of particles $\{\boldsymbol{x}_i\}_{i=1}^I$, some of the particles lie on the boundary. Also introduce $\{r_i\}_{i=1}^I$, $r_i > 0$, and construct functions $\{\Psi_i\}_{i=1}^I$ in the form of (3.7) where $\{b_\alpha(\boldsymbol{x})\}_{|\alpha|\leqslant p}$ are computed from (3.9). The RKP space is

$$V_R = \operatorname{span}\{\Psi_i,\ 1 \leqslant i \leqslant I\} \cap V.$$

Then the RKPM is

$$u^R \in V_R: \quad a(u^R,v) = \ell(v) \quad \forall v \in V_R. \tag{5.2}$$

This problem admits a unique solution $u^R \in V_R$, again following the Lax–Milgram lemma. For error estimates of the RKP solution $u^R \in V_R$ defined in (5.2), we have Céa's inequality

$$\|u - u^R\|_V \leqslant c \inf_{v \in V_R} \|u - v\|_V. \tag{5.3}$$

In the rest of the section, we assume the (r,p)-regularity and Hypothesis (H). Then we can use the error estimates for RKP interpolants derived in the previous section.

5.1. Error estimates for BVP without Dirichlet condition

For a BVP without Dirichlet boundary condition,

$$V_R = \operatorname{span}\{\Psi_i,\ 1 \leqslant i \leqslant I\}.$$

Assume the solution u is continuous. Then its RKP interpolant

$$u^I(\boldsymbol{x}) = \sum_{i=1}^{I} u(\boldsymbol{x}_i)\, \Psi_i(\boldsymbol{x})$$

is well defined and $u^I \in V_R$. Then from (5.3), we have

$$\|u - u^R\|_V \leqslant c\,\|u - u^I\|_V \tag{5.4}$$

and the question of error estimation for the RKP solution u^R is reduced to that for the RKP interpolant u^I. As a sample result, we can state the following.

Theorem 5.1. *Let us employ the RKPM to solve a* $(2n)$*th-order elliptic BVP of the type* (5.1) *without Dirichlet boundary condition. Assume* $\Phi \in C^n$, $p \geqslant n$, $p > d/2 - 1$, *and the* (r,p)*-regularity and Hypothesis* (H) *are valid. Then if* $u \in H^{p+1}(\Omega)$, *we have the error estimate*

$$\|u - u^R\|_{H^n(\Omega)} \leqslant c r^{p+1-n} |u|_{H^{p+1}(\Omega)}. \tag{5.5}$$

5.2. Error estimates for BVP with Dirichlet condition

When the BVP includes a Dirichlet condition, derivation of rigorous error estimates is much more difficult. Since in general $u^I \notin V_R$, and we need to replace (5.4) by

$$\|u - u^R\|_V \leqslant c\,\|u - \tilde{u}^I\|_V, \tag{5.6}$$

where $\tilde{u}^I \in V_R$ is a modification of u^I. This approach cannot be carried out in case $d \geqslant 2$, since a function from V_R does not vanish on a part of the boundary even when it is zero at all the particles on that part of the boundary. Much more needs to be done on employing and analyzing RKPM for solving BVPs involving Dirichlet conditions.

In the one-dimensional case, however, it is possible to derive rigorous error estimates. In the following, we consider a general linear elliptic BVP on $[0, L]$ with Dirichlet boundary conditions

$$u(0) = u_0, \quad u(L) = u_L. \tag{5.7}$$

Let the weak form of the problem be: find $u \in H^1(0, L)$ satisfying (5.7) such that

$$a(u, v) = \ell(v) \quad \forall\, v \in H_0^1(0, L). \tag{5.8}$$

Let the RKP space be

$$V_R = \operatorname{span}\{\Psi_i,\ 1 \leqslant i \leqslant I\}.$$

Then the RKPM for the problem is: find $u^R \in V_R$ satisfying such that $u^R(0) = u_0$, $u^R(L) = u_L$, and

$$a(u^R, v) = \ell(v) \quad \forall\, v \in V_R \cap H_0^1(0, L). \tag{5.9}$$

For an error estimate, Céa's inequality (5.3) is modified to

$$\|u - u^R\|_V \leqslant c \inf\{\|u - v\|_V : v \in V_R,\ v(0) = u_0,\ v(L) = u_L\}. \tag{5.10}$$

