A Brief Introduction to Infinitesimal Calculus

Section 2: Keisler's Axioms

The following presentation of Keisler's foundations for Robinson's Theory of Infinitesimals is explained in more detail in either of the (free .pdf) files: **Foundations of Infinitesimal Calculus** on my web site and the Epilog to Keisler's text, **Elementary Calculus: An Approach Using Infinitesimals** on his web site.

Small, Medium, and Large Hyperreal Numbers

Field Axioms

A "field" of numbers is any set of objects together with two operations, addition and multiplication that satisfy:

- The commutative laws of addition and multiplication,

  $a_1 + a_2 = a_2 + a_1$ & $a_1 \cdot a_2 = a_2 \cdot a_1$

- The associative laws of addition and multiplication,

  $a_1 + (a_2 + a_3) = (a_1 + a_2) + a_3$ & $a_1 \cdot (a_2 \cdot a_3) = (a_1 \cdot a_2) \cdot a_3$

- The distributive law of multiplication over addition,

  $a_1 \cdot (a_2 + a_3) = a_1 \cdot a_2 + a_1 \cdot a_3$

- There is an additive identity, 0,with $0 + a = a$ for every number a.
- There is an multiplicative identity, 1,with $1 \cdot a = a$ for every number $a \neq 0$.
- Each number a has an additive inverse, $-a$,with $a + (-a) = 0$.
- Each nonzero number $a \neq 0$ has a multiplicative inverse, $\frac{1}{a}$,with $a \cdot \frac{1}{a} = 1$.

The binomial expansion that follows is a consequence of the field axioms.

$$(x + \Delta x)^3 = x^3 + 3\,x^2\,\Delta x + ((3\,x + \Delta x) \cdot \Delta x) \cdot \Delta x$$

Hence this formula holds for any pair of numbers x and Δx in a field.

To compare sizes of numbers we need an ordering.

Ordered Field Axioms

A a number system is an ordered field if it satisfies the Field Axioms above and has a relation < that satisfies:

- Every pair of numbers a and b satisfies exactly one of the relations

  $a = b$, $a < b$, or $b < a$

- If $a < b$ and $b < c$,then $a < c$.

- If $a < b$, then $a + c < b + c$.

- If $0 < a$ and $0 < b$,then $0 < a \cdot b$.

In an ordered field the absolute value of a nonzero number is the larger of a and $-a$.

We want to let $\Delta x = \delta x$ be "small" write the differential approximation for $y = f[x] = x^3$,

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

$$(x + \delta x)^3 - x^3 = 3x^2 \, \delta x + \varepsilon \cdot \delta x \text{ , with } \varepsilon = ((3x + \delta x) \cdot \delta x)$$

Once we show that ε is small for limited x, we have proved that $f'[x] = 3x^2$. Moreover, this is equivalent to "epsilon-delta" uniform approximation on compact sets.

Infinitesimal Numbers

A number δ in an ordered field is called *infinitesimal* if it satisfies

$$|\delta| < \frac{1}{m} \text{ for any ordinary natural counting number, } m = 1, 2, 3, \cdots$$

Two hyperreal numbers x and y are said to be infinitely close, or differ by an infinitesimal, if $x - y$ is infinitesimal. In this case we write $x \approx y$.

This definition is intended to include 0 as an infinitesimal.

NOTE: From now on the previously informal "approximately equal" notation "$\approx$" is replaced by this precise definition. An infinitesimal is a number that satisfies $\delta \approx 0$. The point of this section is to show that the technical definition captures the intuitive ideas of Section 1.

