A posteriori error estimates for discontinuous Galerkin methods of obstacle problems

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\textbf{A B S T R A C T}

We present a posteriori error analysis of discontinuous Galerkin methods for solving the obstacle problem, which is a representative elliptic variational inequality of the first kind. We derive reliable error estimators of the residual type. Efficiency of the estimators is theoretically explored and numerically confirmed.

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\section{Introduction}

Since the pioneering work of Babuška and Rheinboldt [1], adaptive finite element methods based on a posteriori error estimation have attracted many researchers, and a variety of different a posteriori error estimators have been proposed and analyzed. A posteriori error estimates and adaptive mesh-refinement techniques are well established for linear partial differential equations, and we refer the reader to [2–4].

The discontinuous Galerkin (DG) method was first introduced for a hyperbolic equation. In recent years, DG methods have been widely used for solving various types of partial differential equations. A historical account of their development can be found in [5]. Advantages of DG methods include the flexibility of mesh-refinements and construction of local shape function spaces (hp-adaptivity), and the increase of locality in discretization, which is of particular interest for parallel computing. For standard finite element methods with conforming and shape-regular meshes, one needs to choose the mesh refinement rule carefully to maintain conformity and shape regularity. In particular, hanging nodes are not allowed. For DG methods, the approximate functions are allowed to be discontinuous across the element boundaries, so general meshes with hanging nodes and elements of different shapes are allowed.

Discontinuous Galerkin methods for elliptic equations were independently proposed in the 1970s. A unified error analysis of DG methods for elliptic problems was given in [6,7]. A unified approach was presented in [8] on a posteriori error control for DG methods. In [9], a unified framework is given on DG methods for elliptic variational inequalities of both
first and second kinds. DG methods for the Signorini problem and the quasistatic contact problem are also studied in [10,11], respectively. In this paper, we focus on a posteriori error analysis of DG methods for solving an obstacle problem.

The obstacle problem. The problem is to find $u \in K := \{v \in V : v \geq \psi \text{ a.e in } \Omega\}$ such that

$$a(u, v - u) \geq (f, v - u)_{\Omega} \quad \forall \, v \in K, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain with boundary $\partial \Omega$, $f \in L^2(\Omega)$, $\psi \in H^1(\Omega)$ with $\psi \leq 0$ on $\partial \Omega$, $V = H^1_0(\Omega)$, $a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx$, and $(\cdot, \cdot)_{\Omega}$ denotes the $L^2$-product in the domain $\Omega$. The obstacle problem is an example of elliptic variational inequalities of the first kind [12] and has a unique solution [13]. This problem arises in various applications, such as the membrane deformation in elasticity theory, and the non-parametric minimal and capillary surfaces as geometrical problems. The elastic-plastic torsion problem and the cavitation problem in the theory of lubrication also can be regarded as obstacle type problems. A variety of numerical methods have been developed to solve the discretized obstacle problem, such as the relaxation method, multilevel projection method, multigrid method and primal–dual active set method [14].

It is difficult to develop a posteriori error estimates to variational inequalities due to the inequality feature. Nevertheless, numerous papers can be found on a posteriori error estimation of finite element methods for obstacle problems, e.g., [15–17]. In [18], Braess showed how to derive a posteriori error estimators for the standard finite element methods of the obstacle problem from the theory for linear equations. We will follow his idea and establish residual type error estimators of discontinuous Galerkin methods for the obstacle problem. For this purpose, we introduce a Lagrange multiplier $\sigma = \sigma(u) \in V^*$ [18,12] by

$$(\sigma, v) := a(u, v) - (f, v) \quad \forall \, v \in V. \tag{1.2}$$

Here $(\cdot, \cdot)$ denotes the duality pairing between $V^* = H^{-1}(\Omega)$ and $V = H^1_0(\Omega)$. We will write $(\sigma, v)$ for $(\sigma, v)$ when $\sigma$ can be regarded as an $L^2$ function. The solution $u$ is then characterized by the linear equation:

$$a(u, v) = (f, v) + (\sigma, v) \quad \forall \, v \in V. \tag{1.3}$$

Let $v = u + \psi$ for all $\psi \in V_\psi$ in the above inequality to get

$$(\sigma, \psi) \geq 0 \quad \forall \, \psi \in V_\psi := \{v \in V : v \geq 0 \text{ a.e in } \Omega\}. \tag{1.4}$$

With the constraint condition $\psi \geq 0$, we define the contact set $C = \{x \in \Omega : u(x) = \psi(x)\}$ and the noncontact set $D = \Omega \setminus C$. Then, $\sigma \geq 0$ in $\Omega$, $\sigma = -\Delta u - f$ in $C$, and $\sigma = 0$ in $D$. For a subset $\omega \subset \Omega$, define

$$|\sigma|_{\omega, \omega} := \sup\{(\sigma, \psi) : v \in H^1_0(\omega), \, |v|_{1,\omega} = 1\},$$

where $(\cdot, \cdot)_\omega$ denotes the duality pairing between $H^{-1}(\omega)$ and $H^1_0(\omega)$, and $|\cdot|_{1,\omega}$ is the semi-norm on $H^1(\omega)$. We omit the subscript $\omega$ if $\omega = \Omega$. Let $a_\omega(w, v) = \int_\omega \nabla w \cdot \nabla v \, dx$. We have $|\sigma|_{\omega, \omega} = |w|_{1,\omega}$, where $w \in H^1_0(\omega)$ is the solution of the auxiliary equation

$$a_\omega(w, v) = (\sigma, v)_{\omega} \quad \forall \, v \in H^1_0(\omega). \tag{1.5}$$

2. Discontinuous Galerkin formulations

Notation. We denote by $\{T\}$ a family of subdivisions of $\overline{\Omega}$ into (closed) triangles such that the minimal angle condition is satisfied. For a triangulation $\mathcal{T}_h$, let $\mathcal{E}_h$ be the union of all edges and $\mathcal{E}^i_h := \mathcal{E}_h \setminus \partial \Omega$ the union of all interior edges. Let $h_K = \text{diam}(K)$ for $K \in \mathcal{T}_h$ and $h_e = \text{length}(e)$ for $e \in \mathcal{E}_h$. We denote by $\mathcal{N}_h$ the set of nodes of $\mathcal{T}_h$. For any element $K \in \mathcal{T}_h$, define the patch set $\mathcal{O}_K := \bigcup\{T \in \mathcal{T}_h, \, T \cap K \neq \emptyset\}$, and for any edge $e$ shared by two elements $K_1$ and $K_2$, define $\mathcal{O}_e := K_1 \cup K_2$. For a scalar-valued function $v$ and a vector-valued function $q$, let $v_l = v|_{\partial K_l}$, $q_l = q|_{\partial K_l}$, and $n_l = n|_{\partial K_l}$ be the unit normal vector external to $\partial K_l$. Then define the average $\{\cdot\}$ and the jump $[\cdot]$ on $e \in \mathcal{E}^i_h$ as follows:

$$[v] = \frac{1}{2}(v_1 + v_2), \quad [v] = v_1 n_1 + v_2 n_2, \quad [q] = \frac{1}{2}(q_1 + q_2), \quad [q] = q_1 \cdot n_1 + q_2 \cdot n_2.$$
Note that $K_h = \{ v_h \in V_h : v_h \geq \psi_i \text{ in } \Omega \}$, $\psi_i$ being the continuous piecewise linear interpolant of $\psi$. Denote by $\nabla_h$ the broken gradient whose restriction to each element $K \in T_h$ is equal to $V$. Define some seminorms and norms by the following relations:

$$|v|_{1,h}^2 = \sum_{K \in T_h} |v|^2_{1,K}, \quad |v|_{1,K}^2 = \| \nabla v \|_{K}^2, \quad \| v \|_K^2 = \int_K v^2 dx, \quad \| v \|_e^2 = \int_e v^2 ds.$$

Throughout this paper, $C$ denotes a generic positive constant dependent on the minimal angle condition but not on the element sizes, which may take different values at different occurrences.