For the RKP interpolant, we can use the ordinary Taylor formula and Property 3.10 to obtain

$$u^I(x) = u(x) + R_p(x),$$

where

$$R_p(x) = \frac{1}{p!} \sum_{i=1}^{I} \Psi_i(x) \int_x^{x_i} (x_i - t)^p u^{(p+1)}(t)\, \mathrm{d}t.$$

In particular,

$$u^I(0) - u(0) = R_p(0) = \frac{1}{p!} \sum_{i=1}^{I} \Psi_i(0) \int_0^{x_i} (x_i - t)^p u^{(p+1)}(t)\, \mathrm{d}t.$$

Then

$$|u^I(0) - u(0)| \leqslant c \sum_{i:|x_i| \leqslant r_i} r_i^p \int_0^{r_i} |u^{(p+1)}(t)|\, \mathrm{d}t.$$

Since there are at most I_0 points x_i with $|x_i| \leqslant r_i$, we have

$$|u^I(0) - u(0)| \leqslant c r^{p+1/2} \|u^{(p+1)}\|_{L^2(0,L)}. \tag{5.11}$$

Similarly,

$$|u^I(L) - u(L)| \leqslant c r^{p+1/2} \|u^{(p+1)}\|_{L^2(0,L)}. \tag{5.12}$$

Define a corrected RKP interpolant,

$$\tilde{u}^I(x) = u^I(x) + \frac{L-x}{L}\,(u(0) - u^I(0)) + \frac{x}{L}\,(u(L) - u^I(L)).$$

We have $\tilde{u}^I(0) = u(0)$, $\tilde{u}^I(L) = u(L)$. Since linear functions can be reproduced, $\tilde{u}^I \in V_R$. By (5.11) and (5.12), we have

$$\|\tilde{u}^I - u^I\|_{W^{l,q}(0,L)} \leqslant c r^{p+1/2} \|u^{(p+1)}\|_{L^2(0,L)}, \quad l \geqslant 0. \tag{5.13}$$

The definition of the corrected RKP interpolant and the related error estimate can be easily modified to adapt to the case with a Dirichlet condition at only one end of the interval $[0, L]$. Then from (5.10), we have

$$\|u - u^R\|_{H^1(0,L)} \leqslant c\,\|u - \tilde{u}^I\|_{H^1(0,L)} \leqslant c\left[\|u - u^I\|_{H^1(0,L)} + \|\tilde{u}^I - u^I\|_{H^1(0,L)}\right]. \tag{5.14}$$

Using (5.14), (5.13) and the estimate for $u - u^I$ from the previous section we get the following result.

Theorem 5.2. *Let us employ the RKPM to solve the BVP* (5.8). *Assume* $\Phi \in C^1$, *and the* (r, p)*-regularity and Hypothesis* (H) *are valid. Then if* $u \in H^{p+1}(\Omega)$, *we have the error estimate*

$$\|u - u^R\|_{H^1(\Omega)} \leqslant c r^p |u|_{H^{p+1}(\Omega)}. \tag{5.15}$$

6. Numerical results

In this section, we present some numerical results on convergence orders of RKPM. The numerical results confirm the theoretical prediction. We thank J.S. Chen and C.T. Wu of the University of Iowa for providing us a preliminary code for solving a one-dimensional model problem by RKPM.

We choose the differential equation

$$-u'' + k\,u = 0 \quad \text{in } (0, 1) \tag{6.1}$$

for our calculations. The general solution of Eq. (6.1) can be expressed in terms of exponential functions and is therefore smooth. The differential equation (6.1) is supplemented by one of the following three sets of boundary conditions:

$$u(0) = u_0, \quad u(1) = u_1, \tag{6.2}$$

$$-u'(0) = q_0, \quad u'(1) = q_1, \tag{6.3}$$

$$u(0) = u_0, \quad u'(1) = q_1. \tag{6.4}$$

The corresponding BVPs are called the Dirichlet, Neumann and mixed BVPs. We choose $k = 1$, $u_0 = q_0 = 0$, and $u_1 = q_1 = 1$ below.

