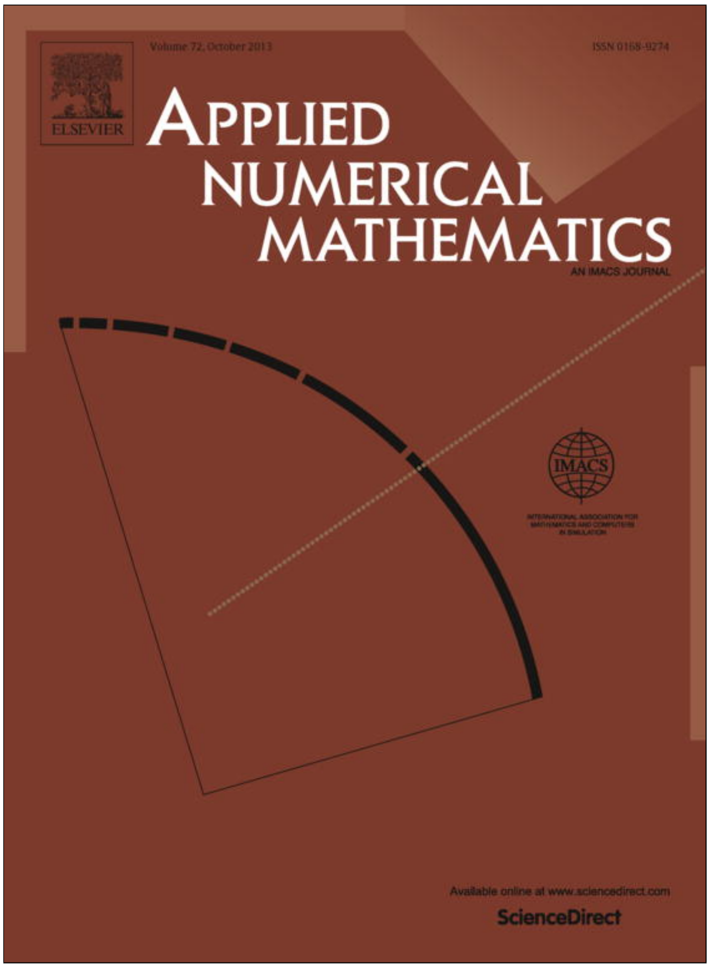
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Another view for a posteriori error estimates for variational inequalities of the second kind

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ABSTRACT

In this paper, we give another view to understand a posteriori error analysis for finite element solutions of elliptic variational inequalities of the second kind. This point of view makes it simpler to derive reliable error estimators in solving variational inequalities of the second kind from the theory for related linear variational equations. Reliable residual-based and gradient recovery-based estimators are deduced. Efficiency of the estimators is also proved.

1. Introduction

Adaptive finite element methods based on a posteriori error estimates are an active research field. Many error estimators can be classified as residual type or recovery type. Various residual quantities are used to capture lost information going from u to u_h, such as residual of the equation, residual from derivative discontinuity and so on. In a gradient recovery error estimator, $\|G_h u_h - \nabla u_h\|$ is used to approximate $\|\nabla u - \nabla u_h\|$, where a gradient recovery operator G_h is applied to the numerical solution u_h to reconstruct the gradient of the true solution u. The theory of a posteriori error estimation is well established for linear equations, and we refer the reader to [1,2,16].

It is more difficult to develop a posteriori error estimators for variational inequalities (VIs) due to the inequality feature. Nevertheless, numerous papers can be found on a posteriori error estimation of finite element methods for obstacle problems, which is a representative elliptic variational inequality (EVI) of the first kind, e.g., [3,10,13–15,17]. For VIs of the second kind, in [4–7], the authors studied a posteriori error estimates and established a framework through the duality theory, but the sharper estimation of one term in the lower bound is still an open problem, i.e., the efficiency was not completely proved. In [8], Braess demonstrated that a posteriori error estimators for finite element solutions of the obstacle problem can be easily derived by applying a posteriori error estimators for a related linear elliptic problem. In this paper, we extend the ideas therein to give another look at a posteriori error analysis for VIs of the second kind. Moreover, we accomplish the proof for the efficiency of the error estimators.

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We take a steady state frictional contact problem as an example to illustrate the derivation process of a posteriori error estimators. The ideas and techniques presented here for this model problem can be extended to other VIs of the second kind.

