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ANALYSIS OF A NUMERICAL METHOD FOR RADIATIVE TRANSFER EQUATION BASED BIOLUMINESCENCE TOMOGRAPHY*

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Abstract

In the bioluminescence tomography (BLT) problem, one constructs quantitatively the bioluminescence source distribution inside a small animal from optical signals detected on the animal's body surface. The BLT problem is ill-posed and often the Tikhonov regularization is used to obtain stable approximate solutions. In conventional Tikhonov regularization, it is crucial to choose a proper regularization parameter to balance the accuracy and stability of approximate solutions. In this paper, a parameter-dependent coupled complex boundary method (CCBM) based Tikhonov regularization is applied to the BLT problem governed by the radiative transfer equation (RTE). By properly adjusting the parameter in the Robin boundary condition, we achieve one important property: the regularized solutions are uniformly stable with respect to the regularization parameter so that the regularization parameter can be chosen based solely on the consideration of the solution accuracy. The discrete-ordinate finite-element method is used to compute numerical solutions. Numerical results are provided to illustrate the performance of the proposed method.

Mathematics subject classification: 92C55, 65F22, 80M10.
Key words: Bioluminescence tomography, radiative transfer equation, Tikhonov regularization, coupled complex boundary method, convergence.

1. Introduction

Bioluminescence tomography (BLT) is a new molecular imaging modality and has shown its potential in monitoring non-invasively physiological and pathological processes *in vivo* at the cellular and molecular level. It is particularly attractive for *in vivo* applications because no external excitation source is needed and thus background noise is low while sensitivity is

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high ([38]). In the BLT problem, one reconstructs quantitatively the bioluminescence source distribution inside a small animal from optical signals detected on the animal's body surface.

A basic prerequisite for the BLT problem is the knowledge about the forward model describing the light propagation in the biological medium. Transmission of the bioluminescent photons through the biological medium is subject to both scattering and absorption, and is accurately described by the radiative transfer equation (RTE) ([2,5]). Since it is very challenging to solve the RTE accurately, diffusion approximation (DA) of the RTE is popularly used as the forward model. Plenty of references can be found in the literature on theoretical analysis and numerical simulations on the DA-based BLT problem, e.g. [11,17,21,25,32,37] and references therein for instance. However, as it is noted in [1], the DA is not always a good approximation of the RTE, especially when the scattering is relatively low. Higher order of approximate equations to the RTE such as $\mathrm{SP_N}$ and differential approximations etc. can be used to increase the approximation accuracy [23,30,39].

In this paper, we consider the more accurate RTE-based BLT problem. Let $X \subset \mathbb{R}^3$ be an open bounded set with a Lipschitz boundary ∂X and Ω be the unit sphere in $\mathbb{R}^3$. Denote by $\Gamma = \partial X \times \Omega$ the boundary of $X \times \Omega$, and by Γ_- and Γ_+ the incoming and outgoing parts of the boundary:

$$\Gamma_- := \Big\{(x,\omega) \in \Gamma \mid \omega\cdot\nu < 0\Big\}, \quad \Gamma_+ := \Big\{(x,\omega) \in \Gamma \mid \omega\cdot\nu > 0\Big\},$$

where $\nu := \nu(x)$ is the unit outward normal vector at $x \in \partial X$. With a normalized non-negative kernel function η:

$$\int_\Omega \eta(x, \omega\cdot\widehat{\omega})d\sigma(\widehat{\omega}) = 1 \quad \forall\, x \in X,\, \omega \in \Omega,$$

define an integral operator S by

$$Su(x,\omega) = \int_\Omega \eta(x,\omega\cdot\widehat{\omega})\, u(x,\widehat{\omega})\, d\sigma(\widehat{\omega}).$$

In most applications, η is chosen to be independent of x. One well-known example is the 3D Henyey-Greestein phase function ([26])

$$\eta(t) = \frac{1-g^2}{4\pi(1+g^2-2gt)^{3/2}}, \quad t := \omega\cdot\widehat{\omega} \in [-1,1],$$

where $g \in (-1,1)$ is the anisotropy factor of the scattering medium: $g = 0$ for isotropic scattering, $g > 0$ for forward scattering, and $g < 0$ for backward scattering.

With an admissible set to be specified later, we consider the following inverse source problem:

Problem 1.1. *Given u_m on Γ_+, find a source function p from the admissible set so that the solution u of the boundary value problem (BVP)*

$$\begin{cases} \omega\cdot\nabla u(x,\omega) + \mu_t(x)u(x,\omega) = \mu_s(x)(Su)(x,\omega) + p(x)\chi_0(x), & (x,\omega) \in X\times\Omega, \\ u(x,\omega) = 0, & (x,\omega) \in \Gamma_- \end{cases} \tag{1.1}$$

matches the boundary measurement u_m for the density of outgoing photons:

$$u(x,\omega) = u_m(x,\omega), \quad (x,\omega) \in \Gamma_+.$$

Here ∇ is the gradient operator with respect to spatial variable x, $\mu_t = \mu_a + \mu_s$ is the total cross-section, μ_s and μ_a are the scattering and absorption cross-sections, χ_0 is the characteristic function of $X_0 \subset X$, i.e., its value is 1 in X_0, and is 0 outside X_0. In what follows, we write $p(x)$ for $p(x)\chi_0(x)$. The inflow boundary condition $u = 0$ on Γ_- indicates that the experiment is carried out in a dark environment. We may equally well consider a general inflow boundary condition $u = u_{in}$ on Γ_- for a possibly non-zero function u_{in}.

The first issue for the RTE-based BLT problem is how to solve the forward BVP (1.1) numerically and effectively. In [31], the BVP (1.1) is solved by the finite-difference discrete-ordinates method where the spatial derivative in RTE is approximated with first-order finite difference while the angular variable is discretized with a set of discrete ordinates. Since the RTE is essentially a hyperbolic-type system, it is natural to apply the discrete-ordinate discontinuous Galerkin (DG) method to solve the BVP (1.1) ([14, 15, 24]). We refer the reader to [6, 13, 16] for details on numerical implementation.

As an inverse source problem, the BLT problem is ill-posed. With only one measurement available on the outgoing boundary, one can not have a unique solution. In [28, 35], under some smoothness assumptions on the optical parameters, unique solvability is shown for the RTE-based BLT problem. In [22], numerical solution of Problem 1.1 is discussed within a Tikhonov regularization framework. In related RTE-based optical tomography problems, where one reconstructs the absorption and/or scattering parameters μ_a and μ_s ([6, 10, 29, 36]).

In this paper, we develop a stable approximation method for Problem 1.1 using the Tikhonov regularization. In the conventional Tikhonov regularization framework, the value of the regularization parameter should be chosen carefully to balance solution accuracy and stability. Based on a parameter dependent coupled complex boundary method (CCBM), we propose a new Tikhonov regularization method for the RTE-based BLT. The parameter dependent CCBM-based Tikhonov regularization framework was first proposed in [19] for the DA-based BLT, with the property that the regularized solutions are insensitive with respect to the small size of the regularization parameter so that we can choose the regularization parameter based solely on the consideration of the solution accuracy. The idea of CCBM is to couple boundary conditions and boundary measurements into a Robin boundary condition in such a way that the Neumann and Dirichlet data are the real and imaginary parts of the Robin boundary condition, respectively. We extend this idea for the RTE-based BLT problem.

The paper is organized as follows. In Section 2, after an introduction of some assumptions and function spaces, a reformulation of the BVP (1.1) as an elliptic BVP is given. A detailed description of the parameter dependent CCBM is proposed in Section 3, where we also apply the Tikhonov regularization to the reformulated inverse problem to obtain stable approximate source functions. In Section 4, we provide a theoretical analysis of the new regularization framework. We discretize the regularized optimization problem with the finite element method in Section 5 and derive a new error estimate. Numerical results are presented in Section 6 to illustrate the performance of the proposed method.

2. The Forward Problem: A Reformulation of the RTE

Let $Q := L^2(X \times \Omega)$ with the inner product $(u, v)_Q := \int_{X\times\Omega} u(x, \omega)\, v(x, \omega)\, dx\, d\sigma(\omega)$ and norm $\|v\|_Q = (v, v)_Q^{1/2}$, $Q_\Gamma := L^2(\Gamma)$ and $Q_{\Gamma_\pm} := L^2(\Gamma_\pm)$ with the inner products

$$(u, v)_{Q_\Gamma} := \int_\Gamma |\omega \cdot \nu|\, u\, v\, d\sigma(x)\, d\sigma(\omega), \quad (u, v)_{Q_{\Gamma_\pm}} := \int_{\Gamma_\pm} |\omega \cdot \nu|\, u\, v\, d\sigma(x)\, d\sigma(\omega),$$

and corresponding norms

$$\|v\|_{Q_\Gamma} = (v,v)_{Q_\Gamma}^{1/2}, \quad \|v\|_{Q_{\Gamma_\pm}} = (v,v)_{Q_{\Gamma_\pm}}^{1/2}.$$

Denote $Q_0 = L^2(X_0)$ and view it as a subspace of Q, i.e., any function $p \in Q_0$ is identified with its extension by zero outside X_0.

We make the following assumption on the data which holds naturally in most applications.

(A1) $\mu_s, \mu_a \in L^\infty(X)$, $\mu_s \geq 0$ and $\mu_a \geq \mu_0 > 0$ a.e. in X, where μ_0 is a constant.

Consider the following operator Σ from Q to Q:

$$\Sigma\, v := \mu_t\, v - \mu_s\, S\, v, \quad v \in Q.$$

It is bounded, self-adjoint and Q-elliptic ([20, Lemma 2.1]):

$$(\Sigma v, v)_Q \geq \mu_0 \|v\|_Q^2, \quad v \in Q. \tag{2.1}$$

Thus, for any $r \in \mathbb{R}$, the power $\Sigma^r : Q \to Q$ is well-defined, and is bounded, self-adjoint and Q-elliptic. Moreover, from (2.1),

$$\|\Sigma^{-1}\|_{Q\to Q} \leq \mu_0^{-1}, \quad \|\Sigma^{-1/2}\|_{Q\to Q} \leq \mu_0^{-1/2}.$$

To obtain a weak form of the forward problem (1.1), define the Hilbert space

$$V := \Big\{ v \in Q \mid \omega \cdot \nabla v \in Q,\ v|_\Gamma \in Q_\Gamma \Big\}$$

with the inner product

$$(u,v)_V := (\Sigma^{-1}(\omega\cdot\nabla u), \omega\cdot\nabla v)_Q + (\Sigma u, v)_Q + (u,v)_{Q_\Gamma} \tag{2.2}$$

and the norm

$$\|v\|_V := \Big(|\Sigma^{-1/2}(\omega\cdot\nabla v)\|_Q^2 + \|\Sigma^{1/2} v\|_Q^2 + \|v\|_{Q_\Gamma}^2 \Big)^{1/2}. \tag{2.3}$$

The norm $\|\cdot\|_V$ is equivalent to the canonical norm $\Big(\|\omega\cdot\nabla v\|_Q^2 + \|v\|_Q^2 + \|v\|_{Q_\Gamma}^2 \Big)^{1/2}$ over V ([20]). The inner product (2.2) is natural in the study of the RTE BVP (1.1) and the norm (2.3) may be viewed as an energy norm.