Archimedes' Axiom is precisely the statement that the (Dedekind) "real" numbers have no positive infinitesimals. Keisler's Algebra Axiom is the following:

Keisler's Algebra Axiom

The hyperreal numbers are an ordered field extension of the real numbers. In particular, there is a positive hyperreal infinitesimal, δ, satisfying

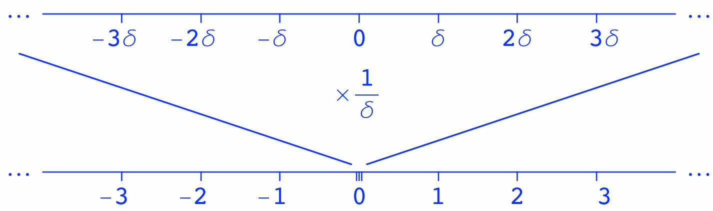
$$0 < \delta < \frac{1}{m} \text{ for any ordinary natural counting number, } m = 1, 2, 3, \cdots$$

It follows from the laws of ordered algebra that there are many different infinitesimals. For example, the law $a < b \Rightarrow a + c < b + c$ applied to $a = 0$ and $b = c = \delta$ says $\delta < 2\delta$. Similarly, all the integer multiples of δ are distinct infinitesimals,

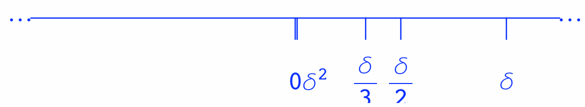
$$\cdots < -3\delta < -2\delta < -\delta < 0 < \delta < 2\delta < 3\delta < \cdots$$

If k is a natural number, $k\delta < \frac{1}{m}$, for any natural m, because $\delta < \frac{1}{k \cdot m}$ when δ is infinitesimal.

Magnifying the line by $1/\delta$ makes integer multiples of δ appear like the integers at unit scale.

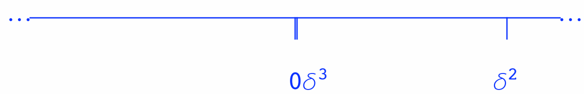


Magnification at center c with power $1/\delta$ is simply the transformation $x \to \frac{1}{\delta}(x-c)$, so by laws of algebra, integer multiples of δ end up the same integers apart for magnification centered at zero. Similar reasoning lets us place $\frac{\delta}{2}, \frac{\delta}{3}, \cdots$ on a magnified line at one half the distance to δ, one third the distance, etc.



Where should we place the numbers $\delta^2, \delta^3 \cdots$? On a scale of δ, they are infinitely near zero, $\frac{1}{\delta}(\delta^2 - 0) = \delta \approx 0$:

Magnification by $1/\delta^2$ reveals δ^2, but moves δ infinitely far to the right, $\frac{1}{\delta^2}(\delta - 0) = \frac{1}{\delta} > m$ for all natural $m = 1, 2, 3, \cdots$



Laws of algebra dictate many "orders of infinitesimal" such as

$$0 < \cdots < \delta^3 < \delta^2 < \delta$$

The laws of algebra show that near every real number there are many hyperreals, say near $\pi = 3.14159\cdots$

$$\cdots < \pi - 3\delta < \pi - 2\delta < \pi - \delta < \pi < \pi + \delta < \pi + 2\delta < \pi + 3\delta < \cdots$$

Medium and Large Numbers

A hyperreal number x is called *limited* (or "finite in magnitude") if there is a natural number m so that $|x| < m$. If there is no natural bound for a hyperreal number it is called *unlimited* (or "infinite").

Infinitesimal numbers are limited, being bounded by 1.

Theorem: Standard Parts of Limited Hyperreal Numbers

Every limited hyperreal number x differs from some real number by an infinitesimal, that is, there is a real r so that $x \approx r$. This number is called the "standard part" of x, $r = \text{st}[x]$.

Proof

Define a Dedekind cut in the real numbers by $A = \{s : s \leq x\}$ and $B = \{s : x < s\}$. st[x] is the real number defined by this cut.

A Curious "Paradox"

The real numbers are Dedekind complete. Sometimes we think of this result as saying the real numbers are the points on a line with no gaps. The Standard Part Theorem says all the limited hyperreals are clustered around real numbers. When we take a line with no gaps and add lots of infinitesimals around each point, we create gaps! The cut in the hyperreals consisting of all numbers that are either negative or infinitesimal on one hand or positive and non-infinitesimal on the other has no number at the cut. There is no largest infinitesimal because twice that number would be infinitesimal and there is no smallest positive non-infinitesimal, because half of it would be infinitesimal, and then twice that also infinitesimal.