**Discontinuous Galerkin schemes.** Here we take the LDG method [20] as an example to demonstrate the a posteriori error analysis of DG methods in solving the obstacle problem (1.1). The derivation and analysis of the a posteriori error estimators for the LDG method can be extended to other DG methods studied in [9]. The LDG method for solving the obstacle problem is: Find $u_h \in K_h$ such that

$$B_h(u_h, v_h - u_h) \geq (f, v_h - u_h) \quad \forall \, v_h \in K_h, \quad (2.1)$$

where

$$B_h(w, v) = (\nabla_h w, \nabla_h v)_{\Omega} - \{ [w], \{ \nabla_h v \} \}_{\partial \Omega} - \{ [\nabla_h w], [v] \}_{\partial \Omega} - (\beta \cdot [w], [\nabla_h v])_{\partial \Omega}$$

$$- (\{ [\nabla_h w], \beta \cdot [v] \})_{\partial \Omega} + (r(w), v)_{\Omega} + \beta \cdot [w], [v])_{\Omega} + \alpha (w, v). \quad (2.2)$$

Here we use the notation $\langle (w, v)_{\Omega}, (w, v)_{\partial \Omega} \rangle$, and $\langle (w, v)_{\partial \Omega} \rangle$ for $\int_\Omega w u dx$, $\int_\Omega w v dx$, and $\int_{\partial \Omega} w v ds$, with $eta \in [L^2(\partial \Omega)]^2$ is a vector-valued function which is constant on each edge; $\alpha (w, v) = \int_{\partial \Omega} \mu [w] \cdot [v] ds$ is the penalty (stabilization) term with the penalty weighting function $\mu : \partial \Omega \to \mathbb{R}$ given by $\eta_e h_e^{-1}$ on each $e \in \partial \Omega$, $\eta_e$ being a positive number on $e$; $r : [L^2(\partial \Omega)]^2 \to W_h$ and $l : L^2(\partial \Omega) \to W_h$ are lifting operators defined by

$$\int_\Omega r(q) \cdot w_h dx = - \int_{\partial \Omega} q \cdot \{ w_h \} ds, \quad \int_\Omega l(v) \cdot w_h dx = - \int_{\partial \Omega} v \{ w_h \} ds \quad \forall \, w_h \in W_h.$$

Let $u \in H^2(\Omega)$ be the solution of (1.1) and $v \in V$. Note that on $e \in \partial \Omega$, $[u] = 0$, $[\nabla u] = 0$, and $[v] = 0$. We have

$$B_h(u, v) = \int_\Omega (f + \sigma) v dx = \int_\Omega (f + \sigma, v) \quad \forall \, v \in V. \quad (2.3)$$

For (2.1), like (1.2), define $\sigma_h = \sigma_h(u_h) \in V_h$ by

$$\langle \sigma_h, v_h \rangle := B_h(u_h, v_h) - (f, v_h) \quad \forall \, v_h \in V_h. \quad (2.4)$$

Note that we can view $\sigma_h$ as an element in $V_h$ and it is determined from

$$\langle \sigma_h, v_h \rangle = B_h(u_h, v_h) - (f, v_h) \quad \forall \, v_h \in V_h. \quad (2.5)$$

Obviously, $u_h$ is also the DG approximation of the solution $z \in V$ of the linear elliptic problem:

$$B_h(z, v) = (f + \sigma, v) \quad \forall \, v \in V. \quad (2.6)$$

Since $f + \sigma_h \in L^2(\Omega)$, $z \in H^2(\Omega)$ [21]. Thus, $-\Delta z = f + \sigma_h$ in $\Omega$. From (2.1) and the definition of $\sigma_h$, we get

$$\langle \sigma_h, v_h - u_h \rangle \geq 0 \quad \forall \, v_h \in K_h. \quad (2.7)$$

Similar to (1.4), we have

$$\langle \sigma_h, v \rangle \geq 0 \quad \forall \, v \in V_h, v \geq 0. \quad (2.8)$$

Since $\langle \sigma_h, v \rangle \geq 0$ does not hold true for all $v \in V_+$, we write $\sigma_h = \sigma_h^+ + \sigma^{ce}$ with $(\sigma_h^+, v) \geq 0$ for all $v \in V_+$, $\sigma^{ce}$ carrying the consistency error. Following the choice of $\bar{\phi}_h$ in [15], we define $\sigma_h^+$ as

$$\sigma_h^+ := \sum_{K \in T_h} \sum_{p \in K} (B_h(u_h, \phi_p^K) - (f, \phi_p^K)) \phi_p^K / (1, \phi_p^K). \quad (2.9)$$

where $\phi_p^K$ is the nodal basis of vertex $p$ in element $K$. From the definition of $\sigma_h$ and (2.7), it is easy to know $\sigma_h^+ \geq 0$ in $\Omega$. Next we show a very important property of $\sigma_h^+$.

$$\sigma_h^+(p)(u_h(p) - \psi_i(p)) = 0 \quad \forall \, p \in N_h. \quad (2.9)$$

First, we know $u_h(p) - \psi_i(p) \geq 0$, so $\sigma_h^+(p)(u_h(p) - \psi_i(p)) \geq 0$. Let $v_h(p) = \psi_i(p)$ and $v_h(x) = u_h(x)$ for all other nodes $x \in N_h$, then

$$(\psi_i(p) - u_h(p)) B_h(u_h, \phi_p^K) = B_h(u_h, v_h - u_h) \geq (f, v_h - u_h) = (\psi_i(p) - u_h(p)) (f, \phi_p^K).$$

After a division by $(1, \phi_p^K)$ on both sides, we obtain $(\psi_i(p) - u_h(p)) \sigma_h^+(p) \geq 0$. So (2.9) is valid.
We group the elements into three parts:
\[ C_h = \cup \{ K \in \mathcal{T}_h : u_h = \psi_1 \text{ on } \omega_K \}, \quad D_h = \cup \{ K \in \mathcal{T}_h : u_h > \psi_1 \text{ on } K \}, \quad F_h = \overline{\mathcal{T}_h} \setminus (C_h \cup D_h). \]

Based on this decomposition \( \overline{\mathcal{T}_h} = C_h \cup D_h \cup F_h \) and the relation (2.9), we obtain
\[ \sigma_n^+ = 0 \quad \text{in } D_h. \tag{2.10} \]

Given \( v_h \in V_h \), written \( v_h = \sum_{K \in \mathcal{T}_h} \sum_{j=1}^3 \alpha_K^{(j)} \phi^{(j)}_k \) , we construct a function \( \chi \in V_h \cap H^1_0(\Omega) \) as follows: At every interior node of the conforming mesh \( \mathcal{T}_h \), the value of \( \chi \) is set to be the average of the values of \( v_h \) computed from all the elements sharing that node, and \( \chi = 0 \) at the boundary nodes. For each \( v \in \mathbb{N}_h \), let \( \omega_v = \{ K \in \mathcal{T}_h : v \in K \} \) and denote its cardinality by \( |\omega_v| \), which is bounded by a constant depending only on the minimal angle condition of the mesh. To each node \( v \), the associated basis function \( \phi^{(v)} \) is given by
\[ \supp \phi^{(v)} = \bigcup_{K \in \omega_v} K, \quad \phi^{(v)}|_K = \phi^{(v)}_K \quad \text{for } \chi^{(v)}_K = v. \]

Then we define \( \chi \in V_h \cap H^1_0(\Omega) \) by \( \chi = \sum_{v \in \mathbb{N}_h} \beta^{(v)} \phi^{(v)} \), where
\[ \beta^{(v)} = \frac{1}{|\omega_v|} \sum_{K \in \omega_v} \alpha_K^{(v)} \quad \text{if } v \in \mathbb{N}_h \cap \Omega, \quad \beta^{(v)} = 0 \quad \text{if } v \in \mathbb{N}_h \cap \partial \Omega. \]

For nonconforming meshes, let \( \mathcal{P}_0^h \) be the set of all hanging nodes. Then we construct \( \chi \) from \( v_h \) same as conforming mesh case on all the nodes \( v \in \mathbb{N}_h \setminus \mathcal{P}_0^h \).

In [22, Theorem 2.2 for the conforming mesh and Theorem 2.3 for the nonconforming mesh], a proof of the following lemma was given. Here we just state the result for the conforming mesh; the same result holds for the nonconforming mesh.

**Lemma 2.1.** Let \( \mathcal{T}_h \) be a conforming mesh consisting of triangles. Then for any \( v_h \in V_h \) there exists \( \chi \in V_h \cap H^1_0(\Omega) \) satisfying
\[ \sum_{K \in \mathcal{T}_h} \| \nabla (v_h - \chi) \|_K^2 \leq C \sum_{e \in \mathcal{E}_h} h_e^{-1} \| \sigma_h \|_e^2. \tag{2.11} \]

### 3. The case of an affine obstacle

Now we follow the ideas in [18] to derive a posteriori error estimators of DG methods for the obstacle problem. We give detailed derivation and analysis of an a posteriori error estimator for the LDG method [20]. Extension to other DG methods can be derived by similar arguments. We distinguish two cases depending on whether the obstacle function is affine. First, we consider the case of an affine obstacle \( \psi \in P_1(\Omega) \); in this case, \( \psi_1 = \psi \).

