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DISCONTINUOUS GALERKIN METHODS FOR SOLVING ELLIPTIC VARIATIONAL INEQUALITIES*

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Abstract. We study discontinuous Galerkin methods for solving elliptic variational inequalities of both the first and second kinds. Analysis of numerous discontinuous Galerkin schemes for elliptic boundary value problems is extended to the variational inequalities. We establish a priori error estimates for the discontinuous Galerkin methods, which reach optimal order for linear elements. Results from some numerical examples are reported.

Key words. variational inequalities, discontinuous Galerkin method, error analysis

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1. Introduction.

In this paper, we study the discontinuous Galerkin methods for solving elliptic variational inequalities.

1.1. Discontinuous Galerkin methods.

Finite element methods are a field of active research in applied mathematics; in particular there has been an active development of discontinuous Galerkin (DG) methods recently. The initial DG method was introduced by Reed and Hill [47] for a hyperbolic equation. In recent years, DG methods have been applied to a wide range of partial differential equations, such as convection-diffusion equations [17, 46], Navier–Stokes equations [6, 19], Hamilton–Jacobi equations [39, 45], the radiative transfer equation [32], and so on. A historical account of the methods can be found in [20].

DG methods differ from the standard finite element methods in that functions are allowed to be discontinuous across the element boundaries. Since no interelement continuity is required, DG methods allow general meshes with hanging nodes and elements of different shapes. The advantages of this include the ease of using polynomial functions of different orders in different elements (p-adaptivity), more flexibility in mesh refinements (h-adaptivity), and the locality of the discretization, which makes them ideally suited for parallel computing. Their compact formulation can be applied near boundaries without special treatment, which greatly increases the robustness and accuracy of any boundary condition implementation. For hp-adaptive strategies and parallel computing of DG methods, see, e.g., [9, 10, 22, 28, 29, 36, 37, 38, 50].

DG methods for elliptic equations were independently proposed in the 1970s. Many variants were introduced and studied, which were generally called interior penalty (IP) methods. Their development was independent of that of the DG methods for hyperbolic equations. There are two basic ways to construct DG methods

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for elliptic problems. The first way is to add a penalty term into the bilinear form, penalizing the interelement discontinuity; see, e.g., [5, 16, 26, 48]. The second one is to choose suitable numerical fluxes to make the DG schemes consistent, conservative, and stable; see, e.g., [6, 18, 21]. In [2, 3], Arnold et al. unified these two families and established a framework which provides a better understanding of their properties, including the differences and the connections between them. In particular, it was shown that the methods of the first family, those based on the choice of the bilinear form, can be obtained as special cases of the second family simply by choosing proper numerical fluxes.

1.2. Elliptic variational inequalities. Variational inequalities form an important family of nonlinear problems. We present here two representative elliptic variational inequalities (EVIs) for which we will develop the DG methods. For more examples of EVIs, we refer the reader to the monograph [27]. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$.

An obstacle problem. Let $f \in L^2(\Omega)$, let $g \in H^1(\Omega)$, and let $\psi \in H^1(\Omega) \cap C(\overline{\Omega})$ be given with $\psi \le g$ on $\partial\Omega$. The obstacle problem is to find $u \in K$ such that

$$a(u, v-u) \ge (f, v-u)_\Omega \quad \forall\, v \in K, \tag{1.1}$$

where

$$K = \{v \in H^1_g(\Omega) : v \ge \psi \text{ a.e in } \Omega\} \tag{1.2}$$

is a closed and convex admissible set, $H^1_g(\Omega) = \{v \in H^1(\Omega) : v = g \text{ on } \partial\Omega\}$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, and $(f,v)_\Omega = \int_\Omega f v \, dx$. The obstacle problem is an example of EVIs of the first kind [30]. This problem arises in a variety of applications, such as membrane deformation in elasticity theory, and nonparametric minimal and capillary surfaces as geometrical problems. The elastic-plastic torsion problem and the cavitation problem in the theory of lubrication can also be regarded as obstacle-type problems.

A simplified friction problem. Let D be an open subset of Ω or $\partial\Omega$, let $f \in L^2(\Omega)$, and let $g \in L^2(D)$ with $g > 0$. Then a simplified friction problem is to find $u \in H^1(\Omega)$ such that

$$a^*(u, v-u) + j(v) - j(u) \ge (f, v-u)_\Omega \quad \forall\, v \in H^1(\Omega), \tag{1.3}$$

where

$$a^*(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u\, v \, dx,$$
$$j(v) = \int_D g\, |v| \, ds.$$

The simplified friction problem is an example of EVIs of the second kind, featured by the presence of nondifferentiable terms in the formulation. Such variational inequalities (VIs) arise in a variety of mechanical problems, e.g., plasticity [33], frictional contact [34, 41].

Both the obstacle problem and the simplified friction problem have a unique solution [27]. A variety of numerical methods have been developed to solve discretized VIs, such as the relaxation method [30], multilevel projection method [53], multigrid method [31, 40, 43, 44], and so on.

To the best of our knowledge, there are few results about DG methods for VIs. It is hard to analyze the behavior of DG methods for VIs because of the nonlinearity of

VIs. In [24, 25], a DG formulation and algorithm of a gradient plasticity problem, in the form of a quasi-static VI of the second kind, was developed and analyzed. In [23], Djoko considered the symmetric and nonsymmetric interior penalty Galerkin (NIPG) methods for solving elliptic VIs and derived a priori error estimates. However, the argument in that paper suffers from a problem related to constraints on the finite element functions. Since there is no stability relation for VIs, we cannot devise stable DG schemes for VIs by first deriving a discrete formulation involving numerical fluxes through integration by parts and then determining the fluxes by a discrete stability identity [18]. In this paper, we follow the unified framework developed in [3] and extend the ideas therein for the study of solving the EVIs (1.1) and (1.3) by DG methods.

This paper is organized as follows. Section 2 introduces notation used in the paper and DG formulations for solving the EVIs. Then we review some properties of the bilinear form B_h, shown in [3], and prove the consistency of DG schemes for the EVIs in section 3. In section 4, we derive a priori error estimates for these DG methods. In the last section, we present some numerical examples, paying particular attention to observed numerical convergence orders.

2. Notation and DG formulations.

2.1. Notation. We assume Ω is a polygonal domain and denote by $\{\mathcal{T}_h\}_h$ a family of subdivisions of $\overline{\Omega}$ into triangles such that the minimal angle condition is satisfied. For $l > 0$ and each $\mathcal{T}_h$, $H^l(\mathcal{T}_h)$ is the space of functions on Ω whose restriction to each element $K \in \mathcal{T}_h$ belongs to the Sobolev space $H^l(K)$. Let $h_K = \operatorname{diam}(K)$ and $h = \max\{h_K : K \in \mathcal{T}_h\}$. Denote by Γ the union of the boundaries of the elements K of $\mathcal{T}_h$, $\Gamma^0 = \Gamma \backslash \partial\Omega$, and also use Γ^∂ for $\partial\Omega$. The traces of functions in $H^1(\mathcal{T}_h)$ belong to $T(\Gamma) := \Pi_{K\in\mathcal{T}_h} L^2(\partial K)$. Note that $v \in T(\Gamma)$ is double-valued on Γ^0 and single-valued on Γ^∂. $L^2(\Gamma)$ can be viewed as the subspace of $T(\Gamma)$ consisting of functions whose two values coincide on all interior edges.

Let e be an edge shared by two elements K_1 and K_2, and let $n_i = n|_{\partial K_i}$ be the unit outward normal vector on ∂K_i. For $v \in T(\Gamma)$, let $v_i = v|_{\partial K_i}$, and we define the average $\{v\}$ and the jump $[v]$ on Γ^0 as follows:

$$\{v\} = \frac{1}{2}(v_1 + v_2), \quad [v] = v_1 n_1 + v_2 n_2 \quad \text{on } e \in \mathcal{E}_h^0,$$

where $\mathcal{E}_h^0$ is the set of interior edges. For $q \in [T(\Gamma)]^2$, we denote $q_i = q|_{\partial K_i}$ and set

$$\{q\} = \frac{1}{2}(q_1 + q_2), \quad [q] = q_1 \cdot n_1 + q_2 \cdot n_2 \quad \text{on } e \in \mathcal{E}_h^0.$$

If $e \in \mathcal{E}_h^\partial$, the set of boundary edges, we set

$$[v] = vn, \quad \{q\} = q \quad \text{on } e \in \mathcal{E}_h^\partial,$$

where n is the unit outward normal on Γ^∂. The collection of all the edges is $\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^\partial$. We do not need $\{v\}$ and $[q]$ on the boundary edges.

Let $p \geq 0$ be an integer, and introduce the following finite element spaces:

$$V_h = \{v_h \in L^2(\Omega) : \ v_h|_K \in P_p(K) \ \forall\, K \in \mathcal{T}_h\},$$
$$W_h = \{w_h \in [L^2(\Omega)]^2 : \ w_h|_K \in [P_p(K)]^2 \ \forall\, K \in \mathcal{T}_h\}.$$

We use the following subsets of the finite element space V_h with $p = 1$ or 2 to approximate the set K of (1.2):

$$K_h^1 = \{v_h \in V_h \text{ with } p = 1 : \ v_h(x) \geq \psi(x) \text{ at all nodes of } \mathcal{T}_h\},$$
$$K_h^2 = \{v_h \in V_h \text{ with } p = 2 : \ v_h(m) \geq \psi(m) \text{ at all midpoints } m \text{ on element edges of } \mathcal{T}_h\}.$$

For $v \in H^1(\mathcal{T}_h)$, $\nabla_h v$ is defined by the relation $\nabla_h v = \nabla v$ on any element $K \in \mathcal{T}_h$.

2.2. DG method formulations. Following [3], we consider the DG methods with a variety of choices of the bilinear forms. We use the shorter notation $(w, v)_\Omega$, $\langle w, v\rangle_\Gamma$, $\langle w, v\rangle_{\Gamma^0}$, and $\langle w, v\rangle_{\Gamma^\partial}$ instead of $\int_\Omega wv\,dx$, $\int_\Gamma wv\,ds$, $\int_{\Gamma^0} wv\,ds$, and $\int_{\Gamma^\partial} wv\,ds$. We first list a variety of bilinear forms for both the obstacle problem and the simplified friction problem. The bilinear form for the obstacle problem is denoted by $B_h : H^2(\mathcal{T}_h) \times H^2(\mathcal{T}_h) \to \mathbb{R}$, whereas that for the simplified friction problem is denoted by $B_h^* : H^2(\mathcal{T}_h) \times H^2(\mathcal{T}_h) \to \mathbb{R}$. The linear form for the obstacle problem is of the form $(f, v)_\Omega + F(v)$, and we also list $F(v) : H^2(\mathcal{T}_h) \to \mathbb{R}$.