Our microscopic pictures of the hyperreal line do not reveal the gaps as long as we view the microscopic images as the image under similarity transformations $(x, y) \to \frac{1}{\delta}(x - a, y - b)$ with hyperreal parameters.

Theorem: Computation Rules for Small, Medium, and Large Numbers

(a) If p and q are limited, so are $p + q$ and $p \cdot q$

(b) If ε and δ are infinitesimal, so is $\varepsilon + \delta$.

(c) If $\delta \approx 0$ and q is finite, then $q \cdot \delta \approx 0$. (finite x infsml = infsml)

(d) $1/0$ is still undefined and $1/x$ is unlimited only when $x \approx 0$.

Proof

These rules are easy to prove as we illustrate with (c). If q is limited, there is a natural number with $|q| < k$. The condition $\delta \approx 0$ means $|\delta| < \frac{1}{k \cdot m}$, so $|q \cdot \delta| < \frac{1}{m}$ proving that $q \cdot \delta \approx 0$.

The uniform derivative of x^3

Let's apply these rules to show that $f[x] = x^3$ satisfies the differential approximation with $f'[x] = 3x^2$ when x is limited. We know by laws of algebra that

$$(x + \delta x)^3 - x^3 = 3x^2 \, \delta x + \varepsilon \cdot \delta x \text{ , with } \varepsilon = ((3x + \delta x) \cdot \delta x)$$

If x is limited and $\delta x \approx 0$, (a) shows that $3x$ is limited and that $3x + \delta x$ is also limited. Condition (b) then shows that $\varepsilon = ((3x + \delta x) \cdot \delta x) \approx 0$ proving that for all limited x

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

with $\varepsilon \approx 0$ whenever $\delta x \approx 0$.

Below we will see that this computation is logically equivalent to the statement that $\underset{\Delta x \to 0}{\text{Lim}} \frac{(x + \Delta x)^3 - x^3}{\Delta x} = 3x^2$, uniformly on compact sets of the real line. It is really no surprise that we can differentiate algebraic functions using algebraic properties of numbers. This does not solve the problem of finding sound foundations for calculus using infinitesimals because we need to treat transcendental functions like sine, cosine, log.

Keisler's Function Extension Axiom

Roughly speaking, Keisler's Function Extension Axiom says that all real functions have extensions to the hyperreal numbers and these "natural" extensions obey the same identities and inequalities as the original function. Some familiar identities are

$$\text{Sin}[\alpha + \beta] = \text{Sin}[\alpha] \, \text{Cos}[\beta] + \text{Cos}[\alpha] \, \text{Sin}[\beta]$$

$\text{Log}[x \cdot y] = \text{Log}[x] + \text{Log}[y]$

The log identity only holds when x and y are positive. Keisler's Function Extension Axiom is formulated so that we can apply it to the Log identity in the form of the implication

$(x > 0 \& y > 0) \Rightarrow \text{Log}[x]$ and $\text{Log}[y]$ are defined and $\text{Log}[x \cdot y] = \text{Log}[x] + \text{Log}[y]$

The Function Extension Axiom guarantees that the natural extension of Log[·] is defined for all positive hyperreals and its identities hold for hyperreal numbers satisfying $x > 0$ and $y > 0$.

We can state the addition formula for sine as the implication

$(\alpha = \alpha \& \ \beta = \beta) \Rightarrow$
$\text{Sin}[\alpha], \text{Sin}[\beta], \text{Sin}[\alpha + \beta], \text{Cos}[\alpha], \text{Cos}[\beta]$ are defined
and
$\text{Sin}[\alpha + \beta] = \text{Sin}[\alpha]\,\text{Cos}[\beta] + \text{Cos}[\alpha]\,\text{Sin}[\beta]$

Logical Real Expressions

Logical real expressions are built up from numbers and variables using functions.

(a) A real number is a real expression.

(b) A variable standing alone is a real expression.