A frictional contact problem. Let $\Omega \subset \mathbb{R}^d$ ($d \geqslant 1$) be a bounded domain with Lipschitz boundary Γ, Γ_1 a relatively closed subset of Γ, and $\Gamma_2 = \Gamma \backslash \Gamma_1$. Assume $f \in L^2(\Omega)$, and $g > 0$ is a constant. Then a frictional contact problem is to find $u \in V = \{v \in H^1(\Omega)\colon\ v = 0 \text{ a.e. on } \Gamma_1\}$ such that

$$a(u, v-u) + j(v) - j(u) \geqslant (f, v-u)_\Omega \quad \forall v \in V, \tag{1.1}$$

where $(\cdot,\cdot)_\Omega$ denotes the L^2 inner product in the domain Ω and

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u v \, dx,$$

$$j(v) = \int_{\Gamma_2} g|v| \, ds.$$

It was proved ([12, Theorem 5.3], [11]) that this problem has a unique solution $u \in V$, and there exists a unique Lagrange multiplier $\lambda \in L^\infty(\Gamma_2)$ such that

$$a(u, v) + \int_{\Gamma_2} g\lambda v \, ds = (f, v)_\Omega \quad \forall v \in V, \tag{1.2}$$

$$|\lambda| \leqslant 1, \quad \lambda\, u = |u| \quad \text{a.e. on } \Gamma_2. \tag{1.3}$$

It follows from (1.2) and (1.3) that the solution u of (1.1) is the weak solution of the boundary value problem

$$-\Delta u + u = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \Gamma_1,$$

$$\left|\frac{\partial u}{\partial n}\right| \leqslant g, \qquad \frac{\partial u}{\partial n} u + g|u| = 0 \quad \text{on } \Gamma_2,$$

where n is the unit outward normal vector. For any $v \in V$, set

$$\ell(v) = (f, v)_\Omega - \int_{\Gamma_2} g\lambda v \, ds.$$

Then (1.2) becomes

$$a(u, v) = \ell(v) \quad \forall v \in V. \tag{1.4}$$

For a Lipschitz subdomain $\omega \subset \Omega$, let

$$\|v\|_{1,\omega}^2 := a_\omega(v, v) = \int_\omega \left(|\nabla v|^2 + v^2\right) dx.$$

For a measurable subset $\gamma \subset \partial\omega \cap \Gamma_2$, define

$$|\lambda|_{*,\gamma} := \sup\left\{ \int_\gamma g\,\lambda\, v \, ds\colon\ v \in H^1(\omega),\ \|v\|_{1,\omega} = 1 \right\}. \tag{1.5}$$

The subscript γ and ω are omitted if $\gamma = \Gamma_2$ and $\omega = \Omega$. We have

$$|\lambda|_{*,\gamma} = \|w\|_{1,\omega}, \tag{1.6}$$

where $w \in H^1(\omega)$ is the solution of the auxiliary equation

$$a_\omega(w, v) = \int_\gamma g\,\lambda\, v \, ds \quad \forall v \in H^1(\omega). \tag{1.7}$$

The relation (1.6) is proved as follows. First,

$$\int_\gamma g\lambda v \, ds = a_\omega(w, v) \leqslant \|w\|_{1,\omega} \|v\|_{1,\omega}.$$

Thus,

$$|\lambda|_{*,\gamma} = \sup_{0\neq v\in H^1(\omega)} \int_\gamma g\,\lambda\, v\, ds/\|v\|_{1,\omega} \leqslant \|w\|_{1,\omega}.$$

Letting $v = w$ in (1.7), we have

$$\|w\|_{1,\omega} = \int_\gamma g\,\lambda\, w\, ds/\|w\|_{1,\omega} \leqslant |\lambda|_{*,\gamma}.$$

We introduce a family of finite element spaces $V_h \subset V$ corresponding to partitions $\mathcal{T}_h$ of $\overline{\Omega}$ into triangular or tetrahedral elements (other kinds of elements, such as quadrilateral elements, or hexahedral or pentahedral elements, can be considered as well). The partitions $\mathcal{T}_h$ are compatible with the decomposition of Γ into Γ_1 and Γ_2. Then the finite element method for the VI (1.1) is: Find $u_h \in V_h$ such that

$$a(u_h, v_h - u_h) + j(v_h) - j(u_h) \geqslant (f, v_h - u_h)_\Omega \quad \forall v_h \in V_h. \tag{1.8}$$

