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Note

Well-posedness of the Fokker–Planck equation in a scattering process[☆]

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ABSTRACT

The Fokker–Planck equation is usually used as an approximation of the linear transport equation for a highly forward peaked scattering process. In this note, we provide a rigorous proof for the solution existence and uniqueness of a boundary value problem to the Fokker–Planck equation. In addition, we present a result on the positivity of the solution.

1. Introduction

The linear transport equation arises in a variety of applications, such as neutron transport, heat transfer, stellar atmospheres, optical molecular imaging, infrared and visible light in space and the atmosphere and so on. We refer the reader to [2,5,6,10,12]. The equation takes into account absorption and scattering due to inhomogeneities and typically models particle densities or energy densities. For the steady-state monoenergetic case, it is an integro-differential equation of the form

$$\boldsymbol{\omega}\cdot\nabla u(\boldsymbol{x},\boldsymbol{\omega})+\mu_t(\boldsymbol{x})u(\boldsymbol{x},\boldsymbol{\omega})=\mu_s(\boldsymbol{x})\int_\Omega \eta(\boldsymbol{\omega}\cdot\boldsymbol{\omega}')u(\boldsymbol{x},\boldsymbol{\omega}')\,\mathrm{d}\boldsymbol{\omega}'+f(\boldsymbol{x},\boldsymbol{\omega}),\quad (\boldsymbol{x},\boldsymbol{\omega})\in X\times\Omega,$$
$$u(\boldsymbol{x},\boldsymbol{\omega})=0,\quad (\boldsymbol{x},\boldsymbol{\omega})\in\Gamma_-.$$

Here $X\subset\mathbb{R}^3$ is a spatial domain and Ω is the unit sphere in $\mathbb{R}^3$. We assume the boundary ∂X of the domain X has the C^1 smoothness and denote by $\boldsymbol{\nu}(\boldsymbol{x})$ the unit outward normal vector at a point $\boldsymbol{x}\in\partial X$. The symbol ∇ stands for the gradient operator with respect to the spatial variable $\boldsymbol{x}$. The optical parameters μ_t, μ_s and the phase function η model the interaction of the propagating particles with underlying media. The quantity $\mu_t=\mu_a+\mu_s$ is the total attenuation coefficient with the absorption coefficient μ_a and the scattering coefficient μ_s; for an example of typical values in optical tomography, $\mu_a=0.1\ \mathrm{cm}^{-1}$, $\mu_s=10\ \mathrm{cm}^{-1}$ [11]. The phase function η is non-negative and is normalized

$$\int_\Omega \eta(\boldsymbol{\omega}\cdot\boldsymbol{\omega}')\,\mathrm{d}\boldsymbol{\omega}'=1\quad\forall\boldsymbol{\omega}\in\Omega.$$

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It indicates the amount of particles scattering from a direction $\boldsymbol{\omega}$ into a direction $\boldsymbol{\omega}'$ after collision. Γ_- is the incoming boundary, defined by

$$\Gamma_- = \{(\boldsymbol{x}, \boldsymbol{\omega}) \in \partial X \times \Omega \mid \boldsymbol{\omega} \cdot \boldsymbol{\nu}(x) < 0\},$$

where $\boldsymbol{\nu}(x)$ is the unit outward normal vector to X at $\boldsymbol{x} \in \partial X$. $f(\boldsymbol{x}, \boldsymbol{\omega})$ is the source function.

The numerical solution of the transport equation is challenging because of its high dimension and of the integro-differential form. In many applications, e.g., light propagation within biological tissues, there is a sharp peak in the forward scattering direction. Forward-peaked scattering corresponds to a sharp peak in the scattering phase function $\eta(\boldsymbol{\omega} \cdot \boldsymbol{\omega}')$ near $\boldsymbol{\omega} \cdot \boldsymbol{\omega}' = 1$. One well-known example is the Henyey–Greenstein phase function

$$\eta_{\mathrm{HG}}(\boldsymbol{\omega} \cdot \boldsymbol{\omega}') = \frac{1 - g^2}{4\pi (1 + g^2 - 2 g\, \boldsymbol{\omega} \cdot \boldsymbol{\omega}')^{3/2}}, \tag{1}$$

where the parameter $g \in (-1, 1)$ is the anisotropy factor of the scattering medium which measures the strength of forward peakedness of the phase function. Typical values in biological tissues for g are around 0.9, which correspond to quite highly forward-peaked scattering. This peak makes solving the transport equation even more difficult since the mesh size in such a calculation must be of the same magnitude as the mean free path, which, in this case, is very small. For this reason, there have been substantial efforts made to develop simpler approximations. The idea is to approximate the integral Boltzmann scattering operator

$$Ku(\boldsymbol{x}, \boldsymbol{\omega}) := \int_\Omega \eta(\boldsymbol{\omega} \cdot \boldsymbol{\omega}') u(\boldsymbol{x}, \boldsymbol{\omega}')\, \mathrm{d}\boldsymbol{\omega}'.$$

