Mathematical Background: Foundations of Infinitesimal Calculus

second edition

by

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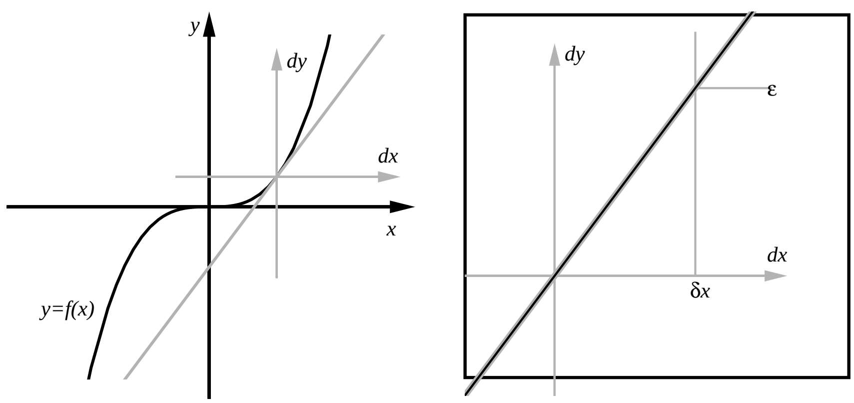

Figure 0.1: A Microscopic View of the Tangent

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Preface to the Mathematical Background

We want you to reason with mathematics. We are not trying to get everyone to give formalized proofs in the sense of contemporary mathematics; 'proof' in this course means 'convincing argument.' We expect you to use correct reasoning and to give careful explanations. The projects bring out these issues in the way we find best for most students, but the pure mathematical questions also interest some students. This book of mathematical "background" shows how to fill in the mathematical details of the main topics from the course. These proofs are completely rigorous in the sense of modern mathematics – technically bulletproof. We wrote this book of foundations in part to provide a convenient reference for a student who might like to see the "theorem - proof" approach to calculus.

We also wrote it for the interested instructor. In re-thinking the presentation of beginning calculus, we found that a simpler basis for the theory was both possible and desirable. The pointwise approach most books give to the theory of derivatives spoils the subject. Clear simple arguments like the proof of the Fundamental Theorem at the start of Chapter 5 below are not possible in that approach. The result of the pointwise approach is that instructors feel they have to either be dishonest with students or disclaim good intuitive approximations. This is sad because it makes a clear subject seem obscure. It is also unnecessary – by and large, the intuitive ideas work provided your notion of derivative is strong enough. This book shows how to bridge the gap between intuition and technical rigor.

A function with a positive derivative ought to be increasing. After all, the slope is positive and the graph is supposed to look like an increasing straight line. How could the function NOT be increasing? Pointwise derivatives make this bizarre thing possible - a positive "derivative" of a non-increasing function. Our conclusion is simple. That definition is WRONG in the sense that it does NOT support the intended idea.

You might agree that the counterintuitive consequences of pointwise derivatives are unfortunate, but are concerned that the traditional approach is more "general." Part of the point of this book is to show students and instructors that nothing of interest is lost and a great deal is gained in the straightforward nature of the proofs based on "uniform" derivatives. It actually is not possible to give a *formula* that is pointwise differentiable and not uniformly differentiable. The pieced together pointwise counterexamples seem contrived and out-of-place in a course where students are learning valuable new rules. It is a theorem that derivatives computed by rules are automatically continuous where defined. We want the course development to emphasize good intuition and positive results. This background shows that the approach is sound.

This book also shows how the pathologies arise in the traditional approach – we left pointwise pathology out of the main text, but present it here for the curious and for comparison. Perhaps only math majors ever need to know about these sorts of examples, but they are fun in a negative sort of way.

