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Optimal Order Error Estimates for Discontinuous Galerkin Methods for the Wave Equation

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Abstract In this paper, we derive optimal order error estimates for spatially semi-discrete and fully discrete schemes to numerically solve the second-order wave equation. The numerical schemes are constructed with the discontinuous Galerkin (DG) discretization for the spatial variable and the centered second-order finite difference approximation for the temporal variable. Under appropriate regularity assumptions on the solution, the schemes are shown to enjoy the optimal order error bounds in terms of both the spatial mesh-size and the time-step. In Grote and Schötzau (J Sci Comput 40:257–272, 2009), a fully discrete DG scheme is studied with an explicit finite difference temporal discretization where a CFL condition is required on the mesh-size and the time-step, and optimal order error estimates are derived in the $L^2(\Omega)$-norm. In comparison, for our fully discrete DG schemes, we do not require a CFL condition on the mesh-size and the time-step, and our optimal order error estimates are derived for the $H^1(\Omega)$-like norm and the $L^2(\Omega)$ norm. Numerical simulation results are reported to illustrate theoretically predicted convergence orders in the $H^1(\Omega)$ and $L^2(\Omega)$ norms.

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1 Introduction

The wave equation plays a fundamental role in the study of acoustic, elastic, electromagnetic, seismic waves. Early references on error estimates of numerical methods based on finite element approximations in the spatial domain include [4,17]. The numerical solution of the wave equation has attracted steady interest in the research community; cf. e.g., [7,15] for mixed finite element methods for the wave equation, [1] on a correction function method to solve the wave equation subject to interface jump conditions, [9] for a scheme that is fourth order accurate in both space and time and is constructed by the method of difference potential at each time step.

In [18], a symmetric interior penalty discontinuous Galerkin (DG) method is applied to solve the wave equation and optimal order error estimates are derived for the spatially semi-discrete scheme. In the sequel [19], a fully discrete scheme for the wave equation is studied, where the symmetric interior penalty DG method is used for the spatial discretization and the centered second-order finite difference approximation is used for the temporal discretization. An optimal order error estimate for the fully discrete solution is derived. In this paper, we analyze spatially semi-discrete schemes and fully discrete schemes for the wave equation, with DG methods for the spatial discretization and the centered second-order finite difference approximation the temporal discretization. The DG method in the first scheme is the same as the symmetric interior penalty DG method considered in [18,19]. For the discrete schemes, we derive optimal order error estimates. Unlike in [19] where a CFL condition is needed for the fully discrete scheme there, we do not require such a CFL condition on the mesh-size and the time-step. Moreover, our optimal order error estimates are derived in the $H^1(\Omega)$-like norm and the $L^2(\Omega)$-norm.

The DG methods discretize differential equations element by element, and neighboring elements are connected through numerical traces, which makes the methods locally conservative. To enforce the continuity requirement on the solution, a penalty term is added in the DG formulation. Due to the locality of the discretization, the DG methods are ideal for parallel computing. In addition, since no inter-element continuity is required in the function spaces, DG methods can handle easily general meshes with hanging nodes and elements of different shapes [3,13]. In the past four decades, DG methods have been developed in solving a variety of problems, such as convection–diffusion equations [11,26], hyperbolic equations [8,18,19,22], Navier–Stokes equations [5,12], Hamilton–Jacobi equations [23,24], the radiative transfer equation [20], variational inequalities [27–31], and so on.

We turn to describe the wave equation problem to be considered in this paper. To simplify the notation, we only discuss the case of two dimensional spatial domains and comment that all the analysis extends to the case of three dimensional spatial domains. Let $\Omega \subset \mathbb{R}^2$ be an open bounded connected domain with a Lipschitz boundary Γ. Let $I = (0, T)$ be the time interval of interest. As in [18,19], the initial-boundary value problem of the wave equation we consider is to find $u(\mathrm{x}, t)$ such that

$$\partial_t^2 u - \nabla\cdot(b\,\nabla u) = f \quad \text{in } \Omega \times I, \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma \times I, \tag{1.2}$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega, \tag{1.3}$$

$$\partial_t u|_{t=0} = v_0 \quad \text{in } \Omega, \tag{1.4}$$

where b, f, u_0 and v_0 are given functions. Throughout the paper, we assume b is a smooth function and for two positive constants $b_{\min}$ and $b_{\max}$,

$$b_{\min} \leq b(\mathrm{x}) \leq b_{\max}, \quad \mathrm{x} \in \overline{\Omega}, \tag{1.5}$$

and moreover,

$$f \in L^2(I; L^2(\Omega)), \quad u_0 \in H_0^1(\Omega), \quad v_0 \in L^2(\Omega). \tag{1.6}$$

The standard weak formulation of the problem (1.1)–(1.4) is to find $u \in L^2(I; H_0^1(\Omega))$ with $\partial_t u \in L^2(I; L^2(\Omega))$ and $\partial_t^2 u \in L^2(I; H^{-1}(\Omega))$ such that

$$\langle \partial_t^2 u, v \rangle + a(u, v) = (f, v) \quad \forall\, v \in H_0^1(\Omega), \text{ a.e. in } I, \tag{1.7}$$

and

$$u|_{t=0} = u_0, \; \partial_t u|_{t=0} = v_0 \quad \text{a.e. in } \Omega. \tag{1.8}$$

Here, the time derivatives are understood in the distributional sense, $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$, and $a(\cdot, \cdot)$ is the bilinear form defined by

$$a(u, v) = \int_\Omega b\, \nabla u \cdot \nabla v \, dx, \quad u, v \in H_0^1(\Omega). \tag{1.9}$$

It is known [25, Chapter III] that the weak formulation has a unique solution and furthermore, $u \in C(\overline{I}; H_0^1(\Omega))$ and $\partial_t u \in C(\overline{I}; L^2(\Omega))$.

In this paper, we study four spatially semi-discrete schemes and their fully discrete counterparts to solve the initial-boundary value problem (1.1)–(1.4) of the wave equation. We use DG discretization in space and finite difference discretization in time. The paper is organized as follows. In Sect. 2 we introduce preliminary materials: notation, DG bilinear forms, as well as $L^2(\Omega)$-projection and Galerkin projection. In Sect. 3, we introduce the spatially semi-discrete schemes and derive optimal order error estimates for the semi-discrete solutions. In Sect. 4, we introduce the fully discrete schemes and present optimal order error estimates for the fully discrete solutions in both $H^1(\Omega)$ and $L^2(\Omega)$ norms. Finally in Sect. 5, we report simulation results on a numerical example to show the numerical convergence orders that match the theoretical predictions.

2 Preliminaries

2.1 Basic Notation

As in [19], we assume Ω is a convex polygon. Let $\{\mathcal{T}_h\}_h$ be a regular family of quasi-uniform triangulations of $\overline{\Omega}$. Corresponding to a triangulation $\mathcal{T}_h$ in the family, denote $h_K = \mathrm{diam}(K)$ and $h = \max\{h_K : K \in \mathcal{T}_h\}$. Let $\mathcal{E}_h$ be the collection of all the edges of $\mathcal{T}_h$, $\mathcal{E}_h^i$ the set of all interior edges, and $\mathcal{E}_h^\partial = \mathcal{E}_h \backslash \mathcal{E}_h^i$ the set of all the edges on the boundary Γ. Consider an edge e shared by two elements K^+ and K^-. Let $\boldsymbol{n}^\pm = \boldsymbol{n}|_{\partial K^\pm}$ be the unit outward normal vector on $\partial K^\pm$. For a piecewise smooth scalar-valued function v, let $v^\pm = v|_{\partial K^\pm}$, and define the average $\{v\}$ and the jump $[\![v]\!]$ on $\mathcal{E}_h^i$ as follows:

$$\{v\} = \frac{1}{2}(v^+ + v^-), \quad [\![v]\!] = v^+ \boldsymbol{n}^+ + v^- \boldsymbol{n}^- \quad \text{on } e \in \mathcal{E}_h^i.$$

For a piecewise smooth vector-valued function $\boldsymbol{w}$, we denote $\boldsymbol{w}^{\pm} = \boldsymbol{w}|_{\partial K^{\pm}}$ and set the average $\{\boldsymbol{w}\}$ and the jump $[\boldsymbol{w}]$ on $\mathcal{E}_h^i$ as follows:

$$\{\boldsymbol{w}\} = \frac{1}{2}(\boldsymbol{w}^+ + \boldsymbol{w}^-), \quad [\boldsymbol{w}] = \boldsymbol{w}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{w}^- \cdot \boldsymbol{n}^- \quad \text{on } e \in \mathcal{E}_h^i.$$

On boundary edges, we define

$$[\![v]\!] = v\boldsymbol{n}, \quad \{\boldsymbol{w}\} = \boldsymbol{w} \quad \text{on } e \in \mathcal{E}_h^\partial,$$

where $\boldsymbol{n}$ is the unit outward normal vector on Γ. A straightforward calculation shows that

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} v\boldsymbol{w} \cdot \boldsymbol{n}_K \, ds = \int_{\mathcal{E}_h} [\![v]\!] \cdot \{\boldsymbol{w}\} \, ds + \int_{\mathcal{E}_h^i} \{v\}[\boldsymbol{w}] \, ds. \tag{2.1}$$

Let $p \geq 1$ be a positive integer that will be used as the local polynomial degree of the DG formulations. We introduce the following discontinuous finite element spaces:

$$V^h = \{v^h \in L^2(\Omega) : \ v^h|_K \in P_p(K) \ \forall K \in \mathcal{T}_h\},$$
$$\boldsymbol{W}^h = \{\boldsymbol{w}^h \in [L^2(\Omega)]^2 : \ \boldsymbol{w}^h|_K \in [P_p(K)]^2 \ \forall K \in \mathcal{T}_h\}.$$

We will need lifting operators $r : [L^2(\mathcal{E}_h)]^2 \rightarrow \boldsymbol{W}^h$, $r_\partial : [L^2(\mathcal{E}_h^\partial)]^2 \rightarrow \boldsymbol{W}^h$, and $r_e : [L^2(e)]^2 \rightarrow \boldsymbol{W}^h$ defined by relations [3]

$$\int_\Omega r(\boldsymbol{q}) \cdot \boldsymbol{w}^h dx = -\int_{\mathcal{E}_h} \boldsymbol{q} \cdot \{\boldsymbol{w}^h\} \, ds,$$
$$\int_\Omega r_\partial(\boldsymbol{q}) \cdot \boldsymbol{w}^h dx = -\int_{\mathcal{E}_h^\partial} \boldsymbol{q} \cdot \{\boldsymbol{w}^h\} \, ds,$$
$$\int_\Omega r_e(\boldsymbol{q}) \cdot \boldsymbol{w}^h dx = -\int_e \boldsymbol{q} \cdot \{\boldsymbol{w}^h\} \, ds$$

for all $\boldsymbol{w}^h \in \boldsymbol{W}^h$.