Let $p \in Q_0$. Following [20, Subsection 2.1], we can formally reformulate the forward problem (1.1) as

$$-\,\omega\cdot\nabla\Sigma^{-1}(\omega\cdot\nabla u) + \Sigma u = p - \omega\cdot\nabla\Sigma^{-1}(p) \qquad \text{in } X \times \Omega, \tag{2.4a}$$

$$u \pm \Sigma^{-1}(\omega\cdot\nabla u) = \pm\Sigma^{-1}(p) \qquad \text{on } \Gamma_\pm. \tag{2.4b}$$

The weak form of the BVP (2.4) is:

$$u \in V, \quad (u,v)_V = (p,v)_Q + (\Sigma^{-1}(p), \omega\cdot\nabla v)_Q \quad \forall\, v \in V. \tag{2.5}$$

Under Assumption (A1), by applying the Lax-Milgram Lemma (cf. [3, Theorem 8.3.4]), we can prove that the problem (2.5) admits a unique solution $u \in V$.

3. The Inverse Problem: the RTE-based BLT

Using the reformulation (2.4), we transform Problem 1.1 to one of finding p from an admissible set $Q_{ad} \subset Q_0$ such that

$$
\begin{aligned}
& -\omega\cdot\nabla\Sigma^{-1}(\omega\cdot\nabla u)+\Sigma u = p-\omega\cdot\nabla\Sigma^{-1}(p) && \text{in } X\times\Omega, && \text{(3.1a)}\\
& u=u_m,\quad \Sigma^{-1}(\omega\cdot\nabla u)=\Sigma^{-1}(p)-u_m && \text{on } \Gamma_+, && \text{(3.1b)}\\
& u-\Sigma^{-1}(\omega\cdot\nabla u) = -\Sigma^{-1}(p) && \text{on } \Gamma_-. && \text{(3.1c)}
\end{aligned}
$$

We assume Q_{ad} is nonempty, closed, and convex.

We need complex versions of the spaces introduced in Section 2. Let $\mathbf{Q}$ be the complex version of Q with the inner product $(u,v)_{\mathbf{Q}} := (u,\bar{v})_Q$ and norm $\|v\|_{\mathbf{Q}} := (v,\bar{v})_Q^{1/2}$. Similarly, let $\mathbf{Q}_\Gamma$, $\mathbf{Q}_{\Gamma_+}$, $\mathbf{Q}_{\Gamma_-}$ and $\mathbf{V}$ be the complex versions of Q_Γ, Q_{Γ_+}, Q_{Γ_-} and V. In particular, the inner product and norm of $\mathbf{V}$ are

$$
\begin{aligned}
&(u,v)_{\mathbf{V}} := (\Sigma^{-1}(\omega\cdot\nabla u),\omega\cdot\nabla v)_{\mathbf{Q}} + (\Sigma u, v)_{\mathbf{Q}} + (u,v)_{\mathbf{Q}_\Gamma},\\
&\|v\|_{\mathbf{V}} := \Big[\|\Sigma^{-1/2}(\omega\cdot\nabla v)\|_{\mathbf{Q}}^2 + \|\Sigma^{1/2}v\|_{\mathbf{Q}}^2 + \|v\|_{\mathbf{Q}_\Gamma}^2\Big]^{1/2}.
\end{aligned}
$$

For a parameter $\alpha>0$, consider a complex BVP

$$
\begin{aligned}
& -\omega\cdot\nabla\Sigma^{-1}(\omega\cdot\nabla u)+\Sigma u = p-\omega\cdot\nabla\Sigma^{-1}(p) && \text{in } X\times\Omega, && \text{(3.2a)}\\
& \Sigma^{-1}(\omega\cdot\nabla u)+i\,\alpha\,u=\Sigma^{-1}(p)-u_m+i\,\alpha\,u_m && \text{on } \Gamma_+, && \text{(3.2b)}\\
& u-\Sigma^{-1}(\omega\cdot\nabla u) = -\Sigma^{-1}(p) && \text{on } \Gamma_-, && \text{(3.2c)}
\end{aligned}
$$

where $i=\sqrt{-1}$ is the imaginary unit. Obviously, if (u,p) satisfy (3.1), then (3.2) holds. Conversely, let (u,p) satisfy (3.2) and write $u=u_1+i\,u_2$, u_1 and u_2 being the real and imaginary parts of u. Then the real-valued functions u_1,u_2 satisfy

$$
\begin{aligned}
& -\omega\cdot\nabla\Sigma^{-1}(\omega\cdot\nabla u_1)+\Sigma u_1 = p-\omega\cdot\nabla\Sigma^{-1}(p) && \text{in } X\times\Omega, && \text{(3.3a)}\\
& \Sigma^{-1}(\omega\cdot\nabla u_1)-\alpha\,u_2=\Sigma^{-1}(p)-u_m && \text{on } \Gamma_+, && \text{(3.3b)}\\
& u_1-\Sigma^{-1}(\omega\cdot\nabla u_1) = -\Sigma^{-1}(p) && \text{on } \Gamma_-, && \text{(3.3c)}
\end{aligned}
$$

and

$$
\begin{aligned}
& -\omega\cdot\nabla\Sigma^{-1}(\omega\cdot\nabla u_2)+\Sigma u_2 = 0 && \text{in } X\times\Omega, && \text{(3.4a)}\\
& \Sigma^{-1}(\omega\cdot\nabla u_2)+\alpha\,u_1=\alpha\,u_m && \text{on } \Gamma_+, && \text{(3.4b)}\\
& u_2-\Sigma^{-1}(\omega\cdot\nabla u_2) = 0 && \text{on } \Gamma_-. && \text{(3.4c)}
\end{aligned}
$$

If $u_2=0$ in $X\times\Omega$, then it follows from (3.3) and (3.4) that $(u,p)=(u_1,p)$ satisfy (3.1). Thus, we have the following reformulation of Problem 1.1.

Problem 3.1. *Given $u_m\in Q_{\Gamma_+}$, find $p\in Q_{ad}$ such that*

$$u_2 = 0 \ \text{in } X\times\Omega,$$

where u_2 is the imaginary part of the solution $u=u_1+i\,u_2$ of the BVP (3.2).

For any $u, v \in \mathbf{V}$, define

$$a(u,v) = (\Sigma^{-1}(\omega\cdot\nabla u), \omega\cdot\nabla v)_{\mathbf{Q}} + (\Sigma u, v)_{\mathbf{Q}} + i\,\alpha(u,v)_{\mathbf{Q}_{\Gamma_+}} + (u,v)_{\mathbf{Q}_{\Gamma_-}}, \tag{3.5a}$$

$$F(v) = (p,v)_{\mathbf{Q}} + (\Sigma^{-1}(p), \omega\cdot\nabla v)_{\mathbf{Q}} + (i\,\alpha - 1)\,(u_m, v)_{\mathbf{Q}_{\Gamma_+}}. \tag{3.5b}$$

Then the weak form of (3.2) is

$$u \in \mathbf{V}, \quad a(u,\bar{v}) = F(\bar{v}) \quad \forall\, v \in \mathbf{V}. \tag{3.6}$$

For a given $p \in Q_0$, by the use of the complex version of Lax-Milgram Lemma ([9, p. 368-369]), the problem (3.6) has a unique solution $u \in \mathbf{V}$. Moreover, we have

$$\|u\|_{\mathbf{V}} \le c\left(\|p\|_{Q_0} + \|u_m\|_{Q_{\Gamma_+}}\right), \tag{3.7}$$

where $c > 0$ is a constant independent of α for $\alpha \le 1$.

Next we apply the Tikhonov regularization to Problem 3.1 for stable approximation of a solution. We allow the measurement on Γ_+ to contain random noise with a known level δ:

$$\|u_m^\delta - u_m\|_{Q_{\Gamma_+}} \le \delta.$$

Then (3.6) is modified to

$$u^\delta \in \mathbf{V}, \quad a(u^\delta, \bar{v}) = F^\delta(\bar{v}) \quad \forall\, v \in \mathbf{V}, \tag{3.8}$$

with

$$F^\delta(v) = (p,v)_{\mathbf{Q}} + (\Sigma^{-1}(p), \omega\cdot\nabla v)_{\mathbf{Q}} + (i\,\alpha - 1)\,(u_m^\delta, v)_{\mathbf{Q}_{\Gamma_+}}. \tag{3.9}$$

For any $p \in Q_0$, denote by $u^\delta(p) = u_1^\delta(p) + i\,u_2^\delta(p) \in \mathbf{V}$ the solution of (3.8). Define an objective functional

$$J_\varepsilon^\delta(p) = \frac{1}{2}\|u_2^\delta(p)\|_Q^2 + \frac{\varepsilon}{2}\|p\|_{Q_0}^2, \quad \varepsilon > 0,$$

and introduce the following Tikhonov regularization framework for Problem 3.1.

Problem 3.2. *Find $p_\varepsilon^\delta \in Q_{ad}$ such that*

$$J_\varepsilon^\delta(p_\varepsilon^\delta) = \inf_{p\in Q_{ad}} J_\varepsilon^\delta(p).$$

It is not difficult to verify that for any $p, q \in Q_0$,

$$(J_\varepsilon^\delta)'(p)\,q = (u_2^\delta(p), u_2^\delta(q) - u_2^\delta(0))_Q + \varepsilon\,(p,q)_{Q_0},$$

$$(J_\varepsilon^\delta)''(p)\,(q,q) = \|u_2^\delta(q) - u_2^\delta(0)\|_Q^2 + \varepsilon\|q\|_{Q_0}^2.$$

Hence, for $\varepsilon > 0$, $J_\varepsilon(\cdot)$ is strictly convex. Recall that Q_{ad} is non-empty, closed and convex. We have the following well-posedness result.