(c) If $E_1, E_2, \cdots, E_n$ are a real expressions and $f[x_1, x_2, \cdots, x_n]$ is a real function of n variables, then $f[E_1, E_2, \cdots, E_n]$ is a real expression.

Logical Real Formulas

A logical real formula is one of the following:

(i) An equation between real expressions, $E_1 = E_2$

(ii) An inequality between real expressions, $E_1 < E_2$, $E_1 \leq E_2$, $E_1 > E_2$, $E_1 \geq E_2$, or $E_1 \neq E_2$

(iii) A statement of the form "*E is defined*" or of the form "*E is undefined*."

Logical Real Statements

Let S and T be finite sets of real formulas. A logical real statement is an implication of the form,

$S \Rightarrow T$

The functional identities for sine and log given above are logical real statements.

Keisler's Function Extension Axiom

Every real function $f[x_1, x_2, \cdots, x_n]$ has a "natural" extension to the hyperreals such that every logical real statement that holds for all real numbers also holds for all hyperreal numbers when the real functions in the statement are replaced by their natural extensions.

There are two general uses of the Function Extension Axiom that underlie most of the theoretical problems in calculus. These involve extension of the discrete maximum and extension of finite summation. The proof of the Extreme Value Theorem below uses a hyperfinite maximum, while the proof of the Fundamental Theorem of Integral Calculus uses hyperfinite summation and a maximum.

Equivalence of infinitesimal conditions and the "epsilon - delta" real number conditions are usually proved by using an auxiliary real function as in the following proof.

Theorem: Simple Equivalency of Limits and Infinitesimals

Let $f[x]$ be a real valued function defined for $0 < |x - a| < \Delta$ with Δ a fixed positive real number. Let b be a real number. Then the following are equivalent:

(a) Whenever the hyperreal number x satisfies $a \neq x \approx a$, the natural extension function satisfies

$$f[x] \approx b$$

(b) For every real accuracy tolerance θ there is a sufficiently small positive real number γ such that if the real number x satisfies $0 < |x - a| < \gamma$,then

$$|f[x] - b| < \theta$$

Condition (b) is the familiar Weierstrass "epsilon-delta" condition (written with θ and γ.) Notice that the condition $f[x] \approx b$ is NOT a logical real statement because the infinitesimal relation is NOT included in the formation rules for forming logical real statements.

Proof

We show that (a) ⇒ (b) by proving that not (b) implies not (a), the contrapositive. Assume (b) fails. Then there is a real $\theta > 0$ such that for every real $\gamma > 0$ there is a real x satisfying $0 < |x - a| < \gamma$ and $|f[x] - b| \geq \theta$. Let $X[\gamma] = x$ be a real function that chooses such an x for a particular γ. Then we have the equivalence

$$\gamma > 0 \Leftrightarrow (X[\gamma] \text{ is defined, } 0 < |X[\gamma] - a| < \gamma,\ |f[X[\gamma]] - b| \geq \theta)$$

By the Function Extension Axiom this equivalence holds for hyperreal numbers and the natural extensions of the real functions $X[\cdot]$ and $f[\cdot]$. In particular, choose a positive infinitesimal γ and apply the equivalence. We have $0 < |X[\gamma] - a| < \gamma$ and $|f[X[\gamma]] - b| > \theta$ and θ is a positive real number. Hence, $f[X[\gamma]]$ is not infinitely close to b, proving not (a) and completing the proof that (a) implies (b).

Conversely, suppose that (b) holds. Then for every positive real θ, there is a positive real γ such that $0 < |x - a| < \gamma$ implies $|f[x] - b| < \theta$. By the Function Extension Axiom, this implication holds for hyperreal numbers. If $\xi \approx a$,then $0 < |\xi - a| < \gamma$ for every real γ, so $|f[\xi] - b| < \theta$ for every real positive θ. In other words, $f[\xi] \approx b$, showing that (b) implies (a) and completing the proof of the theorem.

Other examples of uses of the Function Extension Axiom are given below. We use Keisler's foundations to complete the proofs of the basic results of Section 1.