For our examples, we divide the interval $[0, 1]$ into $N = 20$, 30, 40, 50 and 60 equal parts, and let $h = 1/N$. We use $r = (p + 2.1)h$ as the support size. This choice of the support size guarantees the satisfaction of both (r, p)-regularity and Hypothesis (H). Since r is proportional to h, we show figures for errors compared against h (rather than r itself) in the log–log scale. The generating function is chosen to be

$$\Phi(x) = \begin{cases} (1 - x^2)^9 & \text{if } |x| \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

In Figs. 1 and 2, we report numerical results on the RKPM for the Dirichlet BVP (6.1) and (6.2). We report errors for both the RKP solution and the RKP interpolant, and in both the maximum norm and the L^2 norm for the error as well as its derivative. The numerical results suggest the following empirical error estimates (the parameters h and r are interchangeable in our examples):

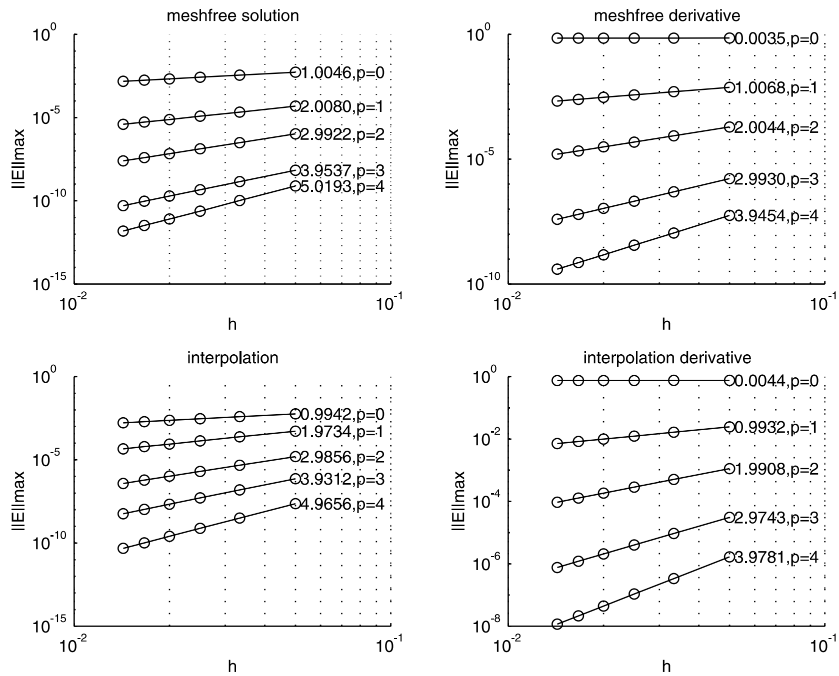

Fig. 1. Errors in maximum norms: the Dirichlet BVP.

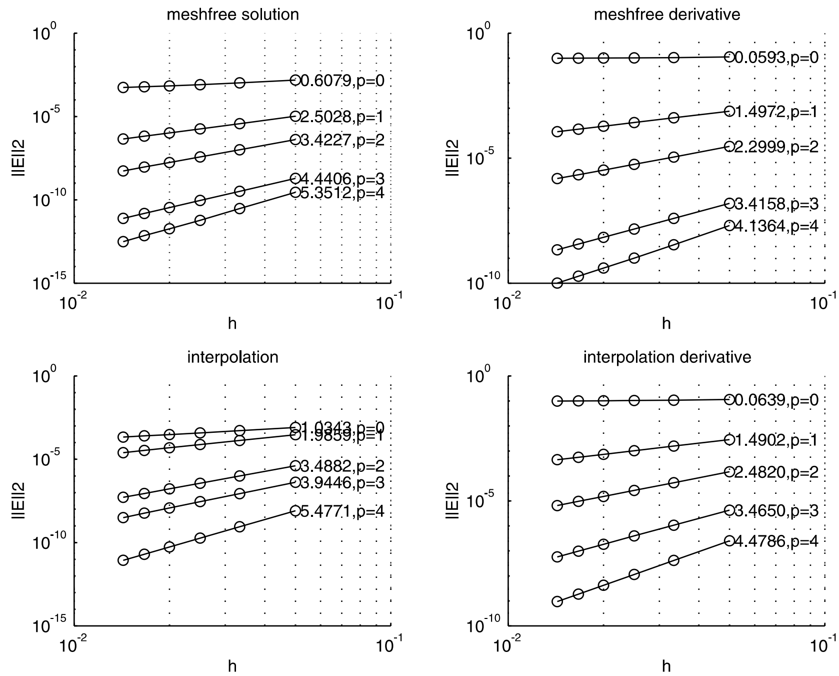

Fig. 2. Errors in integral norms: the Dirichlet BVP.