One well-established example among these is the so-called Fokker–Planck approximation in which the scattering operator is approximated by a second-order differential operator, resulting in the Fokker–Planck equation:

$$-\frac{1}{2}\mu_{tr}(\boldsymbol{x})\Delta^* u(\boldsymbol{x}, \boldsymbol{\omega}) + \boldsymbol{\omega} \cdot \nabla u(\boldsymbol{x}, \boldsymbol{\omega}) + \mu_a(\boldsymbol{x}) u(\boldsymbol{x}, \boldsymbol{\omega}) = f(\boldsymbol{x}, \boldsymbol{\omega}), \tag{2}$$

where Δ^* is the Laplace–Beltrami operator on Ω, $\mu_{tr}(\boldsymbol{x}) = (1 - g)\, \mu_s(\boldsymbol{x})$ with

$$g = 2\pi \int_{-1}^{1} t\, \eta(t)\, dt$$

a measure of the degree of anisotropy. Note that the Henyey–Greenstein phase function η_{HG} is completely determined by the anisotropy parameter g, as is seen from the formula (1). Pomraning [8,9] shows that the Fokker–Planck approximation is an asymptotic limit of the linear transport equation under certain conditions.

In the literature, one can find some papers [3,4,7,13] that discuss properties of the following equation

$$-\Delta u + \nabla \cdot (u \boldsymbol{F}) = f \quad \text{in } X, \tag{3}$$

where X is either a domain or a smooth Riemannian manifold in the Euclidean space, $\boldsymbol{F}$ and f are given vector field and source function. This equation is also called a Fokker–Planck equation. However, the Eq. (2) discussed in this paper is different from (3) in that there are two groups of independent variables: $\boldsymbol{x}$ in a domain and $\boldsymbol{\omega}$ from the unit sphere. In (2), the differential operator Δ^* is with respect to the angular variable $\boldsymbol{\omega}$ whereas ∇ is with respect to the spatial variable $\boldsymbol{x}$. For the Eq. (3), the differential operators Δ and ∇ are both with respect to the same independent variables.

The rest of the paper is as follows. In Section 2, we show rigorously the existence of a unique solution to a boundary value problem of the Fokker–Planck equation of the form (2). In Section 3, we present maximum principles and a positivity property of the solution.

2. Existence and uniqueness

Despite the fact that quite a few papers discuss the Fokker–Planck equation, a rigorous study of its solution existence and uniqueness appears to be missing. The purpose of this section is to fill this gap.

We denote by Γ the boundary of $X \times \Omega$: $\Gamma := \partial(X \times \Omega) = \partial X \times \Omega$. In addition to the incoming boundary Γ_-, we further introduce the boundary subsets

$$\Gamma_+ = \{(\boldsymbol{x}, \boldsymbol{\omega}) \in \Gamma \mid \boldsymbol{\nu}(\boldsymbol{x}) \cdot \boldsymbol{\omega} > 0\}, \qquad \Gamma_0 = \{(\boldsymbol{x}, \boldsymbol{\omega}) \in \Gamma \mid \boldsymbol{\nu}(\boldsymbol{x}) \cdot \boldsymbol{\omega} = 0\};$$

Γ_+ being known as the outgoing boundary. There holds the decomposition $\Gamma = \Gamma_0 \cup \Gamma_+ \cup \Gamma_-$.

We consider the BVP

$$Au = f \quad \text{in } X \times \Omega, \tag{4a}$$

$$u = 0 \quad \text{on } \Gamma_-. \tag{4b}$$

Here A is a partial differential operator of the form given by the left side of (2):

$$Au(\boldsymbol{x}, \omega) = -a_1(\boldsymbol{x})\Delta^* u(\boldsymbol{x}, \omega) + \boldsymbol{\omega} \cdot \nabla u(\boldsymbol{x}, \omega) + a_2(\boldsymbol{x}) u(\boldsymbol{x}, \omega), \tag{5}$$

where Δ^* is the Laplace–Beltrami operator on the unit sphere Ω. We assume

$$a_1, a_2 \in L^\infty(X), \quad a_1, a_2 \ge c_0 \text{ in } X \text{ for some constant } c_0 > 0, \tag{6}$$

and

$$f \in L^2(X \times \Omega).$$

For the Fokker–Planck equation (2), these assumptions are naturally valid; the second part of (6) is automatically satisfied as long as the scattering and absorption effects are not negligible.