For the local discontinuous Galerkin (LDG) method of [21],

$$\begin{aligned}
B_h^{(1)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega - \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma \\
&\quad - \langle \beta \cdot [w], [\nabla_h v]\rangle_{\Gamma^0} - \langle [\nabla_h w], \beta \cdot [v]\rangle_{\Gamma^0} \\
&\quad + (r([w]) + l(\beta \cdot [w]), r([v]) + l(\beta \cdot [v]))_\Omega + \alpha^j(w,v),
\end{aligned} \tag{2.1}$$
$$\begin{aligned}
B_h^{*(1)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + (w,v)_\Omega - \langle [w], \{\nabla_h v\}\rangle_{\Gamma^0} - \langle \{\nabla_h w\}, [v]\rangle_{\Gamma^0} \\
&\quad - \langle \beta \cdot [w], [\nabla_h v]\rangle_{\Gamma^0} - \langle [\nabla_h w], \beta \cdot [v]\rangle_{\Gamma^0} \\
&\quad + (r_0([w]) + l(\beta \cdot [w]), r_0([v]) + l(\beta \cdot [v]))_\Omega + \alpha_0^j(w,v),
\end{aligned}$$
$$F^{(1)}(v) = (r_\partial([g]), r([v]) + l(\beta \cdot [v]))_\Omega + \langle g, \mu v - \nabla v \cdot n\rangle_{\Gamma^\partial}. \tag{2.2}$$

Here $\beta \in [L^2(\Gamma^0)]^2$ is a vector-valued function which is constant on each edge; $\alpha^j(w,v) = \int_\Gamma \mu[w] \cdot [v]\,ds$ and $\alpha_0^j(w,v) = \int_{\Gamma^0} \mu[w] \cdot [v]\,ds$ are the penalty or stabilization terms with the penalty weighting function $\mu : \Gamma \to \mathbb{R}$ given by $\eta_e h_e^{-1}$ on each $e \in \mathcal{E}_h$, η_e being a positive number; $r : [L^2(\Gamma)]^2 \to W_h$, $r_0 : [L^2(\Gamma^0)]^2 \to W_h$, $r_\partial : [L^2(\Gamma^\partial)]^2 \to W_h$, and $l : L^2(\Gamma^0) \to W_h$ are lifting operators defined by

$$\int_\Omega r(q)\cdot w_h\,dx = -\int_\Gamma q\cdot\{w_h\}\,ds, \quad \int_\Omega l(v)\cdot w_h\,dx = -\int_{\Gamma^0} v[w_h]\,ds \quad \forall\, w_h \in W_h, \tag{2.3}$$

$$\int_\Omega r_0(q)\cdot w_h\,dx = -\int_{\Gamma^0} q\cdot\{w_h\}\,ds, \quad \int_\Omega r_\partial(q)\cdot w_h\,dx = -\int_{\Gamma^\partial} q\cdot\{w_h\}\,ds \quad \forall\, w_h \in W_h.$$

For the IP method of [26],

$$\begin{aligned}
B_h^{(2)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega - \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma + \alpha^j(w,v), \\
B_h^{*(2)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + (w,v)_\Omega - \langle [w], \{\nabla_h v\}\rangle_{\Gamma^0} - \langle \{\nabla_h w\}, [v]\rangle_{\Gamma^0} + \alpha_0^j(w,v), \\
F^{(2)}(v) &= \langle g, \mu v - \nabla v \cdot n\rangle_{\Gamma^\partial}.
\end{aligned}$$

For the NIPG method of [48],

$$\begin{aligned}
B_h^{(3)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma + \alpha^j(w,v), \\
B_h^{*(3)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + (w,v)_\Omega + \langle [w], \{\nabla_h v\}\rangle_{\Gamma^0} - \langle \{\nabla_h w\}, [v]\rangle_{\Gamma^0} + \alpha_0^j(w,v), \\
F^{(3)}(v) &= \langle g, \mu v + \nabla v \cdot n\rangle_{\Gamma^\partial}.
\end{aligned}$$

For the method of Brezzi et al. [15],

$$
\begin{aligned}
B_h^{(4)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega - \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma \\
&\quad + (r([w]), r([v]))_\Omega + \alpha^r(w,v), \\
B_h^{*(4)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + (w,v)_\Omega - \langle [w], \{\nabla_h v\}\rangle_{\Gamma^0} - \langle \{\nabla_h w\}, [v]\rangle_{\Gamma^0} \\
&\quad + (r_0([w]), r_0([v]))_\Omega + \alpha_0^r(w,v), \\
F^{(4)}(v) &= -\langle g, \nabla v\cdot n\rangle_{\Gamma^\partial} + (r_\partial([g]), r([v]))_\Omega + \alpha_\partial^r(v).
\end{aligned}
$$

Here,

$$
\begin{aligned}
\alpha^r(u,v) &= \sum_{e\in\mathcal{E}_h} \int_\Omega \eta_e r_e([u])\cdot r_e([v])\,dx, \\
\alpha_0^r(u,v) &= \sum_{e\in\mathcal{E}_h^0} \int_\Omega \eta_e r_e([u])\cdot r_e([v])\,dx, \\
\alpha_\partial^r(v) &= \sum_{e\in\mathcal{E}_h^\partial} \int_\Omega \eta_e r_e([g])\cdot r_e([v])\,dx,
\end{aligned}
$$

and the lift operator $r_e : [L^1(e)]^2 \to W_h$ is given by

$$
\int_\Omega r_e(q)\cdot w_h\,dx = -\int_e q\cdot\{w_h\}\,ds \quad \forall\, w_h\in W_h, \quad q\in[L^1(e)]^2. \tag{2.4}
$$

For the method of Bassi et al. [7],

$$
\begin{aligned}
B_h^{(5)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega - \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma + \alpha^r(w,v), \\
B_h^{*(5)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + (w,v)_\Omega - \langle [w], \{\nabla_h v\}\rangle_{\Gamma^0} - \langle \{\nabla_h w\}, [v]\rangle_{\Gamma^0} + \alpha_0^r(w,v), \\
F^{(5)}(v) &= -\langle g, \nabla v\cdot n\rangle_{\Gamma^\partial} + \alpha_\partial^r(v).
\end{aligned}
$$

For the method of Babuška and Zlámal [5],

$$
\begin{aligned}
B_h^{(6)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + \alpha^j(w,v), \\
B_h^{*(6)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + (w,v)_\Omega + \alpha_0^j(w,v), \\
F^{(6)}(v) &= \langle g, \mu v\rangle_{\Gamma^\partial}.
\end{aligned}
$$

For the method of Brezzi et al. [16],

$$
\begin{aligned}
B_h^{(7)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + \alpha^r(w,v), \\
B_h^{*(7)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + (w,v)_\Omega + \alpha_0^r(w,v), \\
F^{(7)}(v) &= \alpha_\partial^r(v).
\end{aligned}
$$

For the method of Baumann and Oden [8],

$$
\begin{aligned}
B_h^{(8)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma, \\
B_h^{*(8)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + (w,v)_\Omega + \langle [w], \{\nabla_h v\}\rangle_{\Gamma^0} - \langle \{\nabla_h w\}, [v]\rangle_{\Gamma^0}, \\
F^{(8)}(v) &= \langle g, \nabla v\cdot n\rangle_{\Gamma^\partial}.
\end{aligned}
$$

For the method of Bassi and Rebay [6],

$$
\begin{aligned}
B_h^{(9)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega - \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma + (r([w]), r([v]))_\Omega\,,\\
B_h^{*(9)}(w,v) &= (\nabla_h w, \nabla_h v)_\Omega + (w,v)_\Omega - \langle [w], \{\nabla_h v\}\rangle_{\Gamma^0} - \langle \{\nabla_h w\}, [v]\rangle_{\Gamma^0}\\
&\quad + (r_0([w]), r_0([v]))_\Omega\,,\\
F^{(9)}(v) &= -\langle g, \nabla v\cdot n\rangle_{\Gamma^\partial} + (r_\partial([g]), r([v]))_\Omega.
\end{aligned}
$$

Let $B_h(w,v)$ be one of the bilinear forms $B_h^{(j)}(w,v)$ and $F(v) = F^{(j)}(v)$ with $j = 1,\ldots,9$. Then a DG method for the obstacle problem (1.1) is as follows: Find $u_h \in K_h$ such that

$$
B_h(u_h, v_h - u_h) \ge (f, v_h - u_h)_\Omega + F(v_h - u_h) \quad \forall\, v_h \in K_h, \tag{2.5}
$$

where $K_h = K_h^1$ or K_h^2.

Let $B_h^*(w,v)$ be one of the bilinear forms $B_h^{*(j)}(w,v)$ with $j = 1,\ldots,9$. Then a DG method for the simplified friction problem (1.3) is as follows: Find $u_h \in V_h$ such that

$$
B_h^*(u_h, v_h - u_h) + j(v_h) - j(u_h) \ge (f, v_h - u_h)_\Omega \quad \forall\, v_h \in V_h. \tag{2.6}
$$

Here the polynomial degree p in defining V_h is arbitrary.

For the reader's convenience, in Table 1 we summarize the bilinear forms and linear functionals of the DG methods for the obstacle problem. In the table, we let $d = (\nabla_h w, \nabla_h v)_\Omega$, $B_h^{(j)} = B_h^{(j)}(w,v)$, $F^{(j)} = F^{(j)}(v)$, $\alpha^j = \alpha^j(w,v)$, etc. We mention that $F(v) = 0$ for the case of homogeneous Dirichlet boundary $g = 0$. For the simplified friction problem, a similar table can be given but is omitted here.