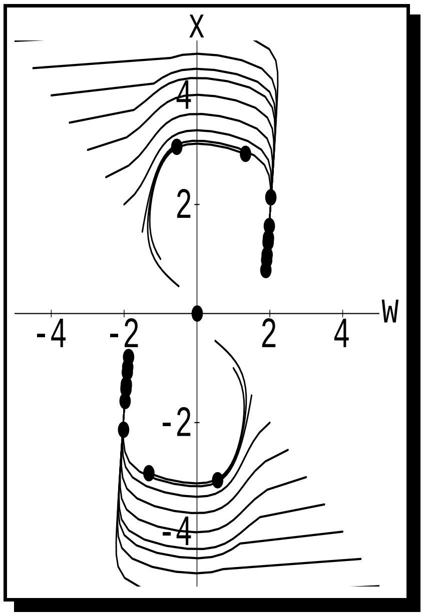
This book also has several theoretical topics that are hard to find in the literature. It includes a complete self-contained treatment of Robinson's modern theory of infinitesimals, first discovered in 1961. Our simple treatment is due to H. Jerome Keisler from the 1970's. Keisler's elementary calculus using infinitesimals is sadly out of print. It used pointwise derivatives, but had many novel ideas, including the first modern use of a microscope to describe the derivative. (The l'Hospital/Bernoulli calculus text of 1696 said curves consist of infinitesimal straight segments, but I do not know if that was associated with a magnifying transformation.) Infinitesimals give us a very simple way to understand the uniform derivatives, although this can also be clearly understood using function limits as in the text by Lax, et al, from the 1970s. Modern graphical computing can also help us "see" graphs converge as stressed in our main materials and in the interesting Uhl, Porta, Davis, *Calculus & Mathematica* text.

Almost all the theorems in this book are well-known old results of a carefully studied subject. The well-known ones are more important than the few novel aspects of the book. However, some details like the converse of Taylor's theorem – both continuous and discrete – are not so easy to find in traditional calculus sources. The microscope theorem for differential equations does not appear in the literature as far as we know, though it is similar to research work of Francine and Marc Diener from the 1980s.

We conclude the book with convergence results for Fourier series. While there is nothing novel in our approach, these results have been lost from contemporary calculus and deserve to be part of it. Our development follows Courant's calculus of the 1930s giving wonderful results of Dirichlet's era in the 1830s that clearly settle some of the convergence mysteries of Euler from the 1730s. This theory and our development throughout is usually easy to apply. "Clean" theory should be the servant of intuition – building on it and making it stronger and clearer.

There is more that is novel about this "book." It is free and it is not a "book" since it is not printed. Thanks to small marginal cost, our publisher agreed to include this electronic text on CD at no extra cost. We also plan to distribute it over the world wide web. We hope our fresh look at the foundations of calculus will stimulate your interest. Decide for yourself what's the best way to understand this wonderful subject. Give your own proofs.

Contents



Part 1

Numbers and Functions

CHAPTER 1 Numbers

This chapter gives the algebraic laws of the number systems used in calculus.

Numbers represent various idealized measurements. Positive integers may count items, fractions may represent a part of an item or a distance that is part of a fixed unit. Distance measurements go beyond rational numbers as soon as we consider the hypotenuse of a right triangle or the circumference of a circle. This extension is already in the realm of imagined "perfect" measurements because it corresponds to a perfectly straight-sided triangle with perfect right angle, or a perfectly round circle. Actual real measurements are always rational and have some error or uncertainty.

The various "imaginary" aspects of numbers are very useful fictions. The rules of computation with perfect numbers are much simpler than with the error-containing real measurements. This simplicity makes fundamental ideas clearer.

Hyperreal numbers have 'teeny tiny numbers' that will simplify approximation estimates. Direct computations with the ideal numbers produce symbolic approximations equivalent to the function limits needed in differentiation theory (that the rules of Theorem 1.12 give a direct way to compute.) Limit theory does not give the answer, but only a way to justify it once you have found it.

1.1 Field Axioms

The laws of algebra follow from the field axioms. This means that algebra is the same with Dedekind's "real" numbers, the complex numbers, and Robinson's "hyperreal" numbers.

