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A posteriori error estimates for discontinuous Galerkin methods of obstacle problems

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ABSTRACT

We present a posteriori error analysis of discontinuous Galerkin methods for solving the obstacle problem, which is a representative elliptic variational inequality of the first kind. We derive reliable error estimators of the residual type. Efficiency of the estimators is theoretically explored and numerically confirmed.

1. Introduction

Since the pioneering work of Babuška and Rheinboldt [1], adaptive finite element methods based on a posteriori error estimation have attracted many researchers, and a variety of different a posteriori error estimators have been proposed and analyzed. A posteriori error estimates and adaptive mesh-refinement techniques are well established for linear partial differential equations, and we refer the reader to [2–4].

The discontinuous Galerkin (DG) method was first introduced for a hyperbolic equation. In recent years, DG methods have been widely used for solving various types of partial differential equations. A historical account of their development can be found in [5]. Advantages of DG methods include the flexibility of mesh-refinements and construction of local shape function spaces (hp-adaptivity), and the increase of locality in discretization, which is of particular interest for parallel computing. For standard finite element methods with conforming and shape-regular meshes, one needs to choose the mesh refinement rule carefully to maintain conformity and shape regularity. In particular, hanging nodes are not allowed. For DG methods, the approximate functions are allowed to be discontinuous across the element boundaries, so general meshes with hanging nodes and elements of different shapes are allowed.

Discontinuous Galerkin methods for elliptic equations were independently proposed in the 1970s. A unified error analysis of DG methods for elliptic problems was given in [6,7]. A unified approach was presented in [8] on a posteriori error control for DG methods. In [9], a unified framework is given on DG methods for elliptic variational inequalities of both

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first and second kinds. DG methods for the Signorini problem and the quasistatic contact problem are also studied in [10,11], respectively. In this paper, we focus on a posteriori error analysis of DG methods for solving an obstacle problem.

The obstacle problem. The problem is to find $u \in K := \{v \in V : v \geq \psi \text{ a.e in } \Omega\}$ such that

$$a(u, v - u) \geq (f, v - u)_\Omega \quad \forall v \in K, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain with boundary $\partial\Omega$, $f \in L^2(\Omega)$, $\psi \in H^1(\Omega)$ with $\psi \leq 0$ on $\partial\Omega$, $V = H_0^1(\Omega)$, $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, and $(\cdot, \cdot)_\Omega$ denotes the L^2-product in the domain Ω. The obstacle problem is an example of elliptic variational inequalities of the first kind [12] and has a unique solution [13]. This problem arises in various applications, such as the membrane deformation in elasticity theory, and the non-parametric minimal and capillary surfaces as geometrical problems. The elastic–plastic torsion problem and the cavitation problem in the theory of lubrication also can be regarded as obstacle type problems. A variety of numerical methods have been developed to solve the discretized obstacle problem, such as the relaxation method, multilevel projection method, multigrid method and primal–dual active set method [14].

It is difficult to develop a posteriori error estimates to variational inequalities due to the inequality feature. Nevertheless, numerous papers can be found on a posteriori error estimation of finite element methods for obstacle problems, e.g., [15–17]. In [18], Braess showed how to derive a posteriori error estimators for the standard finite element methods of the obstacle problem from the theory for linear equations. We will follow his idea and establish residual type error estimators of discontinuous Galerkin methods for the obstacle problem. For this purpose, we introduce a Lagrange multiplier $\sigma = \sigma(u) \in V^*$ [18,12] by

$$\langle \sigma, v \rangle := a(u, v) - (f, v) \quad \forall v \in V. \tag{1.2}$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V^* = H^{-1}(\Omega)$ and $V = H_0^1(\Omega)$. We will write (σ, v) for $\langle \sigma, v \rangle$ when σ can be regarded as an L^2 function. The solution u is then characterized by the linear equation:

$$a(u, v) = (f, v) + \langle \sigma, v \rangle \; \forall v \in V.$$

For the obstacle problem (1.1), a result by Brezis and Stampacchia [19] states that if the domain Ω is smooth and $f \in L^s(\Omega)$, $\psi \in W^{2,s}(\Omega)$ for some $s \in (1, \infty)$, then the solution $u \in W^{2,s}(\Omega)$. We will assume $f \in L^2(\Omega)$ and $\psi \in H^2(\Omega)$, so $u \in H^2(\Omega)$. Then $-\Delta u = f + \sigma$ a.e. in Ω. We can rewrite (1.1) as

$$\langle \sigma, v - u \rangle \geq 0 \quad \forall v \in K. \tag{1.3}$$

Let $v = u + \varphi$ for all $\varphi \in V_+$ in the above inequality to get

$$\langle \sigma, \varphi \rangle \geq 0 \quad \forall \varphi \in V_+ := \{v \in V : v \geq 0 \text{ a.e. in } \Omega\}. \tag{1.4}$$

With the constraint condition $u \geq \psi$, we define the contact set $\mathcal{C} = \{x \in \Omega : u(x) = \psi(x)\}$ and the noncontact set $\mathcal{D} = \Omega \backslash \mathcal{C}$. Then, $\sigma \geq 0$ in Ω, $\sigma = -\Delta u - f$ in $\mathcal{C}$, and $\sigma = 0$ in $\mathcal{D}$. For a subset $\omega \subset \Omega$, define

$$|\sigma|_{*,\omega} := \sup\{\langle \sigma, v \rangle_\omega : v \in H_0^1(\omega), \; |v|_{1,\omega} = 1\},$$

where $\langle \cdot, \cdot \rangle_\omega$ denotes the duality pairing between $H^{-1}(\omega)$ and $H_0^1(\omega)$, and $|\cdot|_{1,\omega}$ is the semi-norm on $H^1(\omega)$. We omit the subscript ω if $\omega = \Omega$. Let $a_\omega(w, v) = \int_\omega \nabla w \cdot \nabla v \, dx$. We have $|\sigma|_{*,\omega} = |w|_{1,\omega}$, where $w \in H_0^1(\omega)$ is the solution of the auxiliary equation

$$a_\omega(w, v) = \langle \sigma, v \rangle_\omega \quad \forall v \in H_0^1(\omega). \tag{1.5}$$

2. Discontinuous Galerkin formulations

Notation. We denote by $\{\mathcal{T}_j\}$ a family of subdivisions of $\overline{\Omega}$ into (closed) triangles such that the minimal angle condition is satisfied. For a triangulation $\mathcal{T}_h$, let $\mathcal{E}_h$ be the union of all edges and $\mathcal{E}_h^i := \mathcal{E}_h \backslash \partial\Omega$ the union of all interior edges. Let $h_K = \text{diam}(K)$ for $K \in \mathcal{T}_h$ and $h_e = \text{length}(e)$ for $e \in \mathcal{E}_h$. We denote by $\mathbb{N}_h$ the set of nodes of $\mathcal{T}_h$. For any element $K \in \mathcal{T}_h$, define the patch set $\omega_K := \cup\{T \in \mathcal{T}_h, T \cap K \neq \emptyset\}$, and for any edge e shared by two elements K_1 and K_2, define $\omega_e := K_1 \cup K_2$. For a scalar-valued function v and a vector-valued function q, let $v_i = v|_{\partial K_i}$, $q_i = q|_{\partial K_i}$, and $n_i = n|_{\partial K_i}$ be the unit normal vector external to ∂K_i. Then define the average $\{\cdot\}$ and the jump $[\cdot]$ on $e \in \mathcal{E}_h^i$ as follows: $\{v\} = \frac{1}{2}(v_1 + v_2)$, $[v] = v_1 n_1 + v_2 n_2$, $\{q\} = \frac{1}{2}(q_1 + q_2)$, $[q] = q_1 \cdot n_1 + q_2 \cdot n_2$. If $e \subset \partial\Omega$, the set of boundary edges, we let $[v] = vn$, $\{q\} = q$, where n is the outward unit normal. We introduce the following linear finite element spaces and set:

$$V_h = \{v_h \in L^2(\Omega) : v_h|_K \in P_1(K) \, \forall K \in \mathcal{T}_h\},$$
$$K_h = \{v_h \in V_h : v_h(x) \geq \psi(x) \text{ at all nodes } x \text{ of } \mathcal{T}_h\},$$
$$W_h = \{w_h \in [L^2(\Omega)]^2 : w_h|_K \in [P_1(K)]^2 \, \forall K \in \mathcal{T}_h\}.$$

Note that $K_h = \{v_h \in V_h : \ v_h \geq \psi_I \text{ in } \Omega\}$, ψ_I being the continuous piecewise linear interpolant of ψ. Denote by ∇_h the broken gradient whose restriction to each element $K \in \mathcal{T}_h$ is equal to ∇. Define some seminorms and norms by the following relations:

$$|v|_{1,h}^2 = \sum_{K\in\mathcal{T}_h} |v|_{1,K}^2, \qquad |v|_{1,K}^2 = \|\nabla v\|_K^2, \qquad \|v\|_K^2 = \int_K v^2 dx, \qquad \|v\|_e^2 = \int_e v^2 ds.$$

Throughout this paper, C denotes a generic positive constant dependent on the minimal angle condition but not on the element sizes, which may take different values at different occurrences.