3. Consistency, boundedness, and stability. In the study of the DG methods for the two representative EVIs, we will need the true solution to be in the space $H^2(\Omega)$. In the literature, one can find some solution regularity results for VIs. For the obstacle problem with $g = 0$, a result by Brezis and Stampacchia [14] states that if the domain Ω is smooth and, for some $s \in (1,\infty)$, $f \in L^s(\Omega)$ and $\psi \in W^{2,s}(\Omega)$, then the solution $u \in W^{2,s}(\Omega)$. From this result, we can conclude that, for our model obstacle problem with a general g, if the domain Ω is smooth and, for some $s \in (1,\infty)$, $f \in L^s(\Omega)$, $g \in W^{2,s}(\Omega)$, and $\psi \in W^{2,s}(\Omega)$, then the solution $u \in W^{2,s}(\Omega)$. We need this result only for the case $s = 2$. For the simplified friction problem with $D = \partial\Omega$, it is proved in [13] that if Ω is smooth, $f \in L^2(\Omega)$, then the solution $u \in H^2(\Omega)$. In the case $D = \Omega$, for an EVI slightly more complicated than the simplified friction problem, it is proved in [12] that if Ω is smooth, $f \in L^2(\Omega)$, then the solution $u \in H^2(\Omega)$. More recent results on solution regularities for VIs can be found in [42, 51].

In general, the solution regularity for VIs is limited no matter how smooth the problem data are. For instance, generally, the solution of the obstacle problem does not belong to the space $H^3(\Omega)$. Therefore, it is usually advisable to use low order elements in applying the DG methods to solve VIs. Nevertheless, theoretically it is of interest to derive error estimates for any polynomial degree where the solution is smooth. Moreover, it may be advantageous to develop *hp* DG methods for solving VIs where the smooth region of the solution is approximated by high order elements, a topic currently under consideration. For these reasons, our error analysis is performed for DG methods of arbitrary polynomial degrees.

TABLE 1
Bilinear forms and linear functionals of DG methods for the obstacle problem.

Methods	Bilinear forms $B_h^{(j)}$ and linear functionals $F^{(j)}$
LDG [21]	$B_h^{(1)} = d - \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma - \langle \beta \cdot [w], [\nabla_h v]\rangle_{\Gamma^0} + \alpha^j$ $\quad -\langle [\nabla_h w], \beta \cdot [v]\rangle_{\Gamma^0} + (r([w]) + l(\beta \cdot [w]), r([v]) + l(\beta \cdot [v]))_\Omega$ $F^{(1)} = (r_\partial([g]), r([v]) + l(\beta \cdot [v]))_\Omega + \langle g, \mu v - \nabla v \cdot n\rangle_{\Gamma^\partial}$
IP [26]	$B_h^{(2)} = d - \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma + \alpha^j$ $F^{(2)} = \langle g, \mu v - \nabla v \cdot n\rangle_{\Gamma^\partial}$
NIPG [48]	$B_h^{(3)} = d + \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma + \alpha^j$ $F^{(3)} = \langle g, \mu v + \nabla v \cdot n\rangle_{\Gamma^\partial}$
Brezzi et al. [15]	$B_h^{(4)} = d - \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma + (r([w]), r([v]))_\Omega + \alpha^r$ $F^{(4)} = -\langle g, \nabla v \cdot n\rangle_{\Gamma^\partial} + (r_\partial([g]), r([v]))_\Omega + \alpha_\partial^r(v)$
Bassi et al. [7]	$B_h^{(5)} = d - \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma + \alpha^r$ $F^{(5)} = -\langle g, \nabla v \cdot n\rangle_{\Gamma^\partial} + \alpha_\partial^r(v)$
Babuška and Zlámal [5]	$B_h^{(6)} = d + \alpha^j$ $F^{(6)} = \langle g, \mu v\rangle_{\Gamma^\partial}$
Brezzi et al. [16]	$B_h^{(7)} = d + \alpha^r$ $F^{(7)} = \alpha_\partial^r(v)$
Baumann and Oden [8]	$B_h^{(8)} = d + \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma$ $F^{(8)} = \langle g, \nabla v \cdot n\rangle_{\Gamma^\partial}$
Bassi and Rebay [6]	$B_h^{(9)} = d - \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma + (r([w]), r([v]))_\Omega$ $F^{(9)} = -\langle g, \nabla v \cdot n\rangle_{\Gamma^\partial} + (r_\partial([g]), r([v]))_\Omega$

For the obstacle problem, if the solution has the regularity $u \in H^2(\Omega)$, then we have the relations (see, e.g., [4])

$$-\triangle u \geq f, \quad u \geq \psi, \quad (-\triangle u - f)(u - \psi) = 0 \quad \text{a.e. in } \Omega. \tag{3.1}$$

Similarly, for the simplified friction problem, assuming the solution $u \in H^2(\Omega)$, the following relations hold (see, e.g., [11, 30]). For $D \subset \partial\Omega$,

$$-\triangle u + u = f \quad \text{a.e. in } \Omega, \quad \nabla u \cdot n + g\lambda\,\chi_D = 0 \quad \text{a.e. on } \partial\Omega, \tag{3.2}$$

and, for $D \subset \Omega$,

$$-\triangle u + u + g\lambda\,\chi_D = f \quad \text{a.e. in } \Omega, \quad \nabla u \cdot n = 0 \quad \text{a.e. on } \partial\Omega, \tag{3.3}$$

where χ_D is the indicator function of the set D, and $\lambda \in L^\infty(D)$ is a Lagrange multiplier, satisfying

$$|\lambda| \leq 1, \quad \lambda u = |u| \quad \text{a.e. in } D. \tag{3.4}$$

We notice that if $u \in H^2(\Omega)$, then, on any interior edge e, $[u] = 0$, $\{u\} = u$, $[\nabla u] = 0$, and $\{\nabla u\} = \nabla u$. The relations (3.1)–(3.4) are useful for showing the consistency of the DG schemes. For all DG methods introduced in the previous section, we have the following result.

LEMMA 3.1 (consistency). *Assume $u \in H^2(\Omega)$ is the solution of* (1.1) *or* (1.3). *Then, for all DG methods $B_h(w,v) = B_h^{(j)}(w,v)$, $B_h^*(w,v) = B_h^{*(j)}(w,v)$, $F(v) = F^{(j)}(v)$ with $j = 1, \dots, 9$, we have*

$$B_h(u, v-u) \geq (f, v-u)_\Omega + F(v-u) \quad \forall\, v \in K \cap H^2(\mathcal{T}_h), \tag{3.5}$$

$$B_h^*(u, v_h - u) + j(v_h) - j(u) \geq (f, v_h - u)_\Omega \quad \forall\, v_h \in V_h. \tag{3.6}$$

Proof. For the obstacle problem (1.1), using an integration by parts and (3.1), we obtain, for any $v \in K \cap H^2(\mathcal{T}_h)$,

$$\begin{aligned}
\int_\Omega \nabla_h u \cdot \nabla_h (v-u)\,dx &= \int_\Omega \nabla u \cdot \nabla (v-u)\,dx = -\int_\Omega \Delta u (v-u)\,dx \\
&= -\int_\Omega \Delta u (v-\psi)\,dx - \int_\Omega (\Delta u + f)(\psi - u)\,dx \\
&\quad + \int_\Omega f(\psi - u)\,dx \\
&= -\int_\Omega \Delta u (v-\psi)\,dx + \int_\Omega f(\psi - u)\,dx \\
&\geq \int_\Omega f(v-\psi)\,dx + \int_\Omega f(\psi-u)\,dx \\
&= \int_\Omega f(v-u)\,dx.
\end{aligned}$$

By the definition of $B_h(u,v)$,

$$\begin{aligned}
B_h(u, v-u) &= \int_\Omega \nabla_h u \cdot \nabla_h (v-u)\,dx + F(v-u) \\
&\geq \int_\Omega f(v-u)\,dx + F(v-u) \\
&= (f, v-u)_\Omega + F(v-u).
\end{aligned}$$

Hence, (3.5) holds.

Similarly, for the solution u of the simplified friction problem (1.3) and $v_h \in V_h$, if $D \subset \Omega$, then, by an integration by parts and (3.3), (3.4), we have

$$\begin{aligned}
\int_\Omega \nabla_h u \cdot \nabla_h (v_h - u)\,dx &= -\int_\Omega \Delta u (v_h - u)\,dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \nabla u \cdot n (v_h - u)\,ds \\
&= \int_\Omega (f - u - g\lambda\chi_D)(v_h - u)\,dx + \int_{\Gamma^0} \nabla u \cdot [v_h - u]\,ds \\
&= \int_\Omega (f-u)(v_h - u)\,dx - \int_D g\lambda (v_h - u)\,dx \\
&\quad + \int_{\Gamma^0} \nabla u \cdot [v_h - u]\,ds \\
&\geq \int_\Omega (f-u)(v_h-u)\,dx + \int_D g(|u| - |v_h|)\,dx \\
&\quad + \int_{\Gamma^0} \nabla u \cdot [v_h - u]\,ds,
\end{aligned}$$

and if $D \subset \partial\Omega$, then, by an integration by parts and (3.2), (3.4), we have

$$\begin{aligned}
\int_\Omega \nabla_h u \cdot \nabla_h (v_h - u)\, dx &= -\int_\Omega \Delta u (v_h - u)\, dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \nabla u \cdot n (v_h - u)\, ds \\
&= -\int_\Omega \Delta u (v_h - u)\, dx + \int_\Gamma \nabla u [v_h - u]\, ds \\
&= \int_\Omega (f - u)(v_h - u)\, dx - \int_D g\lambda (v_h - u)\, dx \\
&\quad + \int_{\Gamma^0} \nabla u [v_h - u]\, ds \\
&\geq \int_\Omega (f - u)(v_h - u)\, dx + \int_D g(|u| - |v_h|)\, dx \\
&\quad + \int_{\Gamma^0} \nabla u [v_h - u]\, ds.
\end{aligned}$$

We obtain (3.6) by the definition of B_h^* and the above inequalities. ∎

To consider the boundedness and stability of the bilinear form B_h, as in [3], let $V(h) = V_h + H^2(\Omega) \cap H_g^1(\Omega) \subset H^2(\mathcal{T}_h)$, and define the seminorms and norm for $v \in V(h)$ by the following relations:

$$|v|_{1,h}^2 = \sum_{K \in \mathcal{T}_h} |v|_{1,K}^2, \quad |v|_{1,*}^2 = \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[v]\|_{0,e}^2, \quad \|v\|_{0,\Omega} = (v, v)_\Omega^{1/2},$$

$$\|\!|v|\!\|^2 = |v|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |v|_{2,K}^2 + |v|_{1,*}^2. \tag{3.7}$$