We now introduce a weak formulation for a boundary value problem (4). For this purpose, let $u \in C^2(\overline{X} \times \Omega)$ be a function satisfying (4). Multiply the Eq. (4a) by a test function $v \in C^1(\overline{X} \times \Omega)$, integrate over $X \times \Omega$, and apply the boundary condition (4b) to find

$$a(u, v) = (f, v) \tag{7}$$

in which,

$$a(u, v) = \int_{X\times\Omega} \left(a_1 \nabla^* u \cdot \nabla^* v - u\, \boldsymbol{\omega} \cdot \nabla v + a_2 uv\right) \mathrm{d}x\, \mathrm{d}\omega + \int_{\Gamma_+} \boldsymbol{\nu} \cdot \boldsymbol{\omega}\, uv\, \mathrm{d}S\, \mathrm{d}\omega,$$

$$(f, v) = \int_{X\times\Omega} f v\, \mathrm{d}x\, \mathrm{d}\omega,$$

where $\mathrm{d}S$ is the infinitesimal area element on ∂X, and $(\cdot,\cdot)$ denotes the inner product in $L^2(X \times \Omega)$. The bilinear form $a(\cdot,\cdot)$ defined on $C^1(\overline{X} \times \Omega) \times C^1(\overline{X} \times \Omega)$ is not symmetric. So we introduce another bilinear form

$$\overline{a}(v, w) = \int_{X\times\Omega} \left(a_1 \nabla^* v \cdot \nabla^* w + a_2 vw\right) \mathrm{d}x\, \mathrm{d}\omega + \frac{1}{2} \int_{\Gamma} |\boldsymbol{\nu} \cdot \boldsymbol{\omega}| vw\, \mathrm{d}S\, \mathrm{d}\omega \tag{8}$$

which symmetries $a(\cdot,\cdot)$:

$$a(v, v) = \overline{a}(v, v) \quad \forall v \in C^1(\overline{X} \times \Omega).$$

Under the condition (6), $\overline{a}(\cdot,\cdot)$ defines an inner product on $C^1(\overline{X} \times \Omega)$. We introduce the completion of the space $C^1(\overline{X} \times \Omega)$ with respect to the inner product $\overline{a}(\cdot,\cdot)$:

$$V_1 := \left\{ v \in L^2(X \times \Omega) \mid |\nabla^* v| \in L^2(X \times \Omega), \int_{\Gamma} |\boldsymbol{\nu} \cdot \boldsymbol{\omega}| v^2\, \mathrm{d}S\, \mathrm{d}\omega < \infty \right\}, \tag{9}$$

with the norm

$$\|v\|_{V_1} = \sqrt{\overline{a}(v, v)}$$

which is equivalent to the standard norm

$$\left[\int_{X\times\Omega} \left(|\nabla^* v|^2 + v^2\right) \mathrm{d}x\, \mathrm{d}\omega + \int_{\Gamma} |\boldsymbol{\nu} \cdot \boldsymbol{\omega}| v^2\, \mathrm{d}S\, \mathrm{d}\omega \right]^{1/2}.$$

We further introduce a subspace of V_1:

$$V_2 := \left\{ v \in V_1 \mid \boldsymbol{\omega} \cdot \nabla v \in L^2(X \times \Omega) \right\}, \tag{10}$$

with the norm

$$\|v\|_{V_2} = \left[\|v\|_{V_1}^2 + \int_{X\times\Omega} |\boldsymbol{\omega} \cdot \nabla v|^2 \mathrm{d}x\, \mathrm{d}\omega \right]^{1/2}.$$

Here, all the derivatives are understood to be the generalized (weak) derivatives.

We can extend $a(\cdot,\cdot)$ continuously with respect to its first argument in V_1 and its second argument in V_2. Denote the extension again by $a(\cdot,\cdot)$. Similarly, the bilinear form $\overline{a}(\cdot,\cdot)$ is extended continuously to $V_1 \times V_1$. Moreover,

$$a(v, v) = \overline{a}(v, v) \quad \forall v \in V_2. \tag{11}$$

With the above preparation, we can define a weak solution of the BVP (4) as follows.