For a non-negative integer m, we will use the notation $\|\cdot\|_m$ for the $H^m(\Omega)$-norm. In particular, $\|\cdot\|_0$ denotes the $L^2(\Omega)$-norm.

2.2 DG Bilinear Forms and Their Properties

We introduce four choices of the DG bilinear form $a_h = a_h^{(j)}$, $1 \leq j \leq 4$, for the approximation of the bilinear form (1.9). The first DG bilinear form corresponds to the interior penalty (IP) method [2,16,32], which is also the bilinear form of the DG method studied in [19]:

$$a_h^{(1)}(u, v) = \int_\Omega b \, \nabla_h u \cdot \nabla_h v \, dx - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{b \, \nabla_h v\} \, ds - \int_{\mathcal{E}_h} \{b \, \nabla_h u\} \cdot [\![v]\!] \, ds$$
$$+ \int_{\mathcal{E}_h} b \, \eta [\![u]\!] \cdot [\![v]\!] \, ds,$$

where the penalty weighting function $\eta : \mathcal{E}_h \rightarrow \mathbb{R}$ is given by $\eta_e h_e^{-1}$ on each $e \in \mathcal{E}_h$ with η_e being a positive number. Here, the broken gradient operator ∇_h is defined piecewise by the relation $\nabla_h v = \nabla v$ on any element $K \in \mathcal{T}_h$. The second bilinear form corresponds to the method in [6],

$$a_h^{(2)}(u, v) = \int_\Omega b \, \nabla_h u \cdot \nabla_h v \, dx - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{b \, \nabla_h v\} \, ds - \int_{\mathcal{E}_h} \{b \, \nabla_h u\} \cdot [\![v]\!] \, ds$$

$$+\sum_{e\in\mathcal{E}_h}\int_\Omega b\,\eta_e r_e([\![u]\!])\cdot r_e([\![v]\!])\,dx.$$

The third bilinear form corresponds to the method in [10],

$$a_h^{(3)}(u,v)=\int_\Omega b\,\nabla_h u\cdot\nabla_h v\,dx-\int_{\mathcal{E}_h}[\![u]\!]\cdot\{b\,\nabla_h v\}\,ds-\int_{\mathcal{E}_h}\{b\,\nabla_h u\}\cdot[\![v]\!]\,ds$$
$$+\int_\Omega b\,r([\![u]\!])\cdot r([\![v]\!])\,dx+\sum_{e\in\mathcal{E}_h}\int_\Omega b\,\eta_e r_e([\![u]\!])\cdot r_e([\![v]\!])\,dx.$$

The fourth bilinear form corresponds to the simplified local DG (LDG) method in [14],

$$a_h^{(4)}(u,v)=\int_\Omega b\,\nabla_h u\cdot\nabla_h v\,dx-\int_{\mathcal{E}_h}[\![u]\!]\cdot\{b\,\nabla_h v\}\,ds-\int_{\mathcal{E}_h}\{b\,\nabla_h u\}\cdot[\![v]\!]\,ds$$
$$+\int_\Omega b\,r([\![u]\!])\cdot r([\![v]\!])dx+\int_{\mathcal{E}_h} b\,\eta[\![u]\!]\cdot[\![v]\!]\,ds.$$

For all the four DG bilinear forms, we have the consistency in the sense of the following result.

Lemma 2.1 (Consistency) *Assume the solution of* (1.1)–(1.4) *has the regularity property* $u\in L^2(0,T;H^2(\Omega))$. *Then for all DG methods* $a_h(w,v)=a_h^{(j)}(w,v)$, $j=1,\ldots,4$, *we have for a.e.* $t\in[0,T]$,

$$\left(\partial_t^2 u, v^h\right)+a_h(u,v^h)=(f,v^h)\quad\forall\, v^h\in V^h. \tag{2.2}$$

Proof Under the assumption $u\in L^2(0,T;H^2(\Omega))$, we know that for a.e. $t\in[0,T]$, $u(t)\in H^2(\Omega)$. Thus for a.e. $t\in[0,T]$, $[\![u(t)]\!]=0$, $\{u(t)\}=u(t)$, $[\nabla u(t)]=0$, and $\{\nabla u(t)\}=\nabla u(t)$ on any interior edge. Thus, for a.e. $t\in[0,T]$, for any $v^h\in V^h$, we perform integration by parts to get

$$\begin{aligned}a_h(u,v^h)&=\int_\Omega b\,\nabla_h u\cdot\nabla_h v^h\,dx-\int_{\mathcal{E}_h}\{b\,\nabla_h u\}\cdot[\![v^h]\!]\,ds\\&=\int_\Omega -\nabla\cdot(b\,\nabla u)\,v^h dx+\sum_{K\in\mathcal{T}_h}\int_{\partial K} b\,\nabla u\cdot\boldsymbol{n}_K\,v^h ds-\int_{\mathcal{E}_h}\{b\,\nabla_h u\}\cdot[\![v^h]\!]\,ds\\&=\int_\Omega -\nabla\cdot(b\,\nabla u)\,v^h dx.\end{aligned}$$

Then,

$$(\partial_t^2 u, v^h)+a_h(u,v^h)=\int_\Omega(\partial_t^2 u-\nabla\cdot(b\,\nabla u))\,v^h dx=\int_\Omega f\,v^h dx,$$

i.e., (2.2) holds. □

As in [3,27], let $V(h)=V^h+H^2(\Omega)\cap H_0^1(\Omega)$ and define a norm for $v\in V(h)$ by the relation

$$\|v\|_h^2=\sum_{K\in\mathcal{T}_h}|v|_{1,K}^2+\sum_{K\in\mathcal{T}_h}h_K^2|v|_{2,K}^2+\sum_{e\in\mathcal{E}_h}h_e^{-1}\|[\![v]\!]\|_{0,e}^2. \tag{2.3}$$

Following [3,27], we have boundedness and stability of the DG bilinear forms.

Lemma 2.2 (Boundedness) *There exists a constant c such that for* $a_h = a_h^{(j)}$, $1 \le j \le 4$,

$$|a_h(u, v)| \le c\, \|u\|_h \|v\|_h \quad \forall\, u, v \in V(h).$$

Lemma 2.3 (Stability) *Let* $\eta_0 = \inf_e \eta_e$ *be sufficiently large for* $j = 1, 2$, *and* $\eta_0 > 0$ *for* $j = 3, 4$. *Then there exists a constant c such that for* $a_h = a_h^{(j)}$, $1 \le j \le 4$,

$$a_h(v, v) \ge c\, \|v\|_h^2 \quad \forall\, v \in V^h.$$

In the rest of the paper, we will assume the conditions stated in Lemma 2.3 are satisfied. Due to the boundedness and stability of the bilinear form $a_h(u, v)$, it makes sense to use the notation

$$\|v^h\|_{a_h} = \Big[a_h(v^h, v^h)\Big]^{1/2}, \quad v^h \in V^h,$$

and this defines a norm that is equivalent to the norm $\|v^h\|_h$. In addition, we notice that

$$\|w\|_h \le c\, \|w\|_2 \quad \forall\, w \in H^2(\Omega). \tag{2.4}$$

2.3 L^2-Projection and Galerkin Projection

We first introduce the $L^2(\Omega)$-projection P^h onto V^h by

$$P^h w \in V^h, \quad (P^h w, v^h) = (w, v^h) \quad \forall\, v^h \in V^h$$

for $w \in L^2(\Omega)$. From [18, Lemmas 4.6 and 4.7], we can derive the following error bounds for the $L^2(\Omega)$-projection:

$$\|w - P^h w\|_0 \le c\, h^{\min\{m, p+1\}} \|w\|_m \quad \forall\, w \in H^m(\Omega) \quad (m \ge 0), \tag{2.5}$$

$$\|w - P^h w\|_h \le c\, h^{\min\{m-1, p\}} \|w\|_m \quad \forall\, w \in H^m(\Omega) \quad (m \ge 2). \tag{2.6}$$

From the equivalence of the two norms $\|\cdot\|_{a_h}$ and $\|\cdot\|_h$, we infer from (2.6) that

$$\|w - P^h w\|_{a_h} \le c\, h^{\min\{m-1, p\}} \|w\|_m \quad \forall\, w \in H^m(\Omega). \tag{2.7}$$

Denote by $\Pi^h : V \to V^h$ the Galerkin projection defined by

$$\Pi^h w \in V^h, \quad a_h(\Pi^h w, v^h) = a_h(w, v^h) \quad \forall\, v^h \in V^h$$

for $w \in V$. From [19, Lemma 4.1],

$$\|w - \Pi^h w\|_0 \le c\, h^{\min\{m, p+1\}} \|w\|_m \quad \forall\, w \in H^m(\Omega), \tag{2.8}$$

$$\|w - \Pi^h w\|_h \le c\, h^{\min\{m-1, p\}} \|w\|_m \quad \forall\, w \in H^m(\Omega). \tag{2.9}$$

Note that the convexity assumption of the domain Ω is used in proving (2.8). We infer from (2.9) that

$$\|w - \Pi^h w\|_{a_h} \le c\, h^{\min\{m-1, p\}} \|w\|_m \quad \forall\, w \in H^m(\Omega). \tag{2.10}$$

Moreover, if $u \in C^l(\overline{I}; H^m(\Omega))$, $l \ge 0$, then by [19, Lemma 4.2],

$$\|\partial_t^l (u(t) - \Pi^h u(t))\|_0 \le c\, h^{\min\{m, p+1\}} \|\partial_t^l u(t)\|_m, \quad t \in \overline{I}. \tag{2.11}$$

Similarly,

$$\|\partial_t^l (u(t) - \Pi^h u(t))\|_{a_h} \le c\, h^{\min\{m-1, p\}} \|\partial_t^l u(t)\|_m, \quad t \in \overline{I}. \tag{2.12}$$

3 Spatially Semi-discrete Schemes

For the spatial discretization, we adopt DG methods. Let a_h be one of the four DG bilinear forms $a_h^{(j)}$, $1 \le j \le 4$. Then the spatially semi-discrete scheme for the problem (1.1)–(1.4) is to find $u^h : [0, T] \to V^h$ such that

$$(\partial_t^2 u^h, v^h) + a_h(u^h, v^h) = (f, v^h) \quad \forall\, v^h \in V^h, \tag{3.1}$$

$$u^h(0) = P^h u_0, \tag{3.2}$$

$$\partial_t u^h(0) = P^h v_0. \tag{3.3}$$

Recall that P^h stands for the $L^2(\Omega)$-projection operator onto V^h.

