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A new general mathematical framework for bioluminescence tomography

Xiaoliang Cheng [a], Rongfang Gong [a,*], Weimin Han [b]

[a] Department of Mathematics, Zhejiang University, Hangzhou 310027, PR China
[b] Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA

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Abstract

Bioluminescence tomography (BLT) is a recently developed area in biomedical imaging. The goal of BLT is to quantitatively reconstruct a bioluminescent source distribution within a small animal from optical signals on the surface of the animal body. While there have been theoretical investigations of the BLT problem in the literature, in this paper, we propose a more general mathematical framework for a study of the BLT problem. For the proposed formulation, we establish a well-posedness result and explore its relation to the formulation studied previously in other papers. We introduce numerical methods for solving the BLT problem, show convergence, and derive error estimates for the discrete solutions. Numerical simulation results are presented showing improvement of solution accuracy with the new general mathematical framework over that with the standard formulation of BLT.

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Keywords: Bioluminescence tomography; Inverse problem; Well-posedness; Numerical solution; Error estimate

1. Introduction

Recently, molecular imaging has been developed rapidly in the study of physiological and pathological processes *in vivo* at the cellular and molecular levels, see, e.g. [3,15,17,18] and references therein. As a recently developed optical imaging technique of molecular imaging modalities, bioluminescence tomography (BLT) provides quantitative and localized analysis on a bioluminescent source distribution in a living object [1,5,9–11]. Without going into detail, we notice that BLT problems reduce to determination of a light source function p in the differential equation

$$-\mathrm{div}(D\nabla u) + \mu u = p\chi_{\Omega_0} \quad \text{in } \Omega \tag{1}$$

with following possible boundary conditions:

$$u + 2D\partial_\nu u = g^- \quad \text{on } \Gamma, \tag{2}$$

$$D\partial_\nu u = -g \equiv g_2 \quad \text{on } \Gamma, \tag{3}$$

$$u = g^- + 2g \equiv g_1 \quad \text{on } \Gamma. \tag{4}$$

Here $D = [3(\mu + \mu')]^{-1}$, μ and μ' are given absorption and reduced scattering coefficients, both the function g and the influx g^- are measurement data, and ∂_ν denotes the outward normal differentiation operator. Moreover, Ω_0 is a measurable subset of Ω or Ω itself, χ_{Ω_0} is the characteristic function of Ω_0, i.e., its value is 1 in Ω_0 and is 0 outside Ω_0.

Note that only two of the three boundary conditions (2)–(4) are independent. As was pointed out in [9], to determine the source function p, we may associate one of the above three boundary conditions (2)–(4) with the differential equation (1) to form a boundary value problem while choosing one of the remaining boundary conditions to form the inverse problem for p. In particular, in [9], discussion of the inverse problem was made for the boundary

* Corresponding author. Tel.: +86 057185922179.
E-mail addresses: xiaoliangcheng@zju.edu.cn (X. Cheng), gongrongfang319@yahoo.com.cn (R. Gong), whan@math.uiowa.edu (W. Han).

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value problem (1) and (2) with the measurement matching for (4). As noted in [9], this pointwise formulation is ill-posed. In general, there are infinitely many solutions. When the form of the source function is pre-specified, there is no solution if data are inconsistent. In this regard, some uniqueness results on the pointwise formulation of the inverse problem are presented in [19]. Also, the source function does not depend continuously on the data. To circumvent these difficulties, a reformulation of the problem can be introduced through weak formulations of boundary value problems and Tikhonov regularization. For this purpose, we first introduce a few symbols for function spaces and sets. For a set G for Ω, Ω_0 or Γ, we denote by $H^s(G)$ the standard Sobolev space with corresponding inner product $(\cdot,\cdot)_{s,G}$ and norm $\|\cdot\|_{s,G}$, and $H^0(G)$ refers to $L^2(G)$. Let $V=H^1(\Omega)$ and $Q=L^2(\Omega_0)$. We also introduce the space $V_0=H^1_0(\Omega)$, and for a given $g_1\in V$, we denote g_1+V_0 for the set $\{g_1+v|v\in V_0\}$.

For any $q\in Q$, we denote by $u_1=u_1(q)\in V$ the solution of the problem

$$\int_\Omega (D\nabla u_1\cdot\nabla v+\mu u_1 v)\mathrm{d}x=\int_{\Omega_0} qv\,\mathrm{d}x+\int_\Gamma g_2 v\,\mathrm{d}s\quad\forall v\in V. \tag{5}$$

Note that this is a weak formulation of the boundary value problem defined by (1) and (3). Suppose the admissible source function p belongs to a closed convex subset denoted Q_{ad} of the space Q. Then introduce the functional

$$J^{(1)}_\varepsilon(q)=\frac{1}{2}\|u_1(q)-g_1\|^2_{0,\Gamma}+\frac{\varepsilon}{2}\|q\|^2_{0,\Omega_0},\quad \varepsilon\geqslant 0,$$

and the following reformulation problem.

Problem 1.1. Find $p^{(1)}_\varepsilon\in Q_{\text{ad}}$ such that

$$J^{(1)}_\varepsilon\left(p^{(1)}_\varepsilon\right)=\inf_{q\in Q_{\text{ad}}} J^{(1)}_\varepsilon(q).$$

In [9], well-posedness of this reformulation is studied. It is shown that the reformulated problem with $\varepsilon>0$ leads to stable and convergent numerical schemes.

Similar discussion of the inverse problem can be made for other choices of boundary value condition and measurement data. As an example, we may switch the roles played by the boundary conditions (3) and (4), i.e., we define the boundary value problem by (1) and (4), and treat (3) as a matching condition. The pointwise formulation of this problem is again ill-posed. So we turn to a regularized formulation. For any $q\in Q$, we denote by $u_2=u_2(q)\in g_1+V_0$ the solution of the problem

$$\int_\Omega (D\nabla u_2\cdot\nabla v+\mu u_2 v)\mathrm{d}x=\int_{\Omega_0} qv\,\mathrm{d}x\quad\forall v\in V_0. \tag{6}$$

This is a weak formulation of the boundary value problem defined by (1) and (4). Introduce the functional

$$J^{(2)}_\varepsilon(q)=\frac{1}{2}\|D\partial_\nu u_2(q)-g_2\|^2_{0,\Gamma}+\frac{\varepsilon}{2}\|q\|^2_{0,\Omega_0},\quad \varepsilon\geqslant 0,$$

and the problem:

Problem 1.2. Find $p^{(2)}_\varepsilon\in Q_{\text{ad}}$ such that

$$J^{(2)}_\varepsilon\left(p^{(2)}_\varepsilon\right)=\inf_{q\in Q_{\text{ad}}} J^{(2)}_\varepsilon(q).$$

Results similar to those for Problem 1.1 are valid for Problem 1.2.

In this paper, we propose a more general mathematical framework for the reconstruction of the source function in BLT. This framework covers both Problems 1.1 and 1.2 as special cases, and it leads to more accurate numerical solutions. In Section 2, we introduce the new general mathematical framework for the BLT problem, and discuss solution existence, uniqueness, and continuous dependence on the data. In Section 3, we explore the limiting behaviors of the solution of the regularized solution as any of the parameters tends to 0. In particular, we show how solutions of Problems 1.1 and 1.2 can be recovered in the limit. Finite elements approximations, including semi-discrete and full-discrete approximations, are introduced and studied in Section 4. In this part, we also obtain convergence and error estimates of numerical solutions. In Section 5, we report some numerical results that show how solution accuracy is improved with proper choices of parameters in the general framework. Some conclusion remarks are stated in the last section.

We comment that the new general mathematical framework in this paper is presented for the BLT problem. It is straightforward to extend this general framework to multi-spectral BLT problems studied in [11].

2. A general framework for BLT

In the rest of the paper, we use c to denote a positive constant taking possibly different values at different places.

We first introduce assumptions on the data for simplicity in the theoretical discussions below. The smoothness assumptions on the data can be substantially weakened for numerical simulations. We use d to denote the space dimension. For applications, $d\leqslant 3$. However, our discussions are valid for any space dimension. Let $\Omega\subset\mathbb{R}^d$ be an open and bounded set with a boundary Γ. We assume either $\Gamma\in C^2$ or Ω is convex. It is shown in [8, Section 1.2] that the boundary of an open, bounded and convex set is Lipschitz continuous. We also assume $D\in C^{0,1}(\Omega)$, $D\geqslant D_0$ a.e. in Ω for some constant $D_0>0$, $\mu\in L^\infty(\Omega)$, $\mu\geqslant 0$ a.e. in Ω and $\mu>0$ a.e. in a subset of Ω with positive measure. Moreover, we assume $g_1\in H^{3/2}(\Gamma)$, $g_2\in L^2(\Gamma)$. Note that the function g_1 is the trace of an $H^2(\Omega)$ function, that will also be denoted as g_1. In other words, we use the same symbol g_1 for both an $H^2(\Omega)$ function and its trace in $H^{3/2}(\Omega)$ such that for some constant $c>0$,

$$\|g_1\|_{2,\Omega}\leqslant c\|g_1\|_{3/2,\Gamma}.$$

See [8] for more details.

Under the above assumptions, we can apply the Lax–Milgram lemma [2,4,6] to show that the solutions $u_1(q)$

and $u_2(q)$ of the boundary value problems (5) and (6) exist and are unique. The following bound follows immediately from the definition (5):

$$\|u_1(q)\|_{1,\Omega} \leqslant c(\|q\|_{0,\Omega_0} + \|g_2\|_{0,\Gamma}). \tag{7}$$

Applying regularity results for elliptic problems ([6] for the case $\Gamma \in C^2$ and [8, Section 3.2] for the case where Ω is convex), we have

$$\|u_2(q)\|_{2,\Omega} \leqslant c(\|q\|_{0,\Omega_0} + \|g_1\|_{3/2,\Gamma}), \tag{8}$$

and if $g_2 \in H^{1/2}(\Gamma)$, which we will assume

$$\|u_1(q)\|_{2,\Omega} \leqslant c(\|q\|_{0,\Omega_0} + \|g_2\|_{1/2,\Gamma}). \tag{9}$$

By the regularity bound (8), we know in particular that $\partial_\nu u_2(q) \in L^2(\Gamma)$ and $\|\partial_\nu u_2(q)\|_{0,\Gamma}$ is well defined.

For fixed constants $r_1, r_2 \geqslant 0$, we define the following functional with a Tikhonov regularization [12,14]

$$J_{\varepsilon,r_1,r_2}(q) = \frac{r_1}{2}\|u_1(q) - g_1\|_{0,\Gamma}^2 + \frac{r_2}{2}\|D\partial_\nu u_2(q) - g_2\|_{0,\Gamma}^2 + \frac{\varepsilon}{2}\|q\|_{0,\Omega_0}^2, \quad \varepsilon \geqslant 0. \tag{10}$$

Then we reformulate the BLT problem as follows.