$$\|u - u^R\|_{L^\infty} \leqslant cr^{p+1},$$

$$\|u - u^I\|_{L^\infty} \leqslant cr^{p+1},$$

$$\|u - u^R\|_{L^2} \leqslant cr^{p+1},$$

$$\|u - u^I\|_{L^2} \leqslant cr^{p+1},$$

$$\|(u - u^R)'\|_{L^\infty} \leqslant cr^{p},$$

$$\|(u - u^I)'\|_{L^\infty} \leqslant cr^{p},$$

$$\|u - u^R\|_{H^1} \leqslant cr^{p},$$

$$\|u - u^I\|_{H^1} \leqslant cr^{p}.$$

The numerical convergence orders match our theoretical results on $\|u - u^R\|_{H^1}$, $\|u - u^I\|_{L^\infty}$, $\|u - u^I\|_{L^2}$ and $\|u - u^I\|_{H^1}$. Nitsche's trick in the theory of the finite element method can be adapted straightforwardly to error estimates in L^2-norm (cf. [18]), yielding $\|u - u^R\|_{L^2} \leqslant cr^{p+1}$.

We observe similar error behaviors on the RKPM for the Neumann BVP (6.1)–(6.3) in Figs. 3 and 4, and the mixed BVP (6.1)–(6.4) in Figs. 5 and 6.

We have also done numerical experiments for higher-dimensional boundary value problems, and the numerical results all confirm the theoretical error estimates. In Figs. 7 and 8, we show the error behavior of the meshfree interpolants and meshfree solutions for the pure Neumann BVP for the differential equation

$$-\Delta u + u = f \quad \text{in } (0,1) \times (0,1).$$

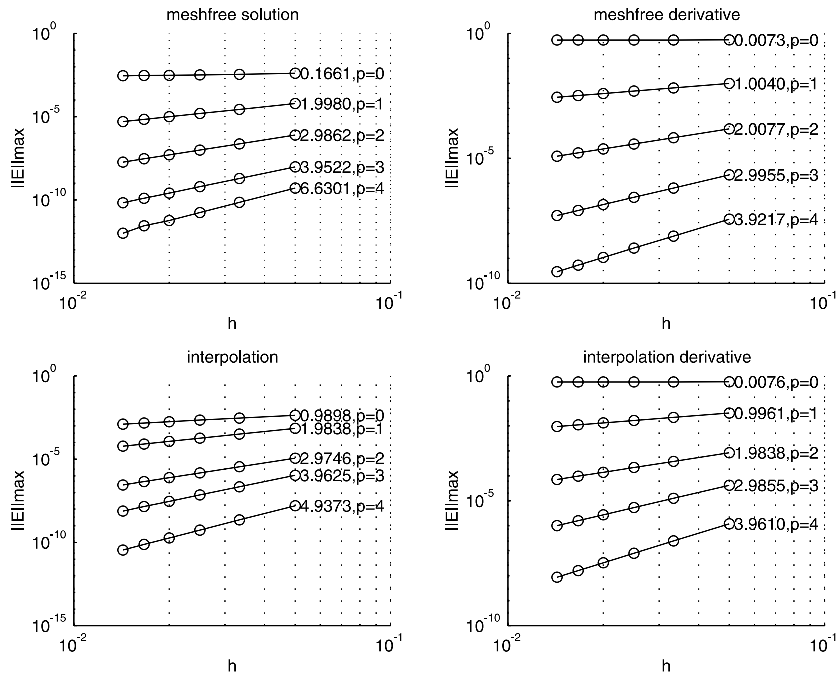

Fig. 3. Errors in maximum norms: the Neumann BVP.