We provide in the next result optimal order error bounds for the semi-discrete solutions.

Theorem 3.1 *Let u and u^h be the solutions of (1.1)–(1.4) and (3.1)–(3.3), respectively. Assume*

$$u \in C(\overline{I}; H^{p+1}(\Omega)), \quad \partial_t u \in L^2(I; H^{p+1}(\Omega)), \quad \partial_t^2 u \in L^2(I; H^p(\Omega)).$$

Then for the spatially semi-discrete schemes with $j = 1, \dots, 4$, we have

$$\max_{0 \le t \le T} \left(\|\partial_t u(t) - \partial_t u^h(t)\|_0 + \|u(t) - u^h(t)\|_h \right) \le c\, h^p, \tag{3.4}$$

where the constant c depends on $\|u\|_{C(\overline{I}; H^{p+1}(\Omega))}$, $\|\partial_t u\|_{L^2(I; H^{p+1}(\Omega))}$, and $\|\partial_t^2 u\|_{L^2(I; H^p(\Omega))}$.

Proof First we notice that the solution regularity assumption implies $\partial_t u \in C(\overline{I}; H^p(\Omega))$. Write the error $e = u - u^h$ as

$$e = e_1 + e_2$$

where

$$e_1 = u - \Pi^h u, \quad e_2 = \Pi^h u - u^h.$$

Then, for $t \in \overline{I}$,

$$\|\partial_t u(t) - \partial_t u^h(t)\|_0 + \|u(t) - u^h(t)\|_h \le \|\partial_t e_1(t)\|_0 + \|e_1(t)\|_h + \|\partial_t e_2(t)\|_0 + \|e_2(t)\|_h. \tag{3.5}$$

By (2.11) and (2.9), we have that for $t \in \overline{I}$,

$$\|\partial_t e_1(t)\|_0 \le c\, h^p \|\partial_t u(t)\|_p, \tag{3.6}$$

$$\|e_1(t)\|_h \le c\, h^p \|u(t)\|_{p+1}. \tag{3.7}$$

Subtract (3.1) from (2.2),

$$(\partial_t^2 e, v^h) + a_h(e, v^h) = 0 \quad \forall\, v^h \in V^h,$$

i.e.,

$$\left(\partial_t^2 e_2, v^h\right) + a_h(e_2, v^h) = -\left(\partial_t^2 e_1, v^h\right) - a_h(e_1, v^h) \quad \forall\, v^h \in V^h. \tag{3.8}$$

We take $v^h = \partial_t e_2$ in (3.8),

$$\left(\partial_t^2 e_2, \partial_t e_2\right) + a_h(e_2, \partial_t e_2) = -\left(\partial_t^2 e_1, \partial_t e_2\right) - a_h(e_1, \partial_t e_2),$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}\left[\|\partial_t e_2\|_0^2 + a_h(e_2, e_2)\right] = -\left(\partial_t^2 e_1, \partial_t e_2\right) - a_h(e_1, \partial_t e_2). \tag{3.9}$$

Integrate (3.9) from 0 to t,

$$\frac{1}{2}\left[\|\partial_t e_2(t)\|_0^2 + \|e_2(t)\|_{a_h}^2\right] = \frac{1}{2}\left[\|\partial_t e_2(0)\|_0^2 + \|e_2(0)\|_{a_h}^2\right] - \int_0^t (\partial_t^2 e_1(s), \partial_t e_2(s))\, ds - \int_0^t a_h(e_1(s), \partial_t e_2(s))\, ds. \tag{3.10}$$

Note that

$$-\int_0^t (\partial_t^2 e_1(s), \partial_t e_2(s))\, ds \le \int_0^t \|\partial_t^2 e_1(s)\|_0 \|\partial_t e_2(s))\|_0 ds \le \frac{1}{2}\int_0^t \|\partial_t^2 e_1(s)\|_0^2 ds + \frac{1}{2}\int_0^t \|\partial_t e_2(s))\|_0^2 ds,$$

and

$$\begin{aligned}-\int_0^t a_h(e_1(s), \partial_t e_2(s))\, ds &= -a_h(e_1(t), e_2(t)) + a_h(e_1(0), e_2(0)) \\ &\quad + \int_0^t a_h(\partial_t e_1(s), e_2(s))\, ds \\ &\le \frac{1}{4}\|e_2(t)\|_{a_h}^2 + \|e_1(t)\|_{a_h}^2 + c\,\|e_1(0)\|_{a_h}\|e_2(0)\|_{a_h} \\ &\quad + \frac{1}{2}\int_0^t \|\partial_t e_1(s)\|_{a_h}^2 ds + \frac{1}{2}\int_0^t \|e_2(s)\|_{a_h}^2 ds.\end{aligned}$$

Thus, from (3.10),

$$\begin{aligned}\|\partial_t e_2(t)\|_0^2 + \|e_2(t)\|_{a_h}^2 &\le c\left(\|\partial_t e_2(0)\|_0^2 + \|e_2(0)\|_{a_h}^2 + \|e_1(0)\|_{a_h}^2 + \|e_1(t)\|_{a_h}^2\right) \\ &\quad + c\int_0^t \left(\|\partial_t^2 e_1(s)\|_0^2 + \|\partial_t e_2(s)\|_0^2 + \|\partial_t e_1(s)\|_{a_h}^2 + \|e_2(s)\|_{a_h}^2\right) ds.\end{aligned}$$

Apply Gronwall's inequality to obtain

$$\begin{aligned}&\max_{0\le t\le T}\left[\|\partial_t e_2(t)\|_0^2 + \|e_2(t)\|_{a_h}^2\right] \\ &\quad \le c\left(\|\partial_t e_2(0)\|_0^2 + \|e_2(0)\|_{a_h}^2 + \|e_1(0)\|_{a_h}^2 + \max_{0\le t\le T}\|e_1(t)\|_{a_h}^2\right) \\ &\qquad + c\int_0^t \left(\|\partial_t^2 e_1(s)\|_0^2 + \|\partial_t e_1(s)\|_{a_h}^2\right) ds.\end{aligned} \tag{3.11}$$

Notice that the assumption $u \in C(\overline{I}; H^{p+1}(\Omega))$ implies $u_0 \in H^{p+1}(\Omega)$. By (2.10),

$$\|e_1(0)\|_{a_h} = \|u_0 - \Pi^h u_0\|_{a_h} \le c\, h^p \|u_0\|_{p+1}. \tag{3.12}$$

Write

$$e_2(0) = \Pi^h u_0 - P^h u_0 = \left(\Pi^h u_0 - u_0\right) + \left(u_0 - P^h u_0\right).$$

By (2.10),

$$\|\Pi^h u_0 - u_0\|_{a_h} \le c\, h^p \|u_0\|_{p+1}.$$

By (2.7),

$$\|u_0 - P^h u_0\|_{a_h} \le c\, h^p \|u_0\|_{p+1}.$$

Thus,

$$\|e_2(0)\|_{a_h} \le c\, h^p \|u_0\|_{p+1}. \tag{3.13}$$

By the assumptions $\partial_t u \in L^2(I; H^{p+1}(\Omega))$ and $\partial_t^2 u \in L^2(I; H^p(\Omega))$, we know $\partial_t u \in C(\overline{I}; H^p(\Omega))$. In particular, this implies $v_0 \in H^p(\Omega)$. From (2.8),

$$\|\Pi^h v_0 - v_0\|_0 \le c\, h^p \|v_0\|_p,$$

and from (2.5),

$$\|v_0 - P^h v_0\|_0 \le c\, h^p \|v_0\|_p.$$

Then by

$$\partial_t e_2(0) = \left(\Pi^h v_0 - v_0\right) + \left(v_0 - P^h v_0\right),$$

we find that

$$\|\partial_t e_2(0)\|_0 \le c\, h^p \|v_0\|_p. \tag{3.14}$$

Again by the solution regularity assumptions, from (3.7), we have

$$\|e_1(t)\|_{a_h} \le c\, h^p \|u(t)\|_{p+1}, \tag{3.15}$$

from (2.11),

$$\|\partial_t^2 e_1(s)\|_0 \le c\, h^p \|\partial_t^2 u(s)\|_p, \tag{3.16}$$

and from (2.12),

$$\|\partial_t e_1(s)\|_{a_h} \le c\, h^p \|\partial_t u(s)\|_{p+1}. \tag{3.17}$$