#### 3.1. Reliable estimator for the LDG method

From (2.3) and (2.6), for all \( v \in V \), we have
\[ B_h(u_h - u, v) = B_h(u_h - z, v) + B_h(z - u, v) = B_h(u_h - z, v) + (\sigma_h - \sigma, v). \]

Denote the error by \( e := u_h - u \). From the definition (2.2) and \( [v] = 0 \) on each \( e \in \mathcal{E}_h \), the above equation becomes
\[ (\nabla_h e, \nabla_h v)_\Omega - \langle [e], [\nabla_h v] \rangle_{\mathcal{E}_h} - \langle \beta \cdot [e], [\nabla_h v] \rangle_{\mathcal{E}_h} = (\nabla_h (u_h - z), \nabla_h v)_\Omega - \langle [u_h - z], [\nabla_h v] \rangle_{\mathcal{E}_h} - \langle \beta \cdot [u_h - z], [\nabla_h v] \rangle_{\mathcal{E}_h} + (\sigma_h - \sigma, v). \]

Then,
\[ \int_\Omega \nabla_h e \cdot \nabla_h v \, dx = \int_\Omega \nabla_h (u_h - z) \cdot \nabla_h v \, dx - \int_{\mathcal{E}_h} [u_h - z - e] \cdot [\nabla_h v] \, ds \]
\[ - \int_{\mathcal{E}_h} \beta \cdot [u_h - z - e] [\nabla_h v] \, ds + (\sigma_h - \sigma, v). \]

Note that \( u_h - z - e = u - z, [u - z] = 0 \) on each \( e \in \mathcal{E}_h \). We have
\[ \int_\Omega \nabla_h e \cdot \nabla_h v \, dx = \int_\Omega \nabla_h (u_h - z) \cdot \nabla_h v \, dx + (\sigma_h - \sigma, v). \tag{3.1} \]
Let \( \chi \in V_h \cap H^1_0(\Omega) \) be the function constructed from \( u_h \), satisfying (2.11) for \( v_h = u_h \). Taking \( v := \chi - u = \chi - u_h + u_h - u \) in (3.1) and using the Cauchy–Schwarz inequality, we have

\[
|e|_{1,h}^2 \leq \| u_h - z \|_{1,h} \| \chi - u \|_{1,h} + |e|_{1,h} \| \chi - u_h \|_{1,h} + (\sigma_h - \sigma, \chi - u) \\
\leq 2|u_h - z|_{1,h}^2 + \frac{1}{2} |e|_{1,h}^2 + \frac{5}{4} |\chi - u_h|_{1,h}^2 + (\sigma_h - \sigma, \chi - u).
\]

With the estimate of \( |\chi - u_h|_{1,h}^2 \) by (2.11), we obtain from (3.2) that

\[
|e|_{1,h}^2 \leq C \left( |u_h - z|_{1,h}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [u_h] \|_e^2 + (\sigma_h - \sigma, \chi - u) \right). \tag{3.3}
\]

We now recall a result from [8]. For the Poisson problem

\[ -\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, \]

rewrite it as the first order system

\[ p = \nabla u, \quad -\nabla \cdot p = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega. \tag{3.4} \]

Then the DG formulation for this problem is

\[
\int_\Omega p_h \cdot \tau_h \, dx = -\int_\Omega u_h \nabla \cdot \tau_h \, dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \hat{u}_h \cdot n_K \cdot \tau_h \, ds \quad \forall \tau_h \in W_h, \tag{3.5}
\]

\[
\int_\Omega p_h \cdot \nabla v_h \, dx = \int_\Omega f \cdot v_h \, dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tilde{p}_h \cdot n_K \cdot v_h \, ds \quad \forall v_h \in V_h, \tag{3.6}
\]

where \( \hat{u}_h \) and \( \tilde{p}_h \) are numerical fluxes. Different choices of the numerical fluxes lead to different DG methods. The following result holds for the LDG method and the methods discussed in [8].

**Theorem 3.1.** Let \( u \in V := H^1_0(\Omega) \) and \( p \in W := L^2(\Omega)^2 \) be the solution of the problem (3.4), and \( u_h \in V_h \) and \( p_h \in W_h \) the solution of the problem (3.5)–(3.6). Then

\[
\| p - p_h \| \leq C (\eta_\ast + \zeta_\ast),
\]

where

\[
\eta_\ast^2 := \sum_{K \in \mathcal{T}_h} h_K^2 \| \text{div} p_h \|_K^2 + \sum_{e \in \mathcal{E}_h} h_e \| [p_h] \|_e^2, \quad \zeta_\ast^2 := \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [u_h] \|_e^2.
\]

**Corollary 3.2.** With the same notation as in Theorem 3.1, we have

\[
\| \nabla u - \nabla_h u_h \| \leq C (\eta + \zeta_\ast),
\]

where

\[
\eta^2 := \sum_{K \in \mathcal{T}_h} h_K^2 \| \Delta u_h \|_K^2 + \sum_{e \in \mathcal{E}_h} h_e \| [\nabla u_h] \|_e^2.
\]

**Proof.** By [8, Lemma 2.1], for all \( v_h \in V_h \), we have

\[
\| r([v_h]) \|^2 \leq C \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [v_h] \|_e^2, \quad \| l([v_h]) \|^2 \leq C \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [v_h] \|_e^2.
\]

From (7, 3.9),

\[ p_h = \nabla u_h - r([\hat{u}_h - u_h]) - l([\hat{u}_h - u_h]). \]

Then

\[
\| \nabla u - \nabla_h u_h \| \leq \| \nabla u - p_h \| + \| p_h - \nabla_h u_h \| \leq C (\eta_\ast + \zeta_\ast) + \| r([\hat{u}_h - u_h]) \| + \| l([\hat{u}_h - u_h]) \|.
\]

From the choices of numerical fluxes \( \hat{u}_h \) in Table 3.1 of [7], we have

\[ [\hat{u}_h - u_h] = -[u_h] \text{ or } 0, \quad ([\hat{u}_h - u_h] = -\beta \cdot [u_h] \text{ or } 0. \]
Then,
\[
\| r((u_h - u_h)) \| \leq C \sum_{e \in E_h} h_e^{-1} \| [u_h]_e \|^2, \quad \| l((u_h - u_h)) \| \leq C \sum_{e \in E_h} h_e^{-1} \| [u_h]_e \|^2,
\]
\[
\| p_h - \nabla_h u_h \| \leq \zeta_+ \quad \text{and} \quad \| \nabla u - \nabla u_h \| \leq C (\eta_+ + \zeta_+). 
\]
Finally, by the inverse inequality and trace inequality, we get
\[
\eta_+^2 \leq 2 \left( \eta^2 + \sum_{K \in \mathcal{T}_h} h_K^2 \| \text{div}(p_h - \nabla u_h) \|_K^2 + \sum_{e \in E_h} h_e \| [p_h - \nabla_h u_h]_e \|_e^2 \right)
\leq 2 \eta^2 + C \sum_{K \in \mathcal{T}_h} \| p_h - \nabla u_h \|_K^2 = 2\eta^2 + C \| p_h - \nabla u_h \| \leq 2\eta^2 + C \zeta_+^2.
\]
Therefore,
\[
\eta_+ \leq C (\eta + \zeta_+)
\]
and the result is proved. ■

Define the interior residuals and edge-based jumps
\[
R_K := \Delta u_h + f + \sigma_h \quad \text{for} \ K \in \mathcal{T}_h, \quad R_e := [\nabla_h u_h] \quad \text{for} \ e \in E_h.
\]
Then the local estimators are
\[
\eta_K := \left( \sum_{K \in \mathcal{T}_h} h_K^2 \| [u_h]_K \|^2 \right)^{1/2}, \quad \eta_{\partial K} := \left( \sum_{e \in \partial K} h_e^{-1} \| [u_h]_e \|^2 \right)^{1/2}.
\]
(3.7)
Applying Corollary 3.2 to \(|u_h - z|_{1,h}\), we obtain from (3.3)
\[
|e|_{1,h}^2 \leq C \left( \sum_{K \in \mathcal{T}_h} \eta_K^2 + \sum_{K \in \mathcal{T}_h} \eta_{\partial K}^2 + (\sigma_h - \sigma \cdot \chi - u) \right).
\]
(3.8)
Before giving an estimate of \((\sigma_h - \sigma \cdot \chi - u)\), we introduce the following result [17].

**Lemma 3.3.** Let \(q_h\) be a continuous piecewise linear function over \(\mathcal{T}_h\), \(p \in \mathcal{N}_h \cap \Omega\) an interior node with \(p \in K, K \in \mathcal{T}_h\). Suppose \(q_h \geq 0\) and \(q_h(p) = 0\). Then
\[
\| q_h \|_K \leq Ch_k \left( \sum_{e \in \mathcal{E}(p)} h_e \| [\nabla q_h]_e \|^2 \right)^{1/2}, \quad \mathcal{E}(p) := \{ e \in E_h : p \in e \}.
\]
\[
\| q_h \|_{\mathcal{F}_h} \leq Ch_k \left( \sum_{K \in \mathcal{F}_h} h_K \| [\nabla q_h]_K \|^2 \right)^{1/2}, \quad \mathcal{F}_h := \{ K \in \mathcal{F}_h : \exists \ p_1, p_2 \in K \cap \mathcal{N}_h, u_h(p_1) > \psi_1(p_1) \text{ and } u_h(p_2) = \psi(p_2) \}.
\]

**Proposition 3.4.** Assume \(\psi\) is an affine function. Then for any fixed \(\epsilon \in (0, 1),\)
\[
(\sigma_h - \sigma \cdot \chi - u) \leq \epsilon \| u_h - u_1 \|_{1,h}^2 + C \left( \sum_{e \in \mathcal{E}(p)} h_e^{-1} \| [u_h]_e \|^2 + \| \sigma_h - \sigma_+ \|^2 + \sum_{K \in \mathcal{F}_h} \sum_{e \in \mathcal{E}(p)} h_e \| [\nabla u_h]_e \|^2 + \eta_{\mathcal{F}_h}^2 \right),
\]
where
\[
\eta_{\mathcal{F}_h}^2 = \sum_{K \in \mathcal{F}_h} h_K^2 \| \nabla \sigma_+ \|^2 + \sum_{K \in \mathcal{F}_h} \int_{\partial K} \sigma_+ (\chi - u_h) \, dx.
\]
\[
\mathcal{F}_h := \{ K \in \mathcal{F}_h : \exists \ p_1, p_2 \in K \cap \mathcal{N}_h, u_h(p_1) > \psi_1(p_1) \text{ and } u_h(p_2) = \psi(p_2) \}.
\]