Discontinuous Galerkin schemes. Here we take the LDG method [20] as an example to demonstrate the a posteriori error analysis of DG methods in solving the obstacle problem (1.1). The derivation and analysis of the a posteriori error estimators for the LDG method can be extended to other DG methods studied in [9]. The LDG method for solving the obstacle problem is: Find $u_h \in K_h$ such that

$$B_h(u_h, v_h - u_h) \geq (f, v_h - u_h) \quad \forall\, v_h \in K_h, \tag{2.1}$$

where

$$\begin{aligned} B_h(w, v) = {} & (\nabla_h w, \nabla_h v)_\Omega - \langle [w], \{\nabla_h v\}\rangle_{\mathcal{E}_h} - \langle \{\nabla_h w\}, [v]\rangle_{\mathcal{E}_h} - \langle \beta \cdot [w], [\nabla_h v]\rangle_{\mathcal{E}_h^i} \\ & - \langle [\nabla_h w], \beta \cdot [v]\rangle_{\mathcal{E}_h^i} + (r([w]) + l(\beta \cdot [w]), r([v]) + l(\beta \cdot [v]))_\Omega + \alpha^j(w, v). \end{aligned} \tag{2.2}$$

Here we use the notation $(w, v)_\Omega$, $\langle w, v\rangle_{\mathcal{E}_h}$, and $\langle w, v\rangle_{\mathcal{E}_h^i}$ for $\int_\Omega wv\,dx$, $\int_{\mathcal{E}_h} wv\,ds$, and $\int_{\mathcal{E}_h^i} wv\,ds$. $\beta \in [L^2(\mathcal{E}_h^i)]^2$ is a vector-valued function which is constant on each edge; $\alpha^j(w, v) = \int_{\mathcal{E}_h} \mu[w] \cdot [v]\,ds$ is the penalty (stabilization) term with the penalty weighting function $\mu : \mathcal{E}_h \to \mathbb{R}$ given by $\eta_e h_e^{-1}$ on each $e \in \mathcal{E}_h$, η_e being a positive number on e; $r : [L^2(\mathcal{E}_h)]^2 \to W_h$ and $l : L^2(\mathcal{E}_h^i) \to W_h$ are lifting operators defined by

$$\int_\Omega r(q) \cdot w_h dx = -\int_{\mathcal{E}_h} q \cdot \{w_h\}\,ds, \qquad \int_\Omega l(v) \cdot w_h\,dx = -\int_{\mathcal{E}_h^i} v\,[w_h]\,ds \quad \forall\, w_h \in W_h.$$

Let $u \in H^2(\Omega)$ be the solution of (1.1) and $v \in V$. Note that on $e \in \mathcal{E}_h^i$, $[u] = 0$, $[\nabla u] = 0$, and $[v] = 0$. We have

$$B_h(u, v) = \int_\Omega -\Delta u\, v\,dx = \int_\Omega (f + \sigma)\, v\,dx = (f + \sigma, v) \quad \forall\, v \in V. \tag{2.3}$$

For (2.1), like (1.2), define $\sigma_h = \sigma_h(u_h) \in V_h^*$ by

$$\langle \sigma_h, v_h\rangle := B_h(u_h, v_h) - (f, v_h) \quad \forall\, v_h \in V_h. \tag{2.4}$$

Note that we can view σ_h as an element in V_h and it is determined from

$$(\sigma_h, v_h) = B_h(u_h, v_h) - (f, v_h) \quad \forall\, v_h \in V_h. \tag{2.5}$$

Obviously, u_h is also the DG approximation of the solution $z \in V$ of the linear elliptic problem:

$$B_h(z, v) = (f + \sigma_h, v) \quad \forall\, v \in V. \tag{2.6}$$

Since $f + \sigma_h \in L^2(\Omega)$, $z \in H^2(\Omega)$ [21]. Thus, $-\Delta z = f + \sigma_h$ in Ω. From (2.1) and the definition of σ_h, we get

$$\langle \sigma_h, v_h - u_h\rangle \geq 0 \quad \forall\, v_h \in K_h.$$

Similar to (1.4), we have

$$\langle \sigma_h, v_h\rangle \geq 0 \quad \forall\, v_h \in V_h, v_h \geq 0. \tag{2.7}$$

Since $\langle \sigma_h, v\rangle \geq 0$ does not hold true for all $v \in V_+$, we write $\sigma_h = \sigma_h^+ + \sigma^{ce}$ with $(\sigma_h^+, v) \geq 0$ for all $v \in V_+$, σ^{ce} carrying the consistency error. Following the choice of ϱ_h in [15], we define σ_h^+ as

$$\sigma_h^+ := \sum_{K\in\mathcal{T}_h} \sum_{p\in K\cap\mathbb{N}_h} \big(B_h(u_h, \phi_K^p) - (f, \phi_K^p)\big)\phi_K^p/(1, \phi_K^p), \tag{2.8}$$

where ϕ_K^p is the nodal basis of vertex p in element K. From the definition of σ_h and (2.7), it is easy to know $\sigma_h^+ \geq 0$ in Ω. Next we show a very important property of σ_h^+,

$$\sigma_h^+(p)(u_h(p) - \psi_I(p)) = 0 \quad \text{for all } p \in \mathbb{N}_h. \tag{2.9}$$

First, we know $u_h(p) - \psi_I(p) \geq 0$, so $\sigma_h^+(p)(u_h(p) - \psi_I(p)) \geq 0$. Let $v_h(p) = \psi_I(p)$ and $v_h(x) = u_h(x)$ for all other nodes $x \in \mathbb{N}_h$, then

$$(\psi_I(p) - u_h(p))B_h(u_h, \phi_K^p) = B_h(u_h, v_h - u_h) \geq (f, v_h - u_h) = (\psi_I(p) - u_h(p))(f, \phi_K^p).$$

After a division by $(1, \phi_K^p)$ on both sides, we obtain $(\psi_I(p) - u_h(p))\sigma_h^+(p) \geq 0$. So (2.9) is valid.

We group the elements into three parts:

$$\mathcal{C}_h = \cup\{K \in \mathcal{T}_h : u_h = \psi_I \text{ on } \omega_K\}, \qquad \mathcal{D}_h = \cup\{K \in \mathcal{T}_h : u_h > \psi_I \text{ on } K\}, \qquad \mathcal{F}_h = \overline{\Omega}\backslash(\mathcal{C}_h \cup \mathcal{D}_h).$$

Based on this decomposition $\overline{\Omega} = \mathcal{C}_h \cup \mathcal{D}_h \cup \mathcal{F}_h$ and the relation (2.9), we obtain

$$\sigma_h^+ = 0 \quad \text{in } \mathcal{D}_h. \tag{2.10}$$

Given $v_h \in V_h$, written $v_h = \sum_{K\in\mathcal{T}_h}\sum_{j=1}^3 \alpha_K^{(j)}\phi_K^{(j)}$, we construct a function $\chi \in V_h \cap H_0^1(\Omega)$ as follows: At every interior node of the conforming mesh $\mathcal{T}_h$, the value of χ is set to be the average of the values of v_h computed from all the elements sharing that node, and $\chi = 0$ at the boundary nodes. For each $\nu \in \mathbb{N}_h$, let $\omega_\nu = \{K \in \mathcal{T}_h : \nu \in K\}$ and denote its cardinality by $|\omega_\nu|$, which is bounded by a constant depending only on the minimal angle condition of the mesh. To each node ν, the associated basis function $\phi^{(\nu)}$ is given by

$$\text{supp}\phi^{(\nu)} = \bigcup_{K\in\omega_\nu} K, \qquad \phi^{(\nu)}|_K = \phi_K^{(j)} \quad \text{for } x_K^{(j)} = \nu.$$

Then we define $\chi \in V_h \cap H_0^1(\Omega)$ by $\chi = \sum_{\nu\in\mathbb{N}_h}\beta^{(\nu)}\phi^{(\nu)}$, where

$$\beta^{(\nu)} = \frac{1}{|\omega_\nu|}\sum_{x_K^{(j)}=\nu}\alpha_K^{(j)} \quad \text{if } \nu \in \mathbb{N}_h \cap \Omega, \qquad \beta^{(\nu)} = 0 \quad \text{if } \nu \in \mathbb{N}_h \cap \partial\Omega.$$

For nonconforming meshes, let $\mathbb{P}_h^0$ be the set of all hanging nodes. Then we construct χ from v_h same as conforming mesh case on all the nodes $\nu \in \mathbb{N}_h\backslash\mathbb{P}_h^0$.

In [22, Theorem 2.2 for the conforming mesh and Theorem 2.3 for the nonconforming mesh], a proof of the following lemma was given. Here we just state the result for the conforming mesh; the same result holds for the nonconforming mesh.

Lemma 2.1. *Let $\mathcal{T}_h$ be a conforming mesh consisting of triangles. Then for any $v_h \in V_h$ there exists $\chi \in V_h \cap H_0^1(\Omega)$ satisfying*

$$\sum_{K\in\mathcal{T}_h}\|\nabla(v_h - \chi)\|_K^2 \le C\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[v_h]\|_e^2. \tag{2.11}$$

3. The case of an affine obstacle

Now we follow the ideas in [18] to derive a posteriori error estimators of DG methods for the obstacle problem. We give detailed derivation and analysis of an a posteriori error estimator for the LDG method [20]. Extension to other DG methods can be derived by same arguments. We distinguish two cases depending on whether the obstacle function is affine. First, we consider the case of an affine obstacle $\psi \in P_1(\Omega)$; in this case, $\psi_I = \psi$.

3.1. Reliable estimator for the LDG method

From (2.3) and (2.6), for all $v \in V$, we have

$$B_h(u_h - u, v) = B_h(u_h - z, v) + B_h(z - u, v) = B_h(u_h - z, v) + (\sigma_h - \sigma, v).$$

Denote the error by $e := u_h - u$. From the definition (2.2) and $[v] = 0$ on each $e \in \mathcal{E}_h$, the above equation becomes

$$\begin{aligned}(\nabla_h e, \nabla_h v)_\Omega - \langle[e], \{\nabla_h v\}\rangle_{\mathcal{E}_h} - \langle\beta\cdot[e], [\nabla_h v]\rangle_{\mathcal{E}_h^i} &= (\nabla_h(u_h - z), \nabla_h v)_\Omega - \langle[u_h - z], \{\nabla_h v\}\rangle_{\mathcal{E}_h} \\ &\quad - \langle\beta\cdot[u_h - z], [\nabla_h v]\rangle_{\mathcal{E}_h^i} + (\sigma_h - \sigma, v).\end{aligned}$$

Then,

$$\begin{aligned}\int_\Omega \nabla_h e\cdot\nabla_h v\,dx &= \int_\Omega \nabla_h(u_h - z)\cdot\nabla_h v\,dx - \int_{\mathcal{E}_h}[u_h - z - e]\cdot\{\nabla_h v\}ds \\ &\quad - \int_{\mathcal{E}_h^i}\beta\cdot[u_h - z - e][\nabla_h v]\,ds + (\sigma_h - \sigma, v).\end{aligned}$$