Applying (3.12)–(3.17) in (3.11), we obtain

$$\|\partial_t e_2(t)\|_0 + \|e_2(t)\|_h \le c\, h^p. \tag{3.18}$$

Combining (3.5), (3.6), (3.7) and (3.18), we obtain the error bound (3.4). □

4 Fully Discrete Scheme

For fully discrete schemes, we need to approximate the temporal derivatives corresponding to a partition of the time interval: $0 = t_0 < t_1 < \cdots < t_N = T$. For simplicity in notation, in the following, we only discuss the case of evenly spaced nodes $t_n = nk$, $0 \le n \le N$, with a uniform time step $k = T/N$. For a continuous function v, we use the notation $v_n = v(t_n)$. We use the symbols γ_k, δ_k, and d_k defined by

$$\gamma_k v_n = \frac{v_{n+1} + v_{n-1}}{2}, \quad \delta_k v_n = \frac{v_{n+1} - v_{n-1}}{2k} \quad \text{and} \quad d_k v_n = \frac{v_{n+1} - 2v_n + v_{n-1}}{k^2}.$$

Let $a_h(\cdot,\cdot)$ be one of the bilinear forms $a_h^{(j)}(\cdot,\cdot)$ with $j = 1, \ldots, 4$, and recall that P^h denotes the $L^2(\Omega)$-projection onto the space V^h. Then a fully discrete approximation of (1.1)–(1.4) is to find $\{u_n^{hk}\}_{n=0}^N \subset V^h$ such that for $1 \le n \le N-1$,

$$\left(d_k u_n^{hk}, v^h\right) + a_h\left(\gamma_k u_n^{hk}, v^h\right) = \left(f_n, v^h\right) \quad \forall\, v^h \in V^h, \tag{4.1}$$

and

$$u_0^{hk} = P^h u_0, \tag{4.2}$$

$$u_1^{hk} = u_0^{hk} + k\,P^h v_0 + \frac{k^2}{2}\,\tilde{u}_0^h, \tag{4.3}$$

where

$$\tilde{u}_0^h \in V^h, \quad (\tilde{u}_0^h, v^h) = (f_0, v^h) - a_h(u_0, v^h) \quad \forall\, v^h \in V^h. \tag{4.4}$$

Now we present optimal order error estimates for the fully discrete schemes.

Theorem 4.1 *Let u and u^{hk} be the solutions of (1.1)–(1.4) and (4.1)–(4.4), respectively. Assume $u \in C^2(\overline{I}; H^{p+1}(\Omega))$, $\partial_t^3 u \in C(\overline{I}; L^2(\Omega)) \cap L^2(I; H^2(\Omega))$, $\partial_t^4 u \in L^2(I; L^2(\Omega))$. Then the following error bound holds*

$$\max_{0\le n\le N-1} k^{-1}\|\left(u_{n+1} - u_{n+1}^{hk}\right) - \left(u_n - u_n^{hk}\right)\|_0 + \max_{0\le n\le N-1} \|u_n - u_n^{hk}\|_h \le c\left(h^p + k^2\right) \tag{4.5}$$

for a constant c depending on $\|u\|_{C^2(\overline{I};H^{p+1}(\Omega))}$, $\|\partial_t^3 u\|_{C(\overline{I};L^2(\Omega))}$, $\|\partial_t^3 u\|_{L^2(I;H^2(\Omega))}$, and $\|\partial_t^4 u\|_{L^2(I;L^2(\Omega))}$.

Proof We write the error $e_n = u_n - u_n^{hk}$ as

$$e_n = e_{1,n} + e_{2,n},$$

where

$$e_{1,n} = u_n - \Pi^h u_n, \quad e_{2,n} = \Pi^h u_n - u_n^{hk}.$$

Then, for $0 \le n \le N-1$,

$$k^{-1}\|\left(u_{n+1} - u_{n+1}^{hk}\right) - \left(u_n - u_n^{hk}\right)\|_0 \le k^{-1}\|e_{1,n+1} - e_{1,n}\|_0 + k^{-1}\|e_{2,n+1} - e_{2,n}\|_0, \tag{4.6}$$

$$\|u_n - u_n^{hk}\|_h \le \|e_{1,n}\|_h + \|e_{2,n}\|_h. \tag{4.7}$$

Since

$$e_{1,n+1} - e_{1,n} = \left(I - \Pi^h\right)(u_{n+1} - u_n) = \left(I - \Pi^h\right)\int_{t_n}^{t_{n+1}} \partial_t u(\cdot, s)\,ds,$$

we have

$$\|e_{1,n+1} - e_{1,n}\|_0 \le \int_{t_n}^{t_{n+1}} \left\|\left(I - \Pi^h\right)\partial_t u(\cdot, s)\right\|_0 ds.$$

Apply (2.8),

$$\|e_{1,n+1} - e_{1,n}\|_0 \le \int_{t_n}^{t_{n+1}} c\,h^p\,\|\partial_t u(\cdot, s)\|_p ds \le c\,k\,h^p\,\|\partial_t u\|_{C(\overline{I};H^p(\Omega))}. \tag{4.8}$$

By (2.9),

$$\|e_{1,n}\|_h \le c\,h^p\,\|u_n\|_{p+1} \le c\,h^p\,\|u\|_{C(\overline{I};H^{p+1}(\Omega))}. \tag{4.9}$$

Consider the quantity

$$A_n = (d_k e_{2,n}, \delta_k e_{2,n}) + a_h(\gamma_k e_{2,n}, \delta_k e_{2,n}).$$

We have

$$A_n = \frac{1}{2\,k^3}\left(\|e_{2,n+1}-e_{2,n}\|_0^2 - \|e_{2,n}-e_{2,n-1}\|_0^2\right) + \frac{1}{4\,k}\left(\|e_{2,n+1}\|_{a_h}^2 - \|e_{2,n-1}\|_{a_h}^2\right). \tag{4.10}$$

Take $v^h = \delta_k e_{2,n}$ in (4.1),

$$\left(d_k u_n^{hk}, \delta_k e_{2,n}\right) + a_h\left(\gamma_k u_n^{hk}, \delta_k e_{2,n}\right) = (f_n, \delta_k e_{2,n}). \tag{4.11}$$

From the consistency Eq. (2.2),

$$\left(\partial_t^2 u_n, \delta_k e_{2,n}\right) + a_h(u_n, \delta_k e_{2,n}) = (f_n, \delta_k e_{2,n}). \tag{4.12}$$

We subtract (4.11) from (4.12) to obtain

$$\left(\partial_t^2 u_n - d_k u_n^{hk}, \delta_k e_{2,n}\right) + a_h(u_n - \gamma_k u_n^{hk}, \delta_k e_{2,n}) = 0.$$

So for $n = 1, \ldots, N-1$,

$$\begin{aligned}(d_k e_{2,n}, \delta_k e_{2,n}) + a_h(\gamma_k e_{2,n}, \delta_k e_{2,n}) &= \left(d_k \Pi^h u_n - \partial_t^2 u_n, \delta_k e_{2,n}\right)\\ &\quad + a_h\left(\gamma_k \Pi^h u_n - u_n, \delta_k e_{2,n}\right).\end{aligned} \tag{4.13}$$

Denote

$$\xi_n = d_k \Pi^h u_n - \partial_t^2 u_n, \quad \eta_n = \gamma_k \Pi^h u_n - u_n, \quad 1 \le n \le N-1. \tag{4.14}$$

Then (4.13) can be rewritten as

$$\begin{aligned}A_n &= (\xi_n, \delta_k e_{2,n}) + a_h(\eta_n, \delta_k e_{2,n})\\ &= \frac{1}{2\,k}\left[(\xi_n, e_{2,n+1}) - (\xi_n, e_{2,n-1})\right] + \frac{1}{2\,k}\left[a_h(\eta_n, e_{2,n+1}) - a_h(\eta_n, e_{2,n-1})\right].\end{aligned} \tag{4.15}$$

Combining (4.10) and (4.15), and multiplying both sides by $2\,k$, we have

$$\begin{aligned}&\frac{1}{k^2}\left(\|e_{2,n+1}-e_{2,n}\|_0^2 - \|e_{2,n}-e_{2,n-1}\|_0^2\right) + \frac{1}{2}\left(\|e_{2,n+1}\|_{a_h}^2 - \|e_{2,n-1}\|_{a_h}^2\right)\\ &\quad = (\xi_n, e_{2,n+1}) - (\xi_n, e_{2,n-1}) + a_h(\eta_n, e_{2,n+1}) - a_h(\eta_n, e_{2,n-1}).\end{aligned} \tag{4.16}$$

Change n to j in (4.16) and make a summation of the relation for $j = 1, \ldots, n-1$,

$$\begin{aligned}&\frac{1}{k^2}\left(\|e_{2,n}-e_{2,n-1}\|_0^2 - \|e_{2,1}-e_{2,0}\|_0^2\right)\\ &\quad + \frac{1}{2}\left(\|e_{2,n}\|_{a_h}^2 - \|e_{2,0}\|_{a_h}^2 + \|e_{2,n-1}\|_{a_h}^2 - \|e_{2,1}\|_{a_h}^2\right)\\ &\quad = \sum_{j=1}^{n-1}\left[(\xi_j, e_{2,j+1}) - (\xi_j, e_{2,j-1})\right] + \sum_{j=1}^{n-1}\left[a_h(\eta_j, e_{2,j+1}) - a_h(\eta_j, e_{2,j-1})\right].\end{aligned} \tag{4.17}$$

Now

$$\begin{aligned}&\sum_{j=1}^{n-1}\left[(\xi_j, e_{2,j+1}) - (\xi_j, e_{2,j-1})\right]\\ &\quad = \sum_{j=1}^{n-1}\left[(\xi_j, e_{2,j+1} - e_{2,j}) + (\xi_j, e_{2,j} - e_{2,j-1})\right]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=2}^{n} (\xi_{j-1}, e_{2,j} - e_{2,j-1}) + \sum_{j=1}^{n-1} (\xi_j, e_{2,j} - e_{2,j-1}) \\
&= (\xi_{n-1}, e_{2,n} - e_{2,n-1}) + \sum_{j=2}^{n-1} (\xi_{j-1} + \xi_j, e_{2,j} - e_{2,j-1}) + (\xi_1, e_{2,1} - e_{2,0}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^{n-1} \left[a_h(\eta_j, e_{2,j+1}) - a_h(\eta_j, e_{2,j-1}) \right] \\
&\quad = \sum_{j=2}^{n} a_h(\eta_{j-1}, e_{2,j}) - \sum_{j=0}^{n-2} a_h(\eta_{j+1}, e_{2,j}) \\
&\quad = a_h(\eta_{n-1}, e_{2,n}) + a_h(\eta_{n-2}, e_{2,n-1}) \\
&\qquad + \sum_{j=2}^{n-2} a_h(\eta_{j-1} - \eta_{j+1}, e_{2,j}) - a_h(\eta_2, e_{2,1}) - a_h(\eta_1, e_{2,0}).
\end{aligned}
$$