Problem 2.1. Find $p_{\varepsilon,r_1,r_2} \in Q_{\rm ad}$ such that

$$J_{\varepsilon,r_1,r_2}(p_{\varepsilon,r_1,r_2}) = \inf_{q\in Q_{\rm ad}} J_{\varepsilon,r_1,r_2}(q). \tag{11}$$

When $r_1 = 1$ and $r_2 = 0$, Problem 2.1 reduces to Problem 1.1, which is the problem discussed in [9]. When $r_1 = 0$ and $r_2 = 1$, we obtain Problem 1.2 from Problem 2.1. Theoretically, it is natural to use $\|u_1(q) - g_1\|_{1/2,\Gamma}$ and $\|D\partial_\nu u_2(q) - g_2\|_{-1/2,\Gamma}$ to replace $\|u_1(q) - g_1\|_{0,\Gamma}$ and $\|D\partial_\nu u_2(q) - g_2\|_{0,\Gamma}$, respectively, in the definition (10). However, it is more convenient to use the $\|\cdot\|_{0,\Gamma}$ norm in the cost function for actual simulation. When g_1 or g_2 is known only on a part Γ_0 of Γ, we can replace $\|u_1(q) - g_1\|_{0,\Gamma}$ or $\|D\partial_\nu u_2(q) - g_2\|_{0,\Gamma}$ by $\|u_1(q) - g_1\|_{0,\Gamma_0}$ or $\|D\partial_\nu u_2(q) - g_2\|_{0,\Gamma_0}$, respectively, in (10).

For any $p, q \in Q$, we can verify that $u_1(p+q) - u_1(p)$ and $u_2(p+q) - u_2(p)$ are linear in q. So we have

$$u_i(p_1 + q) - u_i(p_2) = u_i(p_1) - u_i(p_2 - q) \quad \forall p_1, p_2, q \in Q, \quad i = 1, 2. \tag{12}$$

For the first and second Frechet derivatives of J_{ε,r_1,r_2}, we have the expressions

$$J'_{\varepsilon,r_1,r_2}(p)q = r_1(u_1(p) - g_1, u_1(q) - u_1(0))_{0,\Gamma} + r_2(D\partial_\nu u_2(p) - g_2, D\partial_\nu(u_2(q) - u_2(0)))_{0,\Gamma} + \varepsilon(p,q)_{0,\Omega_0}, \tag{13}$$

$$J''_{\varepsilon,r_1,r_2}(p)q^2 = r_1\|u_1(q) - u_1(0)\|_{0,\Gamma}^2 + r_2\|D\partial_\nu(u_2(q) - u_2(0))\|_{0,\Gamma}^2 + \varepsilon\|q\|_{0,\Omega_0}^2.$$

Hence, for $\varepsilon > 0$, $J_{\varepsilon,r_1,r_2}(\cdot)$ is strictly convex.

Now we are in the position to show the existence and uniqueness of the solution of Problem 2.1 and optimality condition.

Theorem 2.2. *For any $\varepsilon > 0$ and $r_1, r_2 \geqslant 0$ with $r_1 + r_2 > 0$, Problem 2.1 has a unique solution $p_{\varepsilon,r_1,r_2} \in Q_{\rm ad}$. Moreover, the solution $p_{\varepsilon,r_1,r_2} \in Q_{\rm ad}$ is characterized by the following variational inequality*:

$$r_1(u_1(p_{\varepsilon,r_1,r_2}) - g_1, u_1(q) - u_1(p_{\varepsilon,r_1,r_2}))_{0,\Gamma} + r_2(D\partial_\nu u_2(p_{\varepsilon,r_1,r_2}) - g_2, D\partial_\nu(u_2(q) - u_2(p_{\varepsilon,r_1,r_2})))_{0,\Gamma} + \varepsilon(p_{\varepsilon,r_1,r_2}, q - p_{\varepsilon,r_1,r_2})_{0,\Omega_0} \geqslant 0 \quad \forall q \in Q_{\rm ad}. \tag{14}$$

When $Q_{\rm ad}$ is a subspace of Q, the inequality above is reduced to an equation

$$r_1(u_1(p_{\varepsilon,r_1,r_2}) - g_1, u_1(q) - u_1(0))_{0,\Gamma} + r_2(D\partial_\nu u_2(p_{\varepsilon,r_1,r_2}) - g_2, D\partial_\nu(u_2(q) - u_2(0)))_{0,\Gamma} + \varepsilon(p_{\varepsilon,r_1,r_2}, q)_{0,\Omega_0} = 0 \quad \forall q \in Q_{\rm ad}. \tag{15}$$

Proof. Note that $Q_{\rm ad}$ is a closed and convex set of Hilbert space Q, $J_{\varepsilon,r_1,r_2} : Q_{\rm ad} \to \mathbb{R}$ is strictly convex and continuous with the property $J_{\varepsilon,r_1,r_2}(q) \to \infty$ as $\|q\|_{0,\Omega_0} \to \infty$. Then, by a standard result on convex problem [2,7], there is a unique solution $p_{\varepsilon,r_1,r_2} \in Q_{\rm ad}$ to Problem 2.1 and the solution is characterized by the condition

$$J'_{\varepsilon,r_1,r_2}(p_{\varepsilon,r_1,r_2})(q - p_{\varepsilon,r_1,r_2}) \geqslant 0 \quad \forall q \in Q_{\rm ad}.$$

Due to the formula (13), this condition is exactly (14).

If $Q_{\rm ad}$ is a subspace of Q, then we can take $q = 0$ and $q = 2p_{\varepsilon,r_1,r_2}$ in (14) and use the linearity (12) to conclude that

$$r_1(u_1(p_{\varepsilon,r_1,r_2}) - g_1, u_1(0) - u_1(p_{\varepsilon,r_1,r_2}))_{0,\Gamma} + r_2(D\partial_\nu u_2(p_{\varepsilon,r_1,r_2}) - g_2, D\partial_\nu(u_2(0) - u_2(p_{\varepsilon,r_1,r_2})))_{0,\Gamma} + \varepsilon(p_{\varepsilon,r_1,r_2}, -p_{\varepsilon,r_1,r_2})_{0,\Omega_0} = 0.$$

Subtracting this equation from inequality (14), we get

$$r_1(u_1(p_{\varepsilon,r_1,r_2}) - g_1, u_1(q) - u_1(0))_{0,\Gamma} + r_2(D\partial_\nu u_2(p_{\varepsilon,r_1,r_2}) - g_2, D\partial_\nu(u_2(q) - u_2(0)))_{0,\Gamma} + \varepsilon(p_{\varepsilon,r_1,r_2}, q)_{0,\Omega_0} \geqslant 0 \quad \forall q \in Q_{\rm ad}. \tag{16}$$

Replace q by $-q$ in (16) and use (12) again to get

$$r_1(u_1(p_{\varepsilon,r_1,r_2}) - g_1, u_1(q) - u_1(0))_{0,\Gamma} + r_2(D\partial_\nu u_2(p_{\varepsilon,r_1,r_2}) - g_2, D\partial_\nu(u_2(q) - u_2(0)))_{0,\Gamma} + \varepsilon(p_{\varepsilon,r_1,r_2}, q)_{0,\Omega_0} \leqslant 0 \quad \forall q \in Q_{\rm ad}. \tag{17}$$

We then combine (16) and (17) to obtain the variational equation (15). □

As in [9,11], we can show that the solution p_{ε,r_1,r_2} of Problem 2.1 depends continuously on D, μ, g_1, g_2, $r_1 > 0$, $r_2 > 0$ and $\varepsilon > 0$. We omit the detail in this paper.

3. Limiting behaviors

In this section, we analyze limiting behaviors of the solution p_{ε,r_1,r_2} of Problem 2.1 in three cases: $\varepsilon \to 0$, $r_1 \to 0$, or $r_2 \to 0$.

By an argument similar to the proof of Theorem 2.2, we know that a solution $p_{r_1,r_2} \in Q_{\rm ad}$ of Problem 2.1 with $\varepsilon = 0$ is characterized by a variational inequality

$$r_1(u_1(p_{r_1,r_2}) - g_1, u_1(q) - u_1(p_{r_1,r_2}))_{0,\Gamma} + r_2(D\partial_\nu u_2(p_{r_1,r_2}) - g_2, D\partial_\nu(u_2(q) - u_2(p_{r_1,r_2})))_{0,\Gamma} \geqslant 0 \quad \forall q \in Q_{\rm ad}. \tag{18}$$

Denote by $S_{r_1,r_2} \subset Q_{\rm ad}$ the set of solutions of Problem 2.1 with $\varepsilon = 0$. If $S_{r_1,r_2} \neq \emptyset$, it is straightforward to show that S_{r_1,r_2} is closed and convex. We have the following result.

Proposition 3.1. *Assume $S_{r_1,r_2} \neq \emptyset$. Then*

$$p_{\varepsilon,r_1,r_2} \to p_{0,r_1,r_2} \text{ in } Q, \text{ as } \varepsilon \to 0, \tag{19}$$

where $p_{0,r_1,r_2} \in S_{r_1,r_2}$ satisfies

$$\|p_{0,r_1,r_2}\|_{0,\Omega_0} = \inf_{q\in S_{r_1,r_2}} \|q\|_{0,\Omega_0}. \tag{20}$$

Proof. First we note that since S_{r_1,r_2} is non-empty, closed and convex, the element $p_{0,r_1,r_2} \in S_{r_1,r_2}$ is uniquely defined by (20).