Proposition 3.1. *For any $\varepsilon > 0$, Problem 3.2 has a unique solution $p_\varepsilon^\delta \in Q_{ad}$ which depends continuously on all data. Moreover, p_ε^δ is characterized by*

$$(u_{\varepsilon,2}^\delta(p_\varepsilon^\delta), u_{\varepsilon,2}^\delta(q) - u_{\varepsilon,2}^\delta(p_\varepsilon^\delta))_Q + \varepsilon\,(p_\varepsilon^\delta, q - p_\varepsilon^\delta)_{Q_0} \ge 0, \quad \forall\, q \in Q_{ad}, \tag{3.10}$$

or equivalently,

$$p_\varepsilon^\delta = \Pi_{ad}\left[-\frac{1}{\varepsilon}\chi_0\int_\Omega (w_{\varepsilon,2}^\delta + \Sigma^{-1}(\omega\cdot\nabla w_{\varepsilon,2}^\delta))d\sigma(\omega)\right], \tag{3.11}$$

where $u_{\varepsilon,2}^\delta$ is the imaginary part of the solution $u_\varepsilon^\delta := u^\delta(p_\varepsilon^\delta) \in \boldsymbol{V}$ of the BVP (3.8) with p replaced by p_ε^δ, Π_{ad} is the orthogonal projection from Q_0 onto Q_{ad}, and $w_{\varepsilon,2}^\delta$ is the imaginary part of the weak solution $w_\varepsilon^\delta := w^\delta(p_\varepsilon^\delta) \in \boldsymbol{V}$ of the adjoint problem:

$$-\,\omega\cdot\nabla\Sigma^{-1}(\omega\cdot\nabla w_\varepsilon^\delta) + \Sigma w_\varepsilon^\delta = u_{\varepsilon,2}^\delta \qquad \textit{in } X\times\Omega, \tag{3.12a}$$

$$\Sigma^{-1}(\omega\cdot\nabla w_\varepsilon^\delta) + i\,\alpha\, w_\varepsilon^\delta = 0 \qquad \textit{on } \Gamma_+, \tag{3.12b}$$

$$w_\varepsilon^\delta - \Sigma^{-1}(\omega\cdot\nabla w_\varepsilon^\delta) = 0 \qquad \textit{on } \Gamma_-. \tag{3.12c}$$

Proof. The well-posedness of Problem 3.2 follows from a standard argument. The solution p_ε^δ is characterized by

$$(J_\varepsilon^\delta)'(p_\varepsilon^\delta)\,(q - p_\varepsilon^\delta) \ge 0 \quad \forall\, q \in Q_{ad}, \tag{3.13}$$

which is (3.10). For any $q \in Q_0$, denote by $u^\delta(q) = u_1^\delta(q) + i\,u_2^\delta(q)$ the solution of the BVP (3.8), with p replaced by q. Then $\tilde u := u^\delta(q) - u^\delta(0) \in \mathbf{V}$ is the weak solution of the BVP

$$\begin{aligned}
&-\,\omega\cdot\nabla\Sigma^{-1}(\omega\cdot\nabla\tilde u) + \Sigma\tilde u = q - \omega\cdot\nabla\Sigma^{-1}(q) && \text{in } X\times\Omega,\\
&\Sigma^{-1}(\omega\cdot\nabla\tilde u) + i\,\alpha\,\tilde u = \Sigma^{-1}(q) && \text{on } \Gamma_+,\\
&\tilde u - \Sigma^{-1}(\omega\cdot\nabla\tilde u) = -\Sigma^{-1}(q) && \text{on } \Gamma_-.
\end{aligned}$$

Multiply the differential equation in (3.12) with $\tilde u_2$, integrate over $X\times\Omega$, and integrate by parts to get

$$\begin{aligned}
&(u_{\varepsilon,2}^\delta, u_2^\delta(q) - u_2^\delta(0))_Q\\
=&(w_{\varepsilon,2}^\delta, q)_Q + (\omega\cdot\nabla w_{\varepsilon,2}^\delta, \Sigma^{-1}(q))_Q = (w_{\varepsilon,2}^\delta + \Sigma^{-1}(\omega\cdot\nabla w_{\varepsilon,2}^\delta), q)_Q\\
=&\left(\int_\Omega (w_{\varepsilon,2}^\delta + \Sigma^{-1}(\omega\cdot\nabla w_{\varepsilon,2}^\delta))d\sigma(\omega), q\right)_{Q_0}
\end{aligned} \tag{3.15}$$

Substitute (3.15) into (3.13) to obtain

$$\left(\int_\Omega \left(w_{\varepsilon,2}^\delta + \Sigma^{-1}(\omega\cdot\nabla w_{\varepsilon,2}^\delta)\right)d\sigma(\omega) + \varepsilon\, p_\varepsilon^\delta, q - p_\varepsilon^\delta\right)_{Q_0} \ge 0 \quad \forall\, q\in Q_{ad}.$$

Therefore, (3.11) holds. $\square$

4. Theoretical Analysis

We begin with a preparatory lemma.

Lemma 4.1. *For any $p \in Q_0$, denote by $u(p) = u_1(p) + i\,u_2(p)$, $u^\delta(p) = u_1^\delta(p) + i\,u_2^\delta(p) \in \boldsymbol{V}$ the unique solutions of the problems* (3.6) *and* (3.8). *Then we have*

$$\|u_2^\delta(p) - u_2(p)\|_V \le c\,\alpha\,\delta. \tag{4.1}$$

Proof. Subtracting (3.6) from (3.8), we have

$$a(u^\delta(p)-u(p),\bar v)=(i\,\alpha-1)\,(u_m^\delta-u_m,\bar v)_{\mathbf{Q}_{\Gamma_+}}\quad\forall\, v\in\mathbf{V}. \tag{4.2}$$

Applying the complex version of Lax-Milgram Lemma again to (4.2), we know

$$\|u^\delta(p)-u(p)\|_{\mathbf{V}}\le c\|u_m^\delta-u_m\|_{Q_{\Gamma_+}}\le c\,\delta. \tag{4.3}$$

Let $\hat u=\hat u_1+i\,\hat u_2:=u^\delta(p)-u(p)$. From (4.2), we have, for any $v\in V$,

$$(\Sigma^{-1}(\omega\cdot\nabla\hat u_2),\omega\cdot\nabla v)_Q+(\Sigma\hat u_2,v)_Q+(\hat u_2,v)_{Q_{\Gamma_-}}=\alpha\,(u_m^\delta-u_m-\hat u_1,v)_{Q_{\Gamma_+}}. \tag{4.4}$$

Then, take $v=u_2^\delta(p)-u_2(p)$ in (4.4) and use (4.3) to get (4.1). $\square$

Denote by S_0 the solution set of Problem 1.1 or 3.1, and assume it is nonempty. It is straightforward to show that S_0 is closed and convex. Then there is a unique minimal-norm element p^* from S_0 ([3]):

$$\|p^*\|_{Q_0}\le\|p\|_{Q_0}\quad\forall\, p\in S_0.$$

We have the following convergence result.

Theorem 4.1. *Fix $\alpha>0$. For a sequence of noise levels $\{\delta_n\}_{n\ge1}$, $\delta_n\to0$ as $n\to\infty$, let $\varepsilon_n=\varepsilon(\delta_n)$ be chosen with $\varepsilon_n\to0$ and $\delta_n^2/\varepsilon_n\to0$ as $n\to\infty$. Denote by $p_{\varepsilon_n}^{\delta_n}\in Q_{ad}$ the solution of Problem 3.2 with u_m^δ, and ε replaced by $u_m^{\delta_n}$ and ε_n respectively. Then the sequence $\{p_{\varepsilon_n}^{\delta_n}\}_{n\ge1}$ converges to p^* in Q_0 as $n\to\infty$.*

Proof. For simplicity in notation, write $p^n=p_{\varepsilon_n}^{\delta_n}$ and $u_m^n=u_m^{\delta_n}$. Denote by $u^n=u_1^n+i\,u_2^n=u^{\delta_n}(p^n)$ and $u^n(p^*)=u_1^n(p^*)+i\,u_2^n(p^*)$ the unique solutions of (3.8) in $\mathbf{V}$, both with u_m^δ replaced by u_m^n, and with p replaced by p^n, p^* respectively. Moreover, from the definition of p^*, we have $u_2(p^*)=0$, where $u_2(p^*)$ is the imaginary part of the solution of the problem (3.6) with p replaced by p^*. Then, using (4.1),

$$J_{\varepsilon_n}^{\delta_n}(p^n)\le J_{\varepsilon_n}^{\delta_n}(p^*)=\frac12\|u_2^n(p^*)-u_2(p^*)\|_Q^2+\frac{\varepsilon_n}{2}\|p^*\|_{Q_0}^2\le c\,\alpha^2\delta_n^2+\frac12\varepsilon_n\|p^*\|_{Q_0}^2,$$

which gives

$$\|p^n\|_{Q_0}^2\le c\,\alpha^2\,\frac{\delta_n^2}{\varepsilon_n}+\|p^*\|_{Q_0}^2. \tag{4.5}$$

Similar to (3.7), we have a bound on u^n:

$$\|u^n\|_{\mathbf{V}}\le c\left(\|p^n\|_{Q_0}+\|u_m^n\|_{Q_{\Gamma_+}}\right)\le c\left(\|p^n\|_{Q_0}+\delta_n+\|u_m\|_{Q_{\Gamma_+}}\right). \tag{4.6}$$

From (4.5)–(4.6), we see that $\{(p^n,u^n)\}$ is a bounded sequence in $Q_0\times\mathbf{V}$. Thus, there are a subsequence $\{n'\}$ of the sequence $\{n\}$ and elements $p^\infty\in Q_0$, $u^\infty\in\mathbf{V}$ such that as $n'\to\infty$,

$$p^{n'}\rightharpoonup p^\infty \text{ in } Q_0,\quad u^{n'}\rightharpoonup u^\infty \text{ in } \mathbf{V},\ \mathbf{Q},\ \mathbf{Q}_{\Gamma_+} \text{ and } \mathbf{Q}_{\Gamma_-}. \tag{4.7}$$

Let us show that $u^\infty=u(p^\infty)$. From the definition of u^n, we have

$$u^{n'}\in\mathbf{V},\quad a(u^{n'},\bar v)=F^{n'}(\bar v)\quad\forall\, v\in\mathbf{V},$$

where $F^{n'}(\cdot)$ is as defined in (3.9), with u_m^δ and p replaced by $u_m^{n'}$ and $p^{n'}$. Let $n' \to \infty$, and use the convergence relations in (4.7) to get

$$u^\infty \in \mathbf{V}, \quad a(u^\infty, \bar{v}) = F^\infty(\bar{v}) \quad \forall\, v \in \mathbf{V},$$

where $F^\infty(\cdot)$ is as defined in (3.5), with p replaced by p^∞. Thus, $u^\infty = u(p^\infty)$. Then,

$$\frac{1}{2}\|u_2(p^\infty)\|_Q^2 \le \liminf_{n'\to\infty} \frac{1}{2}\|u_2^{n'}\|_Q^2 \le \liminf_{n'\to\infty} J_{\varepsilon_{n'}}^{\delta_{n'}}(p^{n'}).$$

Since

$$J_{\varepsilon_{n'}}^{\delta_{n'}}(p^{n'}) \le J_{\varepsilon_{n'}}^{\delta_{n'}}(p^*) \le c\,\alpha^2\,\delta_{n'}^2 + \frac{1}{2}\,\varepsilon_{n'}\|p^*\|_{0,\Omega_0}^2 \to 0 \quad \text{as } n' \to \infty,$$

we have

$$u_2(p^\infty) = 0 \text{ in } X \times \Omega.$$

As a result, p^∞ is a solution of Problem 3.1 or Problem 1.1. Hence, $p^\infty \in S_0$.