Axiom 1.1. *Field Axioms*

A "field" of numbers is any set of objects together with two operations, addition and multiplication where the operations satisfy:

- *The commutative laws of addition and multiplication,*

$$a_1 + a_2 = a_2 + a_1 \qquad \& \qquad a_1 \cdot a_2 = a_2 \cdot a_1$$

- *The associative laws of addition and multiplication,*

$$a_1 + (a_2 + a_3) = (a_1 + a_2) + a_3 \qquad \& \qquad a_1 \cdot (a_2 \cdot a_3) = (a_1 \cdot a_2) \cdot a_3$$

- *The distributive law of multiplication over addition,*

$$a_1 \cdot (a_2 + a_3) = a_1 \cdot a_2 + a_1 \cdot a_3$$

- *There is an additive identity,* 0, *with* $0 + a = a$ *for every number* a.
- *There is an multiplicative identity,* 1, *with* $1 \cdot a = a$ *for every number* $a \neq 0$.
- *Each number* a *has an additive inverse,* $-a$, *with* $a + (-a) = 0$.
- *Each nonzero number* a *has a multiplicative inverse,* $\frac{1}{a}$, *with* $a \cdot \frac{1}{a} = 1$.

A computation needed in calculus is

Example 1.1. *The Cube of a Binomial*

$$\begin{aligned}(x + \Delta x)^3 &= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 \\ &= x^3 + 3x^2\Delta x + (\Delta x(3x + \Delta x))\,\Delta x\end{aligned}$$

We analyze the term $\varepsilon = (\Delta x(3x + \Delta x))$ in differentiation.

The reader could laboriously demonstrate that only the field axioms are needed to perform the computation. This means it holds for rational, real, complex, or hyperreal numbers. Here is a start. Associativity is needed so that the cube is well defined, or does not depend on the order we multiply. We use this in the next computation, then use the distributive property, the commutativity and the distributive property again, and so on.

$$\begin{aligned}(x + \Delta x)^3 &= (x + \Delta x)(x + \Delta x)(x + \Delta x) \\ &= (x + \Delta x)((x + \Delta x)(x + \Delta x)) \\ &= (x + \Delta x)((x + \Delta x)x + (x + \Delta x)\Delta x) \\ &= (x + \Delta x)((x^2 + x\Delta x) + (x\Delta x + \Delta x^2)) \\ &= (x + \Delta x)(x^2 + x\Delta x + x\Delta x + \Delta x^2) \\ &= (x + \Delta x)(x^2 + 2x\Delta x + \Delta x^2) \\ &= (x + \Delta x)x^2 + (x + \Delta x)2x\Delta x + (x + \Delta x)\Delta x^2) \\ &\vdots\end{aligned}$$

The natural counting numbers $1, 2, 3, \ldots$ have operations of addition and multiplication, but do not satisfy all the properties needed to be a field. Addition and multiplication do satisfy the commutative, associative, and distributive laws, but there is no additive inverse

0 in the counting numbers. In ancient times, it was controversial to add this element that could stand for counting nothing, but it is a useful fiction in many kinds of computations.

The negative integers $-1, -2, -3, \ldots$ are another idealization added to the natural numbers that make additive inverses possible - they are just new numbers with the needed property. Negative integers have perfectly concrete interpretations such as measurements to the left, rather than the right, or amounts owed rather than earned.

The set of all integers; positive, negative, and zero, still do not form a field because there are no multiplicative inverses. Fractions, $\pm 1/2$, $\pm 1/3$, ... are the needed additional inverses. When they are combined with the integers through addition, we have the set of all rational numbers of the form $\pm p/q$ for natural numbers p and $q \neq 0$. The rational numbers are a field, that is, they satisfy all the axioms above. In ancient times, rationals were sometimes considered only "operators" on "actual" numbers like $1, 2, 3, \ldots$.

The point of the previous paragraphs is simply that we often extend one kind of number system in order to have a new system with useful properties. The complex numbers extend the field axioms above beyond the "real" numbers by adding a number $\mathbf{i}$ that solves the equation $x^2 = -1$. (See the CD Chapter 29 of the main text.) Hundreds of years ago this number was controversial and is still called "imaginary." In fact, all numbers are useful constructs of our imagination and some aspects of Dedekind's "real" numbers are much more abstract than $\mathbf{i}^2 = -1$. (For example, since the reals are "uncountable," "most" real numbers have no description what-so-ever.)

The rationals are not "complete" in the sense that the linear measurement of the side of an equilateral right triangle ($\sqrt{2}$) cannot be expressed as p/q for p and q integers. In Section 1.3 we "complete" the rationals to form Dedekind's "real" numbers. These numbers correspond to perfect measurements along an ideal line with no gaps.