Note that $u_h - z - e = u - z$, $[u - z] = 0$ on each $e \in \mathcal{E}_h$. We have

$$\int_\Omega \nabla_h e\cdot\nabla_h v\,dx = \int_\Omega \nabla_h(u_h - z)\cdot\nabla_h v\,dx + (\sigma_h - \sigma, v). \tag{3.1}$$

Let $\chi \in V_h \cap H_0^1(\Omega)$ be the function constructed from u_h, satisfying (2.11) for $v_h = u_h$. Taking $v := \chi - u = \chi - u_h + u_h - u$ in (3.1) and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|e|_{1,h}^2 &\le |u_h - z|_{1,h}|\chi - u|_{1,h} + |e|_{1,h}|\chi - u_h|_{1,h} + (\sigma_h - \sigma, \chi - u)\\ &\le 2|u_h - z|_{1,h}^2 + \frac{1}{2}|e|_{1,h}^2 + \frac{5}{4}|\chi - u_h|_{1,h}^2 + (\sigma_h - \sigma, \chi - u),\end{aligned}$$

$$|e|_{1,h}^2 \le 4|u_h - z|_{1,h}^2 + \frac{5}{2}|\chi - u_h|_{1,h}^2 + 2(\sigma_h - \sigma, \chi - u). \tag{3.2}$$

With the estimate of $|\chi - u_h|_{1,h}^2$ by (2.11), we obtain from (3.2) that

$$|e|_{1,h}^2 \le C\Big(|u_h - z|_{1,h}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2 + (\sigma_h - \sigma, \chi - u)\Big). \tag{3.3}$$

We now recall a result from [8]. For the Poisson problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

rewrite it as the first order system

$$p = \nabla u, \qquad -\nabla\cdot p = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{3.4}$$

Then the DG formulation for this problem is

$$\int_\Omega p_h\cdot\tau_h\,dx = -\int_\Omega u_h\,\nabla_h\cdot\tau_h\,dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\hat{u}_h\,n_K\cdot\tau_h\,ds \quad \forall\,\tau_h\in W_h, \tag{3.5}$$

$$\int_\Omega p_h\cdot\nabla_h v_h\,dx = \int_\Omega f\,v_h\,dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\hat{p}_h\cdot n_K\,v_h\,ds \quad \forall\,v_h\in V_h, \tag{3.6}$$

where $\hat{u}_h$ and $\hat{p}_h$ are numerical fluxes. Different choices of the numerical fluxes lead to different DG methods. The following result holds for the LDG method and the methods discussed in [8].

Theorem 3.1. *Let $u \in V := H_0^1(\Omega)$ and $p \in W := L^2(\Omega)^2$ be the solution of the problem (3.4), and $u_h \in V_h$ and $p_h \in W_h$ the solution of the problem (3.5)–(3.6). Then*

$$\|p - p_h\| \le C\,(\eta_* + \zeta_*)\,,$$

where

$$\eta_*^2 := \sum_{K\in\mathcal{T}_h} h_K^2\|\mathrm{div}p_h + f\|_K^2 + \sum_{e\in\mathcal{E}_h^i} h_e\|[p_h]\|_e^2, \qquad \zeta_*^2 := \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2.$$

Corollary 3.2. *With the same notation as in Theorem 3.1, we have*

$$\|\nabla u - \nabla_h u_h\| \le C(\eta + \zeta_*),$$

where

$$\eta^2 := \sum_{K\in\mathcal{T}_h} h_K^2\|\Delta u_h + f\|_K^2 + \sum_{e\in\mathcal{E}_h^i} h_e\|[\nabla_h u_h]\|_e^2.$$

Proof. By [8, Lemma 2.1], for all $v_h \in V_h$, we have

$$\|r([v_h])\|^2 \le C\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[v_h]\|_e^2, \qquad \|l(\beta\cdot[v_h])\|^2 \le C\sum_{e\in\mathcal{E}_h^i} h_e^{-1}\|[v_h]\|_e^2.$$

From [7, (3.9)],

$$p_h = \nabla_h u_h - r([\hat{u}_h - u_h]) - l(\{\hat{u}_h - u_h\}).$$

Then

$$\|\nabla u - \nabla_h u_h\| \le \|\nabla u - p_h\| + \|p_h - \nabla_h u_h\| \le C\,(\eta_* + \zeta_*) + \|r([\hat{u}_h - u_h])\| + \|l(\{\hat{u}_h - u_h\})\|.$$

From the choices of numerical fluxes $\hat{u}_h$ in Table 3.1 of [7], we have

$$[\hat{u}_h - u_h] = -[u_h] \text{ or } 0, \qquad \{\hat{u}_h - u_h\} = -\beta\cdot[u_h] \text{ or } 0.$$

Then,

$$\|r([\hat{u_h} - u_h])\| \le C \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2, \qquad \|l(\{\hat{u_h} - u_h\})\| \le C \sum_{e\in\mathcal{E}_h^i} h_e^{-1}\|[u_h]\|_e^2,$$

$$\|p_h - \nabla_h u_h\| \le \zeta_* \quad \text{and} \quad \|\nabla u - \nabla_h u_h\| \le C\,(\eta_* + \zeta_*)\,.$$

Finally, by the inverse inequality and trace inequality, we get

$$\begin{aligned}\eta_*^2 &\le 2\Big(\eta^2 + \sum_{K\in\mathcal{T}_h} h_K^2\|\mathrm{div}(p_h - \nabla u_h)\|_K^2 + \sum_{e\in\mathcal{E}_h^i} h_e\|[p_h - \nabla_h u_h]\|_e^2\Big)\\ &\le 2\eta^2 + C\sum_{K\in\mathcal{T}_h}\|p_h - \nabla u_h\|_K^2 = 2\eta^2 + C\|p_h - \nabla_h u_h\|^2 \le 2\eta^2 + C\zeta_*^2.\end{aligned}$$

Therefore,

$$\eta_* \le C\,(\eta + \zeta_*)$$

and the result is proved. ■

Define the interior residuals and edge-based jumps

$$R_K := \Delta u_h + f + \sigma_h = f + \sigma_h \quad \text{for } K \in \mathcal{T}_h, \qquad R_e := [\nabla_h u_h] \quad \text{for } e \in \mathcal{E}_h.$$

Then the local estimators are

$$\eta_K := \Big(h_K^2\|R_K\|_K^2 + \sum_{e\in\partial K\cap\mathcal{E}_h^i} h_e\|R_e\|_e^2\Big)^{1/2}, \qquad \eta_{\partial K} := \Big(\sum_{e\in\partial K} h_e^{-1}\|[u_h]\|_e^2\Big)^{1/2}. \tag{3.7}$$

Applying Corollary 3.2 to $|u_h - z|_{1,h}$, we obtain from (3.3)

$$|e|_{1,h}^2 \le C\Big(\sum_{K\in\mathcal{T}_h}\eta_K^2 + \sum_{K\in\mathcal{T}_h}\eta_{\partial K}^2 + (\sigma_h - \sigma,\, \chi - u)\Big). \tag{3.8}$$

Before giving an estimate of $(\sigma_h - \sigma, \chi - u)$, we introduce the following result [17].

Lemma 3.3. *Let q_h be a continuous piecewise linear function over $\mathcal{T}_h$, $p \in \mathbb{N}_h \cap \Omega$ an interior node with $p \in K$, $K \in \mathcal{T}_h$. Suppose $q_h \ge 0$ and $q_h(p) = 0$. Then*

$$\|q_h\|_K \le Ch_K\Big(\sum_{e\in\mathbb{E}_h(p)} h_e\|[\nabla q_h]\|_e^2\Big)^{1/2}, \qquad \mathbb{E}_h(p) := \{e \in \mathcal{E}_h : p \in e\}.$$

Proposition 3.4. *Assume ψ is an affine function. Then for any fixed $\epsilon \in (0, 1)$,*

$$\langle\sigma_h - \sigma,\, \chi - u\rangle \le \epsilon|u_h - u|_{1,h}^2 + C\Big(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2 + |\sigma_h - \sigma_h^+|_*^2 + \sum_{K\in\mathbb{F}_h, K\cap\partial\Omega=\emptyset}\ \sum_{e\in\mathbb{E}_h(p)} h_e\|[\nabla u_h]\|_e^2 + \eta_{\mathcal{F}_h}^2\Big), \tag{3.9}$$

where

$$\eta_{\mathcal{F}_h}^2 = \sum_{K\in\mathbb{F}_h} h_K^4\|\nabla\sigma_h^+\|_K^2 + \sum_{K\in\mathcal{F}_h\setminus\mathbb{F}_h}\int_K \sigma_h^+(\chi - u_h)\,dx,$$

$$\mathbb{F}_h = \{K \in \mathcal{F}_h : \exists p_1, p_2 \in K \cap \mathbb{N}_h, u_h(p_1) > \psi_I(p_1) \text{ and } u_h(p_2) = \psi_I(p_2)\}.$$

Proof. Note that $\psi_I = \psi$. So $\chi \in K$. From (1.3), $\langle\sigma, \chi - u\rangle \ge 0$. Write

$$(\sigma_h, \chi - u) = (\sigma_h - \sigma_h^+, \chi - u) + (\sigma_h^+, \chi - u). \tag{3.10}$$

Furthermore, we have by (2.11),

$$\begin{aligned}(\sigma_h - \sigma_h^+, \chi - u) &\le |\sigma_h - \sigma_h^+|_*|\chi - u|_{1,\Omega} \le \frac{1}{2}\epsilon|\chi - u|_{1,\Omega}^2 + \frac{1}{2\epsilon}|\sigma_h - \sigma_h^+|_*^2\\ &\le \epsilon|u_h - u|_{1,h}^2 + C\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2 + \frac{1}{2\epsilon}|\sigma_h - \sigma_h^+|_*^2.\end{aligned} \tag{3.11}$$

If $K \in \mathcal{D}_h$, from (2.10), we know $\sigma_h^+ = 0$ in K. For $K \in \mathcal{C}_h$, with the construction of χ, we know $\chi = \psi_I = \psi \le u$, and so $\sigma_h^+(\chi - u) \le 0$ on K. Then

$$(\sigma_h^+, \chi - u) \le \sum_{K\in\mathcal{F}_h} \int_K \sigma_h^+(\chi - u)\,dx. \tag{3.12}$$

We divide $K \in \mathcal{F}_h$ into two kinds to estimate $\int_K \sigma_h^+(\chi - u)\,dx$. First, if $K \in \mathcal{F}_h \backslash \mathbb{F}_h$, we know that $u_h = \psi_I = \psi$ on K. Therefore,

$$\int_K \sigma_h^+(\chi - u)\,dx \le \int_K \sigma_h^+(\chi - u_h)\,dx + \int_K \sigma_h^+(\psi - u)\,dx \le \int_K \sigma_h^+(\chi - u_h)\,dx.$$

Note that if $K \in \mathbb{F}_h$, then $\exists\, p_1 \in \mathbb{N}_h \cap K$ with $u_h(p_1) > \psi_I(p_1)$; so $\sigma_h^+(p_1) = 0$ by (2.9) and consequently,

$$\|\sigma_h^+\|_K \le Ch_K\|\nabla\sigma_h^+\|_K.$$

Now we bound $\int_K \sigma_h^+(\chi - u)\,dx$ with $K \in \mathbb{F}_h$ in two cases.

