SYSTEMS OF NONLINEAR EQUATIONS

Widely used in the mathematical modeling of real world phenomena. We introduce some numerical methods for their solution.

For better intuition, we examine systems of two nonlinear equations and numerical methods for their solution. We then generalize to systems of an arbitrary order.

**The Problem**: Consider solving a system of two nonlinear equations

\[
\begin{align*}
f(x, y) &= 0 \\
g(x, y) &= 0
\end{align*}
\] (1)
Example

Consider solving the system

\[ f(x, y) \equiv x^2 + 4y^2 - 9 = 0 \]
\[ g(x, y) \equiv 18y - 14x^2 + 45 = 0 \]  \hspace{1cm} (2)

Figure: Graph of \( z = x^2 + 4y^2 - 9 \), along with \( z = 0 \) on that surface
To visualize the points \((x, y)\) that satisfy simultaneously both equations in (1), look at the intersection of the zero curves of the functions \(f\) and \(g\). The next figure illustrates this. The zero curves intersect at four points, each of which corresponds to a solution of the system (2). For example, there is a solution near \((1, -1)\).

Figure: Graphs of \(f(x, y) = 0\) and \(g(x, y) = 0\)
The graph of \( z = f(x, y) \) is a surface in \( xyz \)-space.

For \((x, y) \approx (x_0, y_0)\), we approximate \( f(x, y) \) by a plane that is tangent to the surface at the point \((x_0, y_0, f(x_0, y_0))\). The equation of this plane is \( z = p(x, y) \) with

\[
p(x, y) \equiv f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0),
\]

(3)

\[
f_x(x, y) = \frac{\partial f(x, y)}{\partial x}, \quad f_y(x, y) = \frac{\partial f(x, y)}{\partial y},
\]

the partial derivatives of \( f \) with respect to \( x \) and \( y \), respectively.

If \((x, y) \approx (x_0, y_0)\), then

\[
f(x, y) \approx p(x, y)
\]

For functions of two variables, \( p(x, y) \) is the linear Taylor polynomial approximation to \( f(x, y) \).
Let $\alpha = (\xi, \eta)$ denote a solution of the system

\[
\begin{align*}
    f(x, y) &= 0 \\
    g(x, y) &= 0
\end{align*}
\]

Let $(x_0, y_0) \approx (\xi, \eta)$ be an initial guess at the solution.

Approximate the surface $z = f(x, y)$ with the tangent plane at $(x_0, y_0, f(x_0, y_0))$.

If $f(x_0, y_0)$ is sufficiently close to zero, then the zero curve of $p(x, y)$ will be an approximation of the zero curve of $f(x, y)$ for those points $(x, y)$ near $(x_0, y_0)$.

Because the graph of $z = p(x, y)$ is a plane, its zero curve is simply a straight line.
Example

Consider \( f(x, y) \equiv x^2 + 4y^2 - 9 \). Then \( f_x(x, y) = 2x \), \( f_y(x, y) = 8y \). At \((x_0, y_0) = (1, -1)\),

\[
\begin{align*}
  f(x_0, y_0) &= -4, \\
  f_x(x_0, y_0) &= 2, \\
  f_y(x_0, y_0) &= -8
\end{align*}
\]

At \((1, -1, -4)\) the tangent plane to the surface \( z = f(x, y) \) is

\[
z = p(x, y) \equiv -4 + 2(x - 1) - 8(y + 1)
\]

Figure: Zero curves of \( f(x, y) \) and \( p(x, y) \) for \((x, y)\) near \((x_0, y_0)\)
The tangent plane to the surface \( z = g(x, y) \) at 
\( (x_0, y_0, g(x_0, y_0)) \) has the equation \( z = q(x, y) \) with

\[
q(x, y) \equiv g(x_0, y_0) + (x - x_0)g_x(x_0, y_0) + (y - y_0)g_y(x_0, y_0)
\]

Recall that the solution \( \alpha = (\xi, \eta) \) to

\[
f(x, y) = 0 \\
g(x, y) = 0
\]

is the intersection of the zero curves of \( z = f(x, y) \) and 
\( z = g(x, y) \).

Approximate these zero curves by those of the tangent planes 
\( z = p(x, y) \) and \( z = q(x, y) \). The intersection of these latter zero 
curves gives an approximate solution to the above nonlinear 
system.