The Differential Approximation is NOT a logical *real* expression

The implication

$$\delta x \approx 0 \Rightarrow \frac{f[x+\delta x]-f[x]}{\delta x} \approx f'[x]$$

or the differential approximation

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

with $\varepsilon \approx 0$ whenever $\delta x \approx 0$ are not ***real*** expressions because they involve the infinitesimal relation.

Continuity & Extreme Values

We follow the idea of the proof in Section 1 for a real function $f[x]$ on a real interval $[a, b]$. Coding our proof in terms of real functions.

There is a real function $x_M[h]$ so that for each natural number h the maximum of the values $f[x]$ for $x = a + k\,\Delta x$, $k = 1, 2, \cdots, h$ and $\Delta x = (b-a)/h$ occurs at $x_M[h]$. We can express this in terms of real functions using a real function indicating whether a real number is a natural number,

$$I[x] = \begin{cases} 0, & \text{if } x \neq 1, 2, 3, \cdots \\ 1, & \text{if } x = 1, 2, 3, \cdots \end{cases}$$

The maximum of the partition can be described by

$$(a \le x \le b \;\&\; I[h\,\tfrac{x-a}{b-a}] = 1) \Rightarrow f[x] \le f[x_M[h]]$$

We want to extend this function to unlimited "hypernatural" numbers. The greatest integer function Floor[x] satisfies, $I[\text{Floor}[x]] = 1,\ 0 \le x - \text{Floor}[x] \le 1$. The unlimited number $1/\delta$, for $\delta \approx 0$ gives an unlimited $H = \text{Floor}[x]$ with $I[H] = 1$ and

$$(a \le x \le b \;\&\; I[H\,\tfrac{x-a}{b-a}] = 1) \Rightarrow f[x] \le f[x_M[H]]$$

When the natural extension of the indicator function satisfies $I[k] = 1$, we say that k is a hyperinteger. (Every limited hyperinteger is an ordinary positive integer. As you can show with these functions.)

There is a greatest partition point of any number in $[a, b]$, $P[h, x] = a + \text{Floor}[h\,\frac{x-a}{b-a}]\,\frac{b-a}{h}$ with $a \le P[h, x] \le b$ & $I[h\,\frac{P[h,x]-a}{b-a}] = 1$ and $0 \le x - P[h, x] \le 1/h$. When we take the unlimited hypernatural number H we have $x - P[H, x] \le 1/H \approx 0$ and $P[H, x]$ a partition point in the sense that $(a \le x \le b \;\&\; I[H\,\frac{P[H,x]-a}{b-a}] = 1)$, so we have

$$f[P[H, x]] \le f[x_M[H]]$$

Let $r_M = \text{st}[x_M[H]]$, the standard part. Since $a \le x_M[H] \le b$, $a \le r_M \le b$. Continuity of the function in the sense $x_1 \approx x_2 \Rightarrow f[x_1] \approx f[x_2]$ gives

$$f[x] \approx f[P[H, x]] \le f[x_M[H]] \approx f[r_M], \text{ so } f[x] \le f[r_M] \text{ for any real } x \text{ in } [a, b].$$

For more details see **Foundations of Infinitesimal Calculus** p.50.

Microscopic tangency in one variable

One important comment about the proof of the Extreme Value Theorem is this. The simple fact that the standard part of every hyperreal x satisfying $a \le x \le b$ is in the original real interval $[a, b]$ is the form that topological compactness takes in Robinson's theory: A standard topological space is compact if and only if every point in its extension is near a standard point, that is, has a standard part and that standard part is in the original space.