Case 1: $K \cap \partial\Omega \ne \emptyset$. We have

$$\begin{aligned}\int_K \sigma_h^+(\chi - u)\,dx &= \int_K \sigma_h^+\big((\chi - u) - \Lambda_h(\chi - u)\big)dx + \int_K \sigma_h^+\Lambda_h(\chi - u)\,dx \\ &\le \|\sigma_h^+\|_K\big(\|(\chi - u) - \Lambda_h(\chi - u)\|_K + \|\Lambda_h(\chi - u)\|_K\big).\end{aligned}$$

Here $\Lambda_h(\chi - u)$ denotes the Clément interpolant of $(\chi - u)$. By the local approximation property of the Clément interpolant, we have

$$\|(\chi - u) - \Lambda_h(\chi - u)\|_K \le Ch_K\|\nabla(\chi - u)\|_{\omega_K}.$$

Since $(\chi - u) \in H_0^1(\Omega)$ and $K \cap \partial\Omega \ne \emptyset$, there exists a node $p_1 \in K \cap \mathbb{N}_h \cap \partial\Omega$ such that $(\chi - u)(p_1) = 0$. Then since $\Lambda_h(\chi - u)(p_1) = 0$,

$$\|\Lambda_h(\chi - u)\|_K \le Ch_K\|\nabla\Lambda_h(\chi - u)\|_K \le Ch_K\|\nabla(\chi - u)\|_{\omega_K}$$

by the stability property of Λ_h. So we have

$$\begin{aligned}\int_K \sigma_h^+(\chi - u)\,dx &\le \frac{1}{2}\epsilon\|\nabla(\chi - u)\|_{\omega_K}^2 + \frac{C}{\epsilon}h_K^2\|\sigma_h^+\|_K^2 \\ &\le \epsilon\|\nabla_h(\chi - u_h)\|_{\omega_K}^2 + \epsilon\|\nabla_h(u_h - u)\|_{\omega_K}^2 + Ch_K^4\|\nabla\sigma_h^+\|_K^2.\end{aligned} \tag{3.13}$$

Case 2: $K \cap \partial\Omega = \emptyset$. Noting that $\psi_I = \psi$, we get

$$\int_K \sigma_h^+(\chi - u)\,dx = \int_K \sigma_h^+(\chi - \psi_I)\,dx + \int_K \sigma_h^+(\psi - u)\,dx \le \int_K \sigma_h^+(\chi - \psi_I)\,dx$$

by $\sigma_h^+ \ge 0$ and $\psi - u \le 0$ in Ω. First we assume

$$(\chi - \psi_I)(p) = 0 \quad \text{for an interior node } p \in K \cap \mathbb{N}_h. \tag{3.14}$$

Using Lemma 3.3 and $\psi_I = \psi \in P_1(\Omega)$, we obtain

$$\|\chi - \psi_I\|_K \le Ch_K\Big(\sum_{e\in\mathbb{E}_h(p)} h_e\|[\nabla(\chi - \psi_I)]\|_e^2\Big)^{\frac{1}{2}} = Ch_K\Big(\sum_{e\in\mathbb{E}_h(p)} h_e\|[\nabla\chi]\|_e^2\Big)^{\frac{1}{2}}.$$

Then by trace inequality and (2.11), we get

$$\begin{aligned}\int_K \sigma_h^+(\chi - \psi_I)\,dx &\le \|\sigma_h^+\|_K\|\chi - \psi_I\|_K \le Ch_K^2\|\nabla\sigma_h^+\|_K\Big(\sum_{e\in\mathbb{E}_h(p)} h_e\|[\nabla\chi]\|_e^2\Big)^{\frac{1}{2}} \\ &\le C\Big(h_K^4\|\nabla\sigma_h^+\|_K^2 + \sum_{e\in\mathbb{E}_h(p)} h_e(\|[\nabla_h(\chi - u_h)]\|_e^2 + \|[\nabla_h u_h]\|_e^2)\Big) \\ &\le C\Big(h_K^4\|\nabla\sigma_h^+\|_K^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2 + \sum_{e\in\mathbb{E}_h(p)} h_e\|[\nabla_h u_h]\|_e^2\Big).\end{aligned} \tag{3.15}$$

If (3.14) does not hold, there is an interior node $p_2 \in K \cap \mathbb{N}_h$ such that $(u_h - \psi_I)(p_2) = 0$. Then

$$\begin{aligned}\int_K \sigma_h^+(\chi - \psi_I)\,dx &= \int_K \sigma_h^+(\chi - u_h)\,dx + \int_K \sigma_h^+(u_h - \psi_I)\,dx \\ &\le \int_K \sigma_h^+(\chi - u_h)\,dx + C\big(h_K^4\|\nabla\sigma_h^+\|_K^2 + \|\nabla(u_h - \chi)\|_K^2 + \|\nabla(\chi - \psi_I)\|_K^2\big).\end{aligned}$$

Let $\alpha = \min_K\{\chi - \psi_I\} = (\chi - \psi_I)(p)$. By an inverse inequality and Lemma 3.3, we get

$$\|\nabla(\chi - \psi_I)\|_K^2 = \|\nabla(\chi - \psi_I - \alpha)\|_K^2 \le Ch_K^{-2}\|\chi - \psi_I - \alpha\|_K^2 \le C\sum_{e\in\mathbb{E}_h(p)} h_e\|[\nabla\chi]\|_e^2.$$

For $\int_K \sigma_h^+(\chi - u_h)\,dx$, we have

$$\begin{aligned}\int_K \sigma_h^+(\chi - u_h)\,dx &\le \|\sigma_h^+\|_K\|\chi - u_h\|_K \le \|\sigma_h^+\|_K\Big(\sum_{j=1}^3|\beta_K^{(j)} - \alpha_K^{(j)}|\|\phi_K^{(j)}\|_K\Big) \\ &\le h_K\|\sigma_h^+\|_K\Big(C\sum_{j=1}^3|\beta_K^{(j)} - \alpha_K^{(j)}|\Big) \le C\Big(h_K^4\|\nabla\sigma_h^+\|_K^2 + \sum_{j=1}^3|\beta_K^{(j)} - \alpha_K^{(j)}|^2\Big).\end{aligned} \tag{3.16}$$

From the proof of Theorems 2.2 and 2.3 in [22], we know

$$\sum_{K\in\mathcal{T}_h}\sum_{j=1}^3|\beta_K^{(j)} - \alpha_K^{(j)}|^2 \le C\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2,$$

so

$$\sum_{K\in\mathbb{F}_h,\,K\cap\partial\Omega=\emptyset}\int_K \sigma_h^+(\chi - u_h)\,dx \le C\sum_{K\in\mathbb{F}_h,\,K\cap\partial\Omega=\emptyset} h_K^4\|\nabla\sigma_h^+\|_K^2 + C\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2. \tag{3.17}$$

The inequality (3.9) follows from the combination of (3.10)–(3.13) and (3.15)–(3.17). ■

Lemma 3.5.

$$|\sigma_h - \sigma_h^+|_*^2 \le C\sum_{K\in\mathcal{T}_h} h_K^2\|\sigma_h - \sigma_h^+\|_K^2 \le C\sum_{K\in\mathcal{T}_h} h_K^4\|\nabla(\sigma_h - \sigma_h^+)\|_K^2. \tag{3.18}$$

Proof. Let 1_K be the indicator function of $K \in \mathcal{T}_h$. Then

$$(\sigma_h, 1_K) = (\sigma_h, 1_K)_K = B_h(u_h, 1_K) - (f, 1_K).$$

We use the relation $\sum_{p\in K\cap\mathbb{N}_h}\phi_K^p = 1_K$ to get

$$(\sigma_h^+, 1_K) = \sum_{p\in K\cap\mathbb{N}_h}\big(B_h(u_h, \phi_K^p) - (f, \phi_K^p)\big)(1_K, \phi_K^p)/(1, \phi_K^p) = B_h(u_h, 1_K) - (f, 1_K).$$

Hence, for any constant c_K on element K, we get

$$(\sigma_h - \sigma_h^+, c_K 1_K) = 0. \tag{3.19}$$

Then for any piecewise constant c, $c|_K = c_K$, and $v \in H_0^1(\Omega)$, we have

$$|\sigma_h - \sigma_h^+|_* = \sup_{v\in H_0^1(\Omega),\,|v|_1=1}(\sigma_h - \sigma_h^+, v) = \sup_{v\in H_0^1(\Omega),\,|v|_1=1}(\sigma_h - \sigma_h^+, v - c),$$

$$|\sigma_h - \sigma_h^+|_* \le C\sup_{v\in H_0^1(\Omega),\,|v|_1=1}\Big(\sum_{K\in\mathcal{T}_h} h_K^2\|\sigma_h - \sigma_h^+\|_K^2\Big)^{1/2}|v|_1 = C\Big(\sum_{K\in\mathcal{T}_h} h_K^2\|\sigma_h - \sigma_h^+\|_K^2\Big)^{1/2}. \tag{3.20}$$

Using the relation (3.19), we obtain

$$\|\sigma_h - \sigma_h^+\|_K^2 = \inf_{c_K\in\mathbb{R}}(\sigma_h - \sigma_h^+, \sigma_h - \sigma_h^+ - c_K)_K \le Ch_K\|\sigma_h - \sigma_h^+\|_K\|\nabla(\sigma_h - \sigma_h^+)\|_K,$$

$$\|\sigma_h - \sigma_h^+\|_K \le Ch_K\|\nabla(\sigma_h - \sigma_h^+)\|_K. \tag{3.21}$$

A combination of (3.20) and (3.21) completes the proof of (3.18). ■

From (3.8), Proposition 3.4 and Lemma 3.5, we obtain

$$|e|_{1,h}^2 \le C\Big(\sum_{K\in\mathcal{T}_h}\eta_K^2+\sum_{K\in\mathcal{T}_h}\eta_{\partial K}^2+\sum_{K\in\mathcal{T}_h}h_K^2\|\sigma_h-\sigma_h^+\|_K^2+\eta_{\mathcal{F}_h}^2\Big).$$

Recalling (1.5), we have

$$|\sigma-\sigma_h|_* = |u-z|_{1,\Omega} \le |e|_{1,h}+|u_h-z|_{1,h}.$$

Finally, we obtain the following theorem.