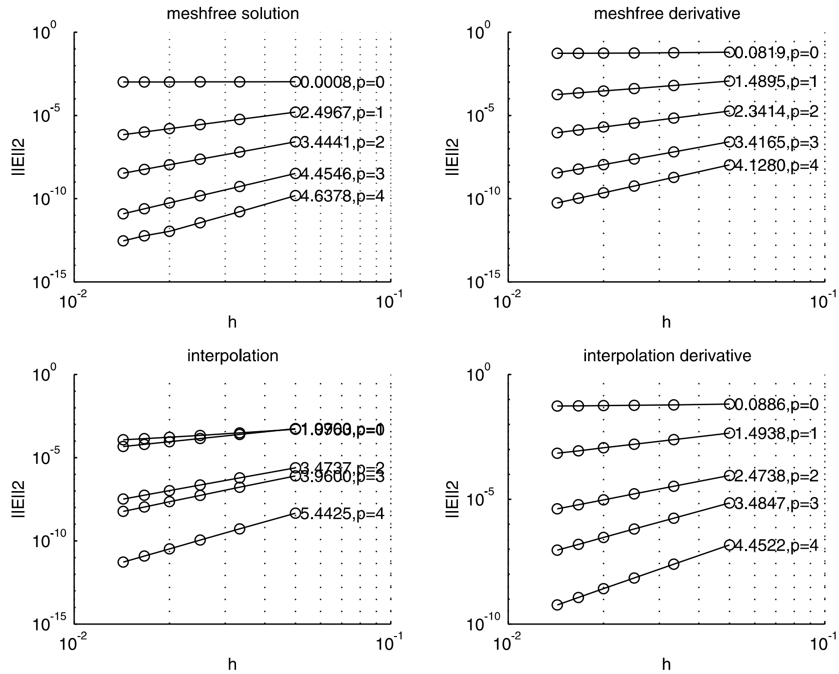

Fig. 4. Errors in integral norms: the Neumann BVP.

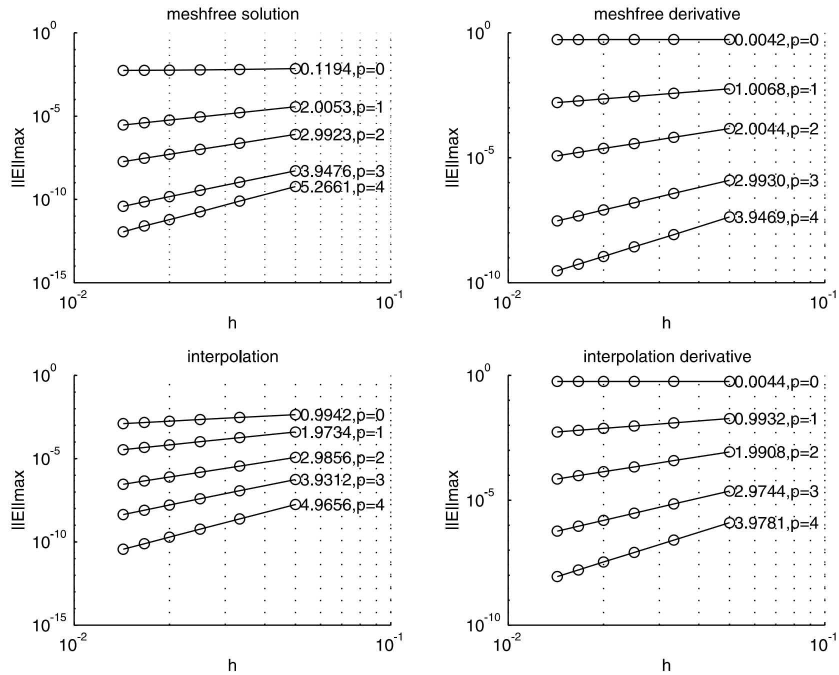

Fig. 5. Errors in maximum norms: the mixed BVP.

We choose the exact solution to be e^{xy}. For the numerical results reported, we use uniform particle distribution by dividing the interval $[0,1]$ into N equal parts. We define $h = 1/N$ and use $r = (p+0.6)\,h$ as the support size. The generating function used is as follows.

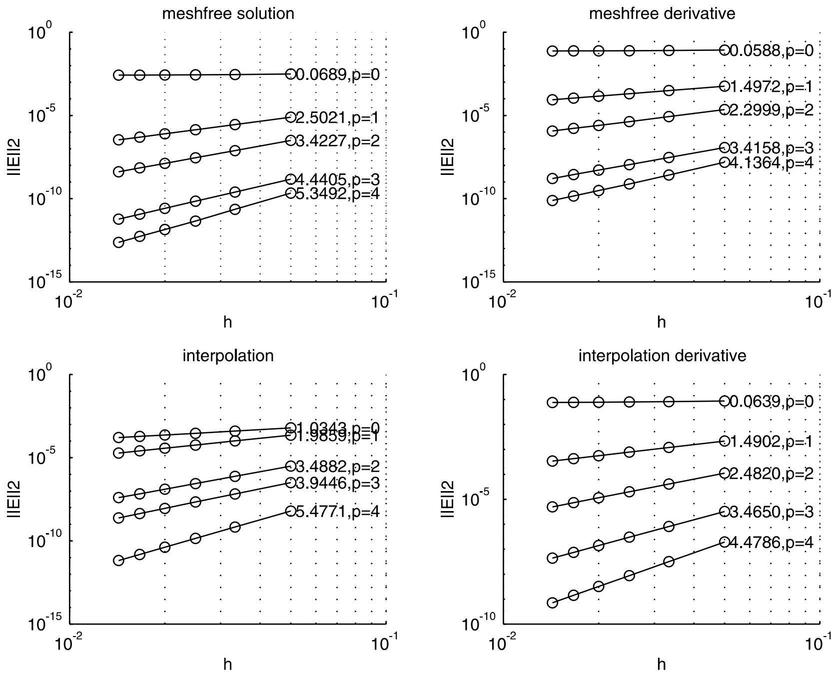

Fig. 6. Errors in integral norms: the mixed BVP.