Take $q = p_{0,r_1,r_2}$ in (14), $q = p_{\varepsilon,r_1,r_2}$ in (18) for $p_{r_1,r_2} = p_{0,r_1,r_2}$, and add the two resulting inequalities to get

$$\varepsilon(p_{\varepsilon,r_1,r_2}, p_{0,r_1,r_2} - p_{\varepsilon,r_1,r_2})_{0,\Omega_0} \geqslant r_1\|u_1(p_{0,r_1,r_2}) - u_1(p_{\epsilon,r_1,r_2})\|^2_{0,\Gamma} + r_2\|D\partial_\nu(u_2(p_{0,r_1,r_2}) - u_2(p_{\varepsilon,r_1,r_2}))\|^2_{0,\Gamma}.$$

Thus, $(p_{\varepsilon,r_1,r_2}, p_{0,r_1,r_2} - p_{\varepsilon,r_1,r_2})_{0,\Omega_0} \geqslant 0$, then $\|p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0} \leqslant \|p_{0,r_1,r_2}\|_{0,\Omega_0}$ and $\{p_{\varepsilon,r_1,r_2}\}_\varepsilon$ is uniformly bounded in Q. So there is a subsequence $\{p_{\varepsilon',r_1,r_2}\}_{\varepsilon'}$ of $\{p_{\varepsilon,r_1,r_2}\}_\varepsilon$, that converges weakly to some element p_{r_1,r_2} in Q. Since S_{r_1,r_2} is closed and convex in Q, $p_{r_1,r_2} \in S_{r_1,r_2}$, and

$$\|p_{r_1,r_2}\|_{0,\Omega_0} \leqslant \varliminf_{\varepsilon'\to 0} \|p_{\varepsilon',r_1,r_2}\|_{0,\Omega_0} \leqslant \|p_{0,r_1,r_2}\|_{0,\Omega_0}.$$

Since $p_{0,r_1,r_2} \in S_{r_1,r_2}$ defined by (20) is unique, $p_{r_1,r_2} = p_{0,r_1,r_2}$. Now that the limit p_{0,r_1,r_2} does not depend on the subsequence selected, the entire sequence $\{p_{\varepsilon,r_1,r_2}\}_\varepsilon$ converges weakly to p_{0,r_1,r_2} as $\varepsilon \to 0$ in Q. By

$$\begin{aligned}\|p_{\varepsilon,r_1,r_2} - p_{0,r_1,r_2}\|^2_{0,\Omega_0} &= \|p_{\varepsilon,r_1,r_2}\|^2_{0,\Omega_0} - 2(p_{\varepsilon,r_1,r_2}, p_{0,r_1,r_2})_{0,\Omega_0} + \|p_{0,r_1,r_2}\|^2_{0,\Omega_0} \\ &\leqslant 2\|p_{0,r_1,r_2}\|^2_{0,\Omega_0} - 2(p_{\varepsilon,r_1,r_2}, p_{0,r_1,r_2})_{0,\Omega_0} \to 0\end{aligned}$$

as $\varepsilon \to 0$, we obtain the strong convergence of p_{ε,r_1,r_2} to p_{0,r_1,r_2} in Q as $\varepsilon \to 0$. □

If $Q_{\rm ad}$ is a bounded set, then S_{r_1,r_2} is non-empty. This can be shown similar to the first part of the proof of Theorem 2.2. However, we cannot ascertain uniqueness of a solution when $\varepsilon = 0$, see [13] in detail. In the case where the solution set $S_{r_1,r_2} = \{p_{r_1,r_2}\}$ is a singleton, we conclude from Proposition 3.1 that $p_{\varepsilon,r_1,r_2} \to p_{r_1,r_2}$, as $\varepsilon \to 0$ in Q.

Next we explore the behavior of the solution p_{ε,r_1,r_2} as $r_1 \to 0$ with $\varepsilon > 0$ and $r_2 > 0$ being fixed. By Theorem 2.2, Problem 2.1 with $r_1 = 0$ has a unique solution $p_{\varepsilon,0,r_2}$. From the inequality

$$J_{\varepsilon,r_1,r_2}(p_{\varepsilon,r_1,r_2}) \leqslant J_{\varepsilon,r_1,r_2}(p_{\varepsilon,0,r_2}),$$

we see that the sequence $\{\|p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0}\}_{r_1}$ is uniformly bounded. From (7) and (8), we have the bounds

$$\|u_1(p_{\varepsilon,r_1,r_2})\|_{1,\Omega} \leqslant c(\|p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0} + \|g_2\|_{0,\Gamma}),$$
$$\|u_2(p_{\varepsilon,r_1,r_2})\|_{2,\Omega} \leqslant c(\|p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0} + \|g_1\|_{3/2,\Gamma}).$$

Thus, $\{\|u_1(p_{\varepsilon,r_1,r_2})\|_{1,\Omega}\}_{r_1}$ and $\{\|u_2(p_{\varepsilon,r_1,r_2})\|_{2,\Omega}\}_{r_1}$ are uniformly bounded. In particular, the sequence $\{\|\partial_\nu u_2(p_{\varepsilon,r_1,r_2})\|_{0,\Gamma}\}_{r_1}$ is also uniformly bounded. So we can select a subsequence $\{p_{\varepsilon,r_1',r_2}\}_{r_1'}$ of $\{p_{\varepsilon,r_1,r_2}\}_{r_1}$ such that

$$p_{\varepsilon,r_1',r_2} \rightharpoonup p_{\varepsilon,r_2} \text{ in } Q, \text{ as } r_1' \to 0$$

for some element $p_{\varepsilon,r_2} \in Q_{\rm ad}$, and $\{u_1(p_{\varepsilon,r_1',r_2})\}_{r_1'}$ and $\{u_2(p_{\varepsilon,r_1',r_2})\}_{r_1'}$ converge weakly in V and $H^2(\Omega)$. Using the definitions (5) and (6), we can show that the limits of $\{u_1(p_{\varepsilon,r_1',r_2})\}_{r_1'}$ and $\{u_2(p_{\varepsilon,r_1',r_2})\}_{r_1'}$ are $u_1(p_{\varepsilon,r_2})$ and $u_2(p_{\varepsilon,r_2})$, respectively. So

$$u_1(p_{\varepsilon,r_1',r_2}) \rightharpoonup u_1(p_{\varepsilon,r_2}) \text{ in } V, \text{ as } r_1' \to 0,$$
$$u_2(p_{\varepsilon,r_1',r_2}) \rightharpoonup u_2(p_{\varepsilon,r_2}) \text{ in } H^2(\Omega), \text{ as } r_1' \to 0.$$

Take the limit $r_1' \to 0$ in (14) with $r_1 = r_1'$ to obtain

$$r_2(D\partial_\nu u_2(p_{\varepsilon,r_2}) - g_2, D\partial_\nu(u_2(q) - u_2(p_{\varepsilon,r_2})))_{0,\Gamma} + \varepsilon(p_{\varepsilon,r_2}, q - p_{\varepsilon,r_2})_{0,\Omega_0} \geqslant 0 \quad \forall q \in Q_{\rm ad}. \tag{21}$$

Thus, $p_{\varepsilon,r_2} = p_{\varepsilon,0,r_2}$ by the uniqueness of a solution of Problem 2.1 with $r_1 = 0$. Since the limit $p_{\varepsilon,0,r_2}$ does not depend on the subsequence we choose in the above argument, the entire family $\{p_{\varepsilon,r_1,r_2}\}_{r_1}$ converges weakly to $p_{\varepsilon,0,r_2}$ in Q as $r_1 \to 0$. Strong convergence can be argued as follows. Take $q = p_{\varepsilon,0,r_2}$ in (14) and $q = p_{\varepsilon,r_1,r_2}$ in (21), and add the two resulting inequalities to obtain

$$r_2\|D\partial_\nu(u_2(p_{\varepsilon,r_1,r_2}) - u_2(p_{\varepsilon,0,r_2}))\|^2_{0,\Gamma} + \varepsilon\|p_{\varepsilon,r_1,r_2} - p_{\varepsilon,0,r_2}\|^2_{0,\Omega_0} \leqslant r_1(u_1(p_{\varepsilon,r_1,r_2}) - g_1, u_1(p_{\varepsilon,0,r_2}) - u_1(p_{\varepsilon,r_1,r_2}))_{0,\Gamma} \to 0 \quad \text{as } r_1 \to 0.$$

We see that both $\|p_{\varepsilon,r_1,r_2} - p_{\varepsilon,0,r_2}\|_{0,\Omega_0}$ and $\|\partial_\nu(u_2(p_{\varepsilon,r_1,r_2}) - u_2(p_{\varepsilon,0,r_2}))\|_{0,\Gamma}$ approach zero as $r_1 \to 0$.

In summary, we have shown the following result.

Proposition 3.2. *For fixed $\varepsilon > 0$ and $r_2 > 0$,*

$$p_{\varepsilon,r_1,r_2} \to p_{\varepsilon,0,r_2} \text{ in } Q, \text{ as } r_1 \to 0,$$

where $p_{\varepsilon,0,r_2}$ is the solution of Problem 2.1 with $r_1 = 0$.

Similarly, we can show the next result.

Proposition 3.3. *For fixed $\varepsilon > 0$ and $r_1 > 0$,*

$$p_{\varepsilon,r_1,r_2} \to p_{\varepsilon,r_1,0} \text{ in } Q, \text{ as} r_2 \to 0,$$

where $p_{\varepsilon,r_1,0}$ is the solution of Problem 2.1 with $r_2 = 0$.

4. Finite element approximation

In this section, we discretize Problem 2.1 and study convergence of the numerical solutions. We use the finite element method to discretize the boundary value problems (5) and (6). For clarity of statement, we consider semi-discrete and full-discrete approximation separately. In the former case, we use linear finite elements to approximate the state variables u_1 and u_2, whereas for full-discrete approximation, additional treatment with piecewise constant space to approximate control variable q is needed. In order to focus on the central ideas in derivation of error bounds for the numerical methods to be introduced, we will assume $\Omega \subset \mathbb{R}^d$ to be a polyhedral convex set. Error analysis of the numerical methods can be performed under the more general assumption that Ω is an open, bounded and convex set, through a rather delicate argument.

4.1. Semi-discrete approximation

Let $\{\mathscr{T}_h\}_h$ be a regular family of finite element partitions of $\overline{\Omega}$ into simplicial elements. Define the linear finite element space

$$V^h = \{v \in C(\overline{\Omega}) | v \text{ linear in } K,\ \forall K \in \mathscr{T}_h\}$$

and its subspace

$$V_0^h = V^h \cap V_0 = \{v \in C(\overline{\Omega}) | v \text{ piecewise linear},\ v = 0 \text{ on } \Gamma\}.$$

Denote by $\Pi_{V^h} v$ for the piecewise linear interpolant of $v \in H^2(\Omega)$. Then we have the existence of a constant $c > 0$ such that [4,16]

$$\|v - \Pi_{V^h} v\|_{0,\Omega} + h\|v - \Pi_{V^h} v\|_{1,\Omega} \leqslant ch^2 \|v\|_{2,\Omega} \quad \forall v \in H^2(\Omega). \tag{22}$$

Let $g_1^h = \Pi_{V^h} g_1 \in V^h$. Then

$$\|g_1 - g_1^h\|_{m,\Omega} \leqslant ch^{2-m} \|g_1\|_{2,\Omega} \leqslant ch^{2-m} \|g_1\|_{3/2,\Gamma}, \quad m = 0, 1. \tag{23}$$

We will use the symbol $g_1^h + V_0^h$ for the set

$$\{v \in V^h | v(a_i) = g_1(a_i)\ \forall \text{ vertex } a_i \in K \cap \Gamma,\ \forall K \in \mathscr{T}_h\}.$$

For any $q \in Q$, denote by $u_1^h = u_1^h(q) \in V^h$ and $u_2^h = u_2^h(q) \in g_1^h + V_0^h$ the solutions of the problems

$$\int_\Omega (D\nabla u_1^h \cdot \nabla v^h + \mu u_1^h v^h) \mathrm{d}x = \int_{\Omega_0} q v^h \,\mathrm{d}x + \int_\Gamma g_2 v^h \,\mathrm{d}s \quad \forall v^h \in V^h \tag{24}$$

and

$$\int_\Omega (D\nabla u_2^h \cdot \nabla v^h + \mu u_2^h v^h) \mathrm{d}x = \int_{\Omega_0} q v^h \,\mathrm{d}x \quad \forall v^h \in V_0^h, \tag{25}$$

respectively.