**Proof.** Note that \(\psi_1 = \psi\). So \(\chi \in K.\) From (1.3), \((\sigma \cdot \chi - u) \geq 0\). Write
\[
(\sigma_h, \chi - u) = (\sigma_h - \sigma_+^+, \chi - u) + (\sigma_+^+, \chi - u).
\]
(3.10)
Furthermore, we have by (2.11),
\[
(\sigma_h - \sigma_+^+, \chi - u) \leq \| \sigma_h - \sigma_+^+ \|_1 \| \chi - u \|_{1,\Omega} \leq \frac{1}{2} \| \chi - u \|^2_{1,\Omega} + \frac{1}{2\epsilon} \| \sigma_h - \sigma_+^+ \|^2 + \frac{1}{2\epsilon} \| \sigma_h - \sigma_+^+ \|^2.
\]
(3.11)
If $K \in \mathcal{D}_h$, from (2.10), we know $\sigma_h^+ = 0$ in $K$. For $K \in \mathcal{E}_h$, with the construction of $\chi$, we know $\chi = \psi_I = \psi \leq u$, and so $\sigma_h^+ (\chi - u) \leq 0$ on $K$. Then

$$
(\sigma_h^+, \chi - u) \leq \sum_{K \in \mathcal{F}_h} \int_K \sigma_h^+ (\chi - u) \, dx.
$$

We divide $K \in \mathcal{F}_h$ into two kinds to estimate $\int_K \sigma_h^+ (\chi - u) \, dx$. First, if $K \in \mathcal{F}_h \setminus \mathcal{F}_h$, we know that $u_h = \psi_I = \psi$ on $K$. Therefore,

$$
\int_K \sigma_h^+ (\chi - u) \, dx \leq \int_K \sigma_h^+ (\chi - u_h) \, dx + \int_K \sigma_h^+ (\psi - u) \, dx \leq \int_K \sigma_h^+ (\chi - u_h) \, dx.
$$

Note that if $K \in \mathcal{F}_h$, then $\exists p_1 \in N_h \cap K$ with $u_h(p_1) > \psi_I(p_1)$; so $\sigma_h^+(p_1) = 0$ by (2.9) and consequently,

$$
\|\sigma_h^+\|_K \leq Ch_k \|\nabla \sigma_h^+\|_K.
$$

Now we bound $\int_K \sigma_h^+ (\chi - u) \, dx$ with $K \in \mathcal{F}_h$ in two cases.

**Case 1:** $K \cap \partial \Omega \neq \emptyset$. We have

$$
\int_K \sigma_h^+ (\chi - u) \, dx = \int_K \sigma_h^+ ((\chi - u) - A_h(\chi - u)) \, dx + \int_K \sigma_h^+ A_h(\chi - u) \, dx
\leq \|\sigma_h^+\|_K \|\chi - u - A_h(\chi - u)\|_K + \|A_h(\chi - u)\|_K.
$$

Here $A_h(\chi - u)$ denotes the Clément interpolant of $(\chi - u)$. By the local approximation property of the Clément interpolant, we have

$$
\|\chi - u - A_h(\chi - u)\|_K \leq Ch_k \|\nabla (\chi - u)\|_{\text{wk}}.
$$

Since $(\chi - u) \in H_0^1(\Omega)$ and $K \cap \partial \Omega \neq \emptyset$, there exists a node $p_1 \in K \cap N_h \cap \partial \Omega$ such that $(\chi - u)(p_1) = 0$. Then since $A_h(\chi - u)(p_1) = 0$,

$$
\|A_h(\chi - u)\|_K \leq Ch_k \|\nabla A_h(\chi - u)\|_K \leq Ch_k \|\nabla (\chi - u)\|_{\text{wk}}
$$

by the stability property of $A_h$. So we have

$$
\int_K \sigma_h^+ (\chi - u) \, dx \leq \frac{1}{2} \epsilon \|\nabla (\chi - u)\|_{\text{wk}}^2 + C \epsilon h_k^2 \|\sigma_h^+\|_K^2
\leq \epsilon \|\nabla_h (\chi - u_h)\|_{\text{wk}}^2 + \epsilon \|\nabla_h (u_h - u)\|_{\text{wk}}^2 + Ch_k^2 \|\nabla \sigma_h^+\|_K^2.
$$

**Case 2:** $K \cap \partial \Omega = \emptyset$. Noting that $\psi_I = \psi$, we get

$$
\int_K \sigma_h^+ (\chi - u) \, dx = \int_K \sigma_h^+ (\chi - \psi_I) \, dx + \int_K \sigma_h^+ (\psi - u) \, dx \leq \int_K \sigma_h^+ (\chi - \psi_I) \, dx
$$

by $\sigma_h^+ \geq 0$ and $\psi - u \leq 0$ in $\Omega$. First we assume

$$
(\chi - \psi_I)(p) = 0 \quad \text{for an interior node } p \in K \cap N_h.
$$

Using Lemma 3.3 and $\psi_I = \psi \in P_1(\Omega)$, we obtain

$$
\|\chi - \psi_I\|_K \leq Ch_k \left( \sum_{e \in \mathcal{E}_h(p)} h_e \|\nabla(\chi - \psi_I)\|_e^2 \right)^{\frac{1}{2}} = Ch_k \left( \sum_{e \in \mathcal{E}_h(p)} h_e \|\nabla \chi\|_e^2 \right)^{\frac{1}{2}}.
$$

Then by trace inequality and (2.11), we get

$$
\int_K \sigma_h^+ (\chi - \psi_I) \, dx \leq \|\sigma_h^+\|_K \|\chi - \psi_I\|_K \leq Ch_k^2 \|\nabla \sigma_h^+\|_K \left( \sum_{e \in \mathcal{E}_h(p)} h_e \|\nabla \chi\|_e^2 \right)^{\frac{1}{2}}
\leq C \left( h_k^2 \|\nabla \sigma_h^+\|_K^2 + \sum_{e \in \mathcal{E}_h(p)} h_e \|\nabla(\chi - u_h)\|_e^2 + \|\nabla u_h\|_e^2 \right).
$$
If (3.14) does not hold, there is an interior node $p_2 \in K \cap N_h$ such that $(u_h - \psi_l)(p_2) = 0$. Then

$$\int_K \sigma_h^+(\chi - \psi_l) \, dx = \int_K \sigma_h^+(\chi - u_h) \, dx + \int_K \sigma_h^+(u_h - \psi_l) \, dx$$

$$\leq \int_K \sigma_h^+(\chi - u_h) \, dx + C \left( h_h^2 \| \nabla \sigma_h^+ \|_K^2 + \| \nabla (u_h - \chi) \|_K^2 + \| \nabla (\chi - \psi_l) \|_K^2 \right).$$

Let $\alpha = \min_{\Omega} \{ \chi - \psi_l \} = (\chi - \psi_l)(p)$. By an inverse inequality and Lemma 3.3, we get

$$\| \nabla (\chi - \psi_l) \|_K^2 = \| \nabla (\chi - \psi_l - \alpha) \|_K^2 \leq C h_h^{-1} \| \chi - \psi_l - \alpha \|_K^2 \leq C \sum_{e \in \partial h(p)} h_e \| \nabla \chi \|_e^2.$$ 

For $\int_K \sigma_h^+(\chi - u_h) \, dx$, we have

$$\int_K \sigma_h^+(\chi - u_h) \, dx \leq \| \sigma_h^+ \|_K \| \chi - u_h \|_K \leq \| \sigma_h^+ \|_K \left( \sum_{j=1}^{3} | \beta_K^{(j)} - \alpha_K^{(j)} | \| \phi_K^{(j)} \|_K \right)$$

$$\leq h_h \| \sigma_h^+ \|_K \left( C \sum_{j=1}^{3} | \beta_K^{(j)} - \alpha_K^{(j)} | \right) \leq C \left( h_h^2 \| \nabla \sigma_h^+ \|_K^2 + \sum_{j=1}^{3} | \beta_K^{(j)} - \alpha_K^{(j)} |^2 \right).$$

(3.16)

From the proof of Theorems 2.2 and 2.3 in [22], we know

$$\sum_{K \in T_h} \sum_{j=1}^{3} | \beta_K^{(j)} - \alpha_K^{(j)} |^2 \leq C \sum_{e \in \partial h} h_e^{-1} \| [u_h] \|_e^2,$$

so

$$\sum_{K \in T_h, K \cap \Omega = \emptyset} \int_K \sigma_h^+(\chi - u_h) \, dx \leq C \sum_{K \in T_h, K \cap \Omega = \emptyset} h_h^2 \| \nabla \sigma_h^+ \|_K^2 + C \sum_{e \in \partial h} h_e^{-1} \| [u_h] \|_e^2.$$ 

(3.17)

The inequality (3.9) follows from the combination of (3.10)–(3.13) and (3.15)–(3.17).