That (3.7) defines a norm can be seen from the next inequality [1, Lemma 2.1]:

$$\|v\|_0 \leq C(|v|_{1,h}^2 + |v|_{1,*}^2)^{1/2} \leq C \ \|\!| v |\!\| \quad \forall\, v \in V(h). \tag{3.8}$$

As in [49, Lemma 7.2], using the definition of the lift operator r_e of (2.4), the Cauchy–Schwarz inequality, the trace inequality, and the inverse inequality, we get, for all $q \in V(h)$,

$$\begin{aligned}
\|r_e([q])\|_{0,\Omega} &= \sup_{w_h \in W_h} \frac{\int_\Omega r_e([q]) \cdot w_h\, dx}{\|w_h\|_{0,\Omega}} = \sup_{w_h \in W_h} \frac{-\int_e [q] \cdot \{w_h\}\, ds}{\|w_h\|_{0,\Omega}} \\
&\leq \sup_{w_h \in W_h} \frac{(\int_e h_e^{-1} |[q]|^2\, ds)^{1/2} (\int_e h_e |\{w_h\}|^2\, ds)^{1/2}}{\|w_h\|_{0,\Omega}} \\
&\leq \sup_{w_h \in W_h} \frac{h_e^{-1/2} \|[q]\|_{0,e} (C \sum_{K \in \mathcal{T}_h} \|w_h\|_{0,K}^2)^{1/2}}{\|w_h\|_{0,\Omega}} \\
&\leq C h_e^{-1/2} \|[q]\|_{0,e}.
\end{aligned} \tag{3.9}$$

Thus, $|v|_* \leq C|v|_{1,*}$, where the seminorm $|v|_* = \left(\sum_{e \in \mathcal{E}_h} \|r_e([v])\|_{0,\Omega}^2\right)^{1/2}$ is defined in [3]. Then the proof of boundedness in [3] holds true for all the nine DG methods by using $|v|_* \leq C|v|_{1,*}$.

LEMMA 3.2 (boundedness). *For* $1 \leq j \leq 9$, $B_h = B_h^{(j)}$ *satisfies*

$$B_h(u, v) \leq C_b \|\!| u |\!\| \ \|\!|v|\!\| \quad \forall\, u, v \in V(h), \tag{3.10}$$

where C_b is a positive constant depending on the angle condition, the polynomial degree, an upper bound on the edge-dependent penalty parameter η for the methods that contain the penalty term α^j or α^r, and, in the case of the LDG method ($j = 1$), an upper bound for the coefficient β.

For the stability, we use the result [3, inequality (4.5)] that there are two constants C_1 and C_2 such that

$$C_1 \sum_{e\in\mathcal{E}_h} \|r_e([v])\|_{0,\Omega}^2 \le |v|_{1,*}^2 \le C_2 \sum_{e\in\mathcal{E}_h} \|r_e([v])\|_{0,\Omega}^2 \quad \forall\, v \in V_h;$$

i.e., the seminorm $|v|_*$ is equivalent to $|v|_{1,*}$ on V_h. Therefore, the proof of stability in [3] is still valid here.

LEMMA 3.3 (stability). *For $1 \le j \le 7$, $B_h = B_h^{(j)}$ satisfies*

$$B_h(v, v) \ge C_s \,|||\, v \,|||^2 \quad \forall\, v \in V_h \tag{3.11}$$

if $\eta_0 = \inf_e \eta_e > 0$ for the methods with $j = 1, 3, 4, 6, 7$, $\eta_0 > 3$ for the method with $j = 5$, and η_0 is large enough for the IP method ($j = 2$), where C_s is a positive constant depending on the angle condition, the polynomial degree, a bound on the edge-dependent penalty parameter η, and, in the case of the LDG method, a bound for the coefficient β.

For the boundedness and stability of the bilinear form B_h^*, let $V^*(h) = V_h + H^2(\Omega)$, and define the seminorms and norm for $v \in V^*(h)$ as follows:

$$|v|_{0,h}^2 = \sum_{K\in\mathcal{T}_h} \|v\|_{0,K}^2, \quad |v|_{0,*}^2 = \sum_{e\in\mathcal{E}_h^0} \|r_e([v])\|_{0,\Omega}^2,$$

$$|||v|||_*^2 = |v|_{1,h}^2 + |v|_{0,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 |v|_{2,K}^2 + |v|_{0,*}^2. \tag{3.12}$$

With arguments similar to those in [3], Lemmas 3.2 and 3.3 hold for the bilinear forms $B_h^*(w, v)$ in terms of the norm $|||\cdot|||_*$.

Notice that (3.11) claims only the coercivity of the bilinear form B_h on V_h. Lack of coercivity of B_h on V is a source of difficulty in studying the DG methods for VIs.

4. Approximation and error estimates. We now turn to error estimations for the DG methods. Write the error as

$$e = u - u_h = (u - u_I) + (u_I - u_h),$$

where $u_I \in V_h$ is a suitable interpolant of the exact solution. If u_I is chosen to be the usual continuous piecewise polynomial interpolant, then the jumps of $(u - u_I)$ will be zero at the interelement boundaries. For the obstacle problem, with the norm defined in (3.7),

$$|||u - u_I|||^2 = |u - u_I|_{1,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 |u - u_I|_{2,K}^2 + \sum_{e\in\mathcal{E}_h^\partial} h_e^{-1} \|[u - u_I]\|_{0,e}^2 \le C_a^2 h^{2p} |u|_{p+1,\Omega}^2. \tag{4.1}$$

To analyze the method of Baumann and Oden ($j = 8$) and extend the analysis to nonconforming meshes, it is convenient to take an interpolant u_I which is discontinuous across the interelement boundaries. As in [3], we just require the local approximation property

$$|u-u_I|_{s,K} \le C h_K^{p+1-s} |u|_{p+1,K};$$

then, for the global approximation error, we have

$$\|\|u-u_I\|\| \le C_a h^p |u|_{p+1,\Omega}. \tag{4.2}$$

For the simplified friction problem, (4.1) and (4.2) hold true with the norm $\|\|u-u_I\|\|_*$ replacing $\|\|u-u_I\|\|$.

4.1. Methods with $1 \le j \le 5$. First we consider solving the obstacle problem with linear elements.

THEOREM 4.1. *Let u and u_h be the solutions of* (1.1) *and* (2.5) *with $K_h = K_h^1$, respectively. Assume $u \in H^2(\Omega)$ and $\psi \in H^2(\Omega)$. Then, for the DG methods with $j = 1, \dots, 5$, we have*

$$\|\|u-u_h\|\| \le Ch, \tag{4.3}$$

where C is a positive constant that depends on $|u|_2$, $|\psi|_2$, the angle condition, a bound on the edge-dependent penalty parameter η, and, in the case of the LDG method, a bound for the coefficient β.

Proof. Let u_I be the usual continuous piecewise linear interpolant of u. Recall the boundedness and stability of the bilinear form B_h. We have

$$C_s \,\|\|\, u_I-u_h\|\|^2 \le B_h(u_I-u_h, u_I-u_h) \equiv T_1 + T_2, \tag{4.4}$$

where

$$T_1 = B_h(u_I-u, u_I-u_h),$$
$$T_2 = B_h(u-u_h, u_I-u_h).$$

We bound T_1 as follows:

$$T_1 \le C_b \,\|\|\, u_I-u \,\|\|\, \|\|u_I-u_h\|\| \le \frac{C_s}{2} \,\|\|\, u_I-u_h \,\|\|^2 + \frac{C_b^2}{2C_s} \,\|\|\, u_I-u \,\|\|^2 . \tag{4.5}$$

To bound T_2, we first recall the relations

$$-\triangle u = f \quad \text{in } \Omega\backslash\Omega^0 = \{x \in \Omega : u(x) > \psi(x)\},$$
$$-\triangle u \ge f \quad \text{in } \Omega^0 = \{x \in \Omega : u(x) = \psi(x)\}.$$

Group the elements of $\mathcal{T}_h$ into three kinds:

$$\mathcal{T}_h^+ = \{K \in \mathcal{T}_h : K \subset \Omega\backslash\Omega^0\},$$
$$\mathcal{T}_h^0 = \{K \in \mathcal{T}_h : K \subset \Omega^0\},$$
$$\mathcal{T}_h^b = \mathcal{T}_h\backslash(\mathcal{T}_h^+ \cup \mathcal{T}_h^0).$$

Note that, on an interior edge, $[u] = 0$, $\{u\} = u$, $[\nabla u] = 0$, $\{\nabla u\} = \nabla u$, and $u = g$ on $\partial\Omega$:

$$\begin{aligned}
B_h(u, u_I - u_h) &= \int_\Omega \nabla u \cdot \nabla_h(u_I - u_h)\,dx - \int_\Gamma \nabla u \cdot [u_I - u_h]\,ds + F(u_I - u_h) \\
&= \sum_{K\in\mathcal{T}_h} \int_K -\Delta u(u_I - u_h)\,dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \nabla u \cdot (u_I - u_h) n\,ds \\
&\quad - \int_\Gamma \nabla u \cdot [u_I - u_h]\,ds + F(u_I - u_h) \\
&= -\int_\Omega \Delta u(u_I - u_h)\,dx + F(u_I - u_h). \qquad (4.6)
\end{aligned}$$

Letting $v_h = u_I$ in (2.5),

$$B_h(u_h, u_I - u_h) \ge (f, u_I - u_h)_\Omega + F(u_I - u_h) = \sum_{K\in\mathcal{T}_h} \int_K f(u_I - u_h)\,dx + F(u_I - u_h). \qquad (4.7)$$

Combining (4.7) and (4.6), we obtain

$$T_2 = B_h(u - u_h, u_I - u_h) \le \sum_{K\in\mathcal{T}_h} \int_K -(\Delta u + f)(u_I - u_h)\,dx \equiv T_3 + T_4 + T_5,$$

where

$$\begin{aligned}
T_3 &= \sum_{K\in\mathcal{T}_h^+} \int_K -(\Delta u + f)(u_I - u_h)\,dx, \\
T_4 &= \sum_{K\in\mathcal{T}_h^0} \int_K -(\Delta u + f)(u_I - u_h)\,dx, \\
T_5 &= \sum_{K\in\mathcal{T}_h^b} \int_K -(\Delta u + f)(u_I - u_h)\,dx.
\end{aligned}$$