Definition 1. We say that $u \in V_1$ is a weak solution of the BVP (4) if

$$a(u, v) = (f, v) \quad \forall v \in V_2.$$

We now study the weak formulation given in Definition 1. First observe that the bilinear form is bounded: for some appropriate constant C,

$$|a(u, v)| \le C\, \|u\|_{V_1} \|v\|_{V_2} \quad \forall u \in V_1, v \in V_2.$$

This is proved by applying the Cauchy–Schwarz inequality. Thus, for any fixed $v \in V_2$, $u \mapsto a(u, v)$ is a bounded linear functional on V_1. Therefore, by the Riesz representation theorem (e.g., [1, Subsection 2.5.2]), there exists a unique element Tv of V_1 such that

$$a(u, v) = \bar{a}(u, Tv) \quad \forall u \in V_1. \tag{12}$$

We claim $T : V_2 \to V_1$ is a bounded linear operator. Indeed if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $v_1, v_2 \in V_2$, we see for each $u \in V_1$ that

$$\begin{aligned}\bar{a}(u, T(\lambda_1 v_1 + \lambda_2 v_2)) &= a(u, \lambda_1 v_1 + \lambda_2 v_2) \quad (\text{by (12)})\\ &= \lambda_1 a(u, v_1) + \lambda_2 a(u, v_2)\\ &= \bar{a}(u, \lambda_1 Tv_1 + \lambda_2 Tv_2).\end{aligned}$$

So T is linear. Furthermore

$$\|Tv\|_{V_1}^2 = \bar{a}(Tv, Tv) = a(Tv, v) \leqslant C\|Tv\|_{V_1}\|v\|_{V_2}.$$

Consequently $\|Tv\|_{V_1} \leqslant C\,\|v\|_{V_2}$ for all $v \in V_2$, and so $T : V_2 \to V_1$ is bounded.

Since $\bar{a}(v, Tv) = \bar{a}(v, v)$ by (11) and (12), we have

$$\|v\|_{V_1} \leq \|Tv\|_{V_1} \quad \text{if } v \in V_2.$$

So, T is a one-to-one mapping from V_2 into V_1.

Furthermore, define subspaces

$$\begin{aligned}&V_{1,T} := \text{completion of } \{Tv \mid v \in V_2\} \text{ in } V_1,\\ &V_{2,T} := \text{completion of } V_2 \text{ with respect to } v \mapsto \|Tv\|_{V_1}.\end{aligned}$$

Then $V_{1,T}$ is a closed subspace of V_1, $V_2 \subset V_{2,T} \subset V_1$, and T can be extended to an isomorphism from $V_{2,T}$ onto $V_{1,T}$ (the extension is again denoted by T). Moreover, the equality

$$a(Tv, w) = \bar{a}(Tv, Tw)$$

is extended from $V_2 \times V_2$ to $V_{2,T} \times V_{2,T}$.

Now, given $f \in L^2(X \times \Omega)$, the linear form $v \mapsto (f, v)$ is a continuous functional on $V_{2,T}$, and $\bar{a}_T(u, v) := \bar{a}(Tu, Tv)$ is an inner product on $V_{2,T}$. So there exists a unique $u' \in V_{2,T}$ such that

$$(f, v) = \bar{a}_T(u', v) = \bar{a}(Tu', Tv) = a(Tu', v) \quad \forall v \in V_{2,T}.$$

Because T is a one-to-one map, we set $u = Tu' \in V_1$ and then

$$(f, v) = a(u, v) \quad \forall v \in V_{2,T}.$$

This implies that $u \in V_1$ is a weak solution of the BVP (4). If $u \in V_2$, then it is also unique. This is shown as follows. Let $\tilde{u} \in V_2$ be another weak solution. Then

$$a(u - \tilde{u}, v) = 0 \quad \forall v \in V_2.$$

Take $v = u - \tilde{u}$ and apply the relation (11), we conclude that $u - \tilde{u} = 0$.

We summarize the above discussion as the following theorem.

Theorem 2. *Assume (6). Then, for an arbitrary $f \in L^2(X \times \Omega)$, there exists $u \in V_1$ satisfying*

$$a(u, v) = (f, v) \quad \forall v \in V_2.$$

Moreover, if $u \in V_2$, then it is unique.