Lemma 3.5.

$$| \sigma_h - \sigma_h^+ |_K^2 \leq C \sum_{K \in T_h} h_h^2 \| \sigma_h - \sigma_h^+ \|_K^2 \leq C \sum_{K \in T_h} h_h^2 \| \nabla (\sigma_h - \sigma_h^+) \|_K^2.$$ 

(3.18)

Proof. Let $1_K$ be the indicator function of $K \in T_h$. Then

$$(\sigma_h, 1_K) = (\sigma_h, 1_K)_K = B_h(u_h, 1_K) - (f, 1_K).$$

We use the relation $\sum_{p \in K \cap \partial h} \phi_K = 1_K$ to get

$$(\sigma_h^+, 1_K) = \sum_{p \in K \cap \partial h} (B_h(u_h, \phi_K^p) - (f, \phi_K^p))(1_K, \phi_K^p)/(1, \phi_K^p) = B_h(u_h, 1_K) - (f, 1_K).$$

Hence, for any constant $c_K$ on element $K$, we get

$$(\sigma_h - \sigma_h^+, c_K 1_K) = 0.$$ 

(3.19)

Then for any piecewise constant $c, c|_K = c_K$, and $v \in H^1_0(\Omega)$, we have

$$| \sigma_h - \sigma_h^+ |_1 = \sup_{v \in H^1_0(\Omega), |v|_1 = 1} (\sigma_h - \sigma_h^+, v) = \sup_{v \in H^1_0(\Omega), |v|_1 = 1} (\sigma_h - \sigma_h^+, v - c),$$

$$| \sigma_h - \sigma_h^+ |_1 \leq C \sup_{v \in H^1_0(\Omega), |v|_1 = 1} \left( \sum_{K \in T_h} h_h^2 \| \sigma_h - \sigma_h^+ \|_K^2 \right)^{1/2} |v|_1 = C \left( \sum_{K \in T_h} h_h^2 \| \sigma_h - \sigma_h^+ \|_K^2 \right)^{1/2}.$$ 

(3.20)

Using the relation (3.19), we obtain

$$\| \sigma_h - \sigma_h^+ \|_K^2 = \inf_{c_K \in \mathbb{R}} (\sigma_h - \sigma_h^+, \sigma_h - \sigma_h^+ - c_K)_K \leq C h_h \| \sigma_h - \sigma_h^+ \|_K \| \nabla (\sigma_h - \sigma_h^+) \|_K,$$

$$\| \sigma_h - \sigma_h^+ \|_K \leq C h_h \| \nabla (\sigma_h - \sigma_h^+) \|_K.$$ 

(3.21)

A combination of (3.20) and (3.21) completes the proof of (3.18).
From (3.8), Proposition 3.4 and Lemma 3.5, we obtain
\[ |e|_{1,h}^2 \leq C \left( \sum_{K \in \mathcal{T}_h} \eta_{eK}^2 + \sum_{K \in \mathcal{T}_h} \eta_{\partial K}^2 + h_k^2 \|\sigma_h - \sigma_h^+\|^2_K + \eta_{\gamma_h}^2 \right). \]

Recalling (1.5), we have
\[ |\sigma - \sigma_h|_* = |u - z|_{1,\Omega} \leq |e|_{1,h} + |u_h - z|_{1,h}. \]

Finally, we obtain the following theorem.

**Theorem 3.6.** Let \( u \in H^2(\Omega) \) and \( u_h \) solve (1.1) and (2.1) respectively, \( \psi_l = \psi \). Then
\[ |e|_{1,h} + |\sigma - \sigma_h|_* \leq C_1 \left( \sum_{K \in \mathcal{T}_h} \eta_{eK}^2 \right)^{1/2} + C_2 \left( \sum_{K \in \mathcal{T}_h} \eta_{\partial K}^2 \right)^{1/2} + C_3 \left( \sum_{K \in \mathcal{T}_h} h_k^2 \|\sigma_h - \sigma_h^+\|^2_K \right)^{1/2} + C_4 \eta_{\gamma_h}. \]

**3.2. Efficiency of the estimator for the LDG method**

Now we consider lower bounds of the estimators. We follow the standard argument of lower bounds of residual error estimators for elliptic problems, see [2, pp. 28–31]. First, we introduce the bubble functions. Let \( K \in \mathcal{T}_h \), and let \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) be the barycentric coordinates on \( K \). Then the interior bubble function \( \varphi_K \) is defined by \( \varphi_K = 27\lambda_1\lambda_2\lambda_3 \) and the three edge bubble functions are given by \( \tau_1 = 4\lambda_2\lambda_3, \tau_2 = 4\lambda_1\lambda_3, \tau_3 = 4\lambda_1\lambda_2 \). We list properties of bubble functions stated in Theorems 2.2 and 2.3 of [2] in the form of a lemma.

**Lemma 3.7.** For each \( K \in \mathcal{T}_h, e \subset \partial K \), let \( \varphi_K \) and \( \tau_e \) be the corresponding interior and edge bubble functions. Let \( P(K) \subset H^1(K) \) and \( P(e) \subset H^1(e) \) be finite-dimensional spaces of functions defined on \( K \) or \( e \). Then there exists a constant \( C \) such that for all \( v \in P(K) \),
\[ C^{-1} \|v\|_K^2 \leq \int_K \varphi_K v^2 \, dx \leq C \|v\|_K^2, \]
\[ C^{-1} \|v\|_K \leq \|\varphi_K v\|_K + h_k |\varphi_K v|_{1,K} \leq C \|v\|_K, \]
\[ C^{-1} \|v\|_e^2 \leq \int_e \tau_e v^2 \, ds \leq C \|v\|_e^2, \]
\[ h_k^{1/2} |\tau_e v|_K + h_k^{1/2} |\tau_e v|_{1,K} \leq C \|v\|_e. \]

Denote \( a_K(u, v) = \int_K \nabla u \cdot \nabla v \, dx \). Then for \( u, v \in H^1(\Omega), a(u, v) = \sum_{K \in \mathcal{T}_h} a_K(u, v) \). For all \( v \in H^1_0(\Omega) \), noting that \( [v] = 0 \) and \( [u - z] = 0 \), we have
\[ \sum_{K \in \mathcal{T}_h} a_K(e, v) = \sum_{K \in \mathcal{T}_h} a_K(u_h - z, v) + a(z - u, v) = \sum_{K \in \mathcal{T}_h} \int_K \nabla(u_h - z) \cdot \nabla v \, dx + (\sigma_h - \sigma, v) \]
\[ = \sum_{K \in \mathcal{T}_h} \int_K (-\Delta u_h - f - \sigma_h) v \, dx + \sum_{e \in \mathcal{E}_h} \int_e [\nabla u_h] \cdot v \, ds + (\sigma_h - \sigma, v). \]

Let \( \tilde{R}_K \) be an approximation to the interior residual \( R_K \) from a suitable finite-dimensional subspace. In (3.23), choose \( v = \tilde{R}_K \varphi_K \) on element \( K \). Since \( \varphi_K \) vanishes on the boundary of \( K \), \( v \) can be extended to be zero on the rest of the domain as a continuous function. Therefore,
\[ a_K(e, \tilde{R}_K \varphi_K) = \int_K R_K \tilde{R}_K \varphi_K \, dx + (\sigma_h - \sigma, \tilde{R}_K \varphi_K)_K. \]

Then
\[ \int_K \tilde{R}_K^2 \varphi_K \, dx = \int_K \tilde{R}_K (\tilde{R}_K - R_K) \varphi_K \, dx + a_K(e, \tilde{R}_K \varphi_K) - (\sigma_h - \sigma, \tilde{R}_K \varphi_K)_K. \]

Applying the Cauchy–Schwarz inequality and Lemma 3.7, we obtain
\[ \int_K \tilde{R}_K (\tilde{R}_K - R_K) \varphi_K \, dx \leq \|\tilde{R}_K \varphi_K\|_K \|\tilde{R}_K - R_K\|_K \leq C \|\tilde{R}_K \|_K \|\tilde{R}_K - R_K\|_K, \]
\[ a_K(e, \tilde{R}_K \varphi_K) \leq |e|_{1,K} \|\tilde{R}_K \varphi_K\|_{1,K} \leq C h_k^{-1} |e|_{1,K} \|\tilde{R}_K\|_K, \]
\[ (\sigma_h - \sigma, \tilde{R}_K \varphi_K)_K \leq |\sigma_h - \sigma|_{*,K} \|\tilde{R}_K \varphi_K\|_{1,K} \leq C h_k^{-1} |\sigma_h - \sigma|_{*,K} \|\tilde{R}_K\|_K. \]
Use Lemma 3.7 again, 
\[ \| \tilde{R}_K \|^2 \leq C \int_K \tilde{R}_K^2 \varphi_K \, dx. \]

Combining the above relations, we obtain 
\[ \| \tilde{R}_K \|_K \leq C (\| \tilde{R}_K - R_K \|_K + h_K^{-1} \| e_{1,K} + h_K^{-1} | \sigma_h - \sigma |_{*K} \|_K). \]

Finally, by the triangle inequality \( \| R_K \|_K \leq \| R_K - \tilde{R}_K \|_K + \| \tilde{R}_K \|_K \), we get 
\[ \| R_K \|_K \leq C (\| \tilde{R}_K - R_K \|_K + h_K^{-1} \| e_{1,K} + h_K^{-1} | \sigma_h - \sigma |_{*K} \|_K). \]

Now choose the finite-dimensional subspace for \( \tilde{R}_K \) to be spanned by the local nodal basis \( \phi_K(i) \), \( i = 1, 2, 3 \). Then \( \| \tilde{R}_K - R_K \|_K \) reduces to \( \| f - f_h \|_K \).

\[ f_h = \sum_{i=1}^3 f_i \phi_K(i) \text{ with } f_i = (f, \phi_K(i))_K / (1, \phi_K(i)). \]

If we choose \( \tilde{R}_K \) as an approximation to the jumps from a suitable finite-dimensional space and let \( v = \tilde{R}_K \tau_e \), we have 
\[ \| \tilde{R}_K \|_e \leq C (h_K^{-1/2} \| e_{1,\text{avg}} + h_K^{-1/2} | \sigma_h - \sigma |_{\text{avg}} + h_K^{1/2} \| f - f_h \|_{\text{avg}}). \]

Then we obtain the efficiency of the local error indicator \( \eta_K \).