It is easy to see that

$$T_3 = 0.$$

On $K \in \mathcal{T}_h^0$, we have $u = \psi$. At any node b of K, we have $u_h(b) \ge \psi(b) = u(b) = u_I(b)$, so $u_I - u_h \le 0$ on $K \in \mathcal{T}_h^0$. Noticing that $-(\Delta u + f) \ge 0$ on $K \in \mathcal{T}_h^0$, we obtain

$$T_4 \le 0.$$

Now consider $K \in \mathcal{T}_h^b$ and $x \in K$. If $x \in \Omega\backslash\Omega^0$, then $-(\Delta u + f)(x) = 0$. For $x \in \Omega^0$, we have $\psi(x) = u(x)$, and so

$$\begin{aligned}
u_I(x) - u_h(x) &= u_I(x) - u(x) + u(x) - u_h(x) \\
&= u_I(x) - u(x) + \psi(x) - u_h(x) \\
&= u_I(x) - u(x) + \psi(x) - \psi_I(x) + \psi_I(x) - u_h(x) \\
&\le u_I(x) - u(x) + \psi(x) - \psi_I(x). \qquad (4.8)
\end{aligned}$$

Thus,

$$
\begin{aligned}
T_5 &= \sum_{K\in\mathcal{T}_h^b} \int_K -(\Delta u + f)(u_I - u_h)\,dx \\
&\le C_1(\|u_I - u\|_{0,\Omega} + \|\psi - \psi_I\|_{0,\Omega}) \\
&\le C_2 h^2(|u|_{2,\Omega} + |\psi|_{2,\Omega}).
\end{aligned}
$$

From the above argument, we obtain

$$
T_2 = B_h(u - u_h, u_I - u_h) \le C_2 h^2(|u|_{2,\Omega} + |\psi|_{2,\Omega}). \tag{4.9}
$$

Combining (4.4), (4.5), and (4.9) and applying (4.1), we have

$$
|||u_I - u_h|||^2 \le C_3 h^2. \tag{4.10}
$$

Finally, from the triangle inequality $|||u - u_h||| \le |||u - u_I\,||| + |||\,u_I - u_h|||$, (4.2), and (4.10), we obtain the error bound (4.3). $\square$

Then we consider solving the obstacle problem with quadratic elements using a technique similar to that in [52].

THEOREM 4.2. *Let u and u_h be the solutions of* (1.1) *and* (2.5) *with $K_h = K_h^2$, respectively. Assume $u \in H^3(\Omega)$, $\psi \in H^3(\Omega)$, and $f \in H^1(\Omega)$. Then, for the DG methods with $j = 1, \ldots, 5$, we have*

$$
|||u - u_h||| \le C h^{3/2},
$$

where C is a positive constant that depends on $\|u\|_3$, $\|\psi\|_3$, and $\|f\|_1$, the angle condition, a bound on the edge-dependent penalty parameter η, and, in the case of the LDG method, a bound for the coefficient β.

Proof. Let u_I be the usual continuous piecewise quadratic interpolant of u. Similar to the proof of Theorem 4.1, we have

$$
C_s\,|||\,u_I - u_h|||^2 \le B_h(u_I - u_h, u_I - u_h) \equiv T_1 + T_2, \tag{4.11}
$$

where

$$
\begin{aligned}
T_1 &= B_h(u_I - u, u_I - u_h), \\
T_2 &= B_h(u - u_h, u_I - u_h).
\end{aligned}
$$

The term T_1 is again bounded by

$$
T_1 \le C_b\,|||\,u_I - u\,|||\;\,|||u_I - u_h||| \le \frac{C_s}{2}\,|||\,u_I - u_h\,|||^2 + \frac{C_b^2}{2C_s}\,|||\,u_I - u\,|||^2\,. \tag{4.12}
$$

For the term T_2,

$$
T_2 = B_h(u - u_h, u_I - u_h) \le \sum_{K\in\mathcal{T}_h} \int_K -(\Delta u + f)(u_I - u_h)\,dx. \tag{4.13}
$$

Let $w := -\Delta u - f$. Then, from (4.13),

$$
T_2 \le \int_\Omega w(u_I - u + \psi - \psi_I)dx + \int_\Omega w(u - \psi)dx + \int_\Omega w(\psi_I - u_h)dx.
$$

We know that $w(u-\psi)=0$ by (3.1) and that

$$\int_\Omega w(u_I - u + \psi - \psi_I)dx \le \|w\|_{0,\Omega}(\|u-u_I\|_{0,\Omega} + \|\psi - \psi_I\|_{0,\Omega}) \\ \le C_1 h^3 \|w\|_{0,\Omega}(|u|_{3,\Omega} + |\psi|_{3,\Omega}).$$

Then

$$T_2 \le C_1 h^3 \|w\|_{0,\Omega}(|u|_{3,\Omega} + |\psi|_{3,\Omega}) + T_3 + T_4 + T_5,$$

where

$$T_3 = \sum_{K\in\mathcal{T}_h^+} \int_K w(\psi_I - u_h)\,dx, \quad T_4 = \sum_{K\in\mathcal{T}_h^0} \int_K w(\psi_I - u_h)\,dx,$$
$$T_5 = \sum_{K\in\mathcal{T}_h^b} \int_K w(\psi_I - u_h)\,dx.$$

Note that $w(x)=0$ if $x \in \Omega\backslash\Omega^0$. So

$$T_3 = 0.$$

To estimate T_4 and T_5, as in [52], we introduce

$$P_0^K v = \frac{1}{|K|}\int_K v\,dx, \qquad R_0^K v = v - P_0^K v.$$

Since $w \ge 0$, $P_0^K w \ge 0$. Since $u_h \in K_h$, $u_h(m) \ge \psi(m)$ for all the midpoints on the edges of the element K, implying that

$$\int_K (\psi_I - u_h)\,dx = \sum_{i=1}^3 (\psi - u_h)(m_i) \le 0.$$

Then we get

$$\int_K w(\psi_I - u_h)\,dx \le \int_K R_0^K w(\psi_I - u_h)\,dx \\ = \int_K R_0^K w\, R_0^K(\psi_I - u_h)\,dx \le \|R_0^K w\|_{0,K}\|R_0^K(\psi_I - u_h)\|_{0,K}.$$

We apply interpolation error estimates to the right-hand side of the above inequality:

$$\int_K w(\psi_I - u_h)\,dx \le C_2 h_K^2 |w|_{1,K}|\psi_I - u_h|_{1,K} \\ \le C_2 h_K^2 |w|_{1,K}(|\psi_I - \psi|_{1,K} + |\psi - u|_{1,K} + |u - u_h|_{1,K}) \\ \le C_3 h_K^2 |w|_{1,K}(h_K^2|\psi|_{3,K} + |\psi - u|_{1,K} + |u - u_h|_{1,K}).$$

Assume $K \in \mathcal{T}_h^0$. Then $u = \psi$ on K, and so

$$T_4 \le C_2 h^2 |w|_{1,\Omega}(h^2|\psi|_{3,\Omega} + |||u - u_h|||). \tag{4.14}$$

Consider the case $K \in \mathcal{T}_h^b$. From the assumption $\psi, u \in H^3(\Omega)$, we know that $\nabla(\psi - u) \in H^2(\Omega) \hookrightarrow C^{0,1}(\overline{\Omega})$. Since $K \in \mathcal{T}_h^b$, there is a point $Q \in K$ such that

$\nabla(\psi - u)(Q) = 0$. Then, for any $x \in K$, we have, for some constant C^* depending on $|\nabla(\psi - u)|_{C^{0,1}(K)}$,

$$|\nabla(\psi - u)(x)| = |\nabla(\psi - u)(x) - \nabla(\psi - u)(Q)| \le C^*|x - Q| \le C^* h_K.$$

Thus,

$$|\nabla(\psi - u)|_{0,K} \le C^* h_K^2$$

and

$$\int_K w(\psi_I - u_h)\, dx \le C_3 h_K^2 |w|_{1,K} (h_K^2 |\psi|_{3,K} + C^* h_K^2 + |u - u_h|_{1,K}).$$

Finally, we obtain

$$T_5 \le C_3 h^2 |w|_{1,\Omega} (h^2 |\psi|_{3,\Omega} + h^2 |\nabla(\psi - u)|_{C^{0,1}(\overline{\Omega})} + \|\|u - u_h\|\|). \tag{4.15}$$

The proof is completed by combining (4.11), (4.12), (4.14), and (4.15). ∎

Next we give error estimates of the DG method with $j = 1, \dots, 5$, for the simplified friction problem.

THEOREM 4.3. *Let u and u_h be the solutions of* (1.3) *and* (2.6)*, respectively. Assume $u \in H^{p+1}(\Omega)$ if $D \subset \Omega$, and assume further $u|_{\partial\Omega} \in H^{p+1}(\partial\Omega)$ if $D \subset \partial\Omega$. Then, for DG methods with $j = 1, \dots, 5$, we have*

$$\|\|u - u_h\|\|_* \le C^* h^{(p+1)/2}, \tag{4.16}$$

where C^* is a positive constant that depends on $|u|_{p+1}$ and $\|g\|_{0,D}$ (also $|u|_{p+1,\partial\Omega}$ if $D \subset \partial\Omega$), the angle condition, the polynomial degree, a bound on the edge-dependent penalty parameter η, and, in the case of the LDG method, a bound for the coefficient β.