Thus, denoting

$$
M = \max_{0 \le n \le N} \|e_{2,n}\|_{a_h}, \tag{4.18}
$$

we find from (4.17) that

$$
\begin{aligned}
&\frac{1}{k^2} \left(\|e_{2,n} - e_{2,n-1}\|_0^2 - \|e_{2,1} - e_{2,0}\|_0^2 \right) \\
&\qquad + \frac{1}{2} \left(\|e_{2,n}\|_{a_h}^2 - \|e_{2,0}\|_{a_h}^2 + \|e_{2,n-1}\|_{a_h}^2 - \|e_{2,1}\|_{a_h}^2 \right) \\
&\quad = (\xi_{n-1}, e_{2,n} - e_{2,n-1}) + \sum_{j=2}^{n-1} (\xi_{j-1} + \xi_j, e_{2,j} - e_{2,j-1}) + (\xi_1, e_{2,1} - e_{2,0}) \\
&\qquad + a_h(\eta_{n-1}, e_{2,n}) + a_h(\eta_{n-2}, e_{2,n-1}) \\
&\qquad + \sum_{j=2}^{n-2} a_h(\eta_{j-1} - \eta_{j+1}, e_{2,j}) - a_h(\eta_2, e_{2,1}) - a_h(\eta_1, e_{2,0}) \\
&\quad \le \frac{1}{2k^2} \|e_{2,n} - e_{2,n-1}\|_0^2 + \frac{k^2}{2} \|\xi_{n-1}\|_0^2 \\
&\qquad + \left(\sum_{j=2}^{n-1} \|\xi_{j-1} + \xi_j\|_0^2 \right)^{1/2} \left(\sum_{j=2}^{n-1} \|e_{2,j} - e_{2,j-1}\|_0^2 \right)^{1/2} \\
&\qquad + \|\xi_1\|_0 \|e_{2,1} - e_{2,0}\|_0 \\
&\qquad + \left(\|\eta_{n-1}\|_{a_h} + \|\eta_{n-2}\|_{a_h} + \sum_{j=2}^{n-2} \|\eta_{j-1} - \eta_{j+1}\|_{a_h} + \|\eta_2\|_{a_h} + \|\eta_1\|_{a_h} \right) M.
\end{aligned}
$$

Then,

$$\frac{1}{2k^2}\|e_{2,n}-e_{2,n-1}\|_0^2+\frac{1}{2}\|e_{2,n}\|_{a_h}^2$$
$$\leq \frac{1}{k^2}\|e_{2,1}-e_{2,0}\|_0^2+\frac{1}{2}\|e_{2,0}\|_{a_h}^2+\frac{1}{2}\|e_{2,1}\|_{a_h}^2+\frac{k^2}{2}\|\xi_{n-1}\|_0^2+\|\xi_1\|_0\|e_{2,1}-e_{2,0}\|_0$$
$$+\frac{k}{2}\sum_{j=2}^{n-1}\|\xi_{j-1}+\xi_j\|_0^2+k\sum_{j=2}^{n-1}\frac{1}{2k^2}\|e_{2,j}-e_{2,j-1}\|_0^2$$
$$+\left(\|\eta_{n-1}\|_{a_h}+\|\eta_{n-2}\|_{a_h}+\sum_{j=2}^{n-2}\|\eta_{j-1}-\eta_{j+1}\|_{a_h}+\|\eta_2\|_{a_h}+\|\eta_1\|_{a_h}\right)M. \quad (4.19)$$

Apply a discrete Gronwall inequality (cf. e.g., [21, Ch. 7]) to (4.19),

$$\frac{1}{k^2}\|e_{2,n}-e_{2,n-1}\|_0^2+\|e_{2,n}\|_{a_h}^2$$
$$\leq c\left(\frac{1}{k^2}\|e_{2,1}-e_{2,0}\|_0^2+\|e_{2,0}\|_{a_h}^2+\|e_{2,1}\|_{a_h}^2+k^2\max_n\|\xi_n\|_0^2+\|\xi_1\|_0\|e_{2,1}-e_{2,0}\|_0\right.$$
$$\left.+k\sum_{j=2}^{N-1}\|\xi_{j-1}+\xi_j\|_0^2\right)$$
$$+c\left(\|\eta_{N-1}\|_{a_h}+\|\eta_{N-2}\|_{a_h}+\sum_{j=2}^{N-2}\|\eta_{j-1}-\eta_{j+1}\|_{a_h}+\|\eta_2\|_{a_h}+\|\eta_1\|_{a_h}\right)M. \quad (4.20)$$

Then from (4.20),

$$M^2\leq c\left(\frac{1}{k^2}\|e_{2,1}-e_{2,0}\|_0^2+\|e_{2,0}\|_{a_h}^2+\|e_{2,1}\|_{a_h}^2\right.$$
$$\left.+k^2\max_n\|\xi_n\|_0^2+\|\xi_1\|_0\|e_{2,1}-e_{2,0}\|_0+k\sum_{j=2}^{N-1}\|\xi_{j-1}+\xi_j\|_0^2\right)$$
$$+c\left(\|\eta_{N-1}\|_{a_h}+\|\eta_{N-2}\|_{a_h}+\sum_{j=2}^{N-2}\|\eta_{j-1}-\eta_{j+1}\|_{a_h}+\|\eta_2\|_{a_h}+\|\eta_1\|_{a_h}\right)M,$$

and thus,

$$M^2\leq c\left(\frac{1}{k^2}\|e_{2,1}-e_{2,0}\|_0^2+\|e_{2,0}\|_{a_h}^2+\|e_{2,1}\|_{a_h}^2+\|\xi_1\|_0\|e_{2,1}-e_{2,0}\|_0+g(\eta)^2\right)$$
$$+c\left(k^2\max_n\|\xi_n\|_0^2+k\sum_{j=2}^{N-1}\|\xi_{j-1}+\xi_j\|_0^2\right), \quad (4.21)$$

where

$$g(\eta)=\|\eta_{N-1}\|_{a_h}+\|\eta_{N-2}\|_{a_h}+\sum_{j=2}^{N-2}\|\eta_{j-1}-\eta_{j+1}\|_{a_h}+\|\eta_2\|_{a_h}+\|\eta_1\|_{a_h}. \quad (4.22)$$

Use (4.21) in (4.20) to obtain

$$\begin{aligned}\frac{1}{k^2}\|e_{2,n}-e_{2,n-1}\|_0^2 &\\ \leq c\left(\frac{1}{k^2}\|e_{2,1}-e_{2,0}\|_0^2+\|e_{2,0}\|_{a_h}^2+\|e_{2,1}\|_{a_h}^2+\|\xi_1\|_0\|e_{2,1}-e_{2,0}\|_0+g(\eta)^2\right)&\\ +c\left(k^2\max_n\|\xi_n\|_0^2+k\sum_{j=2}^{N-1}\|\xi_{j-1}+\xi_j\|_0^2\right).& \quad (4.23)\end{aligned}$$

Combining (4.21) and (4.23), we obtain

$$\begin{aligned}\max_n\frac{1}{k^2}\|e_{2,n}-e_{2,n-1}\|_0^2+\max_n\|e_{2,n}\|_{a_h}^2 &\\ \leq c\left(\frac{1}{k^2}\|e_{2,1}-e_{2,0}\|_0^2+\|e_{2,0}\|_{a_h}^2+\|e_{2,1}\|_{a_h}^2+\|\xi_1\|_0\|e_{2,1}-e_{2,0}\|_0+g(\eta)^2\right)&\\ +c\left(k^2\max_n\|\xi_n\|_0^2+k\sum_{j=2}^{N-1}\|\xi_{j-1}+\xi_j\|_0^2\right).& \quad (4.24)\end{aligned}$$

Note that

$$e_{2,0}=\Pi^h u_0-u_0^{hk}=\left(\Pi^h u_0-u_0\right)+\left(u_0-P^h u_0\right).$$

Thus,

$$\|e_{2,0}\|_{a_h}\leq\|\Pi^h u_0-u_0\|_{a_h}+\|u_0-P^h u_0\|_{a_h}\leq c\,h^p\|u_0\|_{p+1}. \quad (4.25)$$

Similarly,

$$\|e_{2,1}\|_{a_h}\leq\|\Pi^h u_1-u_1\|_{a_h}+\|u_1-u_1^{hk}\|_{a_h}. \quad (4.26)$$

The first term on the right-hand side of (4.26) can be bounded by applying (2.10),

$$\|\Pi^h u_1-u_1\|_{a_h}\leq c\,h^p\|u_1\|_{p+1}.$$

To bound the second term on the right-hand side of (4.26), we start with the Taylor's formula

$$u_1=u_0+kv_0+\frac{k^2}{2}\partial_t^2 u(0)+\frac{1}{2}\int_0^{t_1}(t_1-s)^2\partial_t^3 u(\cdot,s)\,ds.$$

By a_h-consistency (2.2) and (4.4),

$$(\partial_t^2 u(0)-\tilde{u}_0^h,v^h)=0\quad\forall\,v^h\in V^h.$$

Thus,

$$\tilde{u}_0^h=P^h\partial_t^2 u(0).$$

From the definitions (4.2) and (4.3),

$$\begin{aligned}u_1-u_1^{hk}&=u_0-P^h u_0+k\left(v_0-P^h v_0\right)+\frac{k^2}{2}\left(\partial_t^2 u(0)-\tilde{u}_0^h\right)\\ &\quad+\frac{1}{2}\int_0^{t_1}(t_1-s)^2\partial_t^3 u(\cdot,s)\,ds.\end{aligned}$$