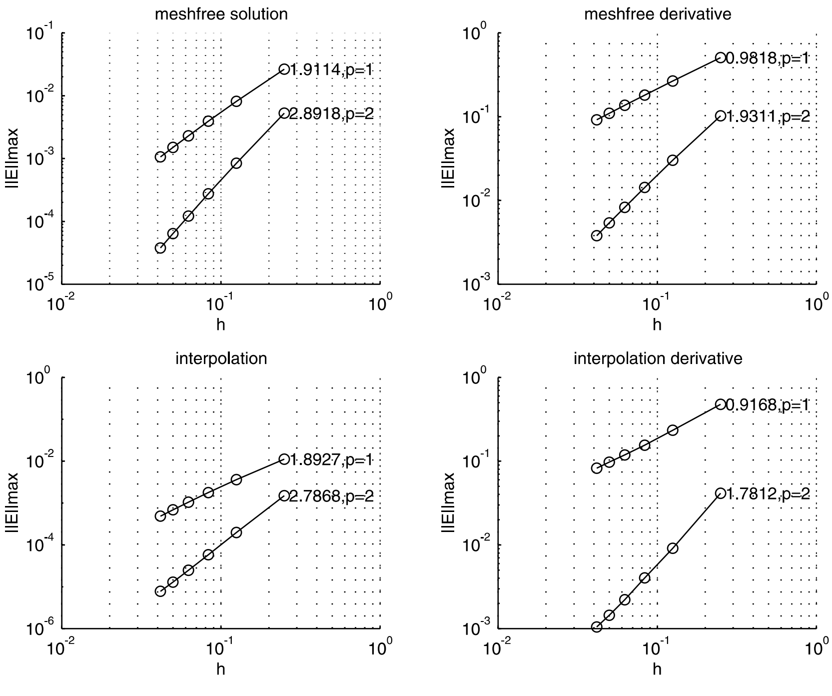

Fig. 7. Errors in maximum norms for a two-dimensional Neumann BVP.

$$\Phi(x,y) = \begin{cases} (1-x^2-y^2)^5 & \text{if } x^2+y^2 \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

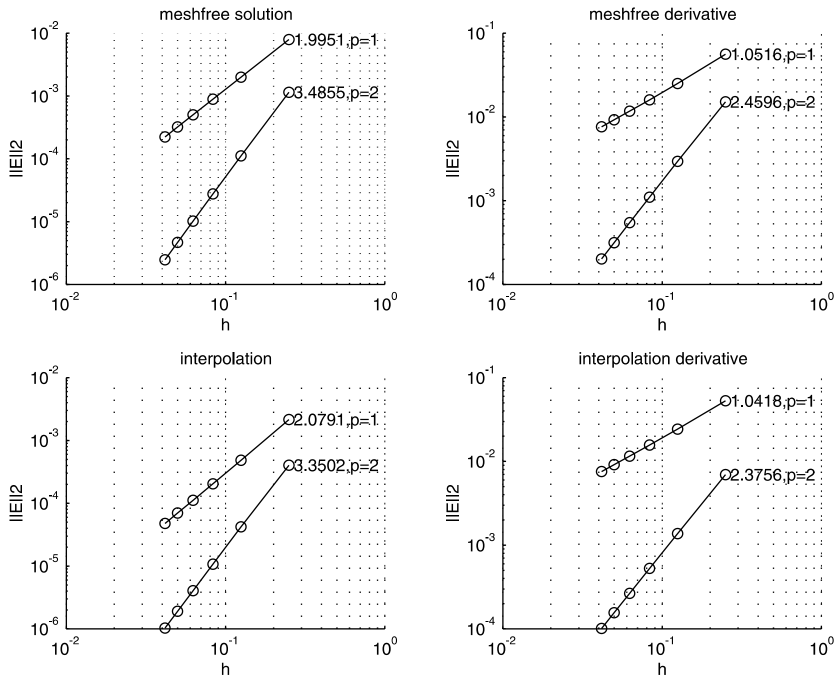

Fig. 8. Errors in integral norms for a two-dimensional Neumann BVP.

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