If the solution u thus obtained is of C^2 on $\overline{X} \times \Omega$, then it can be verified directly that u solves the boundary value problem

$$Au = f \quad \text{in } X \times \Omega, \qquad u = 0 \quad \text{on } \Gamma_-.$$

3. Maximum principle and positivity of the solution

In this section, we present maximum principles for the Fokker–Planck equation (4), and as a consequence, we show a positivity property of the solution. We first introduce a lemma.

Lemma 3. *Let $w \in C^2(\Omega)$. If w achieves its maximum at a point $\boldsymbol{\omega}_0 \in \Omega$, then*

$$\Delta^* w(\boldsymbol{\omega}_0) \leq 0.$$

Proof. Define $w^*(\boldsymbol{x}) = w(\boldsymbol{x}/|\boldsymbol{x}|)$ for $\boldsymbol{x} \in \mathbb{R}^3 \setminus \{0\}$. If w achieves its maximum at $\boldsymbol{\omega}_0 \in \Omega$, then w^* achieves its maximum at any $\boldsymbol{x}_0 \neq \mathbf{0}$ with $\boldsymbol{\omega}_0 = \boldsymbol{x}_0/|\boldsymbol{x}_0|$. Then by the definition of Laplace–Beltrami operator on Ω, we have

$$\Delta^* w(\boldsymbol{\omega}_0) = \Delta w\left(\frac{\boldsymbol{x}_0}{|\boldsymbol{x}_0|}\right) = \Delta w^*(\boldsymbol{x}_0) \leq 0.$$

Therefore, the stated result holds. $\square$

To simplify the notation, we introduce the set

$$X_\Omega = \Gamma_+ \cup \Gamma_0 \cup (X \times \Omega),$$

and define $C^{1,2}(X_\Omega)$ to be the space of all functions $v \in C(X_\Omega)$ such that $\nabla^* v$, $\Delta^* v$ and ∇v are all continuous on X_Ω. We have the following maximum principle.

Theorem 4. *Assume $u \in C^{1,2}(X_\Omega) \cap C(\bar{X} \times \Omega)$ and*

$$a_1(\boldsymbol{x}) > 0, \qquad a_2(\boldsymbol{x}) = 0 \quad \text{for } \boldsymbol{x} \in X.$$

(i) *If*

$$Au < 0 \quad \text{in } X_\Omega, \tag{13}$$

then

$$\max_{\bar{X}\times\Omega} u = \max_{\Gamma_-} u.$$

(ii) *Likewise, if*

$$Au > 0 \quad \text{in } X_\Omega,$$

then

$$\min_{\bar{X}\times\Omega} u = \min_{\Gamma_-} u.$$

Proof. We first prove (i). Let $(\boldsymbol{x}_0, \boldsymbol{\omega}_0) \in X_\Omega$ be a point with $u(\boldsymbol{x}_0, \boldsymbol{\omega}_0) = \max_{\overline{X}\times\Omega} u$. By Lemma 3,

$$-a_1 \Delta^* u(\boldsymbol{x}_0, \boldsymbol{\omega}_0) \geq 0. \tag{14}$$

We distinguish three cases according to the location of the point $(\boldsymbol{x}_0, \boldsymbol{\omega}_0)$.

First, if $(\boldsymbol{x}_0, \boldsymbol{\omega}_0) \in X \times \Omega$, then

$$\nabla u(\boldsymbol{x}_0, \boldsymbol{\omega}_0) = 0. \tag{15}$$

Next, if $(\boldsymbol{x}_0, \boldsymbol{\omega}_0) \in \Gamma_+$, then

$$\boldsymbol{\omega}_0 \cdot \nabla u(\boldsymbol{x}_0, \boldsymbol{\omega}_0) \geq 0. \tag{16}$$

Indeed, the condition $(\boldsymbol{x}_0, \boldsymbol{\omega}_0) \in \Gamma_+$ implies $\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}(\boldsymbol{x}_0) > 0$. For $t > 0$ sufficiently small,

$$u(\boldsymbol{x}_0 - t\boldsymbol{\omega}_0, \boldsymbol{\omega}_0) \leq u(\boldsymbol{x}_0, \boldsymbol{\omega}_0).$$

Rearranging and dividing by t, we get

$$\frac{u(\boldsymbol{x}_0, \boldsymbol{\omega}_0) - u(\boldsymbol{x}_0 - t\boldsymbol{\omega}_0, \boldsymbol{\omega}_0)}{t} \geq 0.$$

Letting $t \to 0^+$, we obtain (16).