Theorem 3.6. *Let $u\in H^2(\Omega)$ and u_h solve (1.1) and (2.1) respectively, $\psi_I=\psi$. Then*

$$|e|_{1,h}+|\sigma-\sigma_h|_* \le C_1\Big(\sum_{K\in\mathcal{T}_h}\eta_K^2\Big)^{\frac{1}{2}}+C_2\Big(\sum_{K\in\mathcal{T}_h}\eta_{\partial K}^2\Big)^{\frac{1}{2}}+C_3\Big(\sum_{K\in\mathcal{T}_h}h_K^2\|\sigma_h-\sigma_h^+\|_K^2\Big)^{\frac{1}{2}}+C_4\eta_{\mathcal{F}_h}. \tag{3.22}$$

3.2. Efficiency of the estimator for the LDG method

Now we consider lower bounds of the estimators. We follow the standard argument of lower bounds of residual error estimators for elliptic problems, see [2, pp. 28–31]. First, we introduce the bubble functions. Let $K\in\mathcal{T}_h$, and let λ_1, λ_2 and λ_3 be the barycentric coordinates on K. Then the interior bubble function φ_K is defined by $\varphi_K=27\lambda_1\lambda_2\lambda_3$ and the three edge bubble functions are given by $\tau_1=4\lambda_2\lambda_3$, $\tau_2=4\lambda_1\lambda_3$, $\tau_3=4\lambda_1\lambda_2$. We list properties of bubble functions stated in Theorems 2.2 and 2.3 of [2] in the form of a lemma.

Lemma 3.7. *For each $K\in\mathcal{T}_h$, $e\subset\partial K$, let φ_K and τ_e be the corresponding interior and edge bubble functions. Let $P(K)\subset H^1(K)$ and $P(e)\subset H^1(e)$ be finite-dimensional spaces of functions defined on K or e. Then there exists a constant C such that for all $v\in P(K)$,*

$$C^{-1}\|v\|_K^2 \le \int_K \varphi_K v^2\,dx \le C\|v\|_K^2,$$
$$C^{-1}\|v\|_K \le \|\varphi_K v\|_K + h_K|\varphi_K v|_{1,K} \le C\|v\|_K,$$
$$C^{-1}\|v\|_e^2 \le \int_e \tau_e v^2\,ds \le C\|v\|_e^2,$$
$$h_K^{-1/2}\|\tau_e v\|_K + h_K^{1/2}|\tau_e v|_{1,K} \le C\|v\|_e.$$

Denote $a_K(u,v)=\int_K\nabla u\cdot\nabla v\,dx$. Then for $u,v\in H^1(\Omega)$, $a(u,v)=\sum_{K\in\mathcal{T}_h}a_K(u,v)$. For all $v\in H_0^1(\Omega)$, noting that $[v]=0$ and $[u-z]=0$, we have

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}a_K(e,v) &= \sum_{K\in\mathcal{T}_h}a_K(u_h-z,v)+a(z-u,v)=\sum_{K\in\mathcal{T}_h}\int_K\nabla(u_h-z)\cdot\nabla v\,dx+(\sigma_h-\sigma,v)\\ &= \sum_{K\in\mathcal{T}_h}\int_K(-\Delta u_h-f-\sigma_h)v\,dx+\sum_{e\in\mathcal{E}_h^i}\int_e[\nabla u_h]\cdot v\,ds+(\sigma_h-\sigma,v).\end{aligned} \tag{3.23}$$

Let $\bar R_K$ be an approximation to the interior residual R_K from a suitable finite-dimensional subspace. In (3.23), choose $v=\bar R_K\varphi_K$ on element K. Since φ_K vanishes on the boundary of K, v can be extended to be zero on the rest of the domain as a continuous function. Therefore,

$$a_K(e,\bar R_K\varphi_K)=\int_K R_K\bar R_K\varphi_K dx+(\sigma_h-\sigma,\bar R_K\varphi_K)_K.$$

Then

$$\int_K\bar R_K^2\varphi_K\,dx=\int_K\bar R_K(\bar R_K-R_K)\varphi_K\,dx+a_K(e,\bar R_K\varphi_K)-(\sigma_h-\sigma,\bar R_K\varphi_K)_K.$$

Applying the Cauchy–Schwarz inequality and Lemma 3.7, we obtain

$$\int_K\bar R_K(\bar R_K-R_K)\varphi_K\,dx \le \|\bar R_K\varphi_K\|_K\|\bar R_K-R_K\|_K \le C\|\bar R_K\|_K\|\bar R_K-R_K\|_K,$$
$$a_K(e,\bar R_K\varphi_K) \le |e|_{1,K}|\bar R_K\varphi_K|_{1,K} \le Ch_K^{-1}|e|_{1,K}\|\bar R_K\|_K,$$
$$(\sigma_h-\sigma,\bar R_K\varphi_K)_K \le |\sigma_h-\sigma|_{*,K}|\bar R_K\varphi_K|_{1,K} \le Ch_K^{-1}|\sigma_h-\sigma|_{*,K}\|\bar R_K\|_K.$$

Use Lemma 3.7 again,

$$\|\bar{R}_K\|_K^2 \le C \int_K \bar{R}_K^2 \varphi_K dx.$$

Combining the above relations, we obtain

$$\|\bar{R}_K\|_K \le C(\|\bar{R}_K - R_K\|_K + h_K^{-1}|e|_{1,K} + h_K^{-1}|\sigma_h - \sigma|_{*,K}).$$

Finally, by the triangle inequality $\|R_K\|_K \le \|R_K - \bar{R}_K\|_K + \|\bar{R}_K\|_K$, we get

$$\|R_K\|_K \le C(\|\bar{R}_K - R_K\|_K + h_K^{-1}|e|_{1,K} + h_K^{-1}|\sigma_h - \sigma|_{*,K}).$$

Now choose the finite-dimensional subspace for $\bar{R}_K$ to be spanned by the local nodal basis $\phi_K^{(i)}$, $i = 1, 2, 3$. Then $\|\bar{R}_K - R_K\|_K$ reduces to $\|f - f_h\|_K$,

$$f_h = \sum_{i=1}^{3} f_h^i \phi_K^{(i)} \quad \text{with } f_h^i = (f, \phi_K^{(i)})_K/(1, \phi_K^{(i)})_K. \tag{3.24}$$

If we choose $\bar{R}_e$ as an approximation to the jumps from a suitable finite-dimensional space and let $v = \bar{R}_e \tau_e$, we have

$$\|R_e\|_e \le C(h_e^{-1/2}|e|_{1,\omega_e} + h_e^{-1/2}|\sigma_h - \sigma|_{*,\omega_e} + h_e^{1/2}\|f - f_h\|_{\omega_e}).$$

Then we obtain the efficiency of the local error indicator η_K.

Theorem 3.8. *Let $u \in H^2(\Omega)$ and u_h be the solutions of (1.1) and (2.1), respectively, and η_K be the estimator (3.7). Then*

$$\eta_K \le C\left(|u - u_h|_{1,\omega_K} + |\sigma - \sigma_h|_{*,\omega_K} + h_K\|f - f_h\|_{\omega_K}\right).$$

To bound the remaining terms in the error estimator (3.22), we first give a lemma.