The complex numbers cannot be ordered with a notion of "smaller than" that is compatible with the field operations. Adding an "ideal" number to serve as the square root of -1 is not compatible with the square of every number being positive. When we make extensions beyond the real number system we need to make choices of the kind of extension depending on the properties we want to preserve.

Hyperreal numbers allow us to compute estimates or limits directly, rather than making inverse proofs with inequalities. Like the complex extension, hyperreal extension of the reals loses a property; in this case completeness. Hyperreal numbers are explained beginning in Section 1.4 below and then are used extensively in this background book to show how many intuitive estimates lead to simple direct proofs of important ideas in calculus.

The hyperreal numbers (discovered by Abraham Robinson in 1961) are still controversial because they contain infinitesimals. However, they are just another extended modern number system with a desirable new property. Hyperreal numbers can help you understand limits of real numbers and many aspects of calculus. Results of calculus could be proved without infinitesimals, just as they could be proved without real numbers by using only rationals. Many professors still prefer the former, but few prefer the latter. We believe that is only because Dedekind's "real" numbers are more familiar than Robinson's, but we will make it clear how both approaches work as a theoretical background for calculus.

There is no controversy concerning the logical soundness of hyperreal numbers. The use of infinitesimals in the early development of calculus beginning with Leibniz, continuing with Euler, and persisting to the time of Gauss was problematic. The founders knew that their use of infinitesimals was logically incomplete and could lead to incorrect results. Hyperreal numbers are a correct treatment of infinitesimals that took nearly 300 years to discover.

With hindsight, they also have a simple description. The Function Extension Axiom 2.1 explained in detail in Chapter 2 was the missing key.

Exercise set 1.1

1. *Show that the identity numbers* 0 *and* 1 *are unique. (HINT: Suppose* $0' + a = a$*. Add* $-a$ *to both sides.)*
2. *Show that* $0 \cdot a = 0$*. (HINT: Expand* $\left(0 + \frac{b}{a}\right) \cdot a$ *with the distributive law and show that* $0 \cdot a + b = b$*. Then use the previous exercise.)*
3. *The inverses* $-a$ *and* $\frac{1}{a}$ *are unique. (HINT: Suppose not,* $0 = a - a = a + b$*. Add* $-a$ *to both sides and use the associative property.)*
4. *Show that* $-1 \cdot a = -a$*. (HINT: Use the distributive property on* $0 = (1-1) \cdot a$ *and use the uniqueness of the inverse.)*
5. *Show that* $(-1) \cdot (-1) = 1$*.*
6. *Other familiar properties of algebra follow from the axioms, for example, if* $a_3 \neq 0$ *and* $a_4 \neq 0$*, then*

$$\frac{a_1 + a_2}{a_3} = \frac{a_1}{a_3} + \frac{a_2}{a_3} \quad , \quad \frac{a_1 \cdot a_2}{a_3 \cdot a_4} = \frac{a_1}{a_3} \cdot \frac{a_2}{a_4} \quad \& \quad a_3 \cdot a_4 \neq 0$$

1.2 Order Axioms

Estimation is based on the inequality $\leq$ *of the real numbers.*

One important representation of rational and real numbers is as measurements of distance along a line. The additive identity 0 is located as a starting point and the multiplicative identity 1 is marked off (usually to the right on a horizontal line). Distances to the right correspond to positive numbers and distances to the left to negative ones. The inequality $<$ indicates which numbers are to the left of others. The abstract properties are as follows.

Axiom 1.2. *Ordered Field Axioms*

A a number system is an ordered field if it satisfies the field Axioms 1.1 *and has a relation* $<$ *that satisfies:*

- *Every pair of numbers* a *and* b *satisfies exactly one of the relations*

  $a = b$, $a < b$, *or* $b < a$
- *If* $a < b$ *and* $b < c$*, then* $a < c$*.*
- *If* $a < b$*, then* $a + c < b + c$*.*
- *If* $0 < a$ *and* $0 < b$*, then* $0 < a \cdot b$*.*

These axioms have simple interpretations on the number line. The first order axiom says that every two numbers can be compared; either two numbers are equal or one is to the left of the other.