Suppose $f[x]$ and $f'[x]$ are real functions defined on the interval (a, b), if we know that for all hyperreal numbers x with $a < x < b$ and $a \not\approx x \not\approx b$, $f[x + \delta x] - f[x] = f'[x]\cdot\delta x + \varepsilon\cdot\delta x$ with $\varepsilon \approx 0$ whenever $\delta x \approx 0$, then arguments like the proof of the simple equivalency of limits and infinitesimals above show that $f'[x]$ is a uniform limit of the difference quotient functions on compact subintervals $[\alpha, \beta] \subset (a, b)$. More generally, we can show:

Theorem: Uniform Differentiability

Suppose $f[x]$ and $f'[x]$ are real functions defined on the open real interval (a, b). The following are equivalent definitions of, "The function $f[x]$ is smooth with continuous derivative $f'[x]$ on (a, b)."

(a) Whenever a hyperreal x satisfies $a < x < b$ and x is not infinitely near a or b, then an infinitesimal increment of the extended dependent variable is approximately linear on a scale of the change, that is, whenever $\delta x \approx 0$

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x \text{ with } \varepsilon \approx 0$$

(b) For every compact subinterval $[\alpha, \beta] \subset (a, b)$, the real limit

$$\lim_{\Delta x \to 0} \frac{f[x+\Delta x] - f[x]}{\Delta x} = f'[x] \text{ uniformly for } \alpha \le x \le b$$

(c) For every pair of hyperreal $x_1 \approx x_2$ with $a < \text{st}[x_i] = c < b$, $\frac{f[x_2] - f[x_1]}{x_2 - x_1} \approx f'[c]$

(d) For every c in (a, b), the real double limit, $\lim_{x_1 \to c,\, x_2 \to c} \frac{f[x_2] - f[x_1]}{x_2 - x_1} = f'[c]$

(e) The traditional pointwise defined derivative $D_x f = \lim_{\Delta x \to 0} \frac{f[x+\Delta x] - f[x]}{\Delta x}$ is continuous on (a, b).

Proof

See **Foundations of Infinitesimal Calculus** p.35 and 36.

Continuity of the derivative follows rigorously from the argument of Section 1, approximating the increment $f[x_1] - f[x_2]$ from both ends of the interval $[x_1, x_2]$.

$$f[x_2] - f[x_1] = f'[x_1] \cdot (x_2 - x_1) + \varepsilon_1 \cdot (x_2 - x_1)$$

$$f[x_1] - f[x_2] = f'[x_2] \cdot (x_1 - x_2) + \varepsilon_2 \cdot (x_1 - x_2)$$

Adding, we obtain

$$0 = ((f'[x_1] - f'[x_2]) + (\varepsilon_1 - \varepsilon_2)) \cdot (x_2 - x_1), \text{ so } (f'[x_1] - f'[x_2]) = (\varepsilon_2 - \varepsilon_1) \approx 0.$$

It certainly is geometrically natural to treat both endpoints equally, but this is a "locally uniform" approximation in real-only terms because the x values are hyperreal. Uniformity gives a non-infiinitesimal explanation why the intuitive proof of the Fundamental Theorem works. We take up the infinitesimal explanation in the next section.

In his *General Investigations of Curved Surfaces* (original in draft of 1825, published in Latin 1827, English translation by Morehead & Hiltebeitel, Princeton, NJ, 1902 and reprinted later by Raven Press), Gauss begins as follows:

> *A curved surface is said to posess continuous curvature at one of its points A, if the directions of all stright lines drawn from A to points of the surface at an infinitely small distance from A are deflected infinitely little from one and the same plane passing trhough A. This plane is said to touch the surface at the point A.*

In *The Handbook of Mathematical Logic*, Jon Barwise (editor), North Holland Studies in Logic, nr. 90, Amsterdam 1977, Chapter A6, we show that this can be interpreted as C^1*-embedded* if we apply the condition to all points in the natural extension of the surface.