By the Lax–Milgram lemma and the assumptions made on the data, the solutions $u_1^h(q)$ and $u_2^h(q)$ uniquely exist.

Define the functional

$$J^h_{\varepsilon,r_1,r_2}(q) = \frac{r_1}{2} \|u_1^h(q) - g_1^h\|^2_{0,\Gamma} + \frac{r_2}{2} \|D\partial_\nu u_2^h(q) - g_2\|^2_{0,\Gamma} + \frac{\varepsilon}{2} \|q\|^2_{0,\Omega_0}, \quad \varepsilon > 0. \tag{26}$$

Then the semi-discrete approximation of Problem 2.1 is the following formulation.

Problem 4.1. Find $p^h_{\varepsilon,r_1,r_2} \in Q_{\mathrm{ad}}$ such that

$$J^h_{\varepsilon,r_1,r_2}\left(p^h_{\varepsilon,r_1,r_2}\right) = \inf_{q \in Q_{\mathrm{ad}}} J^h_{\varepsilon,r_1,r_2}(q). \tag{27}$$

Note that in the standard BLT formulation [9], $r_2 = 0$ and only the finite element problem (24) needs to be formulated and solved. In our proposed general framework, we need to formulate and solve both (24) and (25). However, due to the structure similarity of the two finite element problems, the cost of solving both (24) and (25) can be made only slightly more than that of solving (24) alone.

We summarize in the next theorem some results on Problem 4.1 as discrete analogues of Theorem 2.2 and Propositions 3.1–3.3.

Theorem 4.2. *For any $\varepsilon > 0$, $r_1, r_2 \geqslant 0$ with $r_1 + r_2 > 0$, Problem 4.1 has a unique solution $p^h_{\varepsilon,r_1,r_2} \in Q_{\mathrm{ad}}$, which is characterized by a variational inequality*

$$r_1\left(u_1^h\left(p^h_{\varepsilon,r_1,r_2}\right) - g_1^h, u_1^h(q) - u_1^h\left(p^h_{\varepsilon,r_1,r_2}\right)\right)_{0,\Gamma} + r_2\left(D\partial_\nu u_2^h\left(p^h_{\varepsilon,r_1,r_2}\right) - g_2, D\partial_\nu\left(u_2^h(q) - u_2^h(p^h_{\varepsilon,r_1,r_2})\right)\right)_{0,\Gamma} + \varepsilon\left(p^h_{\varepsilon,r_1,r_2}, q - p^h_{\varepsilon,r_1,r_2}\right)_{0,\Omega_0} \geqslant 0 \quad \forall q \in Q_{\mathrm{ad}}. \tag{28}$$

When Q_{ad} is a subspace of Q, the above inequality is reduced to an equation

$$r_1\left(u_1^h\left(p^h_{\varepsilon,r_1,r_2}\right) - g_1^h, u_1^h(q) - u_1^h(0)\right)_{0,\Gamma} + r_2\left(D\partial_\nu u_2^h\left(p^h_{\varepsilon,r_1,r_2}\right) - g_2, D\partial_\nu\left(u_2^h(q) - u_2^h(0)\right)\right)_{0,\Gamma} + \varepsilon\left(p^h_{\varepsilon,r_1,r_2,\delta}, q\right)_{0,\Omega_0} = 0 \quad \forall q \in Q_{\mathrm{ad}}. \tag{29}$$

The solution $p^h_{\varepsilon,r_1,r_2}$ depends continuously on D, μ, ε, r_1, r_2, g_1 and g_2.

Assume the solution set $S^h_{r_1,r_2}$ for Problem 4.1 with $\varepsilon = 0$ is non-empty. Then for fixed $r_1, r_2 \geqslant 0$ with $r_1 + r_2 > 0$,

$$p^h_{\varepsilon,r_1,r_2} \to p^h_{0,r_1,r_2} \text{ in } Q, \text{ as } \varepsilon \to 0, \tag{30}$$

where $p^h_{0,r_1,r_2} \in S^h_{r_1,r_2}$ is uniquely defined by

$$\left\|p^h_{0,r_1,r_2}\right\|_{0,\Omega_0} = \inf_{q \in S^h_{r_1,r_2}} \|q\|_{0,\Omega_0}. \tag{31}$$

For fixed $\varepsilon > 0$ and $r_2 > 0$, we have

$$p^h_{\varepsilon,r_1,r_2} \to p^h_{\varepsilon,0,r_2} \text{ in } Q, \text{ as } r_1 \to 0,$$

where $p_{\varepsilon,0,r_2} \in Q_{\mathrm{ad}}$ is the unique solution of Problem 4.1 with $r_1 = 0$.

For fixed $\varepsilon > 0$ *and* $r_1 > 0$, *we have*

$$p^h_{\varepsilon,r_1,r_2} \to p^h_{\varepsilon,r_1,0} \text{ in } Q, \text{ as } r_2 \to 0,$$

where $p_{\varepsilon,r_1,0} \in Q_{\rm ad}$ *is the unique solution of Problem 4.1 with* $r_2 = 0$.

For an error analysis of the numerical solution defined by Problem 4.1, we first preset some error bounds for the finite element solutions u_1^h and u_2^h of (24) and (25).

Lemma 4.3. *There is a constant independent of* h, ε, r_1 *and* r_2 *such that for any* $q, q_1, q_2 \in Q$, *the following inequalities hold*:

$$\left\|u_1(q) - u_1^h(q)\right\|_{0,\Gamma} \leqslant ch^{3/2}(\|q\|_{0,\Omega_0} + \|g_2\|_{1/2,\Gamma}), \tag{32}$$

$$\left\|\partial_\nu(u_2(q) - u_2^h(q))\right\|_{0,\Gamma} \leqslant ch^{1/2}(\|q\|_{0,\Omega_0} + \|g_1\|_{3/2,\Gamma}), \tag{33}$$

$$\left\|(u_1(q_1) - u_1(q_2)) - \left(u_1^h(q_1) - u_1^h(q_2)\right)\right\|_{0,\Gamma} \leqslant ch^{3/2}\|q_1 - q_2\|_{0,\Omega_0}, \tag{34}$$

$$\|\partial_\nu(u_2(q_1) - u_2(q_2)) - \partial_\nu\left(u_2^h(q_1) - u_2^h(q_2)\right)\|_{0,\Gamma} \leqslant ch^{1/2}\|q_1 - q_2\|_{0,\Omega_0}. \tag{35}$$

Proof. Proof of the relations (32) and (34) can be found in [9]. Here we prove (33) and (35). By Céa's Lemma [4], (23) and (22),

$$\begin{aligned}
\left\|u_2(q) - u_2^h(q)\right\|_{1,\Omega}
&\leqslant c \inf_{v^h \in g_1^h + V_0^h} \|u_2(q) - v^h\|_{1,\Omega} \\
&\leqslant c\left(\|g_1 - g_1^h\|_{1,\Omega} + \inf_{v^h \in V_0^h} \|(u_2(q) - g_1) - v^h\|_{1,\Omega} \right) \\
&\leqslant ch(\|g_1\|_{3/2,\Gamma} + \|u_2(q)\|_{2,\Omega}).
\end{aligned}$$

Recalling the regularity bound (8), we have

$$\left\|u_2(q) - u_2^h(q)\right\|_{1,\Omega} \leqslant ch(\|q\|_{0,\Omega_0} + \|g_1\|_{3/2,\Gamma}).$$

By the trace inequality

$$\|v\|^2_{0,\Gamma} \leqslant c\|v\|_{1,\Omega}\|v\|_{0,\Omega},$$

we obtain

$$\begin{aligned}
\left\|\partial_\nu\left(u_2(q) - u_2^h(q)\right)\right\|^2_{0,\Gamma} &\leqslant c\|u_2(q)\|_{2,\Omega}\left\|u_2(q) - u_2^h(q)\right\|_{1,\Omega} \\
&\leqslant ch(\|q\|_{0,\Omega_0} + \|g_1\|_{3/2,\Gamma})^2.
\end{aligned}$$

Thus (33) holds.

By noting that $u_2(q_1) - u_2(q_2) \in V_0$ is the solution of Problem 2.1 with $q = q_1 - q_2$ and $g_1 = 0$, and $u_2^h(q_1) - u_2^h(q_2) \in V_0^h$ is the corresponding finite element solution, we obtain (35) from (33). $\square$

With the above preparation, we now present an error estimate. Denote

$$\begin{aligned}
E^h_{\varepsilon,r_1,r_2} &= r_1\left\|u_1(p_{\varepsilon,r_1,r_2}) - u_1^h\left(p^h_{\varepsilon,r_1,r_2}\right)\right\|^2_{0,\Gamma} \\
&\quad + r_2\left\|D\partial_\nu\left(u_2(p_{\varepsilon,r_1,r_2}) - u_2^h\left(p^h_{\varepsilon,r_1,r_2}\right)\right)\right\|^2_{0,\Gamma} \\
&\quad + \varepsilon\left\|p_{\varepsilon,r_1,r_2} - p^h_{\varepsilon,r_1,r_2}\right\|^2_{0,\Omega_0}.
\end{aligned}$$

We have the following error bound.