The second axiom, called transitivity, says that if a is left of b and b is left of c, then a is left of c.

The third axiom says that if a is left of b and we move both by a distance c, then the results are still in the same left-right order.

The fourth axiom is the most difficult abstractly. All the compatibility with multiplication is built from it.

The rational numbers satisfy all these axioms, as do the real and hyperreal numbers. The complex numbers cannot be ordered in a manner compatible with the operations of addition and multiplication.

Definition 1.3. *Absolute Value*
If a is a nonzero number in an ordered field, $|a|$ is the larger of a and $-a$, that is, $|a| = a$ if $-a < a$ and $|a| = -a$ if $a < -a$. We let $|0| = 0$.

Exercise set 1.2

1. *If $0 < a$, show that $-a < 0$ by using the additive property.*
2. *Show that $0 < 1$. (HINT: Recall the exercise that $(-1) \cdot (-1) = 1$ and argue by contradiction, supposing $0 < -1$.)*
3. *Show that $a \cdot a > 0$ for every $a \neq 0$.*
4. *Show that there is no order $<$ on the complex numbers that satisfies the ordered field axioms.*
5. *Prove that if $a < b$ and $c > 0$, then $c \cdot a < c \cdot b$.*
   Prove that if $0 < a < b$ and $0 < c < d$, then $c \cdot a < d \cdot b$.

1.3 The Completeness Axiom

Dedekind's "real" numbers represent points on an ideal line with no gaps.

The number $\sqrt{2}$ is not rational. Suppose to the contrary that $\sqrt{2} = q/r$ for integers q and r with no common factors. Then $2r^2 = q^2$. The prime factorization of both sides must be the same, but the factorization of the squares have an even number distinct primes on each side and the 2 factor is left over. This is a contradiction, so there is no rational number whose square is 2.

A length corresponding to $\sqrt{2}$ can be approximated by (rational) decimals in various ways, for example, $1 < 1.4 < 1.41 < 1.414 < 1.4142 < 1.41421 < 1.414213 < \ldots$. There is no rational for this sequence to converge to, even though it is "trying" to converge. For example, all the terms of the sequence are below $1.41422 < 1.4143 < 1.415 < 1.42 < 1.5 < 2$. Even without remembering a fancy algorithm for finding square root decimals, you can test the successive decimal approximations by squaring, for example, $1.41421^2 = 1.9999899241$ and $1.41422^2 = 2.0000182084$.

It is perfectly natural to add a new number to the rationals to stand for the limit of the better and better approximations to $\sqrt{2}$. Similarly, we could devise approximations to π and make π the number that stands for the limit of such successive approximations. We would like a method to include "all such possible limits" without having to specify the particular approximations. Dedekind's approach is to let the real numbers be the collection of all "cuts" on the rational line.

Definition 1.4. *A Dedekind Cut*

A "cut" in an ordered field is a pair of nonempty sets A and B so that:

- *Every number is either in A or B.*
- *Every a in A is less than every b in B.*

We may think of $\sqrt{2}$ defining a cut of the rational numbers where A consists of all rational numbers a with $a < 0$ or $a^2 < 2$ and B consists of all rational numbers b with $b^2 > 2$. There is a "gap" in the rationals where we would like to have $\sqrt{2}$. Dedekind's "real numbers" fill all such gaps. In this case, a cut of real numbers would have to have $\sqrt{2}$ either in A or in B.

Axiom 1.5. *Dedekind Completeness*

The real numbers are an ordered field such that if A and B form a cut in those numbers, there is a number r such that r is in either A or in B and all other the numbers in A satisfy $a < r$ and in B satisfy $r < b$.

In other words, every cut on the "real" line is made at some specific number r, so there are no gaps. This seems perfectly reasonable in cases like $\sqrt{2}$ and π where we know specific ways to describe the associated cuts. The only drawback to Dedekind's number system is that "every cut" is not a very concrete notion, but rather relies on an abstract notion of "every set." This leads to some paradoxical facts about cuts that do not have specific descriptions, but these need not concern us. Every specific cut has a real number in the middle.