Using properties of the $L^2(\Omega)$-projection P^h onto V^h and (2.7), we deduce that

$$\begin{aligned}\|u_1 - u_1^{hk}\|_{a_h} &\le \|u_0 - P^h u_0\|_{a_h} + k\|v_0 - P^h v_0\|_{a_h} + \frac{k^2}{2}\|\partial_t^2 u(0) - P^h \partial_t^2 u(0)\|_{a_h} \\ &\quad + \frac{k^2}{2}\int_0^{t_1} \|\partial_t^3 u(\cdot, s)\|_{a_h} ds \\ &\le c\, h^p \left(\|u_0\|_{p+1} + k\,\|v_0\|_{p+1} + k^2\|\partial_t^2 u(0)\|_{p+1}\right) \\ &\quad + c\, k^2 \int_0^T \|\partial_t^3 u(\cdot, s)\|_{a_h} ds.\end{aligned}$$

Hence,

$$\begin{aligned}\|e_{2,1}\|_{a_h} &\le c\, h^p \left(\|u_1\|_{p+1} + \|u_0\|_{p+1} + k\,\|v_0\|_{p+1} + k^2\|\partial_t^2 u(0)\|_{p+1}\right) \\ &\quad + c\, k^2 \int_0^T \|\partial_t^3 u(\cdot, s)\|_{a_h} ds.\end{aligned} \tag{4.27}$$

By [19, (35)],

$$\|e_{2,1} - e_{2,0}\|_0 \le c\left(k\, h^{p+1}\|\partial_t u\|_{C(\bar{I};H^{p+1}(\Omega))} + k^3\|\partial_t^3 u\|_{C(\bar{I};L^2(\Omega))}\right). \tag{4.28}$$

By [19, Lemma 4.5],

$$\|\xi_n\|_0 \le c\left(k^{-1}h^{p+1}\int_{t_{n-1}}^{t_{n+1}} \|\partial_t^2 u(\cdot, s)\|_{p+1} ds + k\int_{t_{n-1}}^{t_{n+1}} \|\partial_t^4 u(\cdot, s)\|_0 ds\right). \tag{4.29}$$

Write

$$\begin{aligned}\xi_j + \xi_{j-1} &= \left(d_k \Pi^h u_j - \partial_t^2 u_j\right) + \left(d_k \Pi^h u_{j-1} - \partial_t^2 u_{j-1}\right) \\ &= d_k\left(\Pi^h - I\right) u_j + d_k\left(\Pi^h - I\right) u_{j-1} + \left(d_k u_j - \partial_t^2 u_j\right) \\ &\quad + \left(d_k u_{j-1} - \partial_t^2 u_{j-1}\right).\end{aligned} \tag{4.30}$$

By formulas on page 269 of [19],

$$d_k u_j = \frac{1}{k^2}\int_{t_{j-1}}^{t_{j+1}} \left(k - |s - t_j|\right) \partial_t^2 u(\cdot, s)\, ds, \tag{4.31}$$

$$d_k u_j - \partial_t^2 u_j = \frac{1}{6\,k^2}\int_{t_{j-1}}^{t_{j+1}} \left(k - |s - t_j|\right)^3 \partial_t^4 u(\cdot, s)\, ds. \tag{4.32}$$

So

$$d_k\left(\Pi^h - I\right) u_j = \frac{1}{k^2}\int_{t_{j-1}}^{t_{j+1}} \left(k - |s - t_j|\right)\left(\Pi^h - I\right) \partial_t^2 u(\cdot, s)\, ds$$

and

$$\|d_k\left(\Pi^h - I\right) u_j\|_0 \le \frac{1}{k}\int_{t_{j-1}}^{t_{j+1}} \|\left(\Pi^h - I\right) \partial_t^2 u(\cdot, s)\|_0 ds \le \frac{c\, h^p}{k}\int_{t_{j-1}}^{t_{j+1}} \|\partial_t^2 u(\cdot, s)\|_p ds. \tag{4.33}$$

Similarly,

$$\|d_k\left(\Pi^h - I\right) u_{j-1}\|_0 \le \frac{c\, h^p}{k}\int_{t_{j-2}}^{t_j} \|\partial_t^2 u(\cdot, s)\|_p ds. \tag{4.34}$$

From (4.32),

$$\|d_k u_j - \partial_t^2 u_j\|_0 \le c\, k \int_{t_{j-1}}^{t_{j+1}} \|\partial_t^4 u(\cdot,s)\|_0 ds.$$

Similarly,

$$\|d_k u_{j-1} - \partial_t^2 u_{j-1}\|_0 \le c\, k \int_{t_{j-2}}^{t_j} \|\partial_t^4 u(\cdot,s)\|_0 ds.$$

Thus,

$$\| \left(d_k u_j - \partial_t^2 u_j\right) + \left(d_k u_{j-1} - \partial_t^2 u_{j-1}\right) \|_0 \le c\, k \int_{t_{j-2}}^{t_{j+1}} \|\partial_t^4 u(\cdot,s)\|_0 ds. \tag{4.35}$$

Hence, from (4.30), (4.33), (4.34), and (4.35), we find

$$\|\xi_{j-1} + \xi_j\|_0^2 \le c\, \frac{h^{2p}}{k} \int_{t_{j-2}}^{t_{j+1}} \|\partial_t^2 u(\cdot,s)\|_p^2 ds + c\, k^3 \int_{t_{j-2}}^{t_{j+1}} \|\partial_t^4 u(\cdot,s)\|_0^2 ds.$$

Then,

$$k \sum_{j=2}^{N-1} \|\xi_{j-1} + \xi_j\|_0^2 \le c\, h^{2\,p} \|\partial_t^2 u\|_{L^2(I;H^p(\Omega))}^2 + c\, k^4 \|\partial_t^4 u\|_{L^2(I;L^2(\Omega))}^2. \tag{4.36}$$

We proceed to bound the quantity $g(\eta)$ defined in (4.22). First,

$$\gamma_k u_j - u_j = \frac{1}{2} \int_{t_{j-1}}^{t_{j+1}} (k - |s - t_j|) \partial_t^2 u(\cdot,s)\, ds. \tag{4.37}$$

So,

$$\eta_n = \gamma_k \left(\Pi^h - I\right) u_n + \gamma_k u_n - u_n,$$
$$\|\eta_n\|_{a_h} \le \frac{1}{2} \left[\|(\Pi^h - I) u_{n+1}\|_{a_h} + \|(\Pi^h - I) u_{n-1}\|_{a_h} \right] + \|\gamma_k u_n - u_n\|_{a_h}.$$

Thus,

$$\|\eta_n\|_{a_h} \le c\, h^p \left(\|u_{n+1}\|_{p+1} + \|u_{n-1}\|_{p+1}\right) + c\, k \int_{t_{n-1}}^{t_{n+1}} \|\partial_t^2 u(\cdot,s)\|_{a_h} ds. \tag{4.38}$$

Write

$$\eta_{j+1} - \eta_{j-1} = \gamma_k (\Pi^h - I)(u_{j+1} - u_{j-1}) + \gamma_k (u_{j+1} - u_{j-1}) - (u_{j+1} - u_{j-1}).$$

Note that

$$\begin{aligned} &\|\gamma_k (\Pi^h - I)(u_{j+1} - u_{j-1})\|_{a_h} \\ &\quad \le \frac{1}{2} \left[\|(\Pi^h - I)(u_{j+2} - u_j)\|_{a_h} + \|(\Pi^h - I)(u_j - u_{j-2})\|_{a_h} \right] \\ &\quad \le c\, h^p \int_{t_{j-2}}^{t_{j+2}} \|\partial_t u(\cdot,s)\|_{p+1} ds. \end{aligned} \tag{4.39}$$

From

$$\gamma_k u_{j+1} - u_{j+1} = \frac{1}{2} \int_{t_j}^{t_{j+2}} \left(k - |s - t_{j+1}|\right) \partial_t^2 u(\cdot,s)\, ds,$$

$$\gamma_k u_{j-1} - u_{j-1} = \frac{1}{2}\int_{t_{j-2}}^{t_j} \left(k - |s - t_{j-1}|\right) \partial_t^2 u(\cdot, s)\, ds,$$
$$= \frac{1}{2}\int_{t_j}^{t_{j+2}} \left(k - |s - t_{j+1}|\right) \partial_t^2 u(\cdot, s - 2k)\, ds,$$

we find

$$\begin{aligned}
&\gamma_k(u_{j+1} - u_{j-1}) - (u_{j+1} - u_{j-1})\\
&= \frac{1}{2}\int_{t_j}^{t_{j+2}} \left(k - |s - t_{j+1}|\right) \left(\partial_t^2 u(\cdot, s) - \partial_t^2 u(\cdot, s - 2k)\right) ds\\
&= \frac{1}{2}\int_{t_j}^{t_{j+2}} \left(k - |s - t_{j+1}|\right) \int_{s-2k}^{s} \partial_t^3 u(\cdot, \tau)\, d\tau\, ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\gamma_k(u_{j+1} - u_{j-1}) - (u_{j+1} - u_{j-1})\|_{a_h} &\le c\,k \int_{t_j}^{t_{j+2}} \int_{s-2k}^{s} \|\partial_t^3 u(\cdot, \tau)\|_{a_h} d\tau\, ds\\
&\le c\,k^2 \int_{t_{j-2}}^{t_{j+2}} \|\partial_t^3 u(\cdot, \tau)\|_{a_h} d\tau, \qquad (4.40)
\end{aligned}$$

and then,

$$\|\eta_{j+1} - \eta_{j-1}\|_{a_h} \le c\,h^p \int_{t_{j-2}}^{t_{j+2}} \|\partial_t u(\cdot, s)\|_{p+1} ds + c\,k^2 \int_{t_{j-2}}^{t_{j+2}} \|\partial_t^3 u(\cdot, \tau)\|_{a_h} d\tau. \qquad (4.41)$$

Combining (4.38) and (4.41), we derive

$$\begin{aligned}
&\|\eta_{N-1}\|_{a_h} + \|\eta_{N-2}\|_{a_h} + \sum_{j=2}^{N-2} \|\eta_{j-1} - \eta_{j+1}\|_{a_h} + \|\eta_2\|_{a_h} + \|\eta_1\|_{a_h}\\
&\le c\,h^p \|u\|_{C(\overline{I}; H^{p+1}(\Omega))} + c\,k \int_{t_{n-3}}^{t_n} \|\partial_t^2 u(\cdot, s)\|_{a_h} ds\\
&+ c\,h^p \int_0^T \|\partial_t u(\cdot, s)\|_{p+1} ds + c\,k^2 \int_0^T \|\partial_t^3 u(\cdot, \tau)\|_{a_h} d\tau.
\end{aligned}$$

Hence,

$$\begin{aligned}
g(\eta) \le & c\,h^p \|u\|_{C(\overline{I}; H^{p+1}(\Omega))} + c\,k^2 \|\partial_t^2 u\|_{C(\overline{I}; H^{p+1}(\Omega))}\\
&+ c\,h^p \int_0^T \|\partial_t u(\cdot, s)\|_{p+1} ds + c\,k^2 \int_0^T \|\partial_t^3 u(\cdot, \tau)\|_h d\tau. \qquad (4.42)
\end{aligned}$$

Using (4.6)–(4.9), (4.24)–(4.29), (4.36), (4.42) and noting (2.4), after some simple manipulation, we arrive at the error bound (4.5). □

We proceed to derive an optimal L^2 error estimate for the fully discrete scheme.