Denote by \( (x_1, y_1) \) the solution to

\[
p(x, y) = 0 \\
q(x, y) = 0
\]
Example

Return to the equations

\[ f(x, y) \equiv x^2 + 4y^2 - 9 = 0 \]
\[ g(x, y) \equiv 18y - 14x^2 + 45 = 0 \]

with \((x_0, y_0) = (1, -1)\).

**Figure:** Use of zero curves of tangent plane approximations
Calculating \((x_1, y_1)\). To find the intersection of the zero curves of the tangent planes, we must solve the linear system

\[
\begin{align*}
    f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) &= 0 \\
    g(x_0, y_0) + (x - x_0)g_x(x_0, y_0) + (y - y_0)g_y(x_0, y_0) &= 0
\end{align*}
\]

The solution is denoted by \((x_1, y_1)\). In matrix form,

\[
\begin{bmatrix}
    f_x(x_0, y_0) & f_y(x_0, y_0) \\
    g_x(x_0, y_0) & g_y(x_0, y_0)
\end{bmatrix}
\begin{bmatrix}
    x - x_0 \\
    y - y_0
\end{bmatrix}
=
-\begin{bmatrix}
    f(x_0, y_0) \\
    g(x_0, y_0)
\end{bmatrix}
\]

It is actually computed as follows. Define \(\delta_x\) and \(\delta_y\) to be the solution of the linear system

\[
\begin{bmatrix}
    f_x(x_0, y_0) & f_y(x_0, y_0) \\
    g_x(x_0, y_0) & g_y(x_0, y_0)
\end{bmatrix}
\begin{bmatrix}
    \delta_x \\
    \delta_y
\end{bmatrix}
=
-\begin{bmatrix}
    f(x_0, y_0) \\
    g(x_0, y_0)
\end{bmatrix}
\]

\[
\begin{bmatrix}
    x_1 \\
    y_1
\end{bmatrix}
=
\begin{bmatrix}
    x_0 \\
    y_0
\end{bmatrix}
+
\begin{bmatrix}
    \delta_x \\
    \delta_y
\end{bmatrix}
\]

Usually the point \((x_1, y_1)\) is closer to the solution \(\alpha\) than is the original point \((x_0, y_0)\).
Continue this process, using \((x_1, y_1)\) as a new initial guess. Obtain an improved estimate \((x_2, y_2)\).

This iteration process is continued until a solution with sufficient accuracy is obtained.

The general iteration is given by

\[
\begin{bmatrix}
  f_x(x_k, y_k) & f_y(x_k, y_k) \\
  g_x(x_k, y_k) & g_y(x_k, y_k)
\end{bmatrix}
\begin{bmatrix}
  \delta_{x,k} \\
  \delta_{y,k}
\end{bmatrix}
= -
\begin{bmatrix}
  f(x_k, y_k) \\
  g(x_k, y_k)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  x_{k+1} \\
  y_{k+1}
\end{bmatrix}
= \begin{bmatrix}
  x_k \\
  y_k
\end{bmatrix}
+ \begin{bmatrix}
  \delta_{x,k} \\
  \delta_{y,k}
\end{bmatrix}, \quad k = 0, 1, \ldots
\]

This is *Newton’s method* for solving

\[
f(x, y) = 0 \\
g(x, y) = 0
\]

Many numerical methods for solving nonlinear systems are variations on Newton’s method.
Consider again the system

\[ f(x, y) \equiv x^2 + 4y^2 - 9 = 0 \]
\[ g(x, y) \equiv 18y - 14x^2 + 45 = 0 \]

Newton’s method (4) becomes

\[
\begin{bmatrix}
2x_k & 8y_k \\
-28x_k & 18 \\
\end{bmatrix}
\begin{bmatrix}
\delta_{x,k} \\
\delta_{y,k} \\
\end{bmatrix}
= -
\begin{bmatrix}
x_k^2 + 4y_k^2 - 9 \\
18y_k - 14x_k^2 + 45 \\
\end{bmatrix}
\]

Choose \((x_0, y_0) = (1, -1)\). The resulting Newton iterates are given in Table 1, along with

\[ Error = \|\alpha - (x_k, y_k)\| \equiv \max\{|\xi - x_k|, |\eta - y_k|\} \]
Table: Newton iterates for the system (5)