**Theorem 3.8.** Let \( u \in H^2(\Omega) \) and \( u_h \) be the solutions of (1.1) and (2.1), respectively, and \( \eta_K \) be the estimator (3.7). Then 
\[ \eta_K \leq C (\| u - u_h \|_{1,\text{avg}} + | \sigma - \sigma_h |_{\text{avg}} + h_K \| f - f_h \|_{\text{avg}}). \]

To bound the remaining terms in the error estimator (3.22), we first give a lemma.

**Lemma 3.9.**
\[ h_K \| \sigma_h^+ + f_h \|_K \leq C \left( \sum_{e \in E_h} h_e^{-1} \| [u_h]_e \|_e^2 \right)^{1/2} + C \left( \sum_{e \in \partial K} h_e \| [\nabla u_h]_e \|_e \right)^{1/2}. \]

**Proof.** Using the definitions (2.8) and (2.34), we have 
\[ h_K \| \sigma_h^+ + f_h \|_K = h_K \left\| \sum_{i=1}^3 B_h(u_h, \phi_K(i)) \phi_K(i) / (1, \phi_K(i)) \right\|_K \leq C \sum_{i=1}^3 |B_h(u_h, \phi_K(i))| \]

since \( \| \phi_K(i) / (1, \phi_K(i)) \|_K \leq C h_K^{-1}. \) For each \( \phi_K(i) \), we know \( \phi_K(i)(x) = 0 \) \( \forall x \in \Omega \setminus K \), which implies \( \| \phi_K(i) \| = 0 \) and \( \| \nabla \phi_K(i) \| = 0 \) on \( \partial_h \setminus \partial K \). So

\[ B_h(u_h, \phi_K(i)) = (\nabla u_h, \nabla \phi_K(i))_K - ([u_h], [\nabla \phi_K(i)])_{\partial K} - ([\nabla u_h], [\phi_K(i)])_{\partial K} \]

\[ - (\beta \cdot [u_h], [\nabla \phi_K(i)])_{\partial K} - ([\nabla u_h], \beta \cdot [\phi_K(i)])_{\partial K} \]

\[ + (r([u_h]) + l(\beta \cdot [u_h]), r_{\partial K}([\phi_K(i)]) + l_{\partial K}(\beta \cdot [\phi_K(i)]))_{\Omega} + \int_{\partial K} \mu([u_h] \cdot [\phi_K(i)]). \]

where \( r_{\partial K} : [L^2(\partial K)]^2 \to W_h \) and \( l_{\partial K} : L^2(\partial K) \to W_h \) are lifting operators defined by 
\[ \int_{\partial K} r_{\partial K}(q) \cdot w_h \, dx = - \int_{\partial K} q \cdot [w_h] \, ds, \quad \int_{\partial K} l_{\partial K}(v) \cdot w_h \, dx = \int_{\partial K} v \, [w_h] \, ds, \quad \forall w_h \in W_h. \]

Noting \( \Delta u_h = 0 \) on \( K \), we get by integration by part and the Cauchy–Schwarz inequality 
\[ (\nabla u_h, \nabla \phi_K(i))_K - ([\nabla u_h], [\phi_K(i)])_{\partial K} = \frac{1}{2} \int_{\partial K} [\nabla u_h] \phi_K(i) \, ds \leq C \sum_{e \in \partial K} h_e^{1/2} \| [\nabla u_h]_e \|_e. \]

Similarly, we have 
\[ ([u_h], [\nabla \phi_K(i)])_{\partial K} \leq C \sum_{e \in \partial K} h_e^{-1/2} \| [u_h]_e \|_e, \]
\[ (\beta \cdot [u_h], [\nabla \phi_K(i)])_{\partial K} \leq C \sum_{e \in \partial K} h_e^{-1/2} \| [u_h]_e \|_e, \]
\[ ([\nabla u_h], \beta \cdot [\phi_K(i)])_{\partial K} \leq C \sum_{e \in \partial K} h_e^{1/2} \| [\nabla u_h]_e \|_e, \]
\[ \int_{\partial K} \mu([u_h] \cdot [\phi_K(i)]) \leq C \sum_{e \in \partial K} h_e^{1/2} \| [u_h]_e \|_e. \]
Define the edge lifting operators \( r_e : [L^2(e)]^2 \to W_h \) and \( l_e : L^2(e) \to W_h \) by

\[
\int_{\Omega} r_e(q) \cdot w_h \, dx = -\int_{e} q \cdot \{w_h\} \, ds, \quad \int_{\Omega} l_e(v) \cdot w_h \, dx = -\int_{e} v[w_h] \, ds \quad \forall \, w_h \in W_h.
\]

We recall some results about the lifting operators [7]:

\[
\|r(q)\|_{W_h}^2 = \left\| \sum_{e \in \mathcal{E}_h} r_e(q) \right\|_{W_h}^2 \leq 3 \sum_{e \in \mathcal{E}_h} \|r_e(q)\|_{W_h}^2 \leq C \sum_{e \in \mathcal{E}_h} h_e^{-1} \|\{q\}\|_{L^2(e)}^2.
\]

\[
\|l(q)\|_{W_h}^2 = \left\| \sum_{e \in \mathcal{E}_h^i} l_e(q) \right\|_{W_h}^2 \leq 3 \sum_{e \in \mathcal{E}_h^i} \|l_e(q)\|_{W_h}^2 \leq 12 \sum_{e \in \mathcal{E}_h^i} \|r_e(qn_e)\|_{W_h}^2 \leq C \sum_{e \in \mathcal{E}_h^i} h_e^{-1} \|\{q\}\|_{L^2(e)}^2.
\]

Then

\[
\left( r([u_h]) + l(\beta \cdot [u_h]), r_{\partial K}([\phi_K^{(i)}]) + l_{\partial K}(\beta \cdot [\phi_K^{(i)}]) \right)_{W_h} \leq C \left( \sum_{e \in \mathcal{E}_h} h_e^{-1} \|\{u_h\}\|_{L^2(e)}^2 \right)^{1/2}.
\]

Combination of above inequalities accomplishes the proof. \( \square \)

Now we explore bounds on the remaining terms in the error estimator (3.22). First,

\[
h_K \|\sigma_h^+ - \sigma_h\|_K \leq h_K \|\sigma_h^+ + f_h\|_K + h_K \|f + \sigma_h\|_K + h_K \|f - f_h\|_K.
\]

Note that \( f + \sigma_h = R_K f \) in \( K \). Let \( f_K = \int_{K} f \, dx / |K| \) for \( K \in \mathcal{F}_h \). By inverse inequality, we obtain

\[
h_K^2 \|\nabla \sigma_h^+\|_K \leq h_K^2 \|\nabla (\sigma_h^+ + f_h)\|_K \leq C h_K \|\sigma_h^+ + f_h\|_K \leq C \left( h_K \|\sigma_h^+ + f_h\|_K + h_K \|f - f_h\|_K \right).
\]

For \( \eta_{\partial K} \), by the trace inequality and inverse inequality, we obtain

\[
\eta_{\partial K}^2 = \frac{1}{2} \sum_{e \in \partial K} h_e^{-1} \|\{u_h\}\|_{L^2(e)}^2 = \frac{1}{2} \sum_{e \in \partial K} h_e^{-1} \|\{u_h - u_l\}\|_{L^2(e)}^2
\]

\[
\leq C \left( \sum_{K \in \mathcal{F}_h} h_e^{-1} \|\{u_h - u_l\}\|_{L^2(e)}^2 \right) \leq C |u_h - u_l|_{1,\text{loc}}^2,
\]

where \( u_l \) is a continuous piecewise polynomial interpolant of \( u \). Summarizing the above argument, we get the following theorem.

**Theorem 3.10.** Let \( u \in H^2(\Omega) \) and \( u_h \) be the solutions of (1.1) and (2.1), respectively. Then we have local lower bounds of error estimators in (3.22),

\[
\eta_K \leq C \left( |u - u_h|_{1,\text{loc}} + |\sigma - \sigma_h|_{s,\text{loc}} + h_K \|f - f_h\|_{\text{loc}} \right),
\]

\[
\eta_{\partial K} \leq C |u_h - u_l|_{1,\text{loc}},
\]

\[
h_K \|\sigma_h^+ - \sigma_h\|_K \leq h_K \|\sigma_h^+ + f_h\|_K + h_K \|f + \sigma_h\|_K + h_K \|f - f_h\|_K,
\]

\[
h_K^2 \|\nabla \sigma_h^+\|_K \leq C \left( h_K \|\sigma_h^+ + f_h\|_K + h_K \|f - f_h\|_K \right).
\]

For the term \( \int_{K} \sigma_h^+ (\chi - u_h) \, dx \) with \( K \in \mathcal{F}_h \setminus \mathcal{F}_h \), we are not able to derive a bound of the form

\[
\int_{K} \sigma_h^+ (\chi - u_h) \, dx \leq C \left( |u - u_h|_{\text{loc}} + |\sigma - \sigma_h|_{s,\text{loc}} + h_K \|f - f_h\|_{\text{loc}} \right).
\]

This term is expected to be very small due to the construction of \( \chi \). See Section 5 for numerical confirmation of this.