Theorem 4.4. *There is a constant* $c > 0$ *independent of* ε, r_1, r_2 *and* h *such that*

$$\begin{aligned}
E^h_{\varepsilon,r_1,r_2} &\leqslant cr_1h^3\left(\|p_{\varepsilon,r_1,r_2}\|^2_{0,\Omega_0} + \|g_2\|^2_{1/2,\Gamma}\right) \\
&\quad + cr_2h\left(\|p_{\varepsilon,r_1,r_2}\|^2_{0,\Omega_0} + \|g_1\|^2_{3/2,\Gamma}\right) \\
&\quad + cr_1h^{3/2}\|u_1(p_{\varepsilon,r_1,r_2}) - g_1\|_{0,\Gamma}\|p_{\varepsilon,r_1,r_2} - p^h_{\varepsilon,r_1,r_2}\|_{0,\Omega_0} \\
&\quad + cr_1h^{3/2}\|g_1\|_{3/2,\Gamma}\left\|p_{\varepsilon,r_1,r_2} - p^h_{\varepsilon,r_1,r_2}\right\|_{0,\Omega_0} \\
&\quad + cr_2h^{1/2}\|D\partial_\nu u_2(p_{\varepsilon,r_1,r_2}) - g_2\|_{0,\Gamma} \\
&\quad \times \left\|p_{\varepsilon,r_1,r_2} - p^h_{\varepsilon,r_1,r_2}\right\|_{0,\Omega_0}.
\end{aligned} \tag{36}$$

Proof. We take $q = p_{\varepsilon,r_1,r_2}$ in (28), $q = p^h_{\varepsilon,r_1,r_2}$ in (14), and use the two resulting inequalities to get

$$\begin{aligned}
E^h_{\varepsilon,r_1,r_2} &\leqslant r_1\left(u_1(p_{\varepsilon,r_1,r_2}) - u_1^h\left(p^h_{\varepsilon,r_1,r_2}\right), u_1(p_{\varepsilon,r_1,r_2})\right. \\
&\quad \left.- u_1^h(p_{\varepsilon,r_1,r_2})\right)_{0,\Gamma} + r_1\left(u_1(p_{\varepsilon,r_1,r_2}) - g_1, u_1^h(p_{\varepsilon,r_1,r_2})\right. \\
&\quad \left.- u_1^h\left(p^h_{\varepsilon,r_1,r_2}\right) + u_1\left(p^h_{\varepsilon,r_1,r_2}\right) - u_1(p_{\varepsilon,r_1,r_2})\right)_{0,\Gamma} \\
&\quad + r_1\left(g_1 - g_1^h, u_1^h(p_{\varepsilon,r_1,r_2}) - u_1^h\left(p^h_{\varepsilon,r_1,r_2}\right)\right)_{0,\Gamma} \\
&\quad + r_2\left(D\partial_\nu\left(u_2(p_{\varepsilon,r_1,r_2}) - u_2^h\left(p^h_{\varepsilon,r_1,r_2}\right)\right), D\partial_\nu(u_2(p_{\varepsilon,r_1,r_2})\right. \\
&\quad \left.- u_2^h(p_{\varepsilon,r_1,r_2}))\right)_{0,\Gamma} + r_2\left(D\partial_\nu u_2(p_{\varepsilon,r_1,r_2})\right. \\
&\quad - g_2, D\partial_\nu\left(u_2^h(p_{\varepsilon,r_1,r_2}) - u_2(p_{\varepsilon,r_1,r_2}) + u_2(p^h_{\varepsilon,r_1,r_2})\right. \\
&\quad \left.\left.- u_2^h\left(p^h_{\varepsilon,r_1,r_2}\right)\right)\right)_{0,\Gamma} \\
&\leqslant \frac{r_1}{2}\left\|u_1(p_{\varepsilon,r_1,r_2}) - u_1^h\left(p^h_{\varepsilon,r_1,r_2}\right)\right\|^2_{0,\Gamma} + \frac{r_1}{2}\|u_1(p_{\varepsilon,r_1,r_2}) \\
&\quad - u_1^h(p_{\varepsilon,r_1,r_2})\|^2_{0,\Gamma} + r_1\|u_1(p_{\varepsilon,r_1,r_2}) - g_1\|_{0,\Gamma}\|u_1^h(p_{\varepsilon,r_1,r_2}) \\
&\quad - u_1(p_{\varepsilon,r_1,r_2}) + u_1(p^h_{\varepsilon,r_1,r_2}) - u_1^h\left(p^h_{\varepsilon,r_1,r_2}\right)\|_{0,\Gamma} \\
&\quad + r_1\|g_1 - g_1^h\|_{0,\Gamma}\left\|u_1^h(p_{\varepsilon,r_1,r_2}) - u_1^h\left(p^h_{\varepsilon,r_1,r_2}\right)\right\|_{0,\Gamma} \\
&\quad + \frac{r_2}{2}\left\|D\partial_\nu\left(u_2(p_{\varepsilon,r_1,r_2}) - u_2^h\left(p^h_{\varepsilon,r_1,r_2}\right)\right)\right\|^2_{0,\Gamma} \\
&\quad + \frac{r_2}{2}\|D\partial_\nu\left(u_2(p_{\varepsilon,r_1,r_2}) - u_2^h(p_{\varepsilon,r_1,r_2})\right)\|^2_{0,\Gamma} \\
&\quad + r_2\|D\partial_\nu u_2(p_{\varepsilon,r_1,r_2}) - g_2\|_{0,\Gamma}\left\|D\partial_\nu\left(u_2^h(p_{\varepsilon,r_1,r_2})\right.\right. \\
&\quad \left.\left.- u_2(p_{\varepsilon,r_1,r_2}) + u_2\left(p^h_{\varepsilon,r_1,r_2}\right) - u_2^h\left(p^h_{\varepsilon,r_1,r_2}\right)\right)\right\|_{0,\Gamma}.
\end{aligned}$$

Hence, (36) follows by using relations (8), (23) and (32)–(35) in Lemma 4.3. □

More concrete error bounds can be derived from Theorem 4.4 under additional assumptions. For example, assuming Q_{ad} to be a bounded set in the space Q, we can deduce from (36) that there is a constant $c>0$ independent of $r_1,\ r_2,\ \varepsilon$ and h such that

$$E^h_{\varepsilon,r_1,r_2}\leqslant c(r_1h^{3/2}+r_2h^{1/2}). \tag{37}$$

The error bound (37) indicates a need to choose r_2 to be of the same order as the meshsize h in order to get better solution accuracy.

4.2. Full-discrete approximation

Next, we turn to the full-discrete approximation with finite element to Problem 2.1. We use the linear finite element to discretize state variables and piecewise constant functions to approximate the control variable. We continue to use $\mathscr{T}_h$, V^h, $g_1^h+V_0^h$ and V_0^h defined in the previous subsection. In addition, we assume $\{\mathscr{T}_{0,H}\}_H$ is a regular family of triangulations of $\overline{\Omega_0}$ such that each element T of $\mathscr{T}_{0,H}$ joint to boundary $\partial\Omega_0$ has at most one curved face (for a three-dimensional domain) or side (for a plane domain) with mesh parameter H. Define the space

$$Q^H=\{q\in Q|q|_T\in P_0(T)\ \forall T\in\mathscr{T}_{0,H}\},$$

where $P_0(T)$ is the constant function space, $Q^H_{\mathrm{ad}}=Q^H\cap Q_{\mathrm{ad}}$. Then the full-discrete approximation of Problem 2.1 is the following formulation.

Problem 4.5. Find $p^{h,H}_{\varepsilon,r_1,r_2}\in Q^H_{\mathrm{ad}}$ such that

$$J^h_{\varepsilon,r_1,r_2}\left(p^{h,H}_{\varepsilon,r_1,r_2}\right)=\inf_{q\in Q^H_{\mathrm{ad}}}J^h_{\varepsilon,r_1,r_2}(q).$$

Define the orthogonal projection operator $\Pi^H:Q\to Q^H$ by

$$(\Pi^Hq,q^H)_{0,\Omega_0}=(q,q^H)_{0,\Omega_0}\quad\forall q\in Q,\ \ q^H\in Q^H. \tag{38}$$

Then we have the formula

$$(\Pi^Hq)|_T=\frac{1}{|T|}\int_T q\,\mathrm{d}x\quad\forall T\in\mathscr{T}_{0,H} \tag{39}$$

and the inequalities

$$|\Pi^Hq\|_{0,\Omega_0}\leqslant\|q\|_{0,\Omega_0}\quad\forall q\in Q, \tag{40}$$
$$\|q-\Pi^Hq\|_{0,\Omega_0}\leqslant cH|q|_{1,\Omega_0}\quad\forall q\in H^1(\Omega_0). \tag{41}$$

Similar to Lemma 4.3, we have the following result that will be used in deriving an error bound in Theorem 4.7.

Lemma 4.6. *There is a constant $c>0$ independent of h and H such that*

$$\left\|u^h_1(q)-u^h_1(\Pi^Hq)\right\|_{0,\Gamma}\leqslant cH\|q-\Pi^Hq\|_{0,\Omega_0}, \tag{42}$$

$$\left\|u_1(q)-u^h_1(\Pi^Hq)\right\|_{0,\Gamma}\leqslant cH\|q-\Pi^Hq\|_{0,\Omega_0}+ch^{3/2}(\|q\|_{0,\Omega_0}+\|g_2\|_{1/2,\Gamma}), \tag{43}$$

$$\left\|\partial_\nu\left(u^h_2(q)-u^h_2(\Pi^Hq)\right)\right\|_{0,\Gamma}\leqslant cH^{1/2}\|q-\Pi^Hq\|_{0,\Omega_0}, \tag{44}$$

$$\left\|\partial_\nu\left(u_2(q)-u^h_2(\Pi^Hq)\right)\right\|_{0,\Gamma}\leqslant cH^{1/2}\|q-\Pi^Hq\|_{0,\Omega_0}+ch^{1/2}(\|q\|_{0,\Omega_0}+\|g_1\|_{3/2,\Gamma}). \tag{45}$$

Proof. Denote $e^{h,H}_1(q)=u^h_1(q)-u^h_1(\Pi^Hq)$. Then from definition (24), we have

$$\int_\Omega(D\nabla e^{h,H}_1(q)\cdot\nabla v^h+\mu e^{h,H}_1(q)v^h)\mathrm{d}x=\int_{\Omega_0}(q-\Pi^Hq)v^h\,\mathrm{d}x\quad\forall v^h\in V^h. \tag{46}$$

By (9), the inequality

$$\left\|e^{h,H}_1(q)\right\|_{2,\Omega}\leqslant c\|q-\Pi^Hq\|_{0,\Omega_0} \tag{47}$$

holds.