Similar to the continuous problem, the discrete problem has a unique solution $u_h \in V_h$ and there exists a unique Lagrange multiplier $\lambda_h \in L^\infty(\Gamma_2)$ such that ([6,12])

$$a(u_h, v_h) + \int_{\Gamma_2} g\,\lambda_h v_h\, ds = (f, v_h)_\Omega \quad \forall v_h \in V_h, \tag{1.9}$$

$$|\lambda_h| \leqslant 1, \qquad \lambda_h u_h = |u_h| \quad \text{a.e. on } \Gamma_2. \tag{1.10}$$

For any $v_h \in V_h$, let

$$\ell_h(v_h) = (f, v_h)_\Omega - \int_{\Gamma_2} g\,\lambda_h v_h\, ds.$$

From Hahn–Banach extension theorem, the bounded linear functional ℓ_h, originally defined on V_h, can be extended to a bounded linear functional on V with the norm preserved. Then (1.9) becomes

$$a(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_h. \tag{1.11}$$

Obviously, u_h is also the finite element approximation of the solution $z \in V$ of the linear problem:

$$a(z, v) = \ell_h(v) \quad \forall v \in V, \tag{1.12}$$

which is the weak formulation of the boundary value problem

$$\begin{aligned} &-\Delta z + z = f \quad \text{in } \Omega,\\ &z = 0 \quad \text{on } \Gamma_1,\\ &\frac{\partial z}{\partial n} = -g\lambda_h \quad \text{on } \Gamma_2. \end{aligned} \tag{1.13}$$

Now we present the process to derive a posteriori error estimators for the finite element method (1.8). From (1.4) and (1.12), for all $v \in V$, we have

$$\begin{aligned} a(u_h - u, v) &= a(u_h - z, v) + a(z - u, v)\\ &= a(u_h - z, v) + \ell_h(v) - \ell(v)\\ &= a(u_h - z, v) + \int_{\Gamma_2} g(\lambda - \lambda_h) v\, ds. \end{aligned}$$

Take $v = u_h - u$ in the above relation. Note that by (1.3) and (1.10), we have

$$\begin{aligned} \int_{\Gamma_2} g(\lambda - \lambda_h) v\, ds &= \int_{\Gamma_2} g\,\lambda u_h\, ds - \int_{\Gamma_2} g\,\lambda u\, ds - \int_{\Gamma_2} g\,\lambda_h u_h\, ds + \int_{\Gamma_2} g\,\lambda_h u\, ds\\ &\leqslant \int_{\Gamma_2} g|u_h|\, ds - \int_{\Gamma_2} g|u|\, ds - \int_{\Gamma_2} g|u_h|\, ds + \int_{\Gamma_2} g\,|u|\, ds\\ &= 0. \end{aligned}$$

Then we obtain

$$\|u_h-u\|_1^2=a(u_h-u,u_h-u)\leqslant a(u_h-z,u_h-u)\leqslant\|u_h-z\|_1\|u_h-u\|_1. \tag{1.14}$$

Therefore,

$$\|u_h-u\|_1\leqslant\|u_h-z\|_1. \tag{1.15}$$

Recalling (1.6), we have

$$|\lambda-\lambda_h|_*=\|u-z\|_1\leqslant\|u-u_h\|_1+\|u_h-z\|_1\leqslant 2\|u_h-z\|_1.$$

We summarize the above results in the following theorem.

Theorem 1.1. *Let u and z be the solutions of the problem* (1.1) *and* (1.12), *and let u_h be the finite element solution of u. Then,*

$$\|u_h-u\|_1+|\lambda-\lambda_h|_*\leqslant 3\|u_h-z\|_1.$$

This result is the starting point for derivation of a posteriori error estimators, when combined with the standard results [1] on error estimators for the term $\|u_h-z\|_1$. Based on this observation, we discuss residual type error estimators and gradient recovery type error estimators in the next two sections. Reliability and efficiency are proved for both types of error estimators.