Next we prove $p^\infty = p^*$. From the lower semi-continuity of the norm $\|\cdot\|_{Q_0}$ and the weak convergence of $p^{n'}$ to p^∞, we have

$$\|p^\infty\|_{Q_0} \le \liminf_{n'\to\infty} \|p^{n'}\|_{Q_0}.$$

Therefore, for any fixed $\epsilon > 0$, there exists a positive integer N such that $\forall\, n' > N$,

$$\|p^{n'}\|_{Q_0}^2 \ge \|p^\infty\|_{Q_0}^2 - \epsilon. \tag{4.8}$$

We note that (4.5) also holds when p^* is replaced by p^∞. Therefore, together with (4.8),

$$-\epsilon \le \|p^{n'}\|_{Q_0}^2 - \|p^\infty\|_{Q_0}^2 \le c\,\alpha^2\,\frac{\delta_{n'}^2}{\varepsilon_{n'}}$$

holds for $n' > N$. Let $n' \to \infty$ first and then $\epsilon \to 0$ in the relation above to get

$$\lim_{n'\to\infty} \|p^{n'}\|_{Q_0} = \|p^\infty\|_{Q_0}. \tag{4.9}$$

From the definition of p^*, we have $\|p^*\|_{0,\Omega_0} \le \|p^\infty\|_{0,\Omega_0}$. Combining it with (4.5), for $n' > N$, the following relation holds:

$$\|p^{n'}\|_{Q_0}^2 - \|p^\infty\|_{Q_0}^2 \le \|p^{n'}\|_{Q_0}^2 - \|p^*\|_{Q_0}^2 \le c\,\alpha^2\,\frac{\delta_{n'}^2}{\varepsilon_{n'}}.$$

Letting $n' \to \infty$ in the relation above and using (4.9), we have $\|p^\infty\|_{Q_0} = \|p^*\|_{Q_0}$. Hence, $p^\infty = p^*$ and $p^{n'} \rightharpoonup p^*$ in Q as $n' \to \infty$. Since the limit does not depend on the subsequence selected, the entire sequence $p^n \rightharpoonup p^\infty$ in Q_0, as $n' \to \infty$. The strong convergence follows from $\lim_{n\to\infty} \|p^n\|_{Q_0} = \|p^*\|_{Q_0}$ and the weak convergence. □

We now show a uniform boundedness property. Recall that the solution p_ε^δ is the projection of

$$-\frac{1}{\varepsilon}\chi_0 \int_\Omega \Big(w_{\varepsilon,2}^\delta + \Sigma^{-1}(\omega \cdot \nabla w_{\varepsilon,2}^\delta)\Big) d\sigma(\omega) \tag{4.10}$$

to Q_{ad} (cf. (3.11)).

Theorem 4.2. *Let $\alpha = O(\sqrt{\varepsilon})$. Then, the function (4.10) is uniformly bounded in Q_0 with respect to ε and δ for small $\varepsilon, \delta > 0$.*

Proof. Denote by $u_\varepsilon^\delta \in \mathbf{V}$ the solution of (3.8) with p replaced by p_ε^δ. Then using (4.5)–(4.6) and $\alpha = O(\sqrt{\varepsilon})$, we have

$$\|u_\varepsilon^\delta\|_{\mathbf{V}} \le c\left(\|p_\varepsilon^\delta\|_{Q_0} + \|u_m^\delta\|_{Q_{\Gamma_+}}\right) \le c\left(\delta\,\frac{\alpha}{\sqrt{\varepsilon}} + \|p^*\|_{Q_0} + \delta + \|u_m\|_{Q_{\Gamma_+}}\right) \le c. \tag{4.11}$$

Write $u_\varepsilon^\delta = u_{\varepsilon,1}^\delta + i\,u_{\varepsilon,2}^\delta$. Then $u_{\varepsilon,2}^\delta \in V$ satisfies

$$\begin{aligned}&(\Sigma^{-1}(\omega\cdot\nabla u_{\varepsilon,2}^\delta), \omega\cdot\nabla v)_Q + (\Sigma u_{\varepsilon,2}^\delta, v)_Q + (u_{\varepsilon,2}^\delta, v)_{Q_{\Gamma_-}}\\ &= \alpha\,(u_m^\delta - u_m - u_{\varepsilon,1}^\delta, v)_{Q_{\Gamma_+}} \quad \forall\, v \in V.\end{aligned}$$

Taking $v = u_{\varepsilon,2}^\delta$ and using (4.11), we get

$$\|u_{\varepsilon,2}^\delta\|_V \le c\,\alpha\left(\|u_{\varepsilon,1}^\delta\|_{Q_{\Gamma_+}} + \|u_m^\delta - u_m\|_{Q_{\Gamma_+}}\right) \le c\,\alpha. \tag{4.12}$$

Similarly, from the definition of w_ε^δ in (3.12), we have

$$\|w_\varepsilon^\delta\|_{\mathbf{V}} \le c\,\|u_{\varepsilon,2}^\delta\|_Q \le c\,\alpha. \tag{4.13}$$

Write $w_\varepsilon^\delta = w_{\varepsilon,1}^\delta + i\,w_{\varepsilon,2}^\delta$. Then $w_{\varepsilon,2}^\delta \in V$ satisfies

$$(\Sigma^{-1}(\omega\cdot\nabla w_{\varepsilon,2}^\delta), \omega\cdot\nabla v)_Q + (\Sigma w_{\varepsilon,2}^\delta, v)_Q + (w_{\varepsilon,2}^\delta, v)_{Q_{\Gamma_-}} = \alpha\,(w_{\varepsilon,1}^\delta, v)_{Q_{\Gamma_+}} \quad \forall\, v \in V.$$

Taking $v = u_{\varepsilon,2}^\delta$ and using (4.13), we get

$$\|w_{\varepsilon,2}^\delta\|_V \le c\,\alpha\,\|w_{\varepsilon,1}^\delta\|_{Q_{\Gamma_+}} \le c\,\alpha^2.$$

Therefore, if $\alpha = O(\sqrt{\varepsilon})$,

$$\Big\| -\frac{1}{\varepsilon}\int_\Omega (w_{\varepsilon,2}^\delta + \Sigma^{-1}(\omega\cdot\nabla w_{\varepsilon,2}^\delta))d\sigma(\omega)\Big\|_{Q_0} = O(1),$$

and the proof is completed. □

Theorem 4.2 indicates that reconstruction of the source function can be done for rather small regularization parameter with a properly selected α. It also provides a guidance on how to choose α properly; see the numerical simulation results reported in Section 6.

Finally, we present an improved convergence order result for $p_\varepsilon^\delta - p^*$ as $\varepsilon \to 0$ and $\delta \to 0$. For any $p \in Q_0$, denote $\tilde{u}(p) = \tilde{u}_1(p) + i\,\tilde{u}_2(p) = u^\delta(p) - u^\delta(0) \in \mathbf{V}$. Then $\tilde{u}(\cdot)$ is linear and we have

$$\begin{aligned}(\tilde{u}_2(p), z)_Q &= \left(p, \int_\Omega (\tilde{w}_2 + \Sigma^{-1}(\omega\cdot\nabla\tilde{w}_2))d\sigma(\omega)\right)_{Q_0}\\ &= (p, \tilde{w}_2 + \Sigma^{-1}(\omega\cdot\nabla\tilde{w}_2))_Q, \quad p \in Q_0,\ z \in Q,\end{aligned} \tag{4.14}$$

where $\tilde{w}_2 \in V$ is the imaginary part of the weak solution $\tilde{w} = \tilde{w}_1 + i\,\tilde{w}_2 \in \mathbf{V}$ of the adjoint BVP

$$-\,\omega\cdot\nabla\Sigma^{-1}(\omega\cdot\nabla\tilde{w}) + \Sigma\tilde{w} = z \qquad \text{in } X\times\Omega, \tag{4.15a}$$

$$\Sigma^{-1}(\omega\cdot\nabla\tilde{w}) + i\,\alpha\,\tilde{w} = 0 \qquad \text{on } \Gamma_+, \tag{4.15b}$$

$$\tilde{w} - \Sigma^{-1}(\omega\cdot\nabla\tilde{w}) = 0 \qquad \text{on } \Gamma_-. \tag{4.15c}$$

We assume the following source condition about p^*.

(A2) There is $z^* \in Q$ such that

$$\chi_0 \int_\Omega (\tilde{w}_2^* + \Sigma^{-1}(\omega \cdot \nabla \tilde{w}_2^*)) d\sigma(\omega) = p^*,$$

where $\tilde{w}_2^*$ is the imaginary part of the weak solution $\tilde{w}^* = \tilde{w}_1^* + i\,\tilde{w}_2^* \in \mathbf{V}$ of the problem (4.15) with z replaced by z^*.

Theorem 4.3. *Under Assumption* (A2), *for the solution p_ε^δ of Problem* 3.2*,*

$$\|p_\varepsilon^\delta - p^*\|_{Q_0} \le c\left(\sqrt{\varepsilon} + \frac{\alpha\,\delta}{\sqrt{\varepsilon}}\right). \tag{4.16}$$

In particular, if $\varepsilon = O(\delta^2)$ and $\alpha = O(\sqrt{\varepsilon})$, then

$$\|p_\varepsilon^\delta - p^*\|_{Q_0} \le c\,\delta. \tag{4.17}$$

Proof. From the definitions of p_ε^δ and p^*, we have

$$J_\varepsilon^\delta(p_\varepsilon^\delta) = \frac{1}{2}\|u_2^\delta(p_\varepsilon^\delta)\|_Q^2 + \frac{\varepsilon}{2}\|p_\varepsilon^\delta\|_{Q_0}^2 \le J_\varepsilon^\delta(p^*) = \frac{1}{2}\|u_2^\delta(p^*)\|_Q^2 + \frac{\varepsilon}{2}\|p^*\|_{Q_0}^2,$$

which gives

$$\|u_2^\delta(p_\varepsilon^\delta)\|_Q^2 + \varepsilon\,\|p_\varepsilon^\delta - p^*\|_{Q_0}^2 \le \|u_2^\delta(p^*)\|_Q^2 - 2\,\varepsilon\,(p^*, p_\varepsilon^\delta - p^*)_{Q_0}. \tag{4.18}$$