By (39), we have

$$\int_{\Omega_0}(q-\Pi^Hq)w^H\,\mathrm{d}x=\sum_{T\in\mathscr{T}_{0,H}}\int_T(q-\Pi^Hq)w^H\,\mathrm{d}x=0\quad\forall w^H\in Q^H.$$

Thus, for any $v\in H^1(\Omega)$,

$$\int_{\Omega_0}(q-\Pi^Hq)v\,\mathrm{d}x\leqslant\|q-\Pi^Hq\|_{0,\Omega_0}\inf_{w^H\in Q^H}\|v-w^H\|_{0,\Omega_0}\leqslant cH\|q-\Pi^Hq\|_{0,\Omega_0}\|v\|_{1,\Omega}.$$

Choosing $v^h=e^{h,H}_1(q)$ in (46), we have

$$\left\|e^{h,H}_1(q)\right\|_{1,\Omega}\leqslant cH\|q-\Pi^Hq\|_{0,\Omega_0}. \tag{48}$$

Similarly, denoting $e^{h,H}_2(q)=u^h_2(q)-u^h_2(\Pi^Hq)$, we get

$$\left\|e^{h,H}_2(q)\right\|_{2,\Omega}\leqslant c\|q-\Pi^Hq\|_{0,\Omega_0} \tag{49}$$

and

$$\left\|e^{h,H}_2(q)\right\|_{1,\Omega}\leqslant cH\|q-\Pi^Hq\|_{0,\Omega_0}. \tag{50}$$

Hence, by (48)–(50), we obtain

$$\left\|e^{h,H}_1(q)\right\|_{0,\Gamma}\leqslant c\left\|e^{h,H}_1(q)\right\|^{1/2}_{1,\Omega}\left\|e^{h,H}_1(q)\right\|^{1/2}_{0,\Omega}\leqslant cH\|q-\Pi^Hq\|_{0,\Omega_0}$$

and

$$\left\|\partial_\nu e^{h,H}_2(q)\right\|_{0,\Gamma}\leqslant c\left\|e^{h,H}_2(q)\right\|^{1/2}_{2,\Omega}\left\|e^{h,H}_2(q)\right\|^{1/2}_{1,\Omega}\leqslant cH^{1/2}\|q-\Pi^Hq\|_{0,\Omega_0}.$$

So we have proved (42) and (44).

Noticing

$$\left\|u_1(q)-u^h_1(\Pi^Hq)\right\|_{0,\Gamma}\leqslant\left\|u_1(q)-u^h_1(q)\right\|_{0,\Gamma}+\left\|u^h_1(q)-u^h_1(\Pi^Hq)\right\|_{0,\Gamma}$$

and

$$\left\|\partial_\nu(u_2(q)-u_2^h(\Pi^H q))\right\|_{0,\Gamma} \leqslant \left\|\partial_\nu(u_2(q)-u_2^h(q))\right\|_{0,\Gamma} + \left\|\partial_\nu\left(u_2^h(q)-u_2^h(\Pi^H q)\right)\right\|_{0,\Gamma},$$

by (42), (44) and Lemma 4.3, we conclude relations (43) and (45). □

Denote by

$$E^{h,H}_{\varepsilon,r_1,r_2} = r_1\left\|u_1(p_{\varepsilon,r_1,r_2}) - u_1^h\left(p^{h,H}_{\varepsilon,r_1,r_2}\right)\right\|^2_{0,\Gamma} + r_2\left\|D\partial_\nu\left(u_2(p_{\varepsilon,r_1,r_2}) - u_2^h\left(p^{h,H}_{\varepsilon,r_1,r_2}\right)\right)\right\|^2_{0,\Gamma} + \varepsilon\left\|p_{\varepsilon,r_1,r_2} - p^{h,H}_{\varepsilon,r_1,r_2}\right\|^2_{0,\Omega_0}$$

for the full-discrete error. We have the following error bound.

Theorem 4.7. *There is a constant $c>0$ independent of $\varepsilon, r_1, r_2, h, H$ such that*

$$\begin{aligned} E^{h,H}_{\varepsilon,r_1,r_2} \leqslant{}& cr_1[HE^H(p_{\varepsilon,r_1,r_2}) + h^{3/2}(\|p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0} + \|g_2\|_{1/2,\Gamma})]^2 \\ &+ cr_1\|u_1(p_{\varepsilon,r_1,r_2}) - g_1\|_{0,\Gamma}\left[HE^H(p_{\varepsilon,r_1,r_2}) + h^{3/2}\left\|p_{\varepsilon,r_1,r_2} - p^{h,H}_{\varepsilon,r_1,r_2}\right\|_{0,\Omega_0}\right] \\ &+ cr_1 h^{3/2}\|g_1\|_{3/2,\Omega}\left[E^H(p_{\varepsilon,r_1,r_2}) + \left\|p_{\varepsilon,r_1,r_2} - p^{h,H}_{\varepsilon,r_1,r_2}\right\|_{0,\Omega_0}\right] + cr_2[H^{1/2}E^H(p_{\varepsilon,r_1,r_2}) \\ &+ h^{1/2}(\|p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0} + \|g_1\|_{3/2,\Gamma})]^2 \\ &+ cr_2\|D\partial_\nu u_2(p_{\varepsilon,r_1,r_2}) - g_2\|_{0,\Gamma}\left[H^{1/2}E^H(p_{\varepsilon,r_1,r_2}) + h^{1/2}\left\|p_{\varepsilon,r_1,r_2} - p^{h,H}_{\varepsilon,r_1,r_2}\right\|_{0,\Omega_0}\right], \end{aligned} \tag{51}$$

where

$$E^H(p_{\varepsilon,r_1,r_2}) = \|p_{\varepsilon,r_1,r_2} - \Pi^H p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0} = \inf_{w^H\in Q^H_{\mathrm{ad}}} \|p_{\varepsilon,r_1,r_2} - w^H\|_{0,\Omega_0} \tag{52}$$

is the best approximation error of p_{ε,r_1,r_2} by piecewise constant function in $L^2(\Omega_0)$.

Proof. Take $q = \Pi^H p_{\varepsilon,r_1,r_2}$ in (28) with $p^h_{\varepsilon,r_1,r_2}$ instead by $p^{h,H}_{\varepsilon,r_1,r_2}$ and $q = p^{h,H}_{\varepsilon,r_1,r_2}$ in (14) and add the two resulting inequalities to get

$$E^{h,H}_{\varepsilon,r_1,r_2} \leqslant I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \tag{53}$$

where

$$I_1 = r_1\left(u_1^h\left(p^{h,H}_{\varepsilon,r_1,r_2}\right) - u_1(p_{\varepsilon,r_1,r_2}),\ u_1^h(\Pi^H p_{\varepsilon,r_1,r_2}) - u_1(p_{\varepsilon,r_1,r_2})\right)_{0,\Gamma},$$

$$I_2 = r_1\left(u_1(p_{\varepsilon,r_1,r_2}) - g_1, u_1^h(\Pi^H p_{\varepsilon,r_1,r_2}) - u_1(p_{\varepsilon,r_1,r_2}) + u_1\left(p^{h,H}_{\varepsilon,r_1,r_2}\right) - u_1^h\left(p^{h,H}_{\varepsilon,r_1,r_2}\right)\right)_{0,\Gamma},$$

$$I_3 = r_1\left(g_1 - g_1^h, u_1^h(\Pi^H p_{\varepsilon,r_1,r_2}) - u_1^h\left(p^{h,H}_{\varepsilon,r_1,r_2}\right)\right)_{0,\Gamma},$$

$$I_4 = r_2\left(D\partial_\nu\left(u_2^h\left(p^{h,H}_{\varepsilon,r_1,r_2}\right) - u_2(p_{\varepsilon,r_1,r_2})\right),\ D\partial_\nu\left(u_2^h(\Pi^H p_{\varepsilon,r_1,r_2}) - u_2(p_{\varepsilon,r_1,r_2})\right)\right)_{0,\Gamma},$$

$$I_5 = r_2\left(D\partial_\nu u_2(p_{\varepsilon,r_1,r_2}) - g_2, D\partial_\nu\left(u_2^h(\Pi^H p_{\varepsilon,r_1,r_2}) - u_2(p_{\varepsilon,r_1,r_2}) + u_2\left(p^{h,H}_{\varepsilon,r_1,r_2}\right) - u_2^h\left(p^{h,H}_{\varepsilon,r_1,r_2}\right)\right)\right)_{0,\Gamma},$$

$$I_6 = \varepsilon\left(p^{h,H}_{\varepsilon,r_1,r_2}, \Pi^H p_{\varepsilon,r_1,r_2} - p_{\varepsilon,r_1,r_2}\right)_{0,\Omega_0}.$$

By (43) and (45), we have

$$I_1 \leqslant \frac{r_1}{2}\left\|u_1(p_{\varepsilon,r_1,r_2}) - u_1^h\left(p^{h,H}_{\varepsilon,r_1,r_2}\right)\right\|^2_{0,\Gamma} + cr_1[HE^H(p_{\varepsilon,r_1,r_2}) + h^{3/2}(\|p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0} + \|g_2\|_{1/2,\Gamma})]^2 \tag{54}$$

and

$$I_4 \leqslant \frac{r_2}{2}\left\|D\partial_\nu\left(u_2(p_{\varepsilon,r_1,r_2}) - u_2^h\left(p^{h,H}_{\varepsilon,r_1,r_2}\right)\right)\right\|^2_{0,\Gamma} + cr_2[H^{1/2}E^H(p_{\varepsilon,r_1,r_2}) + h^{1/2}(\|p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0} + \|g_1\|_{3/2,\Gamma})]^2. \tag{55}$$

By (43), (34), (44) and (35), we can prove

$$I_2 \leqslant cr_1\left\|u_1(p_{\varepsilon,r_1,r_2}) - g_1\right\|_{0,\Gamma}\left[HE^H(p_{\varepsilon,r_1,r_2}) + h^{3/2}\left\|p_{\varepsilon,r_1,r_2} - p^{h,H}_{\varepsilon,r_1,r_2}\right\|_{0,\Omega_0}\right], \tag{56}$$

and

$$I_5 \leqslant cr_2\|D\partial_\nu u_2(p_{\varepsilon,r_1,r_2}) - g_2\|_{0,\Gamma}\left[H^{1/2}E^H(p_{\varepsilon,r_1,r_2}) + h^{1/2}\left\|p_{\varepsilon,r_1,r_2} - p^{h,H}_{\varepsilon,r_1,r_2}\right\|_{0,\Omega_0}\right]. \tag{57}$$

By using (8) and (23)

$$I_3 \leqslant cr_1 h^{3/2}\|g_1\|_{3/2,\Omega}\left[E^H(p_{\varepsilon,r_1,r_2}) + \left\|p_{\varepsilon,r_1,r_2} - p^{h,H}_{\varepsilon,r_1,r_2}\right\|_{0,\Omega_0}\right]. \tag{58}$$

Finally, by (38), we get

$$I_6 = 0. \tag{59}$$

Hence, the error bound (51) follows from (53)–(59). □

Regarding the quantity $E^H(p_{\varepsilon,r_1,r_2}) = \|p_{\varepsilon,r_1,r_2} - \Pi^H p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0}$, we note the next result.