Proof. Here we give a proof only for the case $D \subset \Omega$. If $D \subset \partial\Omega$, the proof is similar. As in the proof of Theorem 4.1, let u_I be the usual piecewise polynomial continuous interpolant, and recall the boundedness and stability of the bilinear form B_h^*. We have

$$C_s \,\|\|\, u_I - u_h \|\|_*^2 \le B_h^*(u_I - u_h, u_I - u_h) \equiv T_1 + T_2, \tag{4.17}$$

where

$$T_1 = B_h^*(u_I - u, u_I - u_h),$$
$$T_2 = B_h^*(u - u_h, u_I - u_h).$$

We bound T_1 as follows:

$$T_1 \le C_b \,\|\|\, u_I - u \,\|\|_* \; \|\|u_I - u_h\|\|_* \le \frac{C_s}{2} \,\|\|\, u_I - u_h \,\|\|_*^2 + \frac{C_b^2}{2C_s} \,\|\|\, u_I - u \,\|\|_*^2 . \tag{4.18}$$

To bound T_2, again note that, on an interior edge, $[u] = 0$, $\{u\} = u$, $[\nabla u] = 0$, and $\{\nabla u\} = \nabla u$. We have

$$\begin{aligned} B_h^*(u, u_I - u_h) &= \int_\Omega \nabla u \cdot \nabla_h (u_I - u_h)\, dx + \int_\Omega u(u_I - u_h)\, dx - \int_{\Gamma^0} \nabla u \cdot [u_I - u_h]\, ds \\ &= \sum_{K\in\mathcal{T}_h} \int_K -\Delta u(u_I - u_h)\, dx + \int_\Omega u(u_I - u_h)\, dx \\ &\quad + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \nabla u \cdot n_K (u_I - u_h)\, ds - \int_{\Gamma^0} \nabla u \cdot [u_I - u_h]\, ds \\ &= -\int_\Omega \Delta u(u_I - u_h)\, dx + \int_\Omega u(u_I - u_h)\, dx. \end{aligned} \tag{4.19}$$

Letting $v_h = u_I$ in (2.6),

$$B_h^*(u_h, u_I - u_h) + j(u_I) - j(u_h) \geq (f, u_I - u_h)_\Omega. \tag{4.20}$$

Combining (4.19) and (4.20), we have

$$\begin{aligned} T_2 = B_h^*(u - u_h, u_I - u_h) &\leq j(u_I) - j(u_h) - (f + \Delta u - u, u_I - u_h) \\ &= \int_D g(|u_I| - |u_h|)\, dx - \int_D g\lambda(u_I - u_h)\, dx \\ &= \int_D g(|u_I| - \lambda u_I)\, dx + \int_D g(\lambda u_h - |u_h|)\, dx \\ &\leq \int_D g(|u_I| - \lambda u_I)\, dx = \int_D g(|u_I| - |u| + \lambda u - \lambda u_I)\, dx \\ &\leq 2\int_D g|u - u_I|\, dx \leq 2\, \|g\|_{0,D} \|u - u_I\|_{0,D} \leq C_4 h^{p+1}. \end{aligned} \tag{4.21}$$

From (4.18) and (4.21), the proof is completed. ☐

4.2. Methods with $j = 6, 7$. For the DG methods of Babuška and Zlámal ($j = 6$) and Brezzi et al. ($j = 7$), we cannot get (4.6) or (4.19) as in the proof of Theorem 4.1 or Theorem 4.3, because, for these two methods, the bilinear forms do not contain the term $\int_\Gamma \{\nabla_h w\} \cdot [v]\, ds$ or $\int_{\Gamma^0} \{\nabla_h w\} \cdot [v]\, ds$. Instead of (4.6) and (4.19) we have

$$B_h(u, u_I - u_h) = -\int_\Omega \Delta u(u_I - u_h)\, dx + \int_\Gamma \{\nabla_h u\} \cdot [u_I - u_h]\, ds + F(u_I - u_h), \tag{4.22}$$

$$B_h^*(u, u_I - u_h) = -\int_\Omega \Delta u(u_I - u_h)\, dx + \int_\Omega u(u_I - u_h)\, dx + \int_{\Gamma^0} \{\nabla_h u\} \cdot [u_I - u_h]\, ds, \tag{4.23}$$

implying that we have to bound the terms $\int_\Gamma \{\nabla_h u\} \cdot [u_I - u_h]\, ds$ and $\int_{\Gamma^0} \{\nabla_h u\} \cdot [u_I - u_h]\, ds$. Even though the bilinear forms of the two pure penalty methods are stable and bounded for the norm $|||\cdot|||$ defined in (3.7), the difficulty is in giving good estimates of $\int_\Gamma \{\nabla_h u\} \cdot [u_I - u]\, ds$ dependent on the norm $|||\cdot|||$. Following the ideas of [3], we use superpenalties to reduce the influence that (4.6) or (4.19) does not hold true for these two methods. For the method of Babuška and Zlámal ($j = 6$), take the penalty term for the obstacle problem as

$$\alpha^j(u, v) = \sum_{e\in\mathcal{E}_h} \int_e \eta_e h_e^{-2p-1} [u] \cdot [v]\, ds.$$

The corresponding bilinear form is bounded with respect to the norm $|\!|\!|\cdot|\!|\!|$ defined by

$$|\!|\!|v|\!|\!|_1^2 = |v|_{1,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 |v|_{2,K}^2 + \alpha^j(v,v). \tag{4.24}$$

Then we have, for all $u, v \in V(h)$,

$$\begin{aligned}\sum_{e\in\mathcal{E}_h}\int_e \{\nabla_h u\}\cdot[v]\,ds &= \sum_{e\in\mathcal{E}_h}\int_e (h_e^{2p+1})^{1/2}\{\nabla_h u\}\cdot[v](h_e^{-2p-1})^{1/2}\,ds\\ &\le C\,|\!|\!|\,v\,|\!|\!|_1\left(\sum_{e\in\mathcal{E}_h} h_e^{2p+1}\int_e |\{\nabla_h u\}\cdot n_e|^2 ds\right)^{1/2}\\ &\le C\,h^p\,|\!|\!|\,v\,|\!|\!|_1\,\|u\|_{2,h},\end{aligned}\tag{4.25}$$

where $\|u\|_{2,h}^2 = \sum_K \|u\|_{2,K}^2$. Note that the bilinear form remains stable with respect to the norm in (4.24) if the lower bound for η_e is large enough. For the simplified friction problem, with similar choices and changes, we have

$$\alpha_0^j(u,v) = \sum_{e\in\mathcal{E}_h^0}\int_e \eta_e h_e^{-2p-1}[u]\cdot[v]\,ds,$$

$$|\!|\!|v|\!|\!|_{*1}^2 = |v|_{1,h}^2 + |v|_{0,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2|v|_{2,K}^2 + \alpha_0^j(v,v), \tag{4.26}$$

$$\sum_{e\in\mathcal{E}_h^0}\int_e \{\nabla_h u\}\cdot[v]\,ds \le C\,h^p\,|\!|\!|\,v\,|\!|\!|_{*1}\,\|u\|_{2,h}\quad \forall\, u,v\in V^*(h). \tag{4.27}$$

For the method of Brezzi et al. ($j = 7$), take the penalty term for the obstacle problem as

$$\alpha^r(u,v) = \sum_{e\in\mathcal{E}_h}\int_e h_e^{-2p} r_e([u])\cdot r_e([v])\,ds.$$

As in [16], we define a new norm through the relation

$$|\!|\!|v|\!|\!|_2^2 = |v|_{1,h}^2 + \alpha^r(v,v). \tag{4.28}$$

Boundedness and stability of $B_h^{(7)}$ hold with respect to the norm $|\!|\!|\cdot|\!|\!|_2$. We also have

$$\sum_{e\in\mathcal{E}_h}\int_e \{\nabla_h u\}\cdot[v]\,ds \le C\,h^p\,|\!|\!|\,v\,|\!|\!|_2\,\|u\|_{2,h}\quad \forall\, u,v\in V(h). \tag{4.29}$$

For the simplified friction problem, we have

$$\alpha_0^r(u,v) = \sum_{e\in\mathcal{E}_h^0}\int_e h_e^{-2p} r_e([u])\cdot r_e([v])\,ds,$$

$$|\!|\!|v|\!|\!|_{*2}^2 = |v|_{1,h}^2 + |v|_{0,h}^2 + \alpha_0^r(v,v), \tag{4.30}$$

$$\sum_{e\in\mathcal{E}_h^0}\int_e \{\nabla_h u\}\cdot[v]\,ds \le C\,h^p\,|\!|\!|\,v\,|\!|\!|_{*2}\,\|u\|_{2,h}\quad \forall\, u,v\in V^*(h). \tag{4.31}$$

Through arguments similar to those used in Theorem 4.1, Theorem 4.2, and Theorem 4.3, using (4.25), (4.27), (4.29), and (4.31), we obtain the following theorems.

THEOREM 4.4. *Let u and u_h be the solutions of* (1.1) *and* (2.5) *with $K_h = K_h^p$, $p = 1$ or 2, respectively. Assume $u \in H^{p+1}(\Omega)$ and $\psi \in H^{p+1}(\Omega)$. Then, if the lower bound for the η_e is large enough, for the Babuška–Zlámal DG method ($j = 6$) we have*

$$|||u - u_h|||_1 \leq Ch^{(p+1)/2},$$

and, for the method of Brezzi et al. ($j = 7$), we have

$$|||u - u_h|||_2 \leq Ch^{(p+1)/2},$$

where C is a positive constant that depends on $\|u\|_{p+1}$, $\|\psi\|_{p+1}$, the angle condition, the polynomial degree, and a bound on the edge-dependent penalty parameter η.

THEOREM 4.5. *Let u and u_h be the solutions of* (1.3) *and* (2.6), *respectively. Assume $u \in H^{p+1}(\Omega)$ if $D \subset \Omega$, and assume further $u|_{\partial\Omega} \in H^{p+1}(\partial\Omega)$ if $D \subset \partial\Omega$. Then, if the lower bound for the η_e is large enough, for the Babuška–Zlámal DG method ($j = 6$) we have*

$$|||u - u_h|||_{*1} \leq C^* h^{(p+1)/2},$$

and, for the method of Brezzi et al. ($j = 7$), we have

$$|||u - u_h|||_{*2} \leq C^* h^{(p+1)/2},$$

where C^* is a positive constant that depends on $|u|_{p+1}$ and $\|g\|_{0,D}$ (also $|u|_{p+1,\partial\Omega}$ if $D \subset \partial\Omega$), the angle condition, the polynomial degree, and a bound on the edge-dependent penalty parameter η.

4.3. Methods with $j = 8, 9$. Regarding the two DG methods with the bilinear forms $B_h^{(j)}$ and $B_h^{*(j)}$, $j = 8, 9$, because of instability they could not be analyzed in the same way as for other methods as above. But arguing similarly as in [3] and with the analysis of T_2 in Theorem 4.1, Theorem 4.2, and Theorem 4.3, we can obtain the corresponding results of error estimates.