Finally, suppose $(\boldsymbol{x}_0, \boldsymbol{\omega}_0) \in \Gamma_0$. Let us show that

$$\boldsymbol{\omega}_0 \cdot \nabla u(\boldsymbol{x}_0, \boldsymbol{\omega}_0) = 0. \tag{17}$$

For this purpose, we choose a continuous boundary curve segment $\{\boldsymbol{x}(t) \mid |t| < t_0\} \subset \partial X$, $t_0 > 0$ sufficiently small, such that $\boldsymbol{x}(0) = \boldsymbol{x}_0$, and the direction $\boldsymbol{\omega}(t) := (\boldsymbol{x}(t) - \boldsymbol{x}_0) / \|\boldsymbol{x}(t) - \boldsymbol{x}_0\|$ has the properties that $\boldsymbol{\omega}(t) \cdot \boldsymbol{\omega}_0 > 0$ for $t \in (0, t_0)$ and $\boldsymbol{\omega}(t) \to \boldsymbol{\omega}_0$ as $t \to 0+$, and $\boldsymbol{\omega}(t) \cdot \boldsymbol{\omega}_0 < 0$ for $t \in (-t_0, 0)$ and $\boldsymbol{\omega}(t) \to -\boldsymbol{\omega}_0$ as $t \to 0-$. Since u attains its maximum over $\bar{X} \times \Omega$ at $(\boldsymbol{x}_0, \boldsymbol{\omega}_0)$, we have $u(\boldsymbol{x}(t), \boldsymbol{\omega}_0) \leq u(\boldsymbol{x}_0, \boldsymbol{\omega}_0)$. Thus,

$$\frac{u(\boldsymbol{x}_0 + \|\boldsymbol{x}(t) - \boldsymbol{x}_0\|\boldsymbol{\omega}(t)) - u(\boldsymbol{x}_0, \boldsymbol{\omega}_0)}{\|\boldsymbol{x}(t) - \boldsymbol{x}_0\|} \leq 0.$$

Taking the limits $t \to 0+$ and $t \to 0-$, we get

$$\boldsymbol{\omega}_0 \cdot \nabla u(\boldsymbol{x}_0, \boldsymbol{\omega}_0) \leq 0, \qquad -\boldsymbol{\omega}_0 \cdot \nabla u(\boldsymbol{x}_0, \boldsymbol{\omega}_0) \leq 0,$$

respectively. Therefore, (17) holds.

Combining (14) with (15) or (16) or (17), we know that in any case, $Au(\boldsymbol{x}_0, \boldsymbol{\omega}_0) \geq 0$, contradicting to the condition (13). Therefore, (i) holds. As $A(-u) < 0$ whenever $Au > 0$, assertion (ii) follows from (i). $\square$

Next we include the zeroth-order term. Denote $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$.

Theorem 5. *Assume* $u \in C^{1,2}(X_\Omega) \cap C(\bar{X} \times \Omega)$ *and*

$$a_1(\boldsymbol{x}) > 0, \qquad a_2(\boldsymbol{x}) > 0 \quad \text{for } \boldsymbol{x} \in X.$$

(i) *If*

$$Au \le 0 \quad \text{in } X_\Omega, \tag{18}$$

then

$$\max_{\bar{X} \times \Omega} u \le \max_{\Gamma_-} u^+.$$

(ii) *Likewise, if*

$$Au \ge 0 \quad \text{in } X_\Omega,$$

then

$$\min_{\bar{X} \times \Omega} u \ge \min_{\Gamma_-} u^-.$$

Proof. As in the proof of Theorem 4, we only prove (i). Thus, assume (18) and let u attain a positive maximum at a point $(\boldsymbol{x}_0, \boldsymbol{\omega}_0) \in X_\Omega$. As in the proof of Theorem 4, we have

$$Au(\boldsymbol{x}_0, \boldsymbol{\omega}_0) \ge a_2(\boldsymbol{x}_0)\, u(\boldsymbol{x}_0, \boldsymbol{\omega}_0) > 0.$$

This contradicts to the assumption (18). □

An immediate consequence of the above theorem is the following result, which is an important property for the Fokker–Planck equation to make physical sense.

Corollary 6. *Assume* $a_1(\boldsymbol{x}) > 0$ *and* $a_2(\boldsymbol{x}) > 0$ *for* $\boldsymbol{x} \in X$. *If* $u \in C^{1,2}(X_\Omega) \cap C(\bar{X} \times \Omega)$ *satisfies* (4) *with* $f \ge 0$ *in* $X \times \Omega$, *then* $u \ge 0$ *in* X_Ω.

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