Proposition 4.8. *If S_{r_1,r_2} defined in Section 2 is non-empty, then we have the following convergence result*:

$$\|p_{\varepsilon,r_1,r_2} - \Pi^H p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0} \to 0, \quad \text{as } H, \varepsilon \to 0. \tag{60}$$

Moreover, if $p_{\varepsilon,r_1,r_2} \in H^1(\Omega_0)$,

$$\|p_{\varepsilon,r_1,r_2} - \Pi^H p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0} \leqslant cH\|p_{\varepsilon,r_1,r_2}\|_{1,\Omega_0}. \tag{61}$$

Proof. By Proposition 3.1, $\|p_{\varepsilon,r_1,r_2} - p_{0,r_1,r_2}\|_{0,\Omega_0} \to 0$ as $\varepsilon \to 0$. Using the relation (40), we obtain

$$\begin{aligned}
&\|p_{\varepsilon,r_1,r_2} - \Pi^H p_{\varepsilon,r_1,r_2}\|_{0,\Omega_0} \\
&\quad \leqslant \|p_{\varepsilon,r_1,r_2} - p_{0,r_1,r_2}\|_{0,\Omega_0} + \|p_{0,r_1,r_2} - \Pi^H p_{0,r_1,r_2}\|_{0,\Omega_0} \\
&\qquad + \|\Pi^H(p_{0,r_1,r_2} - p_{\varepsilon,r_1,r_2})\|_{0,\Omega_0} \\
&\quad \leqslant 2\|p_{\varepsilon,r_1,r_2} - p_{0,r_1,r_2}\|_{0,\Omega_0} + \|p_{0,r_1,r_2} - \Pi^H p_{0,r_1,r_2}\|_{0,\Omega_0} \\
&\qquad \to 0 \quad \text{as } H, \varepsilon \to 0.
\end{aligned}$$

Moreover, from (41), we get (61). □

As in the previous subsection, under additional assumptions, we can deduce more concrete error bounds from Theorem 4.7. If Q_{ad} is a bounded set in the space Q, then there is a constant $c > 0$ independent of ε, r_1, r_2, h and H such that

$$\begin{aligned}
E^{h,H}_{\varepsilon,r_1,r_2} \leqslant\ & cr_1\left[h^{3/2} + HE^H(p_{\varepsilon,r_1,r_2})\right] \\
& + cr_2\left[h^{1/2} + H^{1/2}E^H(p_{\varepsilon,r_1,r_2})\right].
\end{aligned}$$

If we further assume $p_{\varepsilon,r_1,r_2} \in H^1(\Omega_0)$, then there is a constant $c > 0$ independent of ε, r_1, r_2, h and H such that

$$E^{h,H}_{\varepsilon,r_1,r_2} \leqslant cr_1(h^{3/2} + H^2) + cr_2(h^{1/2} + H^{3/2}).$$

Both these error bounds suggest the need to choose r_2 in the order of h to achieve better solution accuracy.

With proper choice of ε related to h and H, we can verify the convergence of $p^h_{\varepsilon,r_1,r_2}$ and $p^{h,H}_{\varepsilon,r_1,r_2}$ to p_{0,r_1,r_2}, see [9].

5. A numerical example

In this section, we present numerical results on a model problem. The main purpose is to demonstrate solution accuracy improvement achieved through the new general mathematical framework over that from the standard formulation of the BLT problem studied in [9]. Let $\Omega = (0,1)\times(0,1)$ be the problem domain and $\Omega_0 = (0.5,0.75)\times(0.5,0.75)$ the permissible region. Assume the absorption coefficient $\mu = 0.02$ and the reduced scattering coefficient $\mu\prime = 1.00$ in the whole domain Ω. We take $p \equiv 1$ pW for the true light source in Ω_0 and set $g_2 \equiv 0$ on the boundary $\Gamma = \partial\Omega$. The admissible set is taken to be $Q_{\mathrm{ad}} = \{q \in L^2(\Omega_0) | q \geqslant 0 \text{ a.e. in } \Omega_0\}$. We use uniform square partitions of the regions $\overline{\Omega}$ and $\overline{\Omega_0}$ with mesh parameters $h = H$, where h and H are the maximal diameters of the elements in the partitions $\mathscr{T}_h$ and $\mathscr{T}_{0,H}$. For finite element spaces V^h, V^h_0 and Q^H_{ad}, we use continuous piecewise bilinear functions and piecewise constant functions corresponding to the partitions $\mathscr{T}_h$ and $\mathscr{T}_{0,h}$. The error analysis presented in Section 4 for the linear element is valid also for the bilinear element we use for the numerical example. The boundary value of the finite element solution defined in (24) for a small meshsize is taken as the function g_1, and in this example, we choose 1/512 for the small meshsize.

For a variety of choices of the parameters h, ε, r_1 and r_2, we compute the approximate source function $p^{h,h}_{\varepsilon,r_1,r_2}$. We

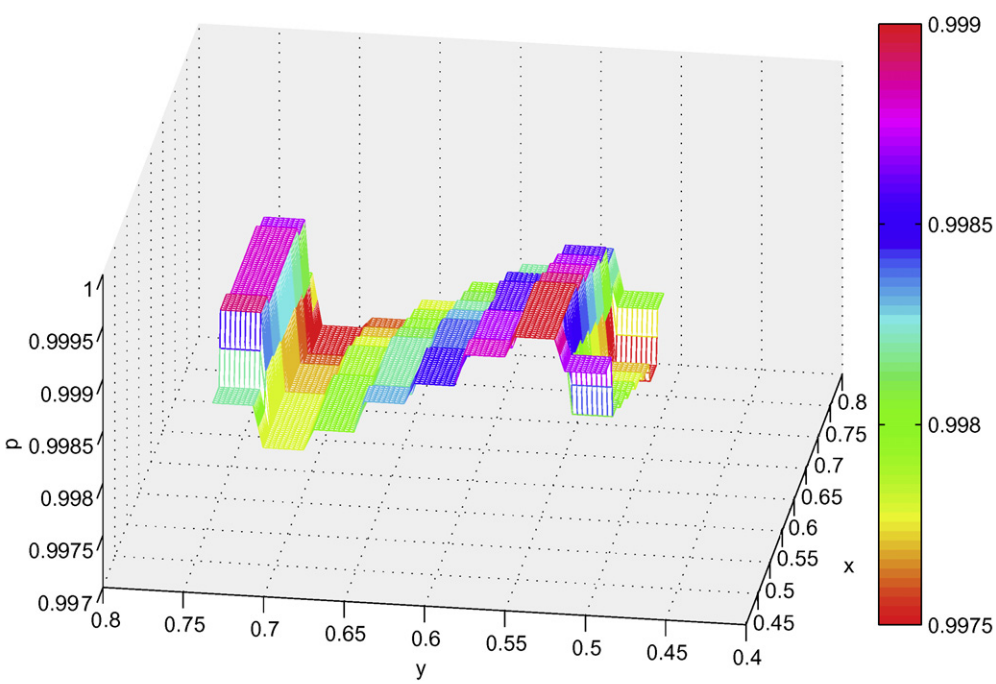

Fig. 1. $p^{h,h}_{\varepsilon,r_1,r_2}$ for $r_1 = r_2 = 0.5$, $\varepsilon = 10^{-4}$ and $h = 1/32$.

distinguish two cases according to whether the measurement data g_1 is noise-free or perturbed by noise at certain level.

Case 1. In this case, we use the exact measurement g_1. We show the reconstructed source function $p^{h,h}_{\varepsilon,r_1,r_2}$ and the error $p - p^{h,h}_{\varepsilon,r_1,r_2}$ for $h = 1/32$ with $r_1 = r_2 = 0.5$ and $h = 1/64$ with $r_1 = 0.8, r_2 = 0.2$ in Figs. 1–4, the regularization parameter $\varepsilon = 10^{-4}$ being used in both. The L^∞ and L^2 norms of the error $p - p^{h,h}_{\varepsilon,r_1,r_2}$ are provided in Tables 1–4. Tables 1 and 2 show the dependence of accuracy of the approximate solution $p^{h,h}_{\varepsilon,r_1,r_2}$ on the regularization parameter ε. We observe that accuracy of discrete solution improves when regularization parameter ε gets smaller. In Tables 3 and 4, we explore numerically improvement in the solution accuracy offered by our generalized formulation when the parameter r_2 is chosen properly, as compared to the standard formulation ($r_2 = 0$) studied in [9]. Moreover, Tables 3 and 4 provide numerical evidence of the theoretical results recorded in Theorems 4.4 and 4.7 in that the accuracy of approximate source function $p^{h,h}_{\varepsilon,r_1,r_2}$ improves as mesh parameter h gets smaller and r_2 gets small accordingly. We also observe that, for fixed r_2 (correspondingly $r_1 = 1 - r_2$) and ε, the error $\left\| p - p^{h,h}_{\varepsilon,r_1,r_2} \right\|$

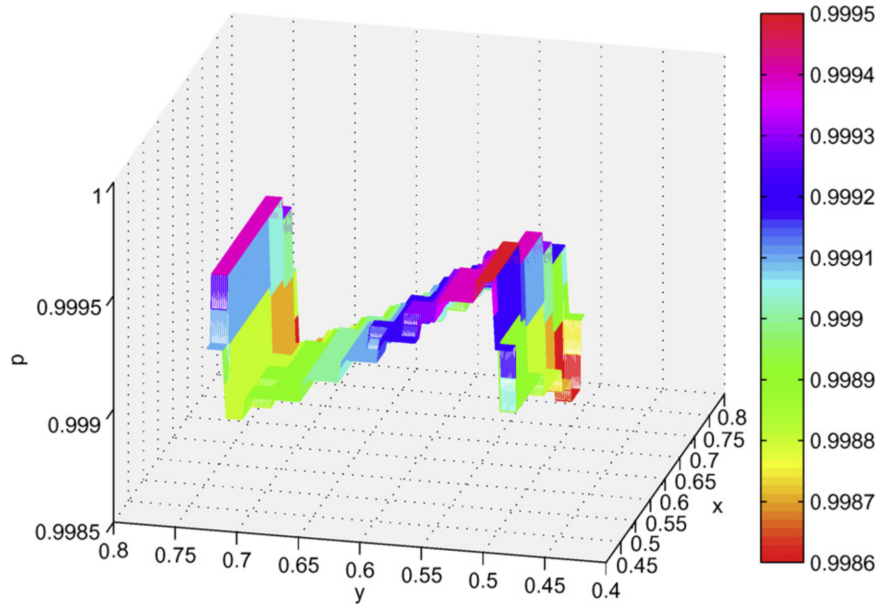

Fig. 2. $p^{h,h}_{\varepsilon,r_1,r_2}$ for $r_1 = 0.8, r_2 = 0.2$, $\varepsilon = 10^{-4}$ and $h = 1/64$.