From (4.1), we obtain

$$\|u_2^\delta(p^*)\|_Q^2 = \|u_2^\delta(p^*) - u_2(p^*)\|_Q^2 \le c\,\alpha^2\delta^2, \tag{4.19}$$

where we used the fact that $u_2(p^*) = 0$ in $X \times \Omega$. In addition, from Assumption (A2) and by using (4.14), we have

$$\begin{aligned}(p^*, p_\varepsilon^\delta - p^*)_{Q_0} &= (\tilde{w}_2^* + \Sigma^{-1}(\omega \cdot \nabla \tilde{w}_2^*), p_\varepsilon^\delta - p^*)_Q \\ &= (z^*, \tilde{u}_2(p_\varepsilon^\delta - p^*))_Q = (z^*, u_2^\delta(p_\varepsilon^\delta) - u_2^\delta(p^*))_Q.\end{aligned} \tag{4.20}$$

Combine (4.18)–(4.20) to give

$$\|u_2^\delta(p_\varepsilon^\delta)\|_Q^2 + 2\,\varepsilon\,(z^*, u_2^\delta(p_\varepsilon^\delta))_Q + \varepsilon\,\|p_\varepsilon^\delta - p^*\|_{Q_0}^2 \le c\,\alpha^2\,\delta^2 + 2\,\varepsilon\,(z^*, u_2^\delta(p^*))_Q. \tag{4.21}$$

By adding $\varepsilon^2\|z^*\|_Q^2$ to both sides of (4.21), we obtain

$$\|u_2^\delta(p_\varepsilon^\delta) + \varepsilon\,z^*\|_Q^2 + \varepsilon\,\|p_\varepsilon^\delta - p^*\|_{Q_0}^2 \le c\,\alpha^2\delta^2 + 2\,\varepsilon\,(z^*, u_2^\delta(p^*))_Q + \varepsilon^2\|z^*\|_Q^2. \tag{4.22}$$

Using (4.1) again,

$$(z^*, u_2^\delta(p^*))_Q = (z^*, u_2^\delta(p^*) - u_2(p^*))_Q \le c\,\alpha\,\delta\,\|z^*\|_Q.$$

Therefore, (4.22) implies

$$\|u_2^\delta(p_\varepsilon^\delta) + \varepsilon\,z^*\|_Q^2 + \varepsilon\,\|p_\varepsilon^\delta - p^*\|_{Q_0}^2 \le c\,\alpha^2\delta^2 + 2\,\varepsilon^2\|z^*\|_Q^2,$$

which leads to (4.16). The estimate (4.17) follows directly from (4.16). $\square$

5. Discretization and Error Estimates

In this section, we discretize Problem 3.2 and study the convergence of the numerical solutions. Note that the following discussion applies to the model with noisy measurement u_m^δ; however, for the conciseness of statements, we omit the symbol δ in most part of this section.

We first discuss discrete-ordinate methods for the angular variable. Let f be a continuous function over the unit sphere Ω and consider a general numerical quadrature formula:

$$\int_\Omega f(\omega)d\sigma(\omega) \approx \sum_{l=1}^{L} w_l f(\omega_l), \tag{5.1}$$

where $\omega_l \in \Omega$ and $w_l, 1 \le l \le L$ are the nodes and weights. Some quadratures can be found in [24] and references therein. Let n_ω be the degree of precision of the quadrature (5.1), i.e., the quadrature integrates exactly all spherical polynomials [4, 12] of a total degree no more than n_ω while it does not integrate exactly some spherical polynomials of a total degree $n_\omega + 1$. Following [27, Corollary 6],

$$\left| \int_\Omega f(\omega)d\sigma(\omega) - \sum_{l=1}^{L} w_l f(\omega_l) \right| \le c_s n_\omega^{-s} \|f\|_{H^s(\Omega)} \quad \forall\, f \in H^s(\Omega),\ s > 1, \tag{5.2}$$

where $H^s(\Omega)$ is the Sobolev space of order s over Ω, see [12] for details about $H^s(\Omega)$; c_s is a constant depending only on s.

Applying (5.1) to the integral operator S leads to an approximation S_d of S:

$$S_d u(x,\omega) = \sum_{l=1}^{L} w_l \eta(x, \omega \cdot \omega_l) u(x, w_l).$$

We assume

$$\mu_t - \mu_s\, m \ge c_0' \quad \text{in } X, \quad m(x) := \max_{1\le k\le L} \sum_{l=1}^{L} w_l \eta(x, \omega_k \cdot \omega_l) \tag{5.3}$$

for some constant $c_0' > 0$. This condition is not restrictive given Assumption (A1). We refer to [24] for a detailed comment on the condition (5.3).

Let $Q_X := L^2(X)$ with the standard inner product and norm. We define the Hilbert space $Q^d := (Q_X)^L$ with the following inner product and norm:

$$(u,v)_{Q^d} := \sum_{l=1}^{L} w_l (u^l, v^l)_{Q_X}, \quad \|v\|_{Q^d} := (v,v)_{Q^d}^{1/2}$$

for any $u = (u^1, u^2, \cdots, u^L)^T$, $v = (v^1, v^2, \cdots, v^L)^T \in Q_X^d$. Let ∂X be the boundary of X,

$$\partial X_+^l := \Big\{x \in \partial X \mid \omega_l \cdot \nu(x) > 0\Big\}, \quad \partial X_-^l := \Big\{x \in \partial X \mid \omega_l \cdot \nu(x) < 0\Big\},$$

and set $Q_{\partial X} := L^2(\partial X)$, $Q_\pm^l := L^2(\partial X_\pm^l)$, all with the standard inner products and norms. Define $Q_{\partial X}^d := (Q_{\partial X})^L$ with the following inner product and norm:

$$(u,v)_{Q_{\partial X}^d} := \sum_{l=1}^{L} w_l \int_{\partial X} |\omega_l \cdot \nu| u^l v^l d\sigma(x), \quad \|v\|_{Q_{\partial X}^d} := (v,v)_{Q_{\partial X}^d}^{1/2},$$

and $Q^d_\pm := (Q^1_\pm, Q^2_\pm, \cdots, Q^L_\pm)^T$ with the following inner products and norms:

$$(u,v)_{Q^d_\pm} := \sum_{l=1}^{L} w_l \int_{\partial X^l_\pm} |\omega_l \cdot \nu| u^l v^l d\sigma(x), \quad \|v\|_{Q^d_\pm} := (v,v)^{1/2}_{Q^d_\pm}.$$

Define the linear operator $\Sigma_{d,k} : Q^d \to Q_X, 1 \le k \le L$, through

$$\Sigma_{d,k} u(x) = \mu_t(x) u^k(x) - \mu_s(x) \sum_{l=1}^{L} w_l \eta(x, \omega_k \cdot \omega_l) u^l(x)$$

and set $\Sigma_d := \Sigma_d(x) = (\Sigma_{d,1}, \Sigma_{d,2}, \cdots, \Sigma_{d,L})^T$. We can represent Σ_d in a matrix of functions:

$$\Sigma_d = \mu_t I_{L\times L} - \mu_s G,$$

with $I_{L\times L}$ being the $L \times L$ identical matrix and

$$G = \big(\eta_{kl}\big)_{L\times L}, \quad \eta_{kl} = w_l \eta(x, \omega_k \cdot \omega_l), \quad 1 \le k, l \le L.$$

Under the assumption (5.3), Σ_d is strictly diagonally dominant in X. Moreover, for any $r \in \mathbb{R}$, the power $\Sigma^r_d : Q^d \to Q^d$ is a bounded, linear, self-adjoint and Q^d-elliptic operator, and

$$\|\Sigma^{-1}_d\|_{Q^d\to Q^d} \le (\mu'_0)^{-1}, \quad \|\Sigma^{-1/2}_d\|_{Q^d\to Q^d} \le (\mu'_0)^{-1/2}.$$

To obtain the angular discretization of the forward BVP (2.4), we start from the angular discretization of the original forward BVP (1.1) in the component form: for $k = 1, 2, \cdots, L$,

$$\omega_k \cdot \nabla u^k_d + \Sigma_{d,k} u_d = p \qquad \text{in } X, \tag{5.4a}$$

$$u^k_d = 0 \qquad \text{on } \partial X^k_-, \tag{5.4b}$$

where $u_d = (u^1_d, u^2_d, \cdots, u^L_d)^T$ and $u^k_d = u^k_d(x)$ is an approximation of $u(x, \omega_k)$, $1 \le k \le L$. We can rewrite (5.4) in the vector form:

$$\omega \odot \nabla u_d + \Sigma_d u_d = f_p \qquad \text{in } X^d, \tag{5.5a}$$

$$u_d = 0 \qquad \text{on } \partial X^d_-, \tag{5.5b}$$

where $f_p := (p, p, \cdots, p)^T$, $X^d := (X, X, \cdots, X)^T$, $\partial X^d_- := (\partial X^1_-, \partial X^2_-, \cdots, \partial X^L_-)^T$, 0 stands for $(0, 0, \cdots, 0)^T$. Here and below, ω is used for an $L \times 3$ matrix with kth row ω^T_k, the gradient ∇u_d of the vector function u_d is an $L \times 3$ matrix with kth row $(\nabla u^k_d)^T$. The result of the dot product $\omega \odot \nabla u_d$ of two $L \times 3$ matrices is a vector $a = (a_1, a_2, \cdots, a_L)^T$ with $a_k = \omega_k \cdot \nabla u^k_d$.

From (5.5), we have

$$u_d = \Sigma^{-1}_d (f_p - \omega \odot \nabla u_d). \tag{5.6}$$

Substitute (5.6) back into the first equation of (5.5), and use the boundary condition to get the discretization in angular direction of the BVP (2.4):

$$-\omega \odot \nabla \Sigma^{-1}_d (\omega \odot \nabla u_d) + \Sigma_d u_d = f_p - \omega \odot \nabla \Sigma^{-1}_d (f_p) \qquad \text{in } X^d, \tag{5.7a}$$

$$u_d \pm \Sigma^{-1}_d (\omega \odot \nabla u_d) = \pm \Sigma^{-1}_d (f_p) \qquad \text{on } \partial X^d_\pm. \tag{5.7b}$$

Define the Hilbert space

$$V^d := \left\{ v \in Q^d \mid \omega \odot \nabla v \in Q^d, v|_{\partial X} \in Q^d_{\partial X} \right\}$$

with the inner product and norm

$$(u,v)_{V^d} := (\Sigma_d^{-1}(\omega \odot \nabla u), \omega \odot \nabla v)_{Q^d} + (\Sigma_d u, v)_{Q^d} + (u,v)_{Q^d_{\partial X}},$$
$$\|v\|_{V^d} := \left[\|\Sigma_d^{-1/2}(\omega \odot \nabla v)\|^2_{Q^d} + \|\Sigma_d^{1/2} v\|^2_{Q^d} + \|v\|^2_{Q^d_{\partial X}} \right]^{1/2}.$$

The weak form of the BVP (5.7) is

$$u_d \in V^d, \quad (u_d, v)_{V^d} = (f_p, v)_{Q^d} + (\Sigma_d^{-1}(f_p), \omega \odot \nabla v)_{Q^d} \quad \forall\, v \in V_d. \tag{5.8}$$

It is easy to verify that (5.8) admits a unique solution in V^d which depends continuously on f_p and thus on p.