<table>
<thead>
<tr>
<th>$k$</th>
<th>$x_k$</th>
<th>$y_k$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>-1.0</td>
<td>3.74E − 1</td>
</tr>
<tr>
<td>1</td>
<td>1.170212765957447</td>
<td>-1.457446808510638</td>
<td>8.34E − 2</td>
</tr>
<tr>
<td>2</td>
<td>1.202158829506705</td>
<td>-1.376760321923060</td>
<td>2.68E − 3</td>
</tr>
<tr>
<td>3</td>
<td>1.203165807091535</td>
<td>-1.374083486949713</td>
<td>2.95E − 6</td>
</tr>
<tr>
<td>4</td>
<td>1.203166963346410</td>
<td>-1.374080534243534</td>
<td>3.59E − 12</td>
</tr>
<tr>
<td>5</td>
<td>1.203166963347774</td>
<td>-1.374080534239942</td>
<td>2.22E − 16</td>
</tr>
</tbody>
</table>

The final iterate is accurate to the precision of the computer arithmetic.
AN ALTERNATIVE NOTATION

We introduce a more general notation for the preceding work. The problem to be solved is

\[ F_1(x_1, x_2) = 0 \]
\[ F_2(x_1, x_2) = 0 \]  \hspace{1cm} (6)

Introduce

\[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad F(x) = \begin{bmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{bmatrix}, \quad F'(x) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix} \]

\( F'(x) \) is called the Frechet derivative of \( F(x) \). It is a generalization to higher dimensions of the ordinary derivative of a function of one variable.

The system (6) can now be written as

\[ F(x) = 0 \]  \hspace{1cm} (7)

A solution of this equation will be denoted by \( \alpha \).
Newton’s method becomes

\[ F'(x^{(k)})\delta^{(k)} = -F(x^{(k)}) \]

\[ x^{(k+1)} = x^{(k)} + \delta^{(k)} \] (8)

for \( k = 0, 1, \ldots \).

A shorter and mathematically equivalent form:

\[ x^{(k+1)} = x^{(k)} - \left[ F'(x^{(k)}) \right]^{-1} F(x^{(k)}) \] (9)

for \( k = 0, 1, \ldots \).

This last formula is often used in discussing and analyzing Newton’s method for nonlinear systems. But (8) is used for practical computations, since it is usually less expensive to solve a linear system than to find the inverse of the coefficient matrix.

Note the analogy of (9) with Newton’s method for a single equation.
Consider the system of $n$ nonlinear equations

\begin{align*}
F_1(x_1, \ldots, x_n) &= 0 \\
& \vdots \\
F_n(x_1, \ldots, x_n) &= 0
\end{align*}

(10)

Define

\[
x = \begin{bmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{bmatrix}, \quad F(x) = \begin{bmatrix}
  F_1(x_1, \ldots, x_n) \\
  \vdots \\
  F_n(x_1, \ldots, x_n)
\end{bmatrix}, \quad F'(x) = \begin{bmatrix}
  \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\
  \vdots & \ddots & \vdots \\
  \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n}
\end{bmatrix}
\]

The nonlinear system (10) can be written as

\[
F(x) = 0
\]

Its solution is denoted by $\alpha \in \mathbb{R}^n$. 
Newton’s method is
\[ F'(x^{(k)})\delta^{(k)} = -F(x^{(k)}) \]
\[ x^{(k+1)} = x^{(k)} + \delta^{(k)}, \quad k = 0, 1, \ldots \] (11)

Alternatively, as before,
\[ x^{(k+1)} = x^{(k)} - \left[ F'(x^{(k)}) \right]^{-1} F(x^{(k)}), \quad k = 0, 1, \ldots \] (12)

This formula is often used in theoretical discussions of Newton’s method for nonlinear systems. But (11) is used for practical computations, since it is usually less expensive to solve a linear system than to find the inverse of the coefficient matrix.

**CONVERGENCE.** Under suitable hypotheses, it can be shown that there is a \( c > 0 \) for which
\[ \|\alpha - x^{(k+1)}\| \leq c \|\alpha - x^{(k)}\|^2, \quad k = 0, 1, \ldots \]

\[ \|\alpha - x^{(k)}\| \equiv \max_{1 \leq i \leq n} |\alpha_i - x_i^{(k)}| \]

Newton’s method is *quadratically convergent*. 