Lemma 3.9.

$$h_K\|\sigma_h^+ + f_h\|_K \le C\Big(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2\Big)^{1/2} + C\Big(\sum_{e\in\partial K} h_e\|[\nabla_h u_h]\|_e^2\Big)^{1/2}. \tag{3.25}$$

Proof. Using the definitions (2.8) and (3.24), we have

$$h_K\|\sigma_h^+ + f_h\|_K = h_K\left\|\sum_{i=1}^{3} B_h(u_h, \phi_K^{(i)})\phi_K^{(i)}/(1, \phi_K^{(i)})\right\|_K \le C\sum_{i=1}^{3}|B_h(u_h, \phi_K^{(i)})|$$

since $\|\phi_K^{(i)}\|_K/(1, \phi_K^{(i)})_K \le Ch_K^{-1}$. For each $\phi_K^{(i)}$, we know $\phi_K^{(i)}(x) = 0\ \forall x \in \Omega\backslash K$, which implies $[\phi_K^{(i)}] = 0$ and $\nabla_h\phi_K^{(i)} = 0$ on $\mathcal{E}_h\backslash\partial K$. So

$$\begin{aligned} B_h(u_h, \phi_K^{(i)}) = {} & (\nabla u_h, \nabla\phi_K^{(i)})_K - \langle[u_h], \{\nabla_h\phi_K^{(i)}\}\rangle_{\partial K} - \langle\{\nabla_h u_h\}, [\phi_K^{(i)}]\rangle_{\partial K} \\ & - \langle\beta\cdot[u_h], [\nabla_h\phi_K^{(i)}]\rangle_{\partial K} - \langle[\nabla_h u_h], \beta\cdot[\phi_K^{(i)}]\rangle_{\partial K} \\ & + \Big(r([u_h]) + l(\beta\cdot[u_h]), r_{\partial K}([\phi_K^{(i)}]) + l_{\partial K}(\beta\cdot[\phi_K^{(i)}])\Big)_\Omega + \int_{\partial K}\mu[u_h]\cdot[\phi_K^{(i)}], \end{aligned}$$

where $r_{\partial K} : [L^2(\partial K)]^2 \to W_h$ and $l_{\partial K} : L^2(\partial K) \to W_h$ are lifting operators defined by

$$\int_\Omega r_{\partial K}(q)\cdot w_h dx = -\int_{\partial K} q\cdot\{w_h\}\, ds, \qquad \int_\Omega l_{\partial K}(v)\cdot w_h dx = -\int_{\partial K} v[w_h]\, ds, \quad \forall\, w_h \in W_h.$$

Noting $\Delta u_h = 0$ on K, we get by integration by part and the Cauchy–Schwarz inequality

$$(\nabla u_h, \nabla\phi_K^{(i)})_K - \langle\{\nabla_h u_h\}, [\phi_K^{(i)}]\rangle_{\partial K} = \frac{1}{2}\int_{\partial K}[\nabla_h u_h]\phi_K^{(i)}\, ds \le C\sum_{e\in\partial K} h_e^{1/2}\|[\nabla_h u_h]\|_e.$$

Similarly, we have

$$\langle[u_h], \{\nabla_h\phi_K^{(i)}\}\rangle_{\partial K} \le C\sum_{e\in\partial K} h_e^{-1/2}\|[u_h]\|_e,$$

$$\langle\beta\cdot[u_h], [\nabla_h\phi_K^{(i)}]\rangle_{\partial K} \le C\sum_{e\in\partial K} h_e^{-1/2}\|[u_h]\|_e,$$

$$\langle[\nabla_h u_h], \beta\cdot[\phi_K^{(i)}]\rangle_{\partial K} \le C\sum_{e\in\partial K} h_e^{1/2}\|[\nabla_h u_h]\|_e,$$

$$\int_{\partial K}\mu[u_h]\cdot[\phi_K^{(i)}] \le C\sum_{e\in\partial K} h_e^{1/2}\|[u_h]\|_e.$$

Define the edge lifting operators $r_e : [L^2(e)]^2 \to W_h$ and $l_e : L^2(e) \to W_h$ by

$$\int_\Omega r_e(q) \cdot w_h \, dx = -\int_e q \cdot \{w_h\} \, ds, \qquad \int_\Omega l_e(v) \cdot w_h \, dx = -\int_e v[w_h] \, ds \quad \forall\, w_h \in W_h.$$

We recall some results about the lifting operators [7]:

$$\|r(q)\|_\Omega^2 = \Big\|\sum_{e\in\mathcal{E}_h} r_e(q)\Big\|_\Omega^2 \le 3\sum_{e\in\mathcal{E}_h} \|r_e(q)\|_\Omega^2 \le C\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[q]\|_e^2,$$

$$\|l(q)\|_\Omega^2 = \Big\|\sum_{e\in\mathcal{E}_h^i} l_e(q)\Big\|_\Omega^2 \le 3\sum_{e\in\mathcal{E}_h^i} \|l_e(q)\|_\Omega^2 \le 12\sum_{e\in\mathcal{E}_h^i} \|r_e(qn_e)\|_\Omega^2 \le C\sum_{e\in\mathcal{E}_h^i} h_e^{-1}\|[q]\|_e^2.$$

Then

$$\Big(r([u_h]) + l(\beta\cdot[u_h]), r_{\partial K}([\phi_K^{(i)}]) + l_{\partial K}(\beta\cdot[\phi_K^{(i)}])\Big)_\Omega \le C\Big(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2\Big)^{1/2}.$$

Combination of above inequalities accomplishes the proof. ∎

Now we explore bounds on the remaining terms in the error estimator (3.22). First,

$$h_K\|\sigma_h^+ - \sigma_h\|_K \le h_K\|\sigma_h^+ + f_h\|_K + h_K\|f + \sigma_h\|_K + h_K\|f - f_h\|_K.$$

Note that $f + \sigma_h = R_K$ in K. Let $f_K = \int_K f\, dx/|K|$ for $K \in \mathbb{F}_h$. By inverse inequality, we obtain

$$h_K^2\|\nabla\sigma_h^+\|_K = h_K^2\|\nabla(\sigma_h^+ + f_K)\|_K \le Ch_K\|\sigma_h^+ + f_K\|_K \le C\left(h_K\|\sigma_h^+ + f_h\|_K + h_K\|f_h - f_K\|_K\right).$$

For $\eta_{\partial K}$, by the trace inequality and inverse inequality, we obtain

$$\begin{aligned}\eta_{\partial K}^2 &= \frac{1}{2}\sum_{e\in\partial K} h_e^{-1}\|[u_h]\|_e^2 = \frac{1}{2}\sum_{e\in\partial K} h_e^{-1}\|[u_h - u_I]\|_e^2\\ &\le C\sum_{\widetilde{K}\in\omega_K} h_e^{-1}(h_e^{-1}\|u_h - u_I\|_{\widetilde{K}}^2 + h_e|u_h - u_I|_{1,\widetilde{K}}^2) \le C|u_h - u_I|_{1,\omega_K}^2,\end{aligned}$$

where u_I is a continuous piecewise polynomial interpolant of u. Summarizing the above argument, we get the following theorem.

Theorem 3.10. *Let $u \in H^2(\Omega)$ and u_h be the solutions of (1.1) and (2.1), respectively. Then we have local lower bounds of error estimators in (3.22),*

$$\begin{aligned}&\eta_K \le C\left(|u - u_h|_{1,\omega_K} + |\sigma - \sigma_h|_{*,\omega_K} + h_K\|f - f_h\|_{\omega_K}\right),\\ &\eta_{\partial K} \le C|u_h - u_I|_{1,\omega_K},\\ &h_K\|\sigma_h^+ - \sigma_h\|_K \le h_K\|\sigma_h^+ + f_h\|_K + h_K\|f + \sigma_h\|_K + h_K\|f - f_h\|_K,\\ &h_K^2\|\nabla\sigma_h^+\|_K \le C\left(h_K\|\sigma_h^+ + f_h\|_K + h_K\|f_h - f_K\|_K\right).\end{aligned}$$

For the term $\int_K \sigma_h^+(\chi - u_h)\, dx$ with $K \in \mathcal{F}_h\backslash\mathbb{F}_h$, we are not able to derive a bound of the form

$$\int_K \sigma_h^+(\chi - u_h)\, dx \le C\left(|u - u_h|_{\omega_K} + |\sigma - \sigma_h|_{*,\omega_K} + h_K\|f - f_h\|_{\omega_K}\right).$$

This term is expected to be very small due to the construction of χ. See Section 5 for numerical confirmation of this.

4. The case of a general obstacle

For the case of a general obstacle, we only need to treat the term $(\sigma_h - \sigma, \chi - u)$ differently. Define $\chi^* = \max\{\chi, \psi\} \in K$ and denote $v^+ = \max\{v, 0\}$. Then with (1.3), we get

$$\begin{aligned}\langle\sigma, u - \chi\rangle &= \langle\sigma, u - \chi^*\rangle + \langle\sigma, \chi^* - \chi\rangle \le \langle\sigma, (\psi - \chi)^+\rangle\\ &\le \epsilon|\sigma - \sigma_h|_*^2 + \frac{1}{4\epsilon}|(\psi - \chi)^+|_{1,\Omega}^2 + (\sigma_h, (\psi - \chi)^+).\end{aligned}$$

Inserting σ_h^+, we have

$$(\sigma_h, \chi - u + (\psi - \chi)^+) = (\sigma_h - \sigma_h^+, \chi - u + (\psi - \chi)^+) + (\sigma_h^+, \chi - u + (\psi - \chi)^+),$$

$$\begin{aligned}(\sigma_h - \sigma_h^+, \chi - u + (\psi - \chi)^+) &\le \frac{1}{\epsilon}|\sigma_h - \sigma_h^+|_*^2 + \frac{\epsilon}{2}|\chi - u|_{1,\Omega}^2 + \frac{\epsilon}{2}|(\psi - \chi)^+|_{1,\Omega}^2 \\ &\le \frac{1}{\epsilon}|\sigma_h - \sigma_h^+|_*^2 + \epsilon|\chi - u_h|_{1,h}^2 + \epsilon|u_h - u|_{1,h}^2 + \frac{\epsilon}{2}|(\psi - \chi)^+|_{1,\Omega}^2.\end{aligned}$$

Noticing $\sigma_h^+ = 0$ on $K \in \mathcal{D}_h$, we obtain

$$(\sigma_h^+, \chi - u + (\psi - \chi)^+) = \sum_{K \in \mathcal{C}_h \cup \mathcal{F}_h} \int_K \sigma_h^+ \big(\chi - u + (\psi - \chi)^+\big)\, dx.$$

We consider three cases of $K \in \mathcal{C}_h \cup \mathcal{F}_h$ to estimate $\int_K \sigma_h^+ \big(\chi - u + (\psi - \chi)^+\big)\, dx$.