Next we bound the difference between the solution $u^* \in V$ of (2.5) and the solution $u_d \in V^d$ of (5.8). Denote $\delta u^k = u_d^k - u^*(\cdot, \omega_k)$, $1 \le k \le L$, and set $\delta u = (\delta u^1, \delta u^2, \cdots, \delta u^L)^T$. Then $\delta u \in V^d$ satisfies

$$(\delta u, v)_{V^d} = (b, v)_{Q^d} + (\Sigma_d^{-1}(b), \omega \odot \nabla v)_{Q^d} \quad \forall\, v \in V^d, \tag{5.9}$$

where $b = (b_1, b_2, \cdots, b_L)^T$ and $b_k = \mu_s(S_d u^*(\cdot, \omega_k) - S u^*(\cdot, \omega_k))$, $1 \le k \le L$. Take $v = \delta u$ in (5.9), recall the definition of Q^d, and apply the Cauchy-Schwarz inequality,

$$\|\delta u\|_{V_d} \le c\,(\|b\|_{Q^d} + \|\Sigma_d^{-1}(b)\|_{Q^d}) \le \tilde{c}\, \|b\|_{Q^d}. \tag{5.10}$$

Assume $\eta(x, w_k \cdot), u^*(x, \cdot) \in H^s(\Omega)$ for any $x \in X$ and $k = 1, 2, \cdots, L$. Then from (5.2),

$$\begin{aligned} |b_k(x)| &= \mu_s(x)\, |S_d u^*(x, \omega_k) - S u^*(x, \omega_k)| \\ &\le c_s\, n_\omega^{-s} \mu_s(x)\, \|\eta(x, \omega_k \cdot) u^*(x, \cdot)\|_{H^s(\Omega)}. \end{aligned} \tag{5.11}$$

Substitute (5.11) into (5.10) to obtain

$$\|\delta u\|_{V^d} \le c(s, \eta, u^*)\, n_\omega^{-s}, \tag{5.12}$$

where

$$c(s, \eta, u^*) := \tilde{c}\, c_s \left(\sum_{k=1}^{L} w_k \int_X \mu_t^2(x)\, \|\eta(x, \omega_k \cdot) u^*(x, \cdot)\|^2_{H^s(\Omega)} dx \right)^{1/2}.$$

Let $\mathbf{Q}^d$ be the complex version of Q^d with the inner product $(u,v)_{\mathbf{Q}^d} := (u, \bar{v})_{Q^d}$ and the norm $\|v\|_{\mathbf{Q}^d} := (v, \bar{v})^{1/2}_{Q^d}$. Similarly, we can define the complex version of spaces $Q^d_{\partial X}$, Q^d_+, Q^d_- and V^d, denoted by $\mathbf{Q}^d_{\partial X}$, $\mathbf{Q}^d_+$, $\mathbf{Q}^d_-$ and $\mathbf{V}^d$, respectively. In particular, the inner product and the norm of $\mathbf{V}_d$ are

$$(u,v)_{\mathbf{V}^d} := (\Sigma^{-1}(\omega \odot \nabla u), \omega \odot \nabla v)_{\mathbf{Q}^d} + (\Sigma u, v)_{\mathbf{Q}^d} + (u,v)_{\mathbf{Q}^d_{\partial X}},$$
$$\|v\|_{\mathbf{V}^d} := \left[\|\Sigma^{-1/2}(\omega \odot \nabla v)\|^2_{\mathbf{Q}^d} + \|\Sigma^{1/2} v\|^2_{\mathbf{Q}^d} + \|v\|^2_{\mathbf{Q}^d_{\partial X}} \right]^{1/2}.$$

The discrete-ordinate method approximation of Problem 3.2 is the following.

Problem 5.1. *Find $p_\varepsilon^d \in Q_{ad}$ such that*

$$J_\varepsilon^d(p_\varepsilon^d) = \inf_{p\in Q_{ad}} J_\varepsilon^d(p),$$

where

$$J_\varepsilon^d(p) = \frac{1}{2}\|u_2^d(p)\|^2_{Q^d} + \frac{\varepsilon}{2}\|p\|^2_{Q_0}$$

and $u_2^d(p)$ is the imaginary part of the weak solution $u^d := u^d(p) \in \boldsymbol{V}^d$ of the problem

$$-\,\omega\odot\nabla\Sigma_d^{-1}(\omega\odot\nabla u^d) + \Sigma_d u^d = f_p - \omega\odot\nabla\Sigma_d^{-1}(f_p) \qquad \text{in } X^d, \tag{5.13a}$$

$$\Sigma_d^{-1}(\omega\odot\nabla u^d) + i\,\alpha\, u^d = \Sigma_d^{-1}(f_p) + (i\,\alpha - 1)\, u_m^d \qquad \text{on } \partial X_+^d, \tag{5.13b}$$

$$u^d - \Sigma_d^{-1}(\omega\odot\nabla u^d) = -\Sigma_d^{-1}(f_p) \qquad \text{on } \partial X_-^d, \tag{5.13c}$$

with $u_m^d = (u_m^\delta(\cdot,\omega_1),\cdots,u_m^\delta(\cdot,\omega_L))^T$ and $\partial X_+^d := (\partial X_+^1,\cdots,\partial X_+^L)^T$.

The weak form of (5.13) is

$$u^d \in \mathbf{V}^d, \quad a^d(u^d,\bar v) = F^d(\bar v) \quad \forall\, v\in\mathbf{V}^d, \tag{5.14}$$

where

$$a^d(u,v) = (\Sigma_d^{-1}(\omega\odot\nabla u), \omega\odot\nabla v)_{\mathbf{Q}^d} + (\Sigma_d u, v)_{\mathbf{Q}^d} + i\,\alpha(u,v)_{\mathbf{Q}_+^d} + (u,v)_{\mathbf{Q}_-^d},$$
$$F^d(v) = (f_p, v)_{\mathbf{Q}^d} + (\Sigma_d^{-1}(f_p), \omega\odot\nabla v)_{\mathbf{Q}^d} + (i\,\alpha - 1)\,(u_m^d, v)_{\mathbf{Q}_+^d}.$$

We can verify that $|\,a^d(\cdot,\cdot)|$ is coercive over $\mathbf{V}^d$, i.e., there is $\alpha_0 > 0$ such that

$$|\,a^d(v,v)| \ge \alpha_0\|v\|^2_{\mathbf{V}^d}.$$

Then, for a given $p\in Q_0$, the problem (5.14) has a unique solution $u^d := u^d(p) \in \mathbf{V}^d$.

Similar to the results on Problem 3.2, we have the following result on Problem 5.1.

Proposition 5.1. *For any $\varepsilon > 0$, Problem 5.1 has a unique solution $p_\varepsilon^d \in Q_{ad}$ which depends continuously on all data. Moreover, p_ε^d is characterized by*

$$(u_{\varepsilon,2}^d(p_\varepsilon^d), u_{\varepsilon,2}^d(q) - u_{\varepsilon,2}^d(p_\varepsilon^d))_{Q^d} + \varepsilon\,(p_\varepsilon^d, q - p_\varepsilon^d)_{Q_0} \ge 0 \quad \forall\, q\in Q_{ad},$$

or equivalently,

$$p_\varepsilon^d = \Pi_{ad}\left[-\frac{1}{\varepsilon}\chi_0\sum_{k=1}^{L} w_k\Big(w_{\varepsilon,2}^{d,k} + (\Sigma_d^{-1}(\omega\odot\nabla w_{\varepsilon,2}^d))_k\Big)\right],$$

where $w_{\varepsilon,2}^d = (w_{\varepsilon,2}^{d,1}, w_{\varepsilon,2}^{d,2},\cdots,w_{\varepsilon,2}^{d,L})^T$ is the imaginary part of the weak solution $w_\varepsilon^d := w^d(p_\varepsilon^d) \in \boldsymbol{V}^d$ of the adjoint problem:

$$-\,\omega\odot\nabla\Sigma_d^{-1}(\omega\odot\nabla w_\varepsilon^d) + \Sigma_d w_\varepsilon^d = u_{\varepsilon,2}^d \qquad \text{in } X,$$

$$\Sigma_d^{-1}(\omega\odot\nabla w_\varepsilon^d) + i\,\alpha\, w_\varepsilon^d = 0 \qquad \text{on } \partial X_+,$$

$$w_\varepsilon^d - \Sigma_d^{-1}(\omega\odot\nabla w_\varepsilon^d) = 0 \qquad \text{on } \partial X_-$$

and $u^d_{\varepsilon,2} = (u^{d,1}_{\varepsilon,2}, u^{d,2}_{\varepsilon,2}, \cdots, u^{d,L}_{\varepsilon,2})^T$ *is the imaginary part of the solution* $u^d_\varepsilon := u^d(p^d_\varepsilon) \in \boldsymbol{V}^d$ *of the BVP* (5.14) *with* p *replaced by* p^d_ε*.*

Let $\alpha = O(\sqrt{\varepsilon})$. Then for any fixed $\delta \geq 0$,

$$-\frac{1}{\varepsilon}\chi_0 \sum_{k=1}^{L} w_k \Big(w^{d,k}_{\varepsilon,2} + (\Sigma_d^{-1}(\omega \odot \nabla w^d_{\varepsilon,2}))_k\Big)$$

is uniformly bounded in Q_0 with respect to ε for small $\varepsilon > 0$.

Assume S^d_0, the solution set of Problem 5.1 for $\varepsilon = 0$, is nonempty. Then S^d_0 is closed and convex, and $p^d_\varepsilon \to p^d_0 := \arg\min_{p \in S^d_0} \|p\|_{Q_0}$ in Q_0 as $\varepsilon \to 0$.