The method of Baumann and Oden. For the method of Baumann and Oden ($j = 8$), there is a weak stability property:

$$B_h^{(8)}(v, v) = |v|_{1,h}^2 \quad \forall\, v \in V(h).$$

Note that $|v|_{1,h}$ is only a seminorm, and $B_h^{(8)}$ cannot be bounded by it. The method is not convergent for the obstacle problem with linear elements. For a polynomial degree $p \geq 2$, [48] gives an approach to do error estimation in the seminorm $|v|_{1,h}$. The key idea of the analysis there is to use an interpolant $u_I \in V_h$ such that $\int_e \{\nabla_h(u - u_I)\}\, ds = 0$ for each $e \in \mathcal{E}_h$, so that

$$B_h^{(8)}(u - u_I, v) = 0$$

for any piecewise constant v with respect to $\mathcal{T}_h$. A straightforward modification of the Morley interpolant for $p = 2$ and the Fraeijs de Veubeke interpolant for $p = 3$ satisfies this property, which is possible only for $p \geq 2$. Let P_0 be the orthogonal projection of $L^2(\Omega)$ onto the space of piecewise constant functions. Using the above equality, we have

$$B_h^{(8)}(u - u_I, v) = B_h^{(8)}(u - u_I, v - P_0 v) \leq C_b\, |||\, u - u_I\, |||\; |||v - P_0 v\, ||| \quad \forall\, v \in V(h).$$

For $v \in V_h$, $|||v - P_0 v||| \le C|v|_{1,h}$. So, letting $v = u_I - u_h$, we have

$$B_h^{(8)}(u - u_I, u_I - u_h) \le C \; ||| \, u - u_I \, ||| \; |u_I - u_h|_{1,h}.$$

Combining the above inequality and an argument similar to that for bounding T_2 in Theorem 4.2, we obtain the following result by using (4.2).

THEOREM 4.6. *Let u and u_h be the solutions of* (1.1) *and* (2.5) *with $K_h = K_h^2$, respectively. Assume $u \in H^3(\Omega)$, $\psi \in H^3(\Omega)$, and $f \in H^1(\Omega)$. Then, for the Baumann–Oden DG method ($j = 8$), we have*

$$|u - u_h|_{1,h} \le Ch^{3/2},$$

where C is a positive constant that depends on $\|u\|_3$, $\|\psi\|_3$, and $\|f\|_1$ and the angle condition.

For the simplified friction problem, we have

$$B_h^{*(8)}(v, v) = |v|_{1,h}^2 + |v|_{0,h}^2 \quad \forall\, v \in V^*(h).$$

Note that $\|v\|_{1,h} := (|v|_{1,h}^2 + |v|_{0,h}^2)^{1/2}$ defines a norm. Then

$$\|u_I - u_h\|_{1,h}^2 = B_h^{*(8)}(u_I - u_h, u_I - u_h) = B_h^{*(8)}(u_I - u, u_I - u_h) + B_h^{*(8)}(u - u_h, u_I - u_h),$$

where the interpolant $u_I \in V_h$ satisfies $\int_e \{\nabla_h (u - u_I)\}\, ds = 0$ for each $e \in \mathcal{E}_h$. Similar to the bounding of the term $B_h^{(8)}(u - u_I, u_I - u_h)$ for the obstacle problem, we get

$$\int_{\Gamma^0} \{\nabla_h (u_I - u)\} \cdot [u_I - u_h]\, ds \le Ch^p |u_I - u_h|_{1,h} \le \frac{1}{4}|u_I - u_h|_{1,h}^2 + Ch^{2p}.$$

Using trace and inverse inequalities, we have

$$\begin{aligned}
&\int_{\Gamma^0} [u_I - u] \cdot \{\nabla_h (u_I - u_h)\}\, ds \\
&\quad = \sum_{e \in \mathcal{E}_h^0} \int_e \{\nabla_h (u_I - u_h)\} \cdot [u_I - u]\, ds \\
&\quad \le C \left[\sum_K (|u_I - u_h|_{1,K}^2 + h_K^2 |u_I - u_h|_{2,K}^2) \right]^{1/2} \left[\sum_{e \in \mathcal{E}_h^0} h_e^{-1} \int_e |[u_I - u]|^2\, ds \right]^{1/2} \\
&\quad \le C|u_I - u_h|_{1,h} \left[\sum_{e \in \mathcal{E}_h^0} (h_e^{-2} |u_I - u|_{0,K}^2 + |u_I - u|_{1,K}^2) \right]^{1/2} \\
&\quad \le Ch^p |u_I - u_h|_{1,h} |u|_{p+1} \\
&\quad \le \frac{1}{4}|u_I - u_h|_{1,h}^2 + Ch^{2p}|u|_{p+1}^2.
\end{aligned}$$

Then

$$
\begin{aligned}
B_h^{*(8)}(u_I-u,u_I-u_h) &= \int_\Omega \nabla_h(u_I-u)\cdot\nabla_h(u_I-u_h)\,dx + \int_\Omega (u_I-u)(u_I-u_h)\,dx \\
&\quad + \int_{\Gamma^0}[u_I-u]\cdot\{\nabla_h(u_I-u_h)\}\,ds \\
&\quad - \int_{\Gamma^0}\{\nabla_h(u_I-u)\}\cdot[u_I-u_h]\,ds \\
&\le \frac{1}{4}\|u_I-u_h\|_{1,h}^2 + \|u_I-u\|_{1,h}^2 + \frac{1}{4}|u_I-u_h|_{1,h}^2 + Ch^{2p}|u|_{p+1}^2 \\
&\quad + \frac{1}{4}|u_I-u_h|_{1,h}^2 + Ch^{2p} \le \frac{3}{4}\|u_I-u_h\|_{1,h}^2 + Ch^{2p}|u|_{p+1}^2.
\end{aligned}
$$

Similar to the proof of Theorem 4.3 for bounding T_2, we have

$$B_h^{*(8)}(u-u_h,u_I-u_h) \le Ch^{p+1}.$$

So we have the following result.

THEOREM 4.7. *Let $p \ge 2$, and let u and u_h be the solutions of* (1.3) *and* (2.6)*, respectively. Assume $u \in H^{p+1}(\Omega)$ if $D \subset \Omega$, and assume further $u|_{\partial\Omega} \in H^{p+1}(\partial\Omega)$ if $D \subset \partial\Omega$. Then, for the Baumann–Oden DG method ($j = 8$), we have*

$$\|u-u_h\|_{1,h} \le C^* h^{(p+1)/2},$$

where C^* is a positive constant that depends on $|u|_{p+1}$ and $\|g\|_{0,D}$ (also $|u|_{p+1,\partial\Omega}$ if $D \subset \partial\Omega$), the angle condition, and the polynomial degree.

The method of Bassi and Rebay. Consider the method of Bassi and Rebay ($j = 9$), in which the bilinear form is

$$B_h^{(9)}(w,v) = (\nabla_h w, \nabla_h v)_\Omega - \langle [w], \{\nabla_h v\}\rangle_\Gamma - \langle \{\nabla_h w\}, [v]\rangle_\Gamma + (r([w]), r([v]))_\Omega\,.$$

By (2.3) for the definition of the lifting operator r, $B_h^{(9)}$ can be rewritten as

$$B_h^{(9)}(w,v) = \int_\Omega (\nabla_h w + r([w]))\cdot(\nabla_h v + r([v]))\,dx.$$

Consequently, a weak stability property is valid:

$$B_h^{(9)}(v,v) = \|\nabla_h v + r([v])\|_{0,h}^2 \quad \forall\, v \in V_h. \tag{4.32}$$

Unfortunately, $B_h^{(9)}(v,v)$ vanishes on the set $Z := \{v \in V_h :\ \nabla_h v + r([v]) = 0\}$, which is not empty [15]. In [3], it is proved that if f is a piecewise polynomial of degree $p-1$, a solution to the discrete problem for the Dirichlet problem of the Poisson equation exists and is unique up to an element of Z. Indeed, over the quotient space V_h/Z, the seminorm $\|\nabla_h v + r([v])\|_{0,h}$ becomes a norm, and the weak stability becomes a strong stability. The same analysis remains true for the obstacle problem.

Let $\rho_h := u_I - u_h$, u_I being the continuous piecewise polynomial interpolant of u. From (4.32), we know

$$\|\nabla_h \rho_h + r([\rho_h])\|_{0,h}^2 = B_h^{(9)}(\rho_h,\rho_h) = B_h^{(9)}(u_I-u,\rho_h) + B_h^{(9)}(u-u_h,\rho_h).$$

First we analyze $B_h^{(9)}(u_I - u, \rho_h)$:

$$\begin{aligned} B_h^{(9)}(u_I - u, \rho_h) &= (\nabla_h(u_I - u) + r([u_I - u]), \nabla_h \rho_h + r([\rho_h]))_\Omega \\ &\le \frac{1}{2}\|\nabla_h(u_I - u) + r([u_I - u])\|_{0,h}^2 + \frac{1}{2}\|\nabla_h \rho_h + r([\rho_h])\|_{0,h}^2 \\ &\le \frac{1}{2}Ch^{2p}|u|_{p+1}^2 + \frac{1}{2}\|\nabla_h \rho_h + r([\rho_h])\|_{0,h}^2. \end{aligned} \tag{4.33}$$

Similar to the bounding of T_2 in the proofs of Theorem 4.1 and Theorem 4.2, we have

$$B_h^{(9)}(u - u_h, \rho_h) \le Ch^{p+1}.$$

Thus,

$$\|\nabla_h \rho_h + r([\rho_h])\|_{0,h} \le Ch^{(p+1)/2}.$$

Let $\sigma_h(u_h) := \nabla_h u_h + r([u_h - g])$, and note that $r([u_I]) = r_\partial([u_I])$. We have

$$\begin{aligned} \|\nabla u - \sigma_h(u_h)\|_{0,h} &\le \|\nabla u - \nabla u_I\|_{0,h} + \|\nabla u_I - \sigma_h(u_h)\|_{0,h} \\ &= \|\nabla u - \nabla u_I\|_{0,h} + \|\nabla_h \rho_h + r([\rho_h]) + r_\partial([g - u_I])\|_{0,h} \\ &\le Ch^{(p+1)/2}. \end{aligned}$$

Summarizing, we have the next result.