The Fundamental Theorem of Integral Calculus

The definite integral $\int_a^b f[x]\, dx$ is approximated in real terms by taking sums of slices of the form

$$f[a] \cdot \Delta x + f[a + \Delta x] \cdot \Delta x + f[a + 2\,\Delta x] \cdot \Delta x + \cdots + f[b'] \cdot \Delta x, \text{ where } b' = a + h \cdot \Delta x \text{ and } a + (h+1) \cdot \Delta x > b$$

Given a real function $f[x]$ defined on $[a, b]$ we can define a new real function $S[a, b, \Delta x]$ by

$$S[a, b, \Delta x] = f[a] \cdot \Delta x + f[a + \Delta x] \cdot \Delta x + f[a + 2\,\Delta x] \cdot \Delta x + \cdots + f[b'] \cdot \Delta x,$$

where $b' = a + h \cdot \Delta x$ and $a + (h+1) \cdot \Delta x > b$. This function has the properties of summation such as

$$|S[a, b, \Delta x]| \le |f[a]| \cdot \Delta x + |f[a + \Delta x]| \cdot \Delta x + |f[a + 2\,\Delta x]| \cdot \Delta x + \cdots + |f[b']| \cdot \Delta x$$

$$|S[a, b, \Delta x]| \le \text{Max}[\,|f[x]| : x = a,\ a + \Delta x,\ a + 2\,\Delta x,\ \cdots,\ b'] \cdot (b - a),$$

We can say we have a sum of infinitesimal slices when we apply this function to an infinitesimal δx,

$$\int_a^b f[x]\,dx \approx \sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x} f[x]\cdot\delta x \text{ or } \int_a^b f[x]\,dx = \text{st}\Big[\sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x} f[x]\cdot\delta x\Big], \text{ when } \delta x \approx 0$$

Officially, we code the various summations with the functions like $S[a, b, \delta x]$ (in order to remove the function $f[x]$ as a variable.) We need to show that this is well-defined, that is, gives the same real standard part for every infinitesimal,

$S[a, b, \delta x] \approx S[a, b, \iota]$ and both are limited (so they have a common standard part.)

When $f[x]$ is continuous, we can show this "existence," but in the case of the Fundamental Theorem, if we know a real function $F[x]$ with $dF[x] = f[x]\, dx$ for all $a \leq x \leq b$, the proof in Section 1 interpreted with the extended summation functions and extended maximum functions proves this "existence" at the same time it shows that the value is $F[b] - F[a]$. The only ingredient needed to make this work is that

$$\text{Max}[\,|\varepsilon[x, \delta x]| : x = a,\ a+\delta x, a+2\,\delta x,\ \cdots,\ b'\,] = \varepsilon[a + k\,\delta x, \delta x] \approx 0$$

This follows from the Uniform Differentiability Theorem above when we take one of the equivalent conditions as the definition of "$dF[x] = f[x]\, dx$ for all $a \leq x \leq b$."

Notice that $\varepsilon[x, \Delta x]$ is the real function $\frac{f[x+\Delta x]-f[x]}{\Delta x} - f'[x]$, so we can define an infinite sum by extending the real function

$$S_\varepsilon[a, b, \Delta x] = \sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x} |\varepsilon| \cdot \delta x$$

For more details see **Foundations of Infinitesimal Calculus** p.51-53.

The Local Inverse Function Theorem

In **A Local Inverse Function Theorem**, *Victoria Symposium on Nonstandard Analysis*, Springer Verlag Lecture Notes in Math, vol 369, 1974) Michael Behrens noticed that the inverse function theorem is true for a function with a uniform derivative even just at one point. (It is NOT true for a pointwise derivative.) Specifically, condition (d) of the Uniform Differentiability Theorem makes the intuitive proof of Section 1 work.

Theorem: The Inverse Function Theorem

If m is a nonzero real number and the real function $f[x]$ is defined for all $x \approx x_0$, a real x_0 with $y_0 = f[x_0]$ and $f[x]$ satisfies

$$\frac{f[x_2]-f[x_1]}{x_2 - x_1} \approx m \text{ whenever } x_1 \approx x_2 \approx x_0$$

then $f[x]$ has an inverse function in a small neighborhood of x_0, that is, there is a real number $\Delta > 0$ and a smooth real function $g[y]$ defined when $|y - y_0| < \Delta$ with $f[g[y]] = y$ and there is a real $\varepsilon > 0$ such that if $|x - x_0| < \varepsilon$, then $|f[x] - y_0| < \Delta$ and $g[f[x]] = x$.