When n_ω goes to ∞, we have $S_d \to S$. Therefore, the following convergence holds:

Proposition 5.2. *Fix* $\delta \geq 0$ *and* $\varepsilon > 0$*. Let* p^δ_ε *and* p^d_ε *be the solutions of Problems* 3.2 *and* 5.1*. Then* $J^d_\varepsilon(p^d_\varepsilon) \to J^\delta_\varepsilon(p^\delta_\varepsilon)$ *and* $p^d_\varepsilon \to p^\delta_\varepsilon$ *in* Q_0 *as* $n_\omega \to +\infty$*.*

We now turn to a finite element discretization for the spatial variable. For simplicity of presentation, we assume X is a polyhedron. Let $\{\mathcal{T}^h\}_h$ be a regular family of finite element partitions of $\overline{X}$, h being the meshsize. Define the linear finite element space:

$$V^h_X = \Big\{v \in C(\overline{X}) \mid v \text{ is linear in } K \ \forall K \in \mathcal{T}^h\Big\}.$$

Set $V^h := (V^h_X)^L$, $\mathbf{V}^h := V^h \oplus iV^h$. Then $\mathbf{V}^h \subset \mathbf{V}^d$, and the finite element approximation of (5.14) is

$$u^h \in \mathbf{V}^h, \quad a^d(u^h, \bar{v^h}) = F^d(\bar{v^h}) \quad \forall v^h \in \mathbf{V}^h. \tag{5.15}$$

The discrete problem (5.15) admits a unique solution $u^h := u^h(p) \in \mathbf{V}^h$ and

$$\|u^h(p_1) - u^h(p_2)\|_{\mathbf{V}^d} \leq c\,\|p_1 - p_2\|_{Q_0}. \tag{5.16}$$

Proposition 5.3. *For any* $p \in Q_0$*, denote by* $u^d \in \boldsymbol{V}^d$ *and* $u^h \in \boldsymbol{V}^h$ *the solutions of* (5.14) *and* (5.15)*. Then*

$$\|u^d - u^h\|_{\boldsymbol{V}^d} \to 0 \ \text{as}\ h \to 0.$$

This result is proved by the standard finite element approximation theory based on the following Cea's inequality:

$$\|u^d - u^h\|_{\mathbf{V}^d} \leq c \inf_{v^h \in \mathbf{V}^h} \|u^d - v^h\|_{\mathbf{V}^d}.$$

Moreover, under the regularity assumption

$$u^d \in (\mathbf{H}^r(X))^L, \quad r > 1, \tag{5.17}$$

we have the error bounds

$$\|u^d - u^h\|_{\mathbf{V}^d} \leq c\,h^{r-1}\|u^d\|_{(\mathbf{H}^r(X))^L}, \tag{5.18}$$

Finally, we study the full discretization of the inverse problem. For any $p \in Q_0$, denote by $u^h(p) = u^h_1(p) + i\,u^h_2(p) \in \mathbf{V}^h$ the solution of (5.15). Define the discrete objective functional

$$J^h_\varepsilon(p) = \frac{1}{2}\|u^h_2(p)\|^2_{Q^d} + \frac{\varepsilon}{2}\|p\|^2_{Q_0}.$$

It is easy to verify that for $\varepsilon > 0$, $J_\varepsilon^h(\cdot)$ is strictly convex.

For a full discretization of Problem 3.2, we approximate the source function p with piecewise constants. Define

$$Q_0^h = \Big\{ p \in Q_0 \mid p \text{ is constant in } K,\ \forall K \in \mathcal{T}_h \text{ and } K \subset \overline{X}_0 \Big\}.$$

Set $Q_{ad}^h = Q_0^h \cap Q_{ad}$ and introduce the following discrete optimization problem:

Problem 5.2. *Find $p_\varepsilon^h \in Q_{ad}^h$ such that*

$$J_\varepsilon^h(p_\varepsilon^h) = \inf_{p^h \in Q_{ad}^h} J_\varepsilon^h(p^h).$$

Similar to Proposition 5.1, we have the following result for Problem 5.2.

Proposition 5.4. *For any $\varepsilon > 0$, Problem 5.2 has a unique solution $p_\varepsilon^h \in Q_{ad}^h$ which depends continuously on all data. Moreover, p_ε^h is characterized by the inequality*

$$\Big(u_{\varepsilon,2}^h(p_\varepsilon^h), u_{\varepsilon,2}^h(q^h) - u_{\varepsilon,2}^h(p_\varepsilon^h) \Big)_{Q^d} + \varepsilon\, (p_\varepsilon^h, q^h - p_\varepsilon^h)_{Q_0} \geq 0 \quad \forall\, q^h \in Q_{ad}^h, \tag{5.19}$$

or equivalently,

$$p_\varepsilon^h = \Pi_{ad}^h \left[-\frac{1}{\varepsilon} \chi_0 \sum_{k=1}^{L} w_k \Big(w_{\varepsilon,2}^{h,k} + (\Sigma_d^{-1}(\omega \odot \nabla w_{\varepsilon,2}^h))_k \Big) \right],$$

where Π_{ad}^h is the orthogonal projection from Q_0 onto Q_{ad}^h, $w_{\varepsilon,2}^h = (w_{\varepsilon,2}^{h,1}, w_{\varepsilon,2}^{h,2}, \cdots, w_{\varepsilon,2}^{h,L})^T$ is the imaginary part of the weak solution $w_\varepsilon^h := w^h(p_\varepsilon^h) \in \boldsymbol{V}^h$ of the adjoint problem:

$$\begin{aligned}
& -\omega \odot \nabla \Sigma_d^{-1}(\omega \odot \nabla w_\varepsilon^h) + \Sigma_d w_\varepsilon^h = u_{\varepsilon,2}^h && \text{in } X,\\
& \Sigma_d^{-1}(\omega \odot \nabla w_\varepsilon^h) + i\, \alpha\, w_\varepsilon^h = 0 && \text{on } \partial X_+,\\
& w_\varepsilon^h - \Sigma_d^{-1}(\omega \odot \nabla w_\varepsilon^h) = 0 && \text{on } \partial X_-,
\end{aligned}$$

and $u_{\varepsilon,2}^h = (u_{\varepsilon,2}^{h,1}, u_{\varepsilon,2}^{h,2}, \cdots, u_{\varepsilon,2}^{h,L})^T$ is the imaginary part of the solution $u_\varepsilon^h := u^h(p_\varepsilon^h) \in \boldsymbol{V}^h$ of the BVP (5.15) with p replaced by p_ε^h.

Let $\alpha = O(\sqrt{\varepsilon})$. Then

$$-\frac{1}{\varepsilon} \chi_0 \sum_{k=1}^{L} w_k \Big(w_{\varepsilon,2}^{h,k} + (\Sigma_d^{-1}(\omega \odot \nabla w_{\varepsilon,2}^h))_k \Big)$$

is uniformly bounded in Q_0 with respect to ε for small $\varepsilon > 0$.

Assume S_0^h, the solution set of Problem 5.2 for $\varepsilon = 0$, is nonempty. Then S_0^h is closed and convex and we have $p_\varepsilon^h \to p_0^h := \arg\min_{p \in S_0^h} \|p\|_{Q_0}$ in Q_0 as $\varepsilon \to 0$.

Define the orthogonal projection operator $\Pi_0^h : Q_0 \to Q_0^h$ by

$$(\Pi_0^h\, p, q^h)_{Q_0} = (p, q^h)_{Q_0} \quad \forall\, p \in Q,\ q^h \in Q_0^h. \tag{5.20}$$

Then, for any $q \in Q_{ad}$, $\Pi_0^h\, q \in Q_{ad}^h$ and

$$\|q - \Pi_0^h\, q\|_{Q_0} \to 0 \text{ as } h \to 0.$$

With an argument similar to the one used in [22, Theorem 4.5], the following convergence result can be shown.

Proposition 5.5. *For any $\varepsilon > 0$, $p_\varepsilon^h \to p_\varepsilon^d$ in Q_0 as $h \to 0$.*

Next we give an error estimate for the light source function p_ε^d with respect to h as follows.

Proposition 5.6. *Let $\varepsilon > 0$ be fixed. Assume (5.17) holds for $u_\varepsilon^d(p_\varepsilon^d)$ and $u_\varepsilon^d(p_\varepsilon^h)$, the solutions of (5.14) with p replaced by p_ε^d and p_ε^h, respectively. Then*

$$\begin{aligned}\|p_\varepsilon^h - p_\varepsilon^d\|_{Q_0} \le c(\varepsilon, \delta, n)\, \alpha^{1/2} \Big(& h^{\frac{r-1}{2}} \|u_{\varepsilon,2}^d(p_\varepsilon^h)\|_{(H^r(X))^L} \\ & + h^{\frac{r-1}{2}} \|u_{\varepsilon,2}^d(p_\varepsilon^d)\|_{(H^r(X))^L} + E^h(p_\varepsilon)^{1/2} \Big),\end{aligned} \tag{5.21}$$

where

$$E^h(p_\varepsilon^d) = \|\Pi_0^h p_\varepsilon^d - p_\varepsilon^d\|_{Q_0} = \inf_{q^h \in Q_{ad}^h} \|q^h - p_\varepsilon^d\|_{Q_0}.$$

Proof. Replace q in (5.1) with p_ε^h and q^h in (5.19) with $\Pi_0^h p_\varepsilon^d$, and add the resulting inequalities, and use (5.20) to get

$$\varepsilon \|p_\varepsilon^h - p_\varepsilon^d\|_{Q_0}^2 + \|u_2^h(p_\varepsilon^h) - u_2^d(p_\varepsilon^d)\|_{Q^d}^2 \le I_1 + I_2 + I_3, \tag{5.22a}$$

where

$$I_1 =: (u_2^h(p_\varepsilon^h), u_2^h(\Pi_0^h p_\varepsilon^d) - u_2^h(p_\varepsilon^d))_{Q^d}, \tag{5.22b}$$

$$I_2 =: (u_2^h(p_\varepsilon^h), u_2^h(p_\varepsilon^d) - u_2^d(p_\varepsilon^d))_{Q^d}, \tag{5.22c}$$

$$I_3 =: (u_2^d(p_\varepsilon^d), u_2^d(p_\varepsilon^h) - u_2^h(p_\varepsilon^h))_{Q^d}. \tag{5.22d}$$

Similar to (4.12), there are constants c_1 and c_2, independent of h, such that

$$\|u_2^h(p_\varepsilon^h)\|_{V^d} \le c_1\, \alpha, \quad \|u_2^d(p_\varepsilon^d)\|_{V^d} \le c_2\, \alpha. \tag{5.23}$$

Then by applying the Cauchy-Schwarz inequality, (5.23) and (5.16), we have

$$|I_1| \le c\, \alpha\, E^h(p_\varepsilon^d). \tag{5.24}$$

Combine Cauchy-Schwarz inequality, (5.23) and (5.18) to give

$$|I_2| \le c\, \alpha\, h^{r-1} \|u_{\varepsilon,2}^d(p_\varepsilon^d)\|_{H^r(X)}, \quad |I_3| \le c\, \alpha\, h^{r-1} \|u_{\varepsilon,2}^d(p_\varepsilon^h)\|_{H^r(X)}. \tag{5.25}$$

Thus, from (5.22), (5.24) and (5.25), we obtain (5.21). □

6. Numerical Results

The numerical method used in this section approximates the solution to the minimization problem described in Problem 5.5. The optimization method is the limited-memory BFGS algorithm. At each step, the finite element system (5.15) is solved, and then the objective function J_ε^h is evaluated.