THEOREM 4.8. *Let u and u_h be the solutions of* (1.1) *and* (2.5) *with $K_h = K_h^p$, $p = 1$ or 2, respectively. Assume $u \in H^{p+1}(\Omega)$ and $\psi \in H^{p+1}(\Omega)$. Then, for the Bassi–Rebay DG method ($j = 9$), we have*

$$\|\nabla u - \sigma_h(u_h)\|_{0,h} \le Ch^{(p+1)/2},$$

where C is a positive constant that depends on $\|u\|_{p+1}$, $\|\psi\|_{p+1}$, the angle condition, and the polynomial degree.

Define $|||v|||_{*3}^2 := \|\nabla_h v + r_0([v])\|_{0,h}^2 + |v|_{0,h}^2$. Then

$$|||\rho_h|||_{*3}^2 = B_h^{*(9)}(\rho_h, \rho_h) = B_h^{*(9)}(u_I - u, \rho_h) + B_h^{*(9)}(u - u_h, \rho_h).$$

Using the argument similar to that of (4.33) and for T_2 in the proof of Theorem 4.3, we get

$$|||\rho_h|||_{*3} \le Ch^{(p+1)/2}.$$

Let $\sigma_h^*(u_h) := \nabla_h u_h + r_0([u_h])$. We have

$$\begin{aligned} \|\nabla u - \sigma_h^*(u_h)\|_{0,h} &\le \|\nabla u - \nabla u_I\|_{0,h} + \|\nabla u_I - \sigma_h(u_h)\|_{0,h} \\ &\le \|\nabla u - \nabla u_I\|_{0,h} + \|\nabla_h \rho_h + r_0([\rho_h])\|_{0,h} \\ &\le \|\nabla u - \nabla u_I\|_{0,h} + |||\rho_h|||_{*3} \le C^* h^{(p+1)/2}. \end{aligned}$$

THEOREM 4.9. *Let u and u_h be the solutions of* (1.3) *and* (2.6), *respectively. Assume $u \in H^{p+1}(\Omega)$ if $D \subset \Omega$, and assume further $u|_{\partial\Omega} \in H^{p+1}(\partial\Omega)$ if $D \subset \partial\Omega$. Then, for the Bassi–Rebay DG method ($j = 9$), we have*

$$\|\nabla u - \sigma_h^*(u_h)\|_{0,h} \le C^* h^{(p+1)/2},$$

where C^* is a positive constant that depends on $|u|_{p+1}$ and $\|g\|_{0,D}$ (also $|u|_{p+1,\partial\Omega}$ if $D \subset \partial\Omega$), the angle condition, and the polynomial degree.

At the end of this section, we comment that a summary of the properties of all nine DG methods for solving the EVIs can be given in the spirit of [3, Table 6.1]. Since the only major difference is in the convergence order of the methods, we do not provide such a table in this paper.

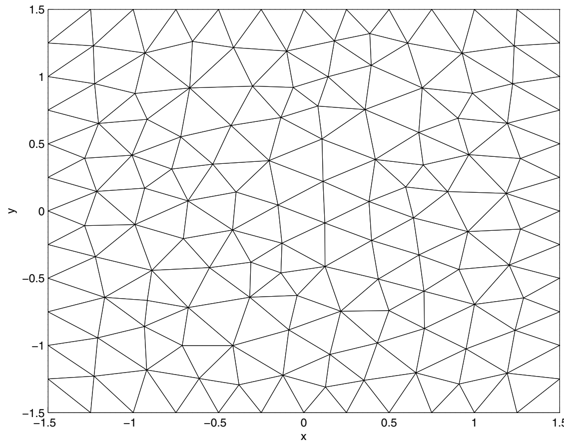

FIG. 1. *Quasi-uniform triangulation with $h = 0.25$ in Example* 1.

5. Numerical examples. We report results from two numerical examples using the LDG method with the constant parameter β being chosen as the unit outward normal vectors n_K of each element K. The discretized problem is solved by a primal-dual active set strategy [35].

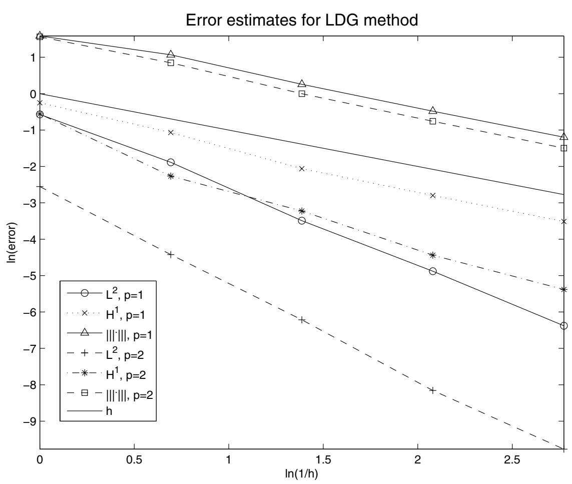

FIG. 2. *Numerical errors for the LDG method in Example* 1.

Example 1. Obstacle problem (1.1) is considered in the domain $\Omega := (-1.5, 1.5)^2$ with a constant right-hand-side term $f \equiv -2$ and the obstacle function $\psi = 0$. The Dirichlet boundary condition g is given as the trace of the exact solution

$$u(x,y) = \begin{cases} \dfrac{r^2}{2} - \ln(r) - \dfrac{1}{2} & \text{if } r \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $r = (x^2 + y^2)^{1/2}$.

We use quasi-uniform triangulations $\mathcal{T}_h$, as shown in Figure 1. Figure 2 and Table 2 show numerical results of the LDG method. We observe that most of the numerical

TABLE 2
Numerical convergence orders for the LDG method in Example 1.

p	Error norms	$h=1$	$h=0.5$	$h=0.25$	$h=0.125$
1	$\|u-u_h\|_{L^2}$	1.8967	2.3134	2.0047	2.1605
	$\|u-u_h\|_{H^1}$	1.1783	1.4325	1.0637	1.0311
	$\vert\vert\vert u-u_h\vert\vert\vert$	0.7496	1.1691	1.0680	1.0305
2	$\|u-u_h\|_{L^2}$	2.6986	2.5850	2.8019	2.3229
	$\|u-u_h\|_{H^1}$	2.4647	1.3830	1.7540	1.3591
	$\vert\vert\vert u-u_h\vert\vert\vert$	1.0345	1.2212	1.0948	1.0669

convergence orders match well with the theoretical predictions. The only exception is for $|||u-u_h|||$ with $p=2$; the numerical convergence rate is $O(h)$ instead of $O(h^{3/2})$. We owe this phenomenon to the lack of sufficient regularity of the exact solution; indeed, $u \notin H^3(\Omega)$.

Example 2. Let $\Omega := (-2,2)^2$, $f=0$, and the obstacle function

$$\psi(x,y) = \sqrt{x^2+y^2} \ \text{ for } x^2+y^2 \le 1, \quad \psi(x,y) = -1 \text{ elsewhere.}$$

The Dirichlet boundary condition is determined from the true solution of problem (1.1):

$$u(x,y) = \begin{cases} \sqrt{1-x^2-y^2}, & r \le r^*, \\ -(r^*)^2 \ln(r/R)/\sqrt{1-(r^*)^2}, & r \ge r^*, \end{cases}$$

where $r = \sqrt{x^2+y^2}$, $R=2$, and $r^* = 0.6979651482\ldots$, which satisfies

$$(r^*)^2(1-\ln(r^*/R)) = 1.$$

We also use quasi-uniform triangulations $\mathcal{T}_h$, as shown in Figure 3. Numerical results of the LDG method are shown in Figure 4 and Table 3. As in the previous example, we observe that most of the numerical convergence orders match well with the theoretical predictions. The only exception is for $|||u-u_h|||$ with $p=2$; the numerical convergence rate is $O(h)$ instead of $O(h^{1.5})$, and we owe this phenomenon to the regularity property $u \notin H^3(\Omega)$.

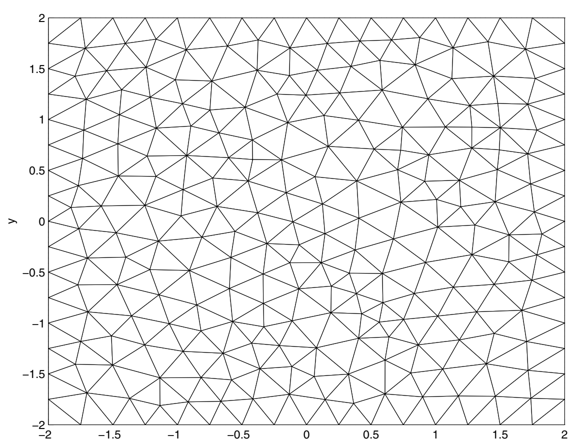

FIG. 3. *Quasi-uniform triangulation with $h=0.25$ in Example 2.*

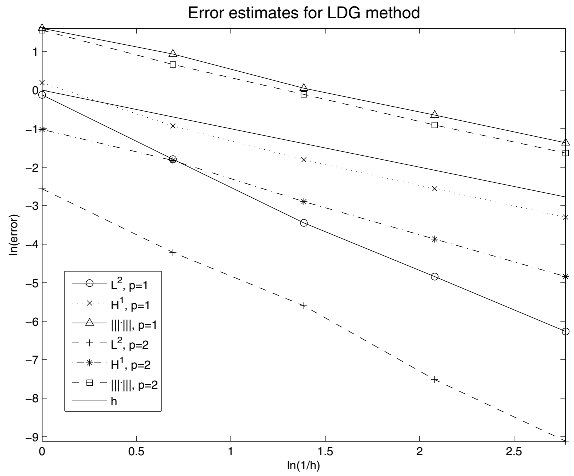

FIG. 4. *Numerical errors for the LDG method in Example* 2.

TABLE 3
Numerical convergence orders for the LDG method in Example 2.

p	Error norms	$h = 1$	$h = 0.5$	$h = 0.25$	$h = 0.125$
1	$\|u - u_h\|_{L^2}$	2.4125	2.3830	2.0136	2.0559
	$\|u - u_h\|_{H^1}$	1.6204	1.2665	1.0849	1.0667
	$\|\|\|u - u_h\|\|\|$	0.9633	1.2826	1.0056	1.0376
2	$\|u - u_h\|_{L^2}$	2.3830	2.0000	2.7677	2.3024
	$\|u - u_h\|_{H^1}$	1.1602	1.5427	1.4064	1.4036
	$\|\|\|u - u_h\|\|\|$	1.2851	1.1201	1.1484	1.0497

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