Completeness of the reals means that "approximation procedures" that are "improving" converge to a number. We need to be more specific later, but for example, bounded increasing or decreasing sequences converge and "Cauchy" sequences converge. We will not describe these details here, but take them up as part of our study of limits below.

Completeness has another important consequence, the Archimedean Property Theorem 1.8. We take that up in the next section. The Archimedean Property says precisely that the real numbers contain no positive infinitesimals. Hyperreal numbers extend the reals by including infinitesimals. (As a consequence the hyperreals are not Dedekind complete.)

1.4 Small, Medium and Large Numbers

Hyperreal numbers give us a way to simplify estimation by adding infinitesimal numbers to the real numbers.

We want to have three different intuitive sizes of numbers, very small, medium size, and very large. Most important, we want to be able to compute with these numbers using the same rules of algebra as in high school and separate the 'small' parts of our computation. Hyperreal numbers give us these computational estimates. Hyperreal numbers satisfy three axioms which we take up separately below, Axiom 1.7, Axiom 1.9, and Axiom 2.1.

As a first intuitive approximation, we could think of these scales of numbers in terms of the computer screen. In this case, 'medium' numbers might be numbers in the range -999 to + 999 that name a screen pixel. Numbers closer than one unit could not be distinguished by different screen pixels, so these would be 'tiny' numbers. Moreover, two medium numbers a and b would be indistinguishably close, $a \approx b$, if their difference was a 'tiny' number less than a pixel. Numbers larger in magnitude than 999 are too big for the screen and could be considered 'huge.'

The screen distinction sizes of computer numbers is a good analogy, but there are difficulties with the algebra of screen - size numbers. We want to have ordinary rules of algebra and the following properties of approximate equality. For now, all you should think of is that $\approx$ means 'approximately equals.'

(a) If p and q are medium, so are $p + q$ and $p \cdot q$.
(b) If ε and δ are tiny, so is $\varepsilon + \delta$, that is, $\varepsilon \approx 0$ and $\delta \approx 0$ implies $\varepsilon + \delta \approx 0$.
(c) If $\delta \approx 0$ and q is medium, then $q \cdot \delta \approx 0$.
(d) $1/0$ is still undefined and $1/x$ is huge only when $x \approx 0$.

You can see that the computer number idea does not quite work, because the approximation rules don't always apply. If $p = 15.37$ and $q = -32.4$, then $p \cdot q = -497.998 \approx -498$, 'medium times medium is medium,' however, if $p = 888$ and $q = 777$, then $p \cdot q$ is no longer screen size...

The hyperreal numbers extend the 'real' number system to include 'ideal' numbers that obey these simple approximation rules as well as the ordinary rules of algebra and trigonometry. Very small numbers technically are called infinitesimals and what we shall assume that is different from high school is that there are positive infinitesimals.

Definition 1.6. *Infinitesimal Number*
A number δ in an ordered field is called infinitesimal if it satisfies

$$\frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \cdots > \frac{1}{m} > \cdots > |\delta|$$

for any ordinary natural counting number $m = 1, 2, 3, \cdots$. We write $a \approx b$ and say a is infinitely close to b if the number $b - a \approx 0$ is infinitesimal.

This definition is intended to include 0 as "infinitesimal."

Axiom 1.7. *The Infinitesimal Axiom*
The hyperreal numbers contain the real numbers, but also contain nonzero infinitesimal numbers, that is, numbers $\delta \approx 0$, positive, $\delta > 0$, but smaller than all the real positive numbers.

This stands in contrast to the following result.

Theorem 1.8. *The Archimedean Property*
The hyperreal numbers are not Dedekind complete and there are no positive infinitesimal numbers in the ordinary reals, that is, if $r > 0$ is a positive real number, then there is a natural counting number m such that $0 < \frac{1}{m} < r$.

PROOF:

We define a cut above all the positive infinitesimals. The set A consists of all numbers a satisfying $a < 1/m$ for every natural counting number m. The set B consists of all numbers b such that there is a natural number m with $1/m < b$. The pair A, B defines a Dedekind cut in the rationals, reals, and hyperreal numbers. If there is a positive δ in A, then there cannot be a number at the gap. In other words, there is no largest positive infinitesimal or smallest positive non-infinitesimal. This is clear because