We present some numerical results illustrating the application of the CCBM method to the inverse RTE problem. The goal is to show the performance of the proposed method and to illustrate some theoretical results presented in the previous sections.

Consider the RTE problem in two dimensions where the spatial domain $X = [0, 1]^2$. The absorption and scattering parameters are chosen so that $\mu_t = 1.1\,\mathrm{mm}^{-1}$, $\mu_s = 1\,\mathrm{mm}^{-1}$, and the

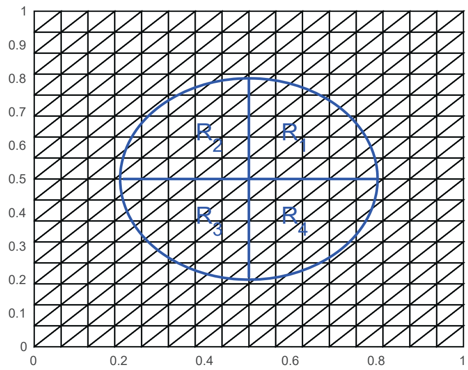

Fig. 6.1. The reconstruction mesh and regions R_1, R_2, R_3, R_4.

scattering phase function is the 2D Henyey-Greenstein phase function with anisotropy factor $g = 0.9$, i.e.

$$\eta(t) = \frac{1-g^2}{2\pi(1+g^2-2gt)}.$$

We take the "true" source term to be

$$p_T = \begin{cases} 1.1 \text{ if } x_1 > 0.5,\ x_2 > 0.5,\ \sqrt{(x_1-0.5)^2+(x_2-0.5)^2} < 0.3, \\ 1.2 \text{ if } x_1 < 0.5,\ x_2 > 0.5,\ \sqrt{(x_1-0.5)^2+(x_2-0.5)^2} < 0.3, \\ 1.3 \text{ if } x_1 < 0.5,\ x_2 < 0.5,\ \sqrt{(x_1-0.5)^2+(x_2-0.5)^2} < 0.3, \\ 1.4 \text{ if } x_1 > 0.5,\ x_2 < 0.5,\ \sqrt{(x_1-0.5)^2+(x_2-0.5)^2} < 0.3. \end{cases}$$

That is, the source term is piecewise constant defined on the regions R_1, R_2, R_3, R_4 shown in Fig. 6.1.

In order to generate "measurement" data, we solve the forward RTE using a discrete-ordinate discontinuous Galerkin method detailed in [24] on a mesh with 40401 vertices and 64 angular nodes. To study the effect of noise in the reconstruction, we artificially add noise to the measurement obtained from the solution of the RTE. Let u_m^0 denote the measurement with no noise added. In the following we refer to "the measurement with z% noise" and write u_m^z. u_m^z is defined on the same mesh as u_m^0, and its value at each node x_j of the mesh is sampled from a Gaussian distribution with mean $u_m^0(x_j)$ and standard deviation $z/100\,\left|u_m^0(x_j)\right|$.

We attempt to reconstruct p_T using the CCBM method and measurements u_m^0, u_m^1, u_m^5, u_m^{10}, and u_m^{20}. In each case, we solve Problem 5.2 on a regular mesh with 289 nodes and 32 angular directions; the reconstruction mesh is shown in Fig. 6.1. The admissible set $Q_{ad} = \{f \mid f|_{R_i} \text{ is constant}, i = 1,2,3,4\}$.

In Tables 6.1 – 6.5 we compare varying values of ε and α in the CCBM reconstruction method across varying noise levels. In each case, we choose $\alpha = C\sqrt{\varepsilon}$ for several choices of C. Denote the reconstructed source as p_R, and let $X_* = R_1 \cup R_2 \cup R_3 \cup R_4$. Since the integral of the source function represents the power, a quantity of interest in biomedical applications, we report the relative L^1 error, $\int_{X_*} |p_T - p_R| / \int_{X_*} |p_T|$. Numerical results for the L^2 norm error are similar. The numerical results demonstrate that the CCBM method performs well in the face of relatively large noise. Further, for any fixed value of C, the reconstruction is fairly stable as a function of ε, as predicted. Thus, even for small values of ε and large noise a reasonable reconstruction can be computed. Moreover, we see that there is a fairly wide range of values of C that work well for reconstruction. We note that an optimization method was used to solve Problem 5.2, and that the starting point for the algorithm represented a function with 30% relative error to p_T. Entries in the tables with relative error near 30% correspond to problems on which the optimization algorithm was not able to take many steps from the starting position.

Table 6.1: The relative L^1 error for different values of $\alpha = C\sqrt{\varepsilon}$. Noise level 0%.

ε	$C = 10^1$	$C = 10^2$	$C = 10^3$	$C = 10^4$
1.000e-14	3.201e-03	1.885e-03	7.346e-06	3.838e-06
1.000e-13	1.884e-03	2.918e-04	7.346e-06	3.840e-06
1.000e-12	1.886e-03	7.074e-06	3.844e-06	3.856e-06
1.000e-11	2.916e-04	7.074e-06	3.845e-06	8.028e-09
1.000e-10	1.157e-04	4.410e-06	3.862e-06	7.416e-09
1.000e-09	1.157e-04	4.412e-06	1.989e-08	7.595e-09
1.000e-08	1.195e-04	1.232e-06	1.964e-08	3.000e-01
1.000e-07	1.194e-04	1.253e-06	2.353e-08	3.000e-01
1.000e-06	1.194e-04	1.242e-06	3.000e-01	3.000e-01
1.000e-05	1.198e-04	1.617e-06	3.000e-01	3.000e-01
1.000e-04	1.236e-04	3.000e-01	3.000e-01	3.000e-01

Table 6.2: The relative L^1 error for different values of $\alpha = C\sqrt{\varepsilon}$. Noise level 1%.

ε	$C = 10^1$	$C = 10^2$	$C = 10^3$	$C = 10^4$
1.000e-14	3.196e-03	2.606e-03	1.660e-03	1.646e-03
1.000e-13	2.514e-03	1.803e-03	1.660e-03	1.646e-03
1.000e-12	2.549e-03	1.659e-03	1.646e-03	1.646e-03
1.000e-11	1.746e-03	1.659e-03	1.646e-03	1.649e-03
1.000e-10	1.603e-03	1.646e-03	1.646e-03	1.633e-03
1.000e-09	1.603e-03	1.646e-03	1.649e-03	1.526e-03
1.000e-08	1.589e-03	1.650e-03	1.633e-03	3.000e-01
1.000e-07	1.594e-03	1.649e-03	1.526e-03	3.000e-01
1.000e-06	1.593e-03	1.632e-03	3.000e-01	3.000e-01
1.000e-05	1.592e-03	1.525e-03	3.000e-01	3.000e-01
1.000e-04	1.574e-03	3.000e-01	3.000e-01	3.000e-01

7. Conclusions

In this work, a parameter-dependent CCBM together with Tikhonov regularization is presented for solving the BLT problem governed by the RTE on a general domain. With the CCBM, the data needed to fit on the boundary is transferred to the inner of the domain. This makes the problem more robust. More importantly, as shown by theory and numerical results, with the introduction of the parameter α, the approximate source functions are uniform with respect to the regularization parameter. This is advantageous because otherwise one will have to pay careful attention on the choice of the regularization parameter for trade off between the

Table 6.3: The relative L^1 error for different values of $\alpha = C\sqrt{\varepsilon}$. Noise level 5%.

ε	$C = 10^1$	$C = 10^2$	$C = 10^3$	$C = 10^4$
1.000e-14	1.131e-02	9.771e-03	8.484e-03	8.479e-03
1.000e-13	9.846e-03	9.771e-03	8.484e-03	8.479e-03
1.000e-12	9.819e-03	8.484e-03	8.479e-03	8.478e-03
1.000e-11	9.819e-03	8.484e-03	8.479e-03	8.465e-03
1.000e-10	8.532e-03	8.480e-03	8.478e-03	8.343e-03
1.000e-09	8.532e-03	8.480e-03	8.465e-03	8.349e-03
1.000e-08	8.528e-03	8.478e-03	8.343e-03	3.000e-01
1.000e-07	8.527e-03	8.466e-03	7.348e-03	3.000e-01
1.000e-06	8.526e-03	8.343e-03	3.000e-01	3.000e-01
1.000e-05	8.514e-03	8.465e-03	3.000e-01	3.000e-01
1.000e-04	8.393e-03	3.000e-01	3.000e-01	3.000e-01

Table 6.4: The relative L^1 error for different values of $\alpha = C\sqrt{\varepsilon}$. Noise level 10%.

ε	$C = 10^1$	$C = 10^2$	$C = 10^3$	$C = 10^4$
1.000e-14	7.830e-03	9.061e-03	1.118e-02	1.119e-02
1.000e-13	9.206e-03	1.119e-02	1.118e-02	1.119e-02
1.000e-12	9.116e-03	1.119e-02	1.119e-02	1.118e-02
1.000e-11	1.124e-02	1.119e-02	1.119e-02	1.114e-02
1.000e-10	1.124e-02	1.119e-02	1.118e-02	1.072e-02
1.000e-09	1.124e-02	1.119e-02	1.114e-02	7.994e-03
1.000e-08	1.124e-02	1.118e-02	1.072e-02	3.000e-01
1.000e-07	1.124e-02	1.114e-02	7.994e-03	3.000e-01
1.000e-06	1.123e-02	1.072e-02	3.000e-01	3.000e-01
1.000e-05	1.119e-02	7.995e-03	3.000e-01	3.000e-01
1.000e-04	1.077e-02	3.000e-01	3.000e-01	3.000e-01

Table 6.5: The relative L^1 error for different values of $\alpha = C\sqrt{\varepsilon}$. Noise level 20%.

ε	$C = 10^1$	$C = 10^2$	$C = 10^3$	$C = 10^4$
1.000e-14	3.294e-02	4.784e-02	4.742e-02	4.740e-02
1.000e-13	3.873e-02	4.742e-02	4.740e-02	4.740e-02
1.000e-12	4.783e-02	4.742e-02	4.740e-02	4.739e-02
1.000e-11	4.740e-02	4.740e-02	4.740e-02	4.728e-02
1.000e-10	4.739e-02	4.740e-02	4.739e-02	4.621e-02
1.000e-09	4.739e-02	4.740e-02	4.728e-02	3.971e-02
1.000e-08	4.739e-02	4.739e-02	4.621e-02	3.000e-01
1.000e-07	4.739e-02	4.728e-02	3.971e-02	3.000e-01
1.000e-06	4.737e-02	4.621e-02	3.000e-01	3.000e-01
1.000e-05	4.726e-02	3.971e-02	3.000e-01	3.000e-01
1.000e-04	4.620e-02	3.000e-01	3.000e-01	3.000e-01

solution accuracy and stability. Also, with the help of the small parameter α, we improve the existing work on the convergence order of the regularized solutions with respect to the noise level.

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