### 4. The case of a general obstacle

For the case of a general obstacle, we only need to treat the term \((\sigma_h - \sigma, \chi - u)\) differently. Define \( \chi^+ = \max\{\chi, \psi\} \in K \) and denote \( v^+ = \max(v, 0) \). Then with (1.3), we get

\[
\langle \sigma, u - \chi \rangle = \langle \sigma, u - \chi^+ \rangle + \langle \sigma, \chi^+ - \chi \rangle \leq \langle \sigma, (\psi - \chi)^+ \rangle
\]

\[
\leq \epsilon |\sigma - \sigma_h|^2 + \frac{1}{4\epsilon} |(\psi - \chi)^+|^2_{1,\Omega} + \langle \sigma_h, (\psi - \chi)^+ \rangle.
\]
Inserting $\sigma_h^+$, we have
\[
(\sigma_h, \chi - u + (\psi - \chi)^+) = (\sigma_h - \sigma_h^+, \chi - u + (\psi - \chi)^+) + (\sigma_h^+, \chi - u + (\psi - \chi)^+)
\]
\[
(\sigma_h, \chi - u + (\psi - \chi)^+) \leq \frac{1}{e} |\sigma_h - \sigma_h^+|_K^2 + \frac{e}{2} |\chi - u|_{1, \Omega}^2 + \frac{e}{2} |(\psi - \chi)^+|_{1, \Omega}^2
\]
\[
\leq \frac{1}{e} |\sigma_h - \sigma_h^+|_K^2 + \frac{e}{2} |\chi - u|_{1, \Omega}^2 + \frac{e}{2} |u_h - u|_{1, \Omega}^2 + \frac{e}{2} |(\psi - \chi)^+|_{1, \Omega}^2.
\]
Noticing $\sigma_h^+ = 0$ on $K \in D_h$, we obtain
\[
(\sigma_h^+, \chi - u + (\psi - \chi)^+) = \sum_{K \in \mathcal{E}_h} \int_K \sigma_h^+(\chi - u + (\psi - \chi)^+) \, dx.
\]
We consider three cases of $K \in \mathcal{E}_h \cup \mathcal{F}_h$ to estimate $\int_K \sigma_h^+(\chi - u + (\psi - \chi)^+) \, dx$.

Case 1: $K \in \mathcal{F}_h$ and $K \cap \partial \Omega = \emptyset$. Observing $(\chi - u), (\psi - \chi)^+ \in V = H_0^1(\Omega)$, we can use the same argument for Case 1 in the proof of Proposition 3.4 to obtain
\[
\int_K \sigma_h^+(\chi - u + (\psi - \chi)^+) \, dx \leq e \|\nabla_h(\chi - u - \psi)\|_{\infty, K}^2 + e \|\nabla_h(u_h - u)\|_{\infty, K}^2 + \frac{1}{2} e \|\nabla(\psi - \chi)^+\|_{1, \Omega}^2 + \frac{C}{e} h^4_K \|\nabla \sigma_h^+\|_{1, \Omega}^2.
\]

For the remaining two cases, noting $\psi - u \leq 0$ and $\sigma_h^+ \geq 0$, we have
\[
\int_K \sigma_h^+(\chi - u) \, dx = \int_K \sigma_h^+(\chi - \psi) \, dx + \int_K \sigma_h^+(\psi - u) \, dx \leq \int_K \sigma_h^+(\chi - \psi) \, dx,
\]
and observing $\chi - \psi + (\psi - \chi)^+ = (\chi - \psi)^+$, we get
\[
\int_K \sigma_h^+(\chi - u + (\psi - \chi)^+) \, dx \leq \int_K \sigma_h^+(\chi - \psi)^+ \, dx.
\]

Case 2: $K \in \mathcal{F}_h$ and $K \cap \partial \Omega = \emptyset$. Since $\chi \geq \psi_h$, we have
\[
(\chi - \psi)^+ = (\chi - \psi_1 + \psi_1 - \psi)^+ \leq (\chi - \psi_1)^+ + (\psi_1 - \psi)^+ = (\chi - \psi_1) + (\psi_1 - \psi)^+.
\]
\[
\int_K \sigma_h^+(\chi - \psi)^+ \, dx \leq \int_K \sigma_h^+(\chi - \psi_1) \, dx + \int_K \sigma_h^+(\psi_1 - \psi) \, dx.
\]

Following the proof of Proposition 3.4 to estimate the first term on the right side,
\[
\sum_{K \in \mathcal{F}_h, K \cap \partial \Omega = \emptyset} \int_K \sigma_h^+(\chi - \psi)^+ \, dx \leq C \sum_{K \in \mathcal{E}_h, K \cap \partial \Omega = \emptyset} h^4_K \|\nabla \sigma_h^+\|_{K}^2 + C \sum_{e \in \mathcal{E}_h} h^{-1}_e \|u_h\|_{e} \|u_h\|_{e}
\]
\[
+ C \sum_{K \in \mathcal{F}_h, K \cap \partial \Omega = \emptyset} \sum_{e \in \mathcal{E}_h} h_e \|\nabla(u_h - \psi_1)\|_{e}^2 + \sum_{K \in \mathcal{F}_h, K \cap \partial \Omega = \emptyset} \int_K \sigma_h^+(\psi_1 - \psi)^+ \, dx.
\]

Case 3: $K \notin \mathcal{F}_h$. We have $u_h = \psi_1$ on $K$ and thus
\[
\int_K \sigma_h^+(\chi - \psi)^+ \, dx \leq \int_K \sigma_h^+(\chi - u_h) \, dx + \int_K \sigma_h^+(\psi_1 - \psi)^+ \, dx.
\]

Note that $\chi = u_h$ on $K \in \mathcal{E}_h$. So
\[
\int_K \sigma_h^+(\chi - \psi)^+ \, dx \leq \int_K \sigma_h^+(\psi_1 - \psi)^+ \, dx \quad \text{for } K \in \mathcal{E}_h.
\]

Therefore, we obtain the following result.

**Proposition 4.1.**
\[
\langle \sigma_h - \sigma, \chi - u \rangle \leq \epsilon |u_h - u|_{1, \Omega}^2 + e |\sigma_h - \sigma|_{1, \Omega}^2 + C \left( \sum_{e \in \mathcal{E}_h} h^{-1}_e \|u_h\|_{e}^2 + |\sigma_h - \sigma|_{1, \Omega}^2 + \eta_{\mathcal{E}_h}^2 \right),
\]
where
\[
\eta_{\mathcal{E}_h}^2 = \sum_{K \in \mathcal{F}_h} h^4_K \|\nabla \sigma_h^+\|_{K}^2 + \sum_{K \in \mathcal{F}_h} \int_K \sigma_h^+(\chi - u_h) \, dx + |(\psi - \chi)^+|_{1, \Omega}^2
\]
\[
+ \sum_{K \in \mathcal{F}_h, K \cap \partial \Omega = \emptyset} \sum_{e \in \mathcal{E}_h} h_e \|\nabla(u_h - \psi_1)\|_{e}^2 + \sum_{K \in \mathcal{F}_h, K \cap \partial \Omega = \emptyset} \int_K \sigma_h^+(\psi_1 - \psi)^+ \, dx.
\]
Here, $0 < \epsilon < 1$ is an arbitrary constant and
\[
\mathcal{F}_h := \{ K \in \mathcal{F}_h : \exists p_1, p_2 \in K \cap \mathbb{N}_h, u_h(p_1) > \psi_1(p_1) \text{ and } u_h(p_2) = \psi_1(p_2) \}.\]
Theorem 4.2. Let \( u \in H^2(\Omega) \) and \( u_h \) solve (1.1) and (2.1), respectively. Then

\[
|e|_{1,h} + |\sigma - \sigma_h|_h \leq C \left[ \left( \sum_{K \in \mathcal{M}_h} \eta_{K}^2 \right)^{\frac{1}{2}} + \left( \sum_{K \in \mathcal{M}_h} \eta_{\partial K}^2 \right)^{\frac{1}{2}} + \left( \sum_{K \in \mathcal{M}_h} h_K^2 \|\sigma_h - \sigma_h^+\|_K \right)^{\frac{1}{2}} + \eta_{\mathcal{E}_h} \right].
\] (4.1)

For the local lower bounds of estimators, the terms \( \eta_K \), \( \eta_{\partial K} \), \( h_K^2 \|\nabla \sigma_h^+\|_K \) and \( h_K \|\sigma_h - \sigma_h^+\|_K \) are bounded in the same way as in Section 3.2. Now consider the other terms in (4.1).