Theorem 4.2 *Let u and u^{hk} be the solutions of* (1.1)–(1.4) *and* (4.1)–(4.4)*, respectively. Assume $u \in C^2(\overline{I}; H^{p+1}(\Omega))$, $\partial_t^3 u \in C(\overline{I}; L^2(\Omega))$, and $\partial_t^4 u \in C(\overline{I}; L^2(\Omega))$. Then for some constant c depending on $\|u\|_{C^2(\overline{I}; H^{p+1}(\Omega))}$, $\|\partial_t^3 u\|_{C(\overline{I}; L^2(\Omega))}$, and $\|\partial_t^4 u\|_{C(\overline{I}; L^2(\Omega))}$, we have the following error bound*

$$\max_{0 \le n \le N-1} \|u_n - u_n^{hk}\|_0 \le c\left(h^{p+1} + k^2\right). \qquad (4.43)$$

Proof Similar to the proof of Theorem 4.1, for $0 \leq n \leq N-1$,

$$\|u_n - u_n^{hk}\|_0 = \|u_n - \Pi^h u_n + \Pi^h u_n - u_n^{hk}\|_0 \leq \|e_{1,n}\|_0 + \|e_{2,n}\|_0. \tag{4.44}$$

By (2.8), we have

$$\|e_{1,n}\|_0 \leq c\, h^{p+1} \|u\|_{C(\bar{I}; H^{p+1}(\Omega))}. \tag{4.45}$$

To bound $\|e_{2,n}\|_0$, we introduce the truncation errors due to the time discretization,

$$r_n = d_k u_n - \nabla \cdot (b\, \nabla \gamma_k u_n) - f_n, \quad n = 1, 2, \ldots, N-1.$$

Then

$$r_n = d_k u_n - \nabla \cdot (b\, \nabla \gamma_k u_n) - \left(\partial_t^2 u_n - \nabla \cdot (b\, \nabla u_n)\right) = d_k u_n - \partial_t^2 u_n - \frac{k^2}{2} \nabla \cdot (b\, \nabla d_k u_n)$$

With the regularity assumed in the theorem, we have

$$\|r_n\|_0 \leq c\, k^2 \left(\|\partial_t^4 u\|_{C(\bar{I}; L^2(\Omega))} + \|\partial_t^2 u\|_{C(\bar{I}; H^2(\Omega))}\right). \tag{4.46}$$

From the definition of r_n, we get

$$\left(d_k u_n, v^h\right) + a_h\left(\gamma_k u_n, v^h\right) = \left(f_n, v^h\right) + \left(r_n, v^h\right) \quad \forall\, v^h \in V^h,\; n = 1, 2, \ldots, N-1. \tag{4.47}$$

Subtracting the Eq. (4.1) from the Eq. (4.47), we have

$$\left(d_k u_n - d_k u_n^{hk}, v^h\right) + a_h\left(\gamma_k u_n - \gamma_k u_n^{hk}, v^h\right) = \left(r_n, v^h\right) \quad \forall\, v^h \in V^h.$$

Since $a_h\left(u_n - \Pi^h u_n, v^h\right) = 0$ by the definition of the Galerkin projection, we obtain

$$\left(d_k e_{2,n}, v^h\right) + a_h\left(\gamma_k e_{2,n}, v^h\right) = \left(r_n, v^h\right) - \left(d_k e_{1,n}, v^h\right). \tag{4.48}$$

We now add up (4.48) from $n = 1$ to $n = m$, for $1 \leq m \leq N-1$. Taking into account cancellation and multiplying both sides by k, we readily see that

$$\begin{aligned} &\left(\frac{e_{2,m+1} - e_{2,m}}{k}, v^h\right) - \left(\frac{e_{2,1} - e_{2,0}}{k}, v^h\right) + k \sum_{n=1}^{m} a_h(\gamma_k e_{2,n}, v^h) \\ &= k \sum_{n=1}^{m} (r_n, v^h) - k \sum_{n=1}^{m} (d_k e_{1,n}, v^h). \end{aligned} \tag{4.49}$$

To simplify the notation, define

$$\begin{aligned} \Phi^m &= k \sum_{n=1}^{m} e_{2,n}, \quad \Phi^0 = 0, \\ R^m &= k \sum_{n=0}^{m} r_n, \quad r_0 = \frac{e_{2,1} - e_{2,0}}{k^2}, \\ A^m &= k \sum_{n=1}^{m} d_k e_{1,n}, \quad A^0 = 0. \end{aligned}$$

Then (4.49) is rewritten as

$$\left(\frac{e_{2,m+1}-e_{2,m}}{k}, v^h\right) + a_h(\gamma_k \Phi^m, v^h)$$
$$= (R^m, v^h) - (A^m, v^h) \quad \forall v^h \in V^h, \quad 0 \le m \le N-1.$$

Take $v^h = e_{2,m+1} + e_{2,m} \in V^h$ and multiply the resulting expression by k,

$$\|e_{2,m+1}\|_0^2 - \|e_{2,m}\|_0^2 + k a_h(\gamma_k \Phi^m, e_{2,m+1} + e_{2,m})$$
$$= \quad k(R^m - A^m, e_{2,m+1} + e_{2,m}), \quad 0 \le m \le N-1.$$

Summing from $m = 0$ to $m = n-1$, for $1 \le n \le N$, we have

$$\|e_{2,n}\|_0^2 - \|e_{2,0}\|_0^2 + k\sum_{m=0}^{n-1} a_h(\gamma_k \Phi^m, e_{2,m+1} + e_{2,m}) = k\sum_{m=0}^{n-1}(R^m - A^m, e_{2,m+1} + e_{2,m}).$$

Since a_h is symmetric, $\Phi^0 = 0$, and

$$\Phi^{m+1} - \Phi^{m-1} = k(e_{2,m+1} + e_{2,m}), \quad m = 1, 2, \ldots, N-1,$$

we get

$$k\sum_{m=0}^{n-1} a_h\left(\gamma_k \Phi^m, e_{2,m+1} + e_{2,m}\right) = k\sum_{m=1}^{n-1} a_h\left(\frac{\Phi^{m+1}+\Phi^{m-1}}{2}, \frac{\Phi^{m+1}-\Phi^{m-1}}{k}\right)$$
$$= \frac{1}{2}\sum_{m=1}^{n-1}\left[a_h\left(\Phi^{m+1}, \Phi^{m+1}\right) - a_h\left(\Phi^{m-1}, \Phi^{m-1}\right)\right]$$
$$= \frac{1}{2}\left(\|\Phi^n\|_{a_h}^2 + \|\Phi^{n-1}\|_{a_h}^2 - \|\Phi^1\|_{a_h}^2\right).$$

Hence, we conclude that for $1 \le n \le N$,

$$\|e_{2,n}\|_0^2 + \frac{1}{2}\|\Phi^n\|_{a_h}^2 + \frac{1}{2}\|\Phi^{n-1}\|_{a_h}^2 = \|e_{2,0}\|_0^2 + \frac{k^2}{2}\|e_{2,1}\|_{a_h}^2$$
$$+ k\sum_{m=0}^{n-1}(R^m - A^m, e_{2,m+1} + e_{2,m}). \tag{4.50}$$

By applying the Cauchy–Schwarz inequality and the inequality $ab \le a^2 + \frac{1}{4}b^2$ to the third term of the right hand side in (4.50), we obtain

$$\|e_{2,n}\|_0^2 \le \|e_{2,0}\|_0^2 + \frac{k^2}{2}\|e_{2,1}\|_{a_h}^2 + k\sum_{m=0}^{n-1}\left(\|R^m\|_0 + \|A^m\|_0\right)\left(\|e_{2,m+1}\|_0 + \|e_{2,m}\|_0\right)$$
$$\le \|e_{2,0}\|_0^2 + \frac{k^2}{2}\|e_{2,1}\|_{a_h}^2 + 2\left(\max_n \|e_{2,n}\|_0\right)\left(k\sum_{m=0}^{n-1}\|R^m\|_0 + k\sum_{m=0}^{n-1}\|A^m\|_0\right)$$
$$\le \|e_{2,0}\|_0^2 + \frac{k^2}{2}\|e_{2,1}\|_{a_h}^2$$

$$+\ 2\left[\frac{1}{4}\max_n \|e_{2,n}\|_0^2+\left(k\sum_{m=0}^{n-1}\|R^m\|_0+k\sum_{m=0}^{n-1}\|A^m\|_0\right)^2\right]$$

$$=\|e_{2,0}\|_0^2+\frac{k^2}{2}\|e_{2,1}\|_{a_h}^2+\frac{1}{2}\max_n \|e_{2,n}\|_0^2+2\left(k\sum_{m=0}^{n-1}\|R^m\|_0+k\sum_{m=0}^{n-1}\|A^m\|_0\right)^2.$$