Case 1: $K \in \mathbb{F}_h$ and $K \cap \partial\Omega \neq \emptyset$. Observing $(\chi - u),\ (\psi - \chi)^+ \in V = H_0^1(\Omega)$, we can use the same argument for Case 1 in the proof of Proposition 3.4 to obtain

$$\int_K \sigma_h^+ \big(\chi - u + (\psi - \chi)^+\big)\, dx \le \epsilon \|\nabla_h(\chi - u_h)\|_{\omega_K}^2 + \epsilon \|\nabla_h(u_h - u)\|_{\omega_K}^2 + \frac{1}{2}\epsilon \|\nabla(\psi - \chi)^+\|_{\omega_K}^2 + \frac{C}{\epsilon} h_K^4 \|\nabla \sigma_h^+\|_K^2.$$

For the remaining two cases, noting $\psi - u \le 0$ and $\sigma_h^+ \ge 0$, we have

$$\int_K \sigma_h^+ \big(\chi - u\big)\, dx = \int_K \sigma_h^+ \big(\chi - \psi\big)\, dx + \int_K \sigma_h^+ \big(\psi - u\big)\, dx \le \int_K \sigma_h^+ \big(\chi - \psi\big)\, dx,$$

and observing $\chi - \psi + (\psi - \chi)^+ = (\chi - \psi)^+$, we get

$$\int_K \sigma_h^+ \big(\chi - u + (\psi - \chi)^+\big)\, dx \le \int_K \sigma_h^+ (\chi - \psi)^+\, dx.$$

Case 2: $K \in \mathbb{F}_h$ and $K \cap \partial\Omega = \emptyset$. Since $\chi \ge \psi_I$, we have

$$(\chi - \psi)^+ = (\chi - \psi_I + \psi_I - \psi)^+ \le (\chi - \psi_I)^+ + (\psi_I - \psi)^+ = (\chi - \psi_I) + (\psi_I - \psi)^+,$$

$$\int_K \sigma_h^+ (\chi - \psi)^+ dx \le \int_K \sigma_h^+ (\chi - \psi_I)\, dx + \int_K \sigma_h^+ (\psi_I - \psi)^+ dx.$$

Following the proof of Proposition 3.4 to estimate the first term on the right side,

$$\begin{aligned}\sum_{K \in \mathbb{F}_h,\ K \cap \partial\Omega = \emptyset} \int_K \sigma_h^+ (\chi - \psi)^+\, dx &\le C \sum_{K \in \mathbb{F}_h,\ K \cap \partial\Omega = \emptyset} h_K^4 \|\nabla \sigma_h^+\|_K^2 + C \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[u_h]\|_e^2 \\ &\quad + C \sum_{K \in \mathbb{F}_h,\ K \cap \partial\Omega = \emptyset} \sum_{e \in \mathbb{E}_h(p)} h_e \|[\nabla(u_h - \psi_I)]\|_e^2 + \sum_{K \in \mathbb{F}_h,\ K \cap \partial\Omega = \emptyset} \int_K \sigma_h^+ (\psi_I - \psi)^+ dx.\end{aligned}$$

Case 3: $K \notin \mathbb{F}_h$. We have $u_h = \psi_I$ on K and thus

$$\int_K \sigma_h^+ (\chi - \psi)^+\, dx \le \int_K \sigma_h^+ (\chi - u_h)\, dx + \int_K \sigma_h^+ (\psi_I - \psi)^+\, dx.$$

Note that $\chi = u_h$ on $K \in \mathcal{C}_h$. So

$$\int_K \sigma_h^+ (\chi - \psi)^+\, dx \le \int_K \sigma_h^+ (\psi_I - \psi)^+\, dx \quad \text{for } K \in \mathcal{C}_h.$$

Therefore, we obtain the following result.

Proposition 4.1.

$$\langle \sigma_h - \sigma, \chi - u \rangle \le \epsilon |u_h - u|_{1,\Omega}^2 + \epsilon |\sigma_h - \sigma|_{*,\Omega}^2 + C\Big(\sum_{e \in \mathcal{E}_h} h_e^{-1} \|[u_h]\|_e^2 + |\sigma_h - \sigma_h^+|_*^2 + \eta_{\mathcal{C}_h}^2\Big),$$

where

$$\begin{aligned}\eta_{\mathcal{C}_h}^2 &= \sum_{K \in \mathbb{F}_h} h_K^4 \|\nabla \sigma_h^+\|_K^2 + \sum_{K \in \mathcal{F}_h \setminus \mathbb{F}_h} \int_K \sigma_h^+ (\chi - u_h)\, dx + |(\psi - \chi)^+|_{1,\Omega}^2 \\ &\quad + \sum_{K \in \mathbb{F}_h,\ K \cap \partial\Omega = \emptyset} \sum_{e \in \mathbb{E}_h(p)} h_e \|[\nabla(u_h - \psi_I)]\|_e^2 + \sum_{K \in \mathcal{F}_h \cup \mathcal{C}_h} \int_K \sigma_h^+ (\psi_I - \psi)^+\, dx.\end{aligned}$$

Here, $0 < \epsilon < 1$ *is an arbitrary constant and*

$$\mathbb{F}_h := \{K \in \mathcal{F}_h : \exists\, p_1, p_2 \in K \cap \mathbb{N}_h,\ u_h(p_1) > \psi_I(p_1) \text{ and } u_h(p_2) = \psi_I(p_2)\}.$$

Theorem 4.2. *Let $u \in H^2(\Omega)$ and u_h solve (1.1) and (2.1), respectively. Then*

$$|e|_{1,h} + |\sigma - \sigma_h|_* \le C\left[\Big(\sum_{K\in\mathcal{T}_h}\eta_K^2\Big)^{\frac12} + \Big(\sum_{K\in\mathcal{T}_h}\eta_{\partial K}^2\Big)^{\frac12} + \Big(\sum_{K\in\mathcal{T}_h}h_K^2\|\sigma_h - \sigma_h^+\|_K^2\Big)^{\frac12} + \eta_{\mathcal{C}_h}\right]. \tag{4.1}$$

For the local lower bounds of estimators, the terms η_K, $\eta_{\partial K}$, $h_K^2\|\nabla\sigma_h^+\|_K$ and $h_K\|\sigma_h - \sigma_h^+\|_K$ are bounded in the same way as in Section 3.2. Now consider the other terms in (4.1).

$$h_e^{1/2}\|[\nabla(u_h - \psi_I)]\|_e \le h_e^{1/2}\|[\nabla u_h]\|_e + h_e^{1/2}\|[\nabla\psi_I]\|_e, \tag{4.2}$$

$$|(\psi-\chi)^+|_{1,K} = |\psi - \chi|_{1,K\cap\{\psi>\chi\}} \le |\psi - \psi_I|_{1,K\cap\{\psi>\chi\}} + |\chi - \psi_I|_{1,K\cap\{\psi>\chi\}}. \tag{4.3}$$

Let $\alpha = \min_K\{\chi - \psi_I\} = (\chi - \psi_I)(p)$. By inverse inequality and Lemma 3.3, we get

$$\begin{aligned}\|\nabla(\chi - \psi_I)\|_K^2 &= \|\nabla(\chi - \psi_I - \alpha)\|_K^2 \le Ch_K^{-2}\|\chi - \psi_I - \alpha\|_K^2 \le C\sum_{e\in\mathbb{E}_h(p)} h_e\|[\nabla(\chi - \psi_I)]\|_e^2 \\ &\le C\Big(\sum_{e\in\mathbb{E}_h(p)} h_e\|[\nabla u_h]\|_e^2 + \sum_{K\cap p\neq\emptyset}|\chi - u_h|_{1,K}^2 + \sum_{e\in\mathbb{E}_h(p)} h_e\|[\nabla\psi_I]\|_e^2\Big).\end{aligned} \tag{4.4}$$

Finally, we bound $\int_K \sigma_h^+(\psi_I - \psi)^+\,dx$. We have

$$\int_K \sigma_h^+(\psi_I - \psi)^+\,dx = \int_K(\sigma_h^+ + f)(\psi_I - \psi)^+\,dx + \int_K -f(\psi_I - \psi)^+\,dx,$$

$$\int_K \sigma_h^+(\psi_I - \psi)^+dx \le \frac12\|h_K(\sigma_h^+ + f)\|_K^2 + \frac12\|h_K^{-1}(\psi_I - \psi)^+\|_K^2 + \int_K -f(\psi_I - \psi)^+\,dx. \tag{4.5}$$

Here (4.2)–(4.5) give local lower bounds for these terms. Notice that these lower bounds will be zero or be absorbed by η_K if $\psi \in P_1(\Omega)$. Summarizing the above argument, we get the following theorem.

Theorem 4.3. *Let $u \in H^2(\Omega)$ and u_h be the solutions of (1.1) and (2.1), respectively. Then we have local lower bounds of error estimators in (4.1),*

$$\begin{aligned}&\eta_K \le C\left(|u - u_h|_{1,\omega_K} + |\sigma - \sigma_h|_{*,\omega_K} + h_K\|f - f_h\|_{\omega_K}\right),\\ &\eta_{\partial K} \le C|u_h - u_I|_{1,\omega_K},\\ &h_K\|\sigma_h^+ - \sigma_h\|_K \le h_K\|\sigma_h^+ + f_h\|_K + h_K\|f + \sigma_h\|_K + h_K\|f - f_h\|_K,\\ &h_K^2\|\nabla\sigma_h^+\|_K \le C\left(h_K\|\sigma_h^+ + f_h\|_K + h_K\|f_h - f_K\|_K\right),\end{aligned}$$

and (4.2)–(4.5) hold true.

We comment that for other DG methods studied in [9] for solving the obstacle problem, it is easy to see that (3.1), (3.23) and (3.25) hold true. So Theorems 3.6, 4.2, 3.10 and 4.3 hold for all of them by the similar arguments.

5. Implementation and numerical example

Each loop of the adaptive algorithm consists of four steps,

Solve $\longrightarrow$ **Estimate** $\longrightarrow$ **Mark** $\longrightarrow$ **Refine**.