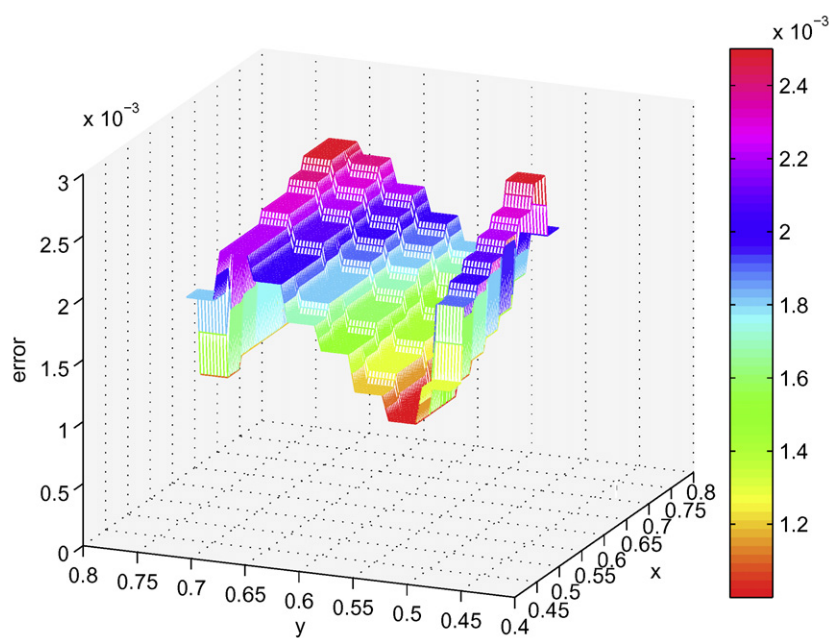

Fig. 3. $p - p^{h,h}_{\varepsilon,r_1,r_2}$ for $r_1 = r_2 = 0.5$, $\varepsilon = 10^{-4}$ and $h = 1/32$.

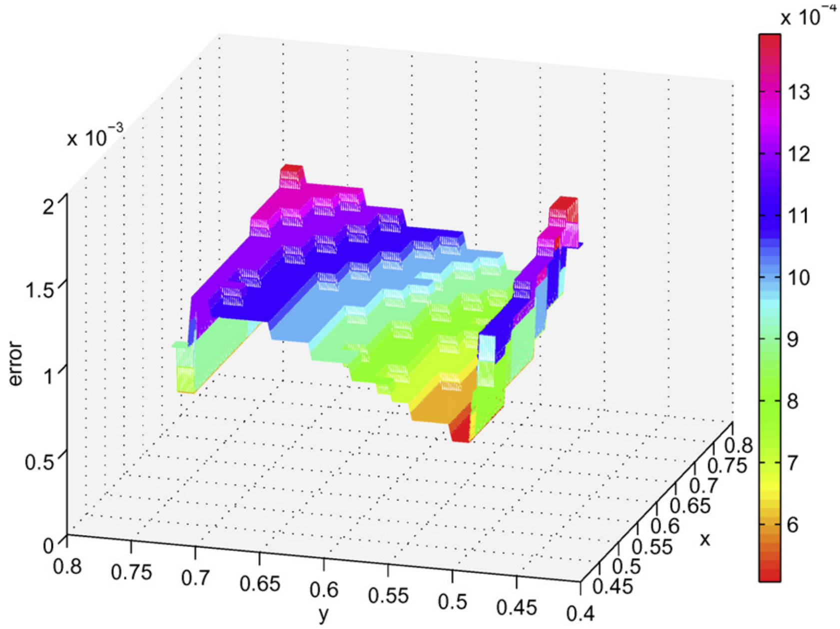

Fig. 4. $p - p^{h,h}_{\varepsilon,r_1,r_2}$ for $r_1 = 0.8, r_2 = 0.2$, $\varepsilon = 10^{-4}$ and $h = 1/64$.

Table 1
$\|p - p^{h,h}_{\varepsilon,r_1,r_2}\|_\infty$ for $r_1 = r_2 = 0.5$

ε	1	1×10^{-1}	1×10^{-2}	1×10^{-3}	1×10^{-4}
$h = 1/4$	2.3884e−2	2.1180e−2	2.0909e−2	2.0882e−2	2.0879e−2
$h = 1/8$	3.2687e−1	1.3853e−2	1.3839e−2	1.3837e−2	1.3837e−2
$h = 1/16$	5.7368e−1	7.9307e−3	7.8020e−3	7.7893e−3	7.7881e−3
$h = 1/32$	6.2180e−1	3.0223e−3	2.9281e−3	2.4965e−3	2.9389e−3
$h = 1/64$	8.3383e−1	4.9602e−3	4.2834e−3	4.6899e−3	4.2158e−3

Table 2
$\|p - p^{h,h}_{\varepsilon,r_1,r_2}\|_0$ for $r_1 = r_2 = 0.5$

ε	1	1×10^{-1}	1×10^{-2}	1×10^{-3}	1×10^{-4}
$h = 1/4$	5.9709e−3	5.2950e−3	5.2273e−3	5.2204e−3	5.2197e−3
$h = 1/8$	6.4347e−2	2.9377e−3	2.8698e−3	2.8632e−3	2.8626e−3
$h = 1/16$	7.4374e−2	1.4438e−3	1.3779e−3	1.3714e−3	1.3707e−3
$h = 1/32$	8.5888e−2	4.7031e−4	4.1167e−4	4.2203e−4	4.0359e−4
$h = 1/64$	8.1559e−2	5.4279e−4	5.1204e−4	5.5401e−4	4.8618e−4

stabilizes as h becomes small. This is expected from our theoretical analysis. In fact, from Theorems 4.4 or 4.7, a relation of the form $r_2 = O(r_1 h)$ is needed if we want to get the best bound for $\left\|p_{\varepsilon,r_1,r_2} - p^{h,h}_{\varepsilon,r_1,r_2}\right\|$, that is, smaller r_2 is needed corresponding to smaller h for better solution accuracy.

Case 2. In applications, the measured data g_1 is subject to noise. The effect of the noise on the accuracy of the approximate solution $p^{h,h}_{\varepsilon,r_1,r_2}$ for different meshes is reported in Table 5. For each grid parameter h, we consider the noise level δ at 1%, 10%, and 20%, respectively. Because we use random noise, at the same noise level δ, we compute the discrete solution $p^{h,h}_{\varepsilon,r_1,r_2}$ 10 times and use their average value in computing the L^2-norm errors. We observe that the

Table 3
$\|p - p^{h,h}_{\varepsilon,r_1,r_2}\|_\infty$ for $\varepsilon = 1\times10^{-4}$

h	1/4	1/8	1/16	1/32	1/64
$r_1 = 0.1,\ r_2 = 0.9$	2.0994e−2	9.6667e−3	2.3319	3.3488	5.1812e−2
$r_1 = 0.2,\ r_2 = 0.8$	2.0489e−2	1.2257e−2	4.0590e−3	7.8846e−3	2.2117e−3
$r_1 = 0.5,\ r_2 = 0.5$	2.0879e−2	1.3837e−2	7.7881e−3	2.4831e−3	4.2158e−3
$r_1 = 0.8,\ r_2 = 0.2$	2.0976e−2	1.4250e−2	8.7734e−3	4.3573e−3	1.4211e−3
$r_1 = 0.9,\ r_2 = 0.1$	2.0994e−2	1.4329e−2	8.9622e−3	4.5554e−3	2.0181e−3
$r_1 = 1,\ r_2 = 0$	2.1009e−2	1.4395e−2	9.1163e−3	4.8785e−3	2.5156e−3

Table 4
$\|p - p^{h,h}_{\varepsilon,r_1,r_2}\|_0$ for $\varepsilon = 1\times10^{-4}$

h	1/4	1/8	1/16	1/32	1/64
$r_1 = 0.1,\ r_2 = 0.9$	5.2486e−3	2.0291e−3	2.5410e−1	3.1858e−1	6.8901e−3
$r_1 = 0.2,\ r_2 = 0.8$	5.1224e−3	2.5484e−3	7.8245e−4	9.3063e−4	2.9745e−3
$r_1 = 0.5,\ r_2 = 0.5$	5.2197e−3	2.8626e−3	1.3707e−3	4.0359e−4	4.8618e−4
$r_1 = 0.8,\ r_2 = 0.2$	5.2441e−3	2.9412e−3	1.5256e−3	7.3534e−4	2.4890e−4
$r_1 = 0.9,\ r_2 = 0.1$	5.2486e−3	2.9558e−3	1.5545e−3	7.8147e−4	3.4250e−4
$r_1 = 1,\ r_2 = 0$	5.2522e−3	2.9675e−3	1.5778e−3	8.1757e−4	4.2404e−4

Table 5
$\|p - p^{h,h}_{\varepsilon,r_1,r_2}\|_0$ for $r_1 = r_2 = 0.5$, $\varepsilon = 1 \times 10^{-4}$

δ	0	1%	10%	20%
$h = 1/4$	5.2197e−3	5.4890e−3	1.3006e−2	2.4969e−2
$h = 1/8$	2.8626e−3	3.1263e−3	1.3386e−2	2.4202e−2
$h = 1/16$	1.3707e−3	1.7355e−3	1.1974e−2	2.3674e−2
$h = 1/32$	4.0359e−4	1.2963e−3	1.1979e−2	2.3882e−2
$h = 1/64$	4.8618e−4	1.2076e−3	1.1100e−2	2.2164e−2

numerical solution is quite stable with respect to the perturbation in the measurement data g_1.

6. Conclusion remarks

Bioluminescence tomography is a new modality in optical imaging and is attracting more and more attention. In this paper, we present a new general reconstruction method for BLT based on finite element discretization. Because of the ill-posedness of BLT, we adopt Tikhonov regularization by introducing a parameter ε and explore the solution behavior as $\varepsilon \to 0$. We analyze the theoretical properties of BLT including the existence, uniqueness and stability of light source function in our new general framework. Finite element methods are applied to the practical reconstruction and numerical results are reported to illustrate improvement of solution accuracy obtainable from the proposed general framework over the standard formulation for BLT. The improvement is achieved with a slight increment in the amount of work as compared to the standard formulation.

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