2. Residual type error estimators

First, we introduce some notations. Given a bounded set $D\subset\mathbb{R}^d$ and a positive integer m, $H^m(D)$ is the usual Sobolev space with the corresponding norm $\|\cdot\|_{m,D}$ and semi-norm $|\cdot|_{m,D}$, which are abbreviated by $\|\cdot\|_m$ and $|\cdot|_m$, respectively, when D coincides with Ω. For convenience, we rewrite $\|\cdot\|_{0,D}$ as $\|\cdot\|_D$. We assume Ω is a polyhedral domain and denote by $\{\mathcal{T}_h\}_h$ a family of partitions of $\overline{\Omega}$. For a partition $\mathcal{T}_h$, denote all the edges of $\mathcal{T}_h$ by $\mathcal{E}_h$, and $\mathcal{E}_h^i=\mathcal{E}_h\backslash\Gamma$, $\mathcal{E}_{h,\Gamma_2}=\mathcal{E}_h\cap\Gamma_2$. Let $h_K=\operatorname{diam}(K)$ for $K\in\mathcal{T}_h$ and $h_e=\operatorname{diam}(e)$ for $e\in\mathcal{E}_h$. For any element $K\in\mathcal{T}_h$, define the patch set $\omega_K:=\cup\{T\in\mathcal{T}_h,\ T\cap K\neq\emptyset\}$, and for any edge e shared by two elements K and $\widetilde{K}$, define $\omega_e:=K\cup\widetilde{K}$. For a given element $K\in\mathcal{T}_h$, $\mathcal{N}(K)$ and $\mathcal{E}(K)$ denote the sets of the nodes of K and sides of K, respectively; n_K denotes the unit outward normal vector to the boundary of K and n_e a unit vector on e. Throughout the paper, C denotes a generic positive constant independent of the element size, which may take different values at different occurrences.

Define the interior residuals and edge-based jumps

$$R_K:=-\Delta u_h+u_h-f\quad\text{for each } K\in\mathcal{T}_h,$$

$$R_e:=\begin{cases}\left[\frac{\partial u_h}{\partial n}\right] & \text{if } e\in\mathcal{E}_h^i,\\ \frac{\partial u_h}{\partial n}+g\lambda_h & \text{if } e\in\mathcal{E}_{h,\Gamma_2}.\end{cases}$$

Here $[\frac{\partial u_h}{\partial n}]=\nabla u_h|_K\cdot n_K+\nabla u_h|_{\widetilde{K}}\cdot n_{\widetilde{K}}$ represents the discontinuity of the gradient of u_h across the edge e shared by the neighboring elements K and $\widetilde{K}$. They lead to the local estimators

$$\eta_{R,K}=\left(h_k^2\|R_K\|_K^2+\frac{1}{2}\sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_h^i}h_e\|R_e\|_e^2+\sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_{h,\Gamma_2}}h_e\|R_e\|_e^2\right)^{1/2}\quad\text{for any } K\in\mathcal{T}_h. \tag{2.1}$$

It follows from [1] that the residual-type a posteriori error estimator for the elliptic equation (1.14) satisfies

$$\|u_h-z\|_1\leqslant C\left(\sum_{K\in\mathcal{T}_h}\eta_{R,K}^2\right)^{1/2}.$$

Hence, we have the following theorem.

Theorem 2.1. *Let $u\in V$ and $u_h\in V_h$ be the solutions of the problems* (1.1) *and* (1.8). *Then,*

$$\|u_h-u\|_1+|\lambda-\lambda_h|_*\leqslant C\eta_R,\qquad \eta_R^2:=\sum_{K\in\mathcal{T}_h}\eta_{R,K}^2.$$

Now we turn to consider lower bounds with residual error estimators. This can be achieved by following the standard argument for lower bounds with residual error estimators for linear elliptic problems, see [1, pp. 28–32]. Define

$$a_K(u,v)=\int_K(\nabla u\cdot\nabla v+uv)\,dx,$$

so that for $u, v \in H^1(\Omega)$,

$$a(u,v) = \sum_{K \in \mathcal{T}_h} a_K(u,v).$$

For any $v \in V$, by integration by parts, we have

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} a_K(u_h - u, v) &= \sum_{K \in \mathcal{T}_h} a_K(u_h - z, v) + a(z - u, v) \\ &= \sum_{K \in \mathcal{T}_h} \int_K R_K v \, dx + \sum_{e \in \mathcal{E}_h^i \cup \mathcal{E}_{h,\Gamma_2}} \int_e R_e v \, ds + \sum_{e \in \mathcal{E}_{h,\Gamma_2}} \int_e g(\lambda - \lambda_h) v \, ds. \end{aligned} \tag{2.2}$$

We will use the bubble functions. For each $K \in \mathcal{T}_h$, let λ_1, λ_2 and λ_3 be the barycentric coordinates on K. Then the interior bubble function φ_K is defined by

$$\varphi_K = 27\lambda_1\lambda_2\lambda_3,$$

and the three edge bubble functions are given by

$$\tau_1 = 4\lambda_2\lambda_3, \qquad \tau_2 = 4\lambda_1\lambda_3, \qquad \tau_3 = 4\lambda_1\lambda_2.$$

We recall some properties of the bubble functions [1, Theorems 2.2 and 2.3].