Let $\mathcal{T}_0$ be the initial mesh. At each loop, first solve the obstacle problem by the LDG method on a mesh $\mathcal{T}_j$. Then based on the analysis in Section 3, for the affine obstacle case, we choose the error indicator ξ_K of element K as

$$\xi_K = \begin{cases}\left(\eta_K^2 + \eta_{\partial K}^2 + h_K^2\|\sigma_h^+ - \sigma_h\|_K^2\right)^{\frac12}, & \text{if } K \in \mathcal{T}_h\backslash\mathcal{F}_h,\\ \left(\eta_K^2 + \eta_{\partial K}^2 + h_K^2\|\sigma_h^+ - \sigma_h\|_K^2 + h_K^4\|\nabla\sigma_h^+\|_K^2\right)^{\frac12}, & \text{if } K \in \mathbb{F}_h,\\ \left(\eta_K^2 + \eta_{\partial K}^2 + h_K^2\|\sigma_h^+ - \sigma_h\|_K^2 + \|\sigma_h^+\|_K\|u_h - \chi\|_K\right)^{\frac12}, & \text{if } K \in \mathcal{F}_h\backslash\mathbb{F}_h.\end{cases}$$

With the error estimators, we still need to mark the elements to be refined. Here, we use the bulk criterion strategy

$$\sum_{K\in\mathcal{M}_h}\xi_K \ge \theta\sum_{K\in\mathcal{T}_h}\xi_K = \theta\cdot(\text{Total Error}), \quad 0 < \theta < 1. \tag{5.1}$$

In this strategy, the elements are marked according to the sizes of element errors. Therefore, elements with larger errors are put into the marked set $\mathcal{M}_h$ until the inequality (5.1) is satisfied. Last, we refine the marked elements and get the new mesh $\mathcal{T}_{j+1}$. For DGMs, refinement allows hanging nodes. To refine the marked elements, we connect the midpoints on the three edges to divide the element into four new elements. In the numerical example, the discretized problem is solved by a primal–dual active set strategy [14].

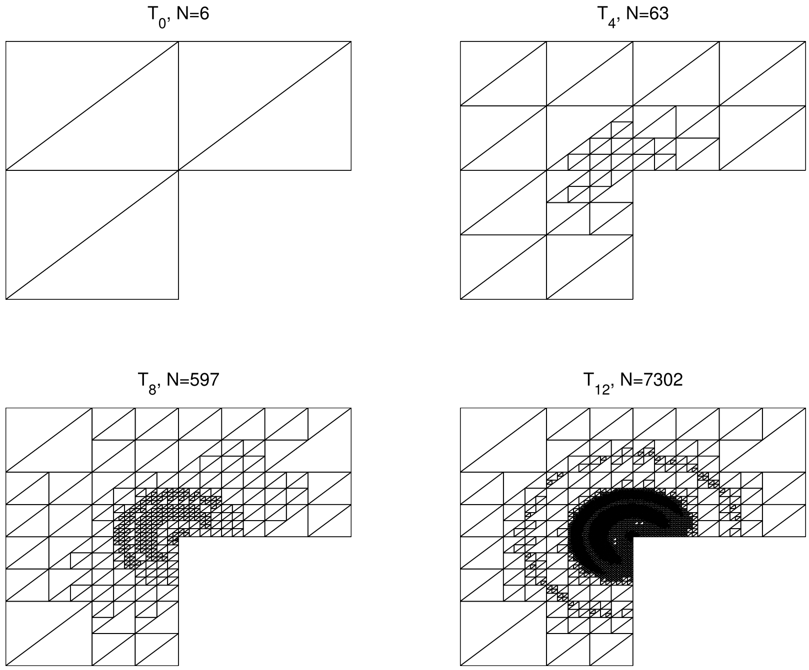

Fig. 1. The refined meshes in Example.

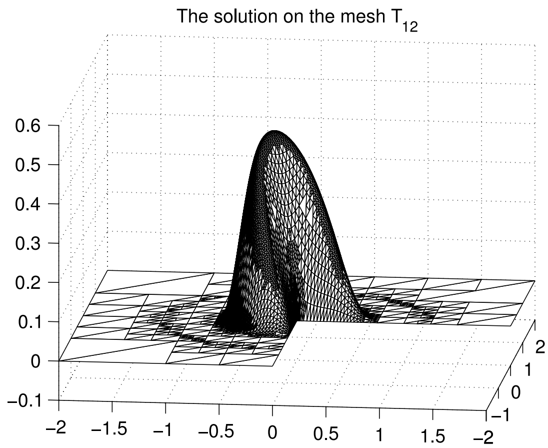

Fig. 2. The solution on the mesh $\mathcal{T}_{12}$.

Example. Let $\Omega := (-2, 2)^2 \backslash (0, 2) \times (-2, 0)$, $u_D = 0$ on $\partial\Omega$, $\psi = 0$ and,

$$f(r, \varphi) := -r^{2/3} \sin(2\varphi/3)(\nu_1'(r)/r + \nu_1''(r)) - \frac{4}{3} r^{-1/3} \nu_1'(r) \sin(2\varphi/3) - \nu_2(r)$$

where the polar coordinate system (r, φ) is used, and with $\bar{r} = 2(r - 1/4)$,

$$\nu_1(r) = \begin{cases} 1 & \text{if } \bar{r} < 0, \\ -6\bar{r}^5 + 15\bar{r}^4 - 10\bar{r}^3 + 1 & \text{if } 0 \le \bar{r} < 1, \\ 0 & \text{if } \bar{r} \ge 1, \end{cases} \qquad \nu_2(r) = \begin{cases} 0 & \text{if } r \le 5/4, \\ 1 & \text{otherwise.} \end{cases}$$

The exact solution is $u(r, \varphi) = r^{2/3} \nu_1(r) \sin(2\varphi/3)$.

In this example, we choose $\theta = 0.5$. Fig. 1 shows the adaptive meshes with hanging nodes, N denoting the number of elements. Fig. 2 shows the numerical solution on the mesh $\mathcal{T}_{12}$. In Fig. 3, the L^2 norm and H^1 norm errors are given respectively on uniformly refined meshes and adaptively refined meshes. For the final uniformly refined mesh, the number of elements $N = 24\,576$, H^1 norm and L^2 norm error are 0.097020 and 0.001293, respectively. To achieve the same level of accuracy, with adaptivity, we only need to compute the solution on the mesh $\mathcal{T}_{11}$; correspondingly, $N = 3975$, and H^1

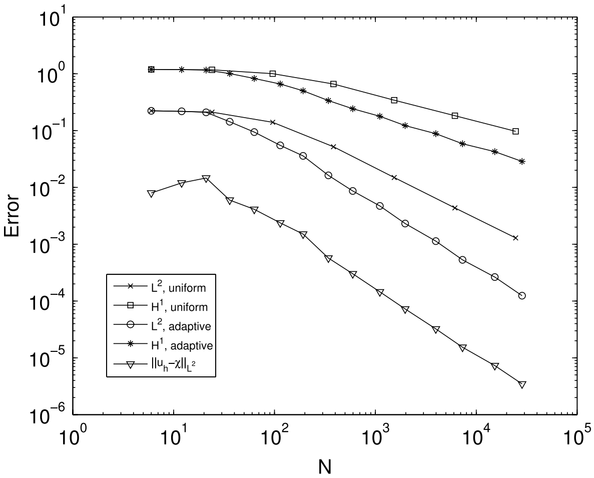

Fig. 3. The error on the uniform and adaptive refined meshes.

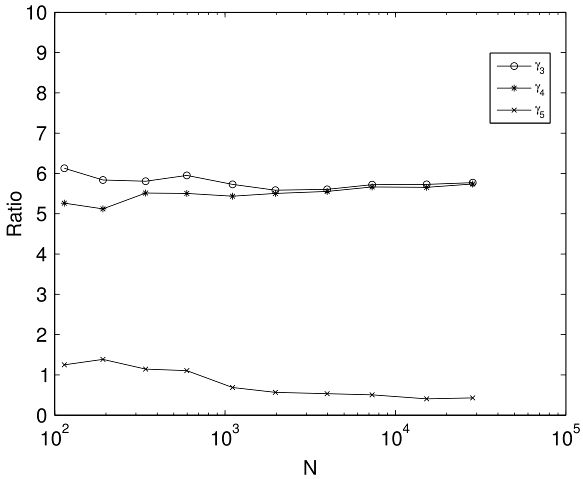

Fig. 4. The ratio of error indicators to the real error $\|\nabla e\|$.

Table 1
The maximum ratio of $(\|\sigma_h^+\|_K \|u_h - \chi\|_K)^{1/2}$ to ξ_K and η_K.

N	192	342	597	1107	1974	3975	7302	15 261	28 434
γ_1	0.01184	0.00450	0.00719	0.00497	0.00611	0.00824	0.01209	0.00563	0.00640
γ_2	0.04982	0.01738	2.12665	0.01357	0.01419	0.01747	0.02735	0.01246	0.01459

norm and L^2 norm errors being 0.088184 and 0.001134, respectively. Therefore comparing to the uniform refinement, with smaller memory, the adaptive strategy save lots of time even though it needs to spend a little time to solve the problem on the coarse meshes. In this figure, we also provide the quantity of $\|u_h - \chi\|_\Omega$, which is very small.

We compare the quantity $(\|\sigma_h^+\|_K \|u_h - \chi\|_K)^{1/2}$ with ξ_K and η_K in Table 1. In this table,

$$\gamma_1 = \max_{K \in \mathcal{F}_h \setminus \mathbb{F}_h} \left[\frac{(\|\sigma_h^+\|_K \|u_h - \chi\|_K)^{\frac{1}{2}}}{\xi_K} \right], \qquad \gamma_2 = \max_{K \in \mathcal{F}_h \setminus \mathbb{F}_h} \left[\frac{(\|\sigma_h^+\|_K \|u_h - \chi\|_K)^{\frac{1}{2}}}{\eta_K} \right].$$

In addition, we examine the quality of the error estimators provided above, let

$$\gamma_3^2 = \frac{\sum\limits_{K \in \mathcal{T}_j} \xi_K^2}{\sum\limits_{K \in \mathcal{T}_j} \|\nabla e\|_K^2}, \qquad \gamma_4^2 = \frac{\sum\limits_{K \in \mathcal{T}_j} \eta_K^2}{\sum\limits_{K \in \mathcal{T}_j} \|\nabla e\|_K^2}, \qquad \gamma_5^2 = \frac{\sum\limits_{K \in \mathcal{T}_j} h_K^2 \|\sigma_h^+ - \sigma_h\|_K^2}{\sum\limits_{K \in \mathcal{T}_j} \|\nabla e\|_K^2}.$$

Then γ_3, γ_4 and γ_5 are the ratios of the error indicators to the exact error, see Fig. 4. The error indicator efficiency index γ_3 stabilizes to a constant around 5.77.

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