\[
h_K^{1/2} \|\nabla (u_h - \psi)\|_e \leq h_K^{1/2} \|\nabla u_h\|_e + h_K^{1/2} \|\nabla \psi_t\|_e.
\] (4.2)

\[
|\psi - \chi|^+_{1,K} = |\psi - \chi|_{1,K\cap\{\psi > \chi\}} \leq |\psi - \psi_t|_{1,K\cap\{\psi > \chi\}} + |\chi - \psi_t|_{1,K\cap\{\psi > \chi\}}.
\] (4.3)

Let \( \alpha = \min_{K} \{ \chi - \psi_t \} = (\chi - \psi_t)(p) \). By inverse inequality and Lemma 3.3, we get

\[
\|\nabla (\chi - \psi_t)\|_K^2 = \|\nabla (\chi - \psi_t - \alpha)\|_K^2 \leq C \sum_{e \in \mathcal{E}_h(p)} h_e \|\nabla (\chi - \psi_t - \alpha)\|^2_e \leq C \sum_{e \in \mathcal{E}_h(p)} h_e \|\nabla (\chi - \psi_t)\|^2_e.
\] (4.4)

Finally, we bound \( \int_K \sigma_h^+ (\psi - \psi^+ \psi) dx \) have

\[
\int_K \sigma_h^+ (\psi - \psi^+ \psi) dx = \int_K (\sigma_h^+ + f) (\psi_t - \psi^+ \psi) dx + \int_K -f (\psi_t - \psi^+ \psi) dx,
\]

\[
\int_K \sigma_h^+ (\psi - \psi^+ \psi) dx \leq \frac{1}{2} h_K \|\sigma_h^+ + f\|_K^2 + \frac{1}{2} h_K^{-1} \|\psi_t - \psi^+ \psi\|_K^2 + \int_K -f (\psi_t - \psi^+ \psi) dx.
\] (4.5)

Here (4.2)–(4.5) give local lower bounds for these terms. Notice that these lower bounds will be zero or be absorbed by \( \eta_K \) if \( \psi \in P_1(\Omega) \). Summarizing the above argument, we get the following theorem.

Theorem 4.3. Let \( u \in H^2(\Omega) \) and \( u_h \) be the solutions of (1.1) and (2.1), respectively. Then we have local lower bounds of error estimators in (4.1),

\[
\eta_K \leq C \left( |u - u_h|_{1,\text{H}} + |\sigma - \sigma_h|_{1,\text{H}} + h_K \|f - f_h\|_{\text{H}} \right),
\]

\[
\eta_{\partial K} \leq C \left( |u_h|_{1,\text{H}} \right),
\]

\[
h_K \|\sigma_h^+ - \sigma_h\|_K \leq h_K \|\sigma_h^+ + f_h\|_K + h_K \|\sigma_h + f_h\|_K + h_K \|f - f_h\|_K,
\]

\[
h_K^2 \|\nabla \sigma_h^+\|_K \leq C \left( h_K \|\sigma_h^+ + f_h\|_K + h_K \|f - f_h\|_K \right),
\]

and (4.2)–(4.5) hold true.

We comment that for other DG methods studied in [9] for solving the obstacle problem, it is easy to see that (3.1), (3.23) and (3.25) hold true. So Theorems 3.6, 4.2, 3.10 and 4.3 hold for all of them by the similar arguments.

5. Implementation and numerical example

Each loop of the adaptive algorithm consists of four steps,

Solve \( \rightarrow \) Estimate \( \rightarrow \) Mark \( \rightarrow \) Refine.

Let \( \mathcal{T}_0 \) be the initial mesh. At each loop, first solve the obstacle problem by the LDG method on a mesh \( \mathcal{T}_j \). Then based on the analysis in Section 3, for the affine obstacle case, we choose the error indicator \( \xi_K \) of element \( K \) as

\[
\xi_K = \begin{cases} 
(n_K^2 + n_{\partial K}^2 + h_K^2 \|\sigma_h^+ - \sigma_h\|_K^2)^{\frac{1}{2}}, & \text{if } K \in \mathcal{T}_h \setminus \mathcal{F}_h, \\
(n_K^2 + n_{\partial K}^2 + h_K^2 \|\sigma_h^+ - \sigma_h\|_K^2 + h_K^2 \|\nabla \sigma_h^+\|_K^2)^{\frac{1}{2}}, & \text{if } K \in \mathcal{F}_h, \\
(n_K^2 + n_{\partial K}^2 + h_K^2 \|\sigma_h^+ - \sigma_h\|_K^2 + \|\sigma_h^+\|_K \|u_h - \chi\|_K)^{\frac{1}{2}}, & \text{if } K \in \mathcal{F}_h \setminus \mathcal{E}_h.
\end{cases}
\]

With the error estimators, we still need to mark the elements to be refined. Here, we use the bulk criterion strategy

\[
\sum_{K \in \mathcal{M}_h} \xi_K \geq \theta \sum_{K \in \mathcal{E}_h} \xi_K = \theta \cdot (\text{Total Error}), \quad 0 < \theta < 1.
\] (5.1)

In this strategy, the elements are marked according to the sizes of element errors. Therefore, elements with larger errors are put into the marked set \( \mathcal{M}_h \) until the inequality (5.1) is satisfied. Last, we refine the marked elements and get the new mesh \( \mathcal{T}_{j+1} \). For DGMs, refinement allows hanging nodes. To refine the marked elements, we connect the midpoints on the three edges to divide the element into four new elements. In the numerical example, the discretized problem is solved by a primal–dual active set strategy [14].
Example. Let \( \Omega := (-2, 2)^2 \setminus (0, 2) \times (-2, 0) \), \( u_D = 0 \) on \( \partial \Omega \), \( \psi = 0 \) and,

\[
f(r, \varphi) := -r^{2/3} \sin(2\varphi/3)(v_1'(r)/r + v_2''(r)) - \frac{4}{3} r^{-1/3} v_1'(r) \sin(2\varphi/3) - v_2(r)
\]

where the polar coordinate system \((r, \varphi)\) is used, and with \( \bar{r} = 2(r - 1/4) \),

\[
v_1(r) = \begin{cases} 
1 & \text{if } \bar{r} < 0, \\
-6\bar{r}^5 + 15\bar{r}^4 - 10\bar{r}^3 + 1 & \text{if } 0 \leq \bar{r} < 1, \\
0 & \text{if } \bar{r} \geq 1,
\end{cases}
\]

\[
v_2(r) = \begin{cases} 
0 & \text{if } r \leq 5/4, \\
1 & \text{otherwise}.
\end{cases}
\]

The exact solution is \( u(r, \varphi) = r^{2/3} v_1(r) \sin(2\varphi/3) \).

 Fig. 1. The refined meshes in Example. Fig. 2. The solution on the mesh \( T_{12} \). \]
Fig. 3. The error on the uniform and adaptive refined meshes.

Fig. 4. The ratio of error indicators to the real error $\| \nabla e \|$.

**Table 1**
The maximum ratio of $(\| \sigma_h^+ \|_K \| u_h - \chi \|_K )^{1/2}$ to $\xi_K$ and $\eta_K$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>192</th>
<th>342</th>
<th>597</th>
<th>1107</th>
<th>1974</th>
<th>3975</th>
<th>7302</th>
<th>15261</th>
<th>28434</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1$</td>
<td>0.01184</td>
<td>0.00450</td>
<td>0.00719</td>
<td>0.00497</td>
<td>0.00611</td>
<td>0.00824</td>
<td>0.01209</td>
<td>0.00563</td>
<td>0.00640</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>0.04982</td>
<td>0.01738</td>
<td>2.12665</td>
<td>0.01357</td>
<td>0.01419</td>
<td>0.01747</td>
<td>0.02735</td>
<td>0.01246</td>
<td>0.01459</td>
</tr>
</tbody>
</table>

We compare the quantity $(\| \sigma_h^+ \|_K \| u_h - \chi \|_K )^{1/2}$ with $\xi_K$ and $\eta_K$ in Table 1. In this table,

$$\gamma_1 = \max_{K \in \mathcal{F}_h \setminus \mathcal{F}_h} \frac{\left( \| \sigma_h^+ \|_K \| u_h - \chi \|_K \right)^{1/2}}{\xi_K}, \quad \gamma_2 = \max_{K \in \mathcal{F}_h \setminus \mathcal{F}_h} \frac{\left( \| \sigma_h^+ \|_K \| u_h - \chi \|_K \right)^{1/2}}{\eta_K}.$$

In addition, we examine the quality of the error estimators provided above, let

$$\gamma_3 = \frac{\sum_{K \in \mathcal{F}_j} \xi_K^2}{\sum_{K \in \mathcal{F}_j} \| \nabla e \|_K^2}, \quad \gamma_4 = \frac{\sum_{K \in \mathcal{F}_j} \eta_K^2}{\sum_{K \in \mathcal{F}_j} \| \nabla e \|_K^2}, \quad \gamma_5 = \frac{\sum_{K \in \mathcal{F}_j} \| \sigma_h^+ - \sigma_h \|_K^2}{\sum_{K \in \mathcal{F}_j} \| \nabla e \|_K^2}.$$

Then $\gamma_3$, $\gamma_4$ and $\gamma_5$ are the ratios of the error indicators to the exact error, see Fig. 4. The error indicator efficiency index $\gamma_3$ stabilizes to a constant around 5.77.
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