Lemma 2.2. *For each $K \in \mathcal{T}_h$, $e \in \mathcal{E}(K)$, let φ_K and τ_e be the corresponding interior and edge bubble functions. Let $P(K) \subset H^1(K)$ and $P(e) \subset H^1(e)$ be finite-dimensional spaces of functions defined on K and e. Then there exists a constant C, independent of h_K, such that for all $v \in P(K)$,*

$$C^{-1}\|v\|_K^2 \leqslant \int_K \varphi_K v^2 \, dx \leqslant C\|v\|_K^2, \tag{2.3}$$

$$C^{-1}\|v\|_K \leqslant \|\varphi_K v\|_K + h_K|\varphi_K v|_{1,K} \leqslant C\|v\|_K, \tag{2.4}$$

$$C^{-1}\|v\|_e^2 \leqslant \int_e \tau_e v^2 \, ds \leqslant C\|v\|_e^2, \tag{2.5}$$

$$h_K^{-1/2}\|\tau_e v\|_K + h_K^{1/2}|\tau_e v|_{1,K} \leqslant C\|v\|_e. \tag{2.6}$$

For each $K \in \mathcal{T}_h$, φ_K and τ_e are respectively the interior and edge bubble functions on K or $e \in \mathcal{E}_h^i \cup \mathcal{E}_{h,\Gamma_2}$, and $\overline{R}_K$ is an approximation to the interior residual R_K from a suitable finite element space containing u_h and Δu_h. In (2.2), choosing $v = \overline{R}_K\varphi_K$ on element K and using an argument similar to that in [1, pp. 28–32], we obtain

$$\|R_K\|_K \leqslant C\big(\|R_K - \overline{R}_K\|_K + h_K^{-1}\|u_h - u\|_{1,K}\big).$$

For $e \in \mathcal{E}_h^i$, let $\overline{R}_e$ be an approximation to the jump R_e from a suitable finite-dimensional space and let $v = \overline{R}_e\tau_e$ in (2.2). We have

$$\|R_e\|_e \leqslant C\big(h_e^{-1/2}\|u_h - u\|_{1,\omega_e} + h_e^{1/2}\|R_K - \overline{R}_K\|_{\omega_e} + \|R_e - \overline{R}_e\|_e\big).$$

For $e \in \mathcal{E}_{h,\Gamma_2}$, we obtain

$$a_{\omega_e}(u_h - u, \overline{R}_e\tau_e) = \int_{\omega_e} R_K\overline{R}_e\tau_e \, dx + \int_e R_e\overline{R}_e\tau_e \, ds + \int_e g(\lambda - \lambda_h)\overline{R}_e\tau_e \, ds.$$

Therefore,

$$\int_e \overline{R}_e^2\tau_e \, ds = \int_e \overline{R}_e(\overline{R}_e - R_e)\tau_e \, ds + a_{\omega_e}(u_h - u, \overline{R}_e\tau_e) - \int_{\omega_e} R_K\overline{R}_e\tau_e \, dx - \int_e g(\lambda - \lambda_h)\overline{R}_e\tau_e \, ds.$$

Applying Lemma 2.2, we bound the terms in the above relation as follows:

$$\int_e \overline{R}_e^2\tau_e \, ds \geqslant C^{-1}\|\overline{R}_e\|_e^2,$$

$$\int_e \overline{R}_e(\overline{R}_e - R_e)\tau_e \, ds \leqslant \|\overline{R}_e\tau_e\|_e\|\overline{R}_e - R_e\|_e \leqslant C\|\overline{R}_e\|_e\|\overline{R}_e - R_e\|_e,$$

$$a_{\omega_e}(u_h - u, \overline{R}_e\tau_e) \leqslant \|u_h - u\|_{1,\omega_e}\|\overline{R}_e\tau_e\|_{1,\omega_e} \leqslant Ch_e^{-1/2}\|u_h - u\|_{1,\omega_e}\|\overline{R}_e\|_e,$$

$$\int\limits_{\omega_e} R_K\overline{R}_e\tau_e\,dx \leqslant \|R_K\|_{\omega_e}\|\overline{R}_e\tau_e\|_{\omega_e} \leqslant Ch_e^{1/2}\|R_K\|_{\omega_e}\|\overline{R}_e\|_e,$$

$$\int\limits_e g(\lambda - \lambda_h)\overline{R}_e\tau_e\,ds \leqslant |\lambda - \lambda_h|_{*,e}\|\overline{R}_e\tau_e\|_{1,\omega_e} \leqslant Ch_e^{-1/2}|\lambda - \lambda_h|_{*,e}\|\overline{R}_e\|_e.$$

Hence,

$$\begin{aligned}\|R_e\|_e &\leqslant \|\overline{R}_e\|_e + \|R_e - \overline{R}_e\|_e \\ &\leqslant C\big(h_e^{-1/2}\|u_h - u\|_{1,\omega_e} + h_e^{-1/2}\|\lambda - \lambda_h\|_* + h_e^{1/2}\|R_K - \overline{R}_K\|_{\omega_e} + \|R_e - \overline{R}_e\|_e\big).\end{aligned} \tag{2.7}$$

Note that $\Delta u_h + u_h$ in K and $\partial u_h/\partial n_e$ on e are polynomials. Hence, the terms $\|R_K - \overline{R}_K\|_K$ and $\|R_e - \overline{R}_e\|_e$ can be replaced by $\|f - \bar{f}\|_K$ and $\|\lambda_h - \bar{\lambda}_h\|_e$, with discontinuous piecewise polynomial approximations $\bar{f}$ and $\bar{\lambda}_h$. Then we obtain the efficiency bound of the local error indicator $\eta_{R,K}$ (see also [5,6]).

Theorem 2.3. *Let u and u_h be the solutions of (1.1) and (1.8), respectively, and let $\eta_{R,K}$ be the estimator (2.1). Then*

$$\eta_{R,K}^2 \leqslant C\left(\|u - u_h\|_{1,\omega_K}^2 + |\lambda - \lambda_h|_{*,e}^2 + h_K^2\|f - \bar{f}\|_{\omega_K}^2 + \sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_{h,\Gamma_2}} h_e\|\lambda_h - \bar{\lambda}_h\|_e^2\right). \tag{2.8}$$

Due to the inequality nature of the variational inequalities, in the efficiency bound (2.8) of $\eta_{R,K}$, there is a term involving λ and λ_h. In [5,6], because of the presence of this term, the efficiency of the estimators was not proved completely. From Theorem 2.1, we see that the involvement of the term $|\lambda - \lambda_h|_{*,e}$ in the bound (2.8) is very natural. This comment is also valid for the case of gradient recovery type error estimators.

3. Gradient recovery type error estimators

In this section, we study a gradient recovery type error estimator for the linear finite element solution of the frictional contact problem (1.1). Some additional notations are needed in this section. We denote by $\mathcal{N}_h$ the set of nodes of $\mathcal{T}_h$, and $\mathcal{N}_{h,0}$ is the set of free nodes, i.e., those nodes that do not lie on Γ_1. Let $\mathcal{N}_v \subset \mathcal{N}_h$ be the set of the element vertices of the partition $\mathcal{T}_h$, $\mathcal{N}_{v,\Gamma_1} \subset \mathcal{N}_v$ the subset of the element vertices lying on Γ_1, $\mathcal{N}_{v,i} \subset \mathcal{N}_v$ the set of the interior vertices, and $\mathcal{N}_{v,0} = \mathcal{N}_v \cap \mathcal{N}_{h,0}$. Let $\{\varphi_a : a \in \mathcal{N}_v\}$ denote the nodal basis functions of the linear elements for all the vertices. Define an equivalence relation

$$\xi(a) := \begin{cases} a, & \text{if } a \in \mathcal{N}_{v,0}; \\ b,\ b \in \mathcal{N}_{v,i} \text{ and } \exists K \in \mathcal{T}_h,\ \text{s.t. } a, b \in K & \text{if } a \in \mathcal{N}_{v,\Gamma_1}. \end{cases}$$

Then we can classify the set of vertices $\mathcal{N}_v$ into card($\mathcal{N}_{v,0}$) classes of equivalence, that is, $I(a) = \{\tilde{a} \in \mathcal{N}_v : \xi(\tilde{a}) = a\}$ for each node $a \in \mathcal{N}_{v,0}$. We set

$$\psi_a = \sum_{\tilde{a}\in I(a)} \varphi_{\tilde{a}} \quad \text{for every node } a \in \mathcal{N}_{v,0}.$$

Note that $\{\psi_a\colon a \in \mathcal{N}_{v,0}\}$ is a partition of unity. Let $\widetilde{K}_a = \operatorname{supp}(\psi_a)$ and $h_a = \operatorname{diam}(\widetilde{K}_a)$. For a given $v \in L^1(\Omega)$, let

$$v_a = \frac{\int_{\widetilde{K}} v\psi_a\,dx}{\int_{\widetilde{K}} \varphi_a\,dx}, \quad a \in \mathcal{N}_{v,0}.$$

Then a Clément type interpolation operator $\Pi_h : V \to V_h$ is defined as follows:

$$\Pi_h v = \sum_{a\in\mathcal{N}_{v,0}} v_a\varphi_a.$$

The next theorem summarizes some basic estimates for Π_h. Its proof can be found in [9].

Theorem 3.1. *There exists an h-independent positive constant C such that for all $v \in V$ and $f \in L^2(\Omega)$,*

$$
\begin{aligned}
&|v-\Pi_h v|_{1,\Omega}^2 \leqslant C|v|_{1,\Omega}^2,\\
&\int_\Omega f(v-\Pi_h v)\,dx \leqslant C|v|_{1,\Omega}\Bigl(\sum_{a\in\mathcal{N}_{v,0}} h_a^2 \min_{f_a\in\mathbb{R}}\|f-f_a\|_{0,\widetilde{K}_a}^2\Bigr)^{1/2},\\
&\sum_{K\in\mathcal{T}_h}\bigl\|h_K^{-1}(v-\Pi_h v)\bigr\|_K^2 \leqslant C|v|_{1,\Omega}^2,\\
&\sum_{e\in\mathcal{E}_h}\bigl\|h_e^{-1/2}(v-\Pi_h v)\bigr\|_e^2 \leqslant C|v|_{1,\Omega}^2.
\end{aligned}
$$

There are many types of gradient recovery operators G_h. For $G_h u_h$ to be a good approximation of the true gradient ∇u, a set of sufficient conditions can be found in [1, Lemma 4.5]. Consider a gradient recovery operator $G_h : V_h \to (V_h)^d$ defined as follows:

$$
G_h v_h(x) = \sum_{a\in\mathcal{N}_v} G_h v_h(a)\varphi_a(x), \qquad G_h v_h(a) = \frac{1}{|\widetilde{K}_a|}\int_{\widetilde{K}_a}\nabla v_h\,dx.
$$

From (1.11) and (1.12), we get the Galerkin orthogonality

$$
a(u_h - z, v_h) = 0 \quad \forall v_h \in V_h.
$$

Using the above equation, for any $v \in V$, we get

$$
\begin{aligned}
a(u_h - z, v) &= a(u_h - z, v - \Pi_h v)\\
&= I_0 + \int_\Omega G_h u_h\cdot\nabla(v-\Pi_h v)\,dx + \int_\Omega u_h(v-\Pi_h v)\,dx - a(z, v-\Pi_h v)
\end{aligned}
$$

where

$$
I_0 = \int_\Omega (\nabla u_h - G_h u_h)\cdot\nabla(v-\Pi_h v)\,dx \leqslant C\|\nabla u_h - G_h u_h\|_\Omega |v|_{1,\Omega}.
$$

Perform element-wise integration by parts,

$$
\begin{aligned}
\int_\Omega G_h u_h\cdot\nabla(v-\Pi_h v)\,dx &= \sum_{K\in\mathcal{T}_h}\int_K G_h u_h\cdot\nabla(v-\Pi_h v)\,dx\\
&= \sum_{K\in\mathcal{T}_h}\int_K -\operatorname{div}(G_h u_h)(v-\Pi_h v)\,dx + \sum_{K\in\mathcal{T}_h}\int_{\mathcal{E}(K)}(G_h u_h\cdot n_K)(v-\Pi_h v)\,ds.
\end{aligned}
$$

The first summation is rewritten as

$$
\sum_{K\in\mathcal{T}_h}\int_K \operatorname{div}(\nabla u_h - G_h u_h)(v-\Pi_h v)\,dx + \sum_{K\in\mathcal{T}_h}\int_K -\Delta u_h\,(v-\Pi_h v)\,dx.
$$

Since $G_h u_h$ is continuous across the element boundaries,

$$
\sum_{K\in\mathcal{T}_h}\int_{\mathcal{E}(K)}(G_h u_h\cdot n_K)(v-\Pi_h v)\,ds = \sum_{e\in\mathcal{E}_{h,\Gamma_2}}\int_e (G_h u_h\cdot n_e)(v-\Pi_h v)\,ds.
$$

Applying the above relations and using Eq. (1.12), we obtain that

$$
a(u_h - z, v) = I_0 + I_1 + I_2 + I_3, \tag{3.1}
$$

where

$$
\begin{aligned}
I_1 &= \sum_{K\in\mathcal{T}_h}\int_K \operatorname{div}(\nabla u_h - G_h u_h)(v-\Pi_h v)\,dx,\\
I_2 &= \sum_{K\in\mathcal{T}_h}\int_K(-\Delta u_h + u_h - f)(v-\Pi_h v)\,dx = \sum_{K\in\mathcal{T}_h}\int_K R_K(v-\Pi_h v)\,dx,\\
I_3 &= \sum_{e\in\mathcal{E}_{h,\Gamma_2}}\int_e (G_h u_h\cdot n_e + g\lambda_h)(v-\Pi_h v)\,ds.
\end{aligned}
$$

It is shown in [6] (see also [4]) that

$$I_1 \leqslant C|v|_{1,\Omega}\left(\sum_{K\in\mathcal{T}_h}\|\nabla u_h - G_h u_h\|_K^2\right)^{1/2},$$

$$I_2 \leqslant C|v|_{1,\Omega}\sum_{a\in\mathcal{N}_{v,0}}\left(h_a^4\|\nabla u_h\|_{\widetilde{K}_a}^2 + h_a^2\min_{f_a\in\mathbb{R}}\|f-f_a\|_{\widetilde{K}_a}^2\right),$$

$$I_3 \leqslant C|v|_{1,\Omega}\left(\sum_{e\in\mathcal{E}_{h,\Gamma_2}} h_e\|G_h u_h\cdot n_e + g\lambda_h\|_e^2\right)^{1/2}.$$

Taking $v = u_h - z$ in (3.1) and recalling Theorem 1.1, we obtain the next result.

Theorem 3.2. *Let u and u_h be the solutions of (1.1) and (1.8), respectively. Then*

$$\|u-u_h\|_1^2 + |\lambda-\lambda_h|_*^2 \leqslant C\eta_G^2 + C\sum_{a\in\mathcal{N}_{h,0}}\left(h_a^4\|\nabla u_h\|_{\widetilde{K}_a}^2 + h_a^2\min_{f_a\in\mathbb{R}}\|f-f_a\|_{\widetilde{K}_a}^2\right), \tag{3.2}$$

where

$$\eta_G^2 = \sum_{K\in\mathcal{T}_h}\eta_{G,K}^2, \qquad \eta_{G,K}^2 = \|\nabla u_h - G_h u_h\|_K^2 + \sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_{h,\Gamma_2}} h_e\|G_h u_h\cdot n_e + g\lambda_h\|_e^2. \tag{3.3}$$

The term $(\sum_{a\in\mathcal{N}_{h,0}} h_a^4\|\nabla u_h\|_{\widetilde{K}_a}^2)^{1/2}$ is bounded by $O(h^2)$, and $(\sum_{a\in\mathcal{N}_{h,0}} h_a^2\min_{f_a\in\mathbb{R}}\|f-f_a\|_{\widetilde{K}_a}^2)^{1/2}$ is bounded by $o(h)$ if $f\in L^2(\Omega)$ or bounded by $O(h^2)$ if $f\in H^1(\Omega)$ (see [6]), which guarantees the reliability of estimator η_G.

For the efficiency of the estimator, it is shown in Lemma 3.1 in [6] that

$$\eta_{G,K}^2 \leqslant C\left(\sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_{h,\Gamma_2}} h_e\|R_e\|_e^2 + \sum_{e'\in\mathcal{E}_{\omega_K}} h_{e'}\|R_{e'}\|_{e'}^2\right),$$

where $\mathcal{E}_{\omega_K}$ denotes the set of inner sides of the patch ω_K corresponding to the element K. Using the relation (2.7), we obtain the following results.

Theorem 3.3. *Let u and u_h be the solutions of (1.1) and (1.8), respectively, and let $\eta_{G,K}$ be the estimator (3.3). Then*

$$\eta_{G,K}^2 \leqslant C\left(\|u-u_h\|_{1,\omega_K}^2 + |\lambda-\lambda_h|_{*,e}^2 + h_K^2\|f-\bar{f}\|_{\omega_K}^2 + \sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_{h,\Gamma_2}} h_e\|\lambda_h-\bar{\lambda}_h\|_e^2\right). \tag{3.4}$$

This theorem shows the efficiency of gradient recovery type estimators $\eta_{G,K}$. The inequality (3.4) is comparable to (2.8).

4. Summary

In this paper, we study a posteriori error estimation of finite element methods for a frictional contact problem, and establish a compact framework to derive reliable residual type and gradient recovery type error estimators by applying a posteriori error analysis for a related linear elliptic problem. Furthermore, we prove the efficiency of the error estimators, which was an open problem stated in [6]. This framework can also be used to derive reliable and efficient a posteriori error estimators for other variational inequalities of the second kind.

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