Proof

This proof introduces a "permanence principle." When a logical real formula is true for all infinitesimals, it must remain true out to some positive real number. We know that the statement

$$|x - x_0| < \delta \Rightarrow f[x] \text{ is defined}$$

is true whenever $\delta \approx 0$. Suppose that for every positive real number Δ there was a real point r with $|r - x_0| < \Delta$ where $f[r]$ was not defined. We could define a real function $U[\Delta] = r$. Then the logical real statement

$$\Delta > 0 \Rightarrow (r = U[\Delta],\ |r - x_0| < \Delta,\ f[r] \text{ is undefined})$$

is true. The Function Extension Axiom means it must also be true with $\Delta = \delta \approx 0$, a contradiction, hence, there is a positive real Δ so that $f[x]$ is defined whenever $|x - x_0| < \Delta$.

We complete the proof of the Inverse Function Theorem by a permanence principle on the domain of y-values where we can invert $f[x]$. The intuitive proof of Section 1 shows that whenever $|y - y_0| < \delta \approx 0$, we have $|x_1 - x_0| \approx 0$, and for every natural n and k,

$$|x_n - x_0| < 2|x_1 - x_0|,\ |x_{n+k} - x_k| < \frac{1}{2^{k-1}}|x_1 - x_0|,\ f[x_n] \text{ is defined, } |y - f[x_{n+1}]| < \frac{1}{2}|y - f[x_n]|$$

Recall that we re-focus our infinitesimal microscope after each step in the recursion. The term $y - f[x_{n+1}]$ is the error at the n^{th} step of solving the linear equation rather than the nonlinear one, and we can't see this error at the scale of our microscope, $|y - f[x_n]|$. Technically we write the differential approximation

$$f[x_2] = f[x_1 + \frac{1}{m}(y - f[x_n])] = f[x_1] + m\cdot(\frac{1}{m}(y - f[x_n])) + \iota\cdot(\frac{1}{m}(y - f[x_n]))$$

$$f[x_2] - y = \iota\cdot(\frac{1}{m}(y - f[x_n])), \text{ with } \iota \approx 0$$

Now by the permanence principle, there is a real $\Delta > 0$ so that whenever $|y - y_0| < \Delta$, the properties above hold, making the sequence x_n convergent. Define $g[y] = \operatorname{Lim}_{n\to\infty} x_n$.

For more details see **Foundations of Infinitesimal Calculus** p.66 - 68.

Second Differences and Higher Order Smoothness

In Section 1 we derived Leibniz' second derivative formula for the radius of curvature of a curve. We actually used infinitesimal second differences, rather than second derivatives and a complete justification requires some more work. We conclude this Section with a result connecting higher order infinitesimal differences and interated derivatives.

One way to re-state the Uniform First Derivative Theorem above is: The curve $y = f[x]$ is smooth if and only if the line through any two pairs of infinitely close points on the curve is near the same real line,

$$x_1 \approx x_2 \Rightarrow \frac{f[x_1] - f[x_2]}{x_1 - x_2} \approx m$$

A natural way to extend this is to ask: What is the parabola through three infinitely close points? Is the (standard part) of it independent of the choice of the triple? In **A Discrete Condition for Higher-Order Smoothness**, *Boletim da Sociedade Portugesa de Matematica*, n.35, Outtubro de 1996, p. 81-94, Vitor Neves and I show:

Theorem: Theorem on Higher Order Smoothness

Let $f[x]$ be a real function defined on a real open interval (α, ω). Then $f[x]$ is n-times continuously differentiable on (α, ω) if and only if the n^{th}-order differences δ^nf are S-continuous on (α, ω). In this case, the coefficients of the interpolating polynomial are near the coefficients of the Taylor polynomial,

$$\delta^n f[x_0, \ldots, x_n] \approx \frac{1}{n!} f^{(n)}[b]$$

whenever the interpolating points satisfy $x_1 \approx \cdots \approx x_n \approx b$.

For more details see **Foundations of Infinitesimal Calculus** p.108.