So

$$\max_n \|e_{2,n}\|_0\le\sqrt{2}\,\|e_{2,0}\|_0+\frac{1}{2}k^2+\frac{1}{2}\|e_{2,1}\|_{a_h}^2+2\left(k\sum_{m=0}^{N-1}\|R^m\|_0+k\sum_{m=0}^{N-1}\|A^m\|_0\right). \tag{4.51}$$

Note that

$$e_{2,0}=\Pi^h u_0-u_0^{hk}=\left(\Pi^h u_0-u_0\right)+\left(u_0-P^h u_0\right),$$

Consequently,

$$\|e_{2,0}\|_0\le\|\Pi^h u_0-u_0\|_0+\|u_0-P^h u_0\|_0\le c\,h^{p+1}\|u_0\|_{p+1}. \tag{4.52}$$

Now

$$\|R^m\|_0=\|k\sum_{n=1}^{m}r_n+kr_0\|_0\le k\sum_{n=1}^{m}\|r_n\|_0+k\|r_0\|_0,$$

and by (4.28),

$$k\|r_0\|_0=\|\frac{e_{2,1}-e_{2,0}}{k}\|_0\le c\left(h^{p+1}\|\partial_t u\|_{C(\bar{I};H^{p+1}(\Omega))}+k^2\|\partial_t^3 u\|_{C(\bar{I};L^2(\Omega))}\right). \tag{4.53}$$

In addition,

$$\|A^m\|_0\le k\sum_{n=1}^{m}\|d_k e_{1,n}\|_0=k\sum_{n=1}^{m}\|d_k(I-\Pi^h)u_n\|_0.$$

Similar to (4.33),

$$\begin{aligned}\|d_k\left(I-\Pi^h\right)u_n\|_0&\le\frac{1}{k}\int_{t_{n-1}}^{t_{n+1}}\|\left(I-\Pi^h\right)\partial_t^2 u(\cdot,s)\|_0 ds\\&\le\frac{c\,h^{p+1}}{k}\int_{t_{n-1}}^{t_{n+1}}\|\partial_t^2 u(\cdot,s)\|_{p+1}ds.\end{aligned} \tag{4.54}$$

The error bound (4.43) now follows from (4.27), (4.44)–(4.46), (4.51)–(4.54), and some simple manipulation. □

5 Numerical Results

In this section, we report numerical results on convergence orders. For the numerical simulation, we solve the initial-boundary value problem (1.1)–(1.4) with a spatial domain $\Omega := (0,1)^2$. We let $b = 1$ and choose the exact solution $u(x, y, t) = t^2\sin(\pi x)\sin(\pi y)$. Then, $u_0 = 0$, $v_0 = 0$, and we determine the source function f from the Eq. (1.1). We use

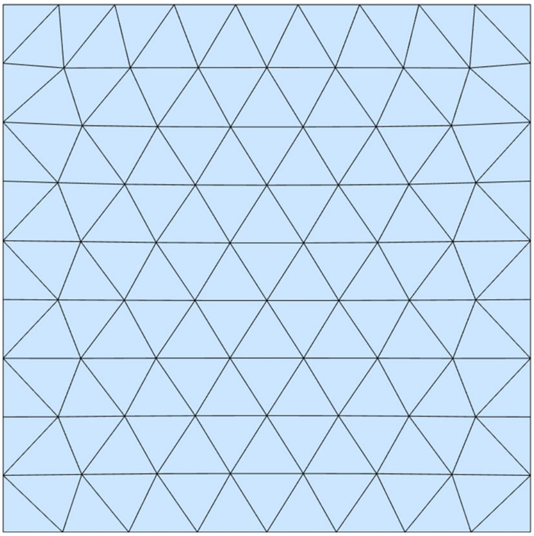

Fig. 1 Quasi-uniform triangulation with $h = 0.125$

Table 1 Numerical convergence orders of H^1 norm at $t = 1$ for $p = 1, 2, 3$

h	Errors for $p = 1$	Order	Errors for $p = 2$	Order	Errors for $p = 3$	Order
2^{-2}	5.5951e−1	–	9.8658e−2	–	6.4255e−3	–
2^{-3}	2.2692e−1	1.3020	2.1356e−2	2.2078	6.1471e−4	3.3858
2^{-4}	1.0079e−1	1.1709	5.0522e−3	2.0796	6.5592e−5	3.2283
2^{-5}	4.7023e−2	1.0999	1.2246e−3	2.0446	7.4433e−6	3.1395

Table 2 Numerical convergence orders of L^2 norm at $t = 1$ for $p = 1, 2, 3$

h	Errors for $p = 1$	Order	Errors for $p = 2$	Order	Errors for $p = 3$	Order
2^{-2}	1.0697e−1	–	3.3758e−3	–	3.2712e−4	–
2^{-3}	2.5095e−2	2.0917	3.3914e−4	3.3153	1.4351e−5	4.5106
2^{-4}	5.9196e−3	2.0838	3.8018e−5	3.1571	7.4579e−7	4.2662
2^{-5}	1.4710e−3	2.0087	4.5919e−6	3.0495	4.0691e−8	4.1960

a sequence of quasi-uniform triangulations $\mathcal{T}_h$ of the type shown in Fig. 1 to partition $\overline{\Omega}$. The continuous wave equation is discretized by the fully discrete scheme (4.1)–(4.4) with the IPDG method, and the penalty parameter $\eta_e = 200\,(p+1)^2$.

To illustrate the dependence of the numerical errors on the mesh size h, we use the time step $k = 10^{-2}$ for $p = 1$, $k = 10^{-3}$ for $p = 2$ and $k = 2 \times 10^{-4}$ for $p = 3$. Then we take $h = 2^{-2}, \ldots, 2^{-5}$ and compute numerical solutions. The numerical errors and convergence orders in the $H^1(\Omega)$-norm and $L^2(\Omega)$-norm at $t = 1$ are summarized in Tables 1 and 2, respectively. The numerical convergence orders are also shown in Figs. 2 and 3. We observe that the numerical convergence orders for H^1-norm and L^2-norm are around p and $p + 1$, respectively, which matches well with the theoretical prediction.

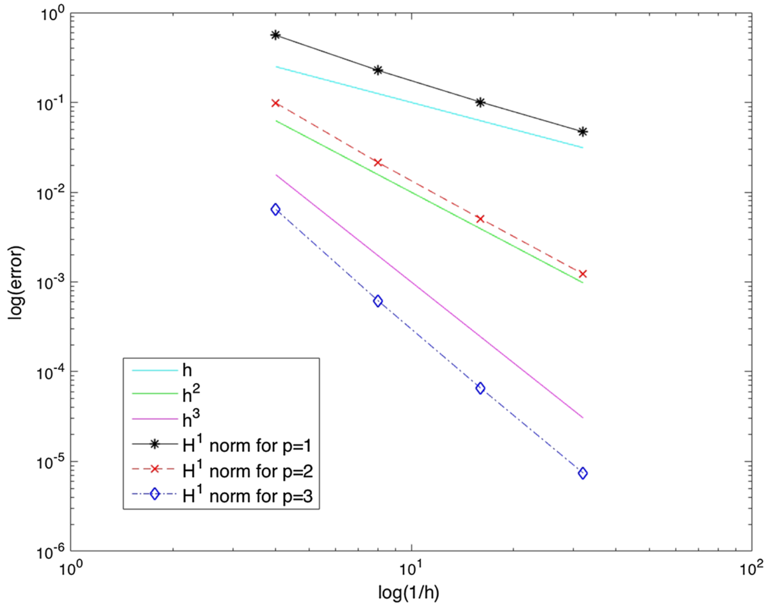

Fig. 2 Numerical convergence orders of H^1 norm at $t = 1$ for $p = 1, 2, 3$

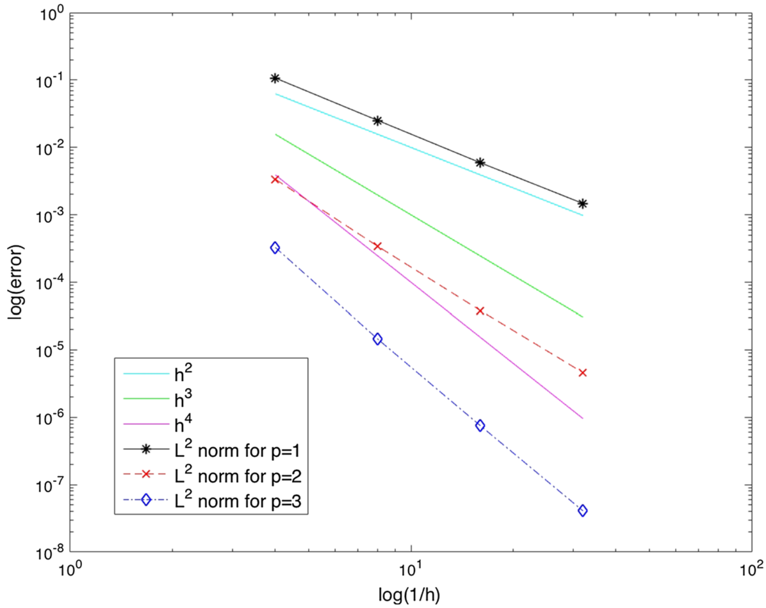

Fig. 3 Numerical convergence orders of L^2 norm at $t = 1$ for $p = 1, 2, 3$

Table 3 Numerical convergence orders at $t = 1$ for fixed $h = 0.03$ and $p = 2$

k	2^{-2}	2^{-3}	2^{-4}	2^{-5}
H^1 errors	3.0048e−1	6.1773e−2	1.3055e−2	3.2043e−2
Order	–	2.2822	2.2424	2.0265
L^2 errors	6.5980e−2	1.3563e−2	2.8564e−3	6.5927e−4
Order	–	2.2824	2.2474	2.1153

Next, we use quadratic element ($p = 2$) and fix a mesh size $h = 0.03$, and examine the dependence of the numerical solution errors on the time step size k, cf. the results in Table 3. We observe that the numerical convergence orders are nearly 2 with respect to k, which supports the theoretical results in Theorems 4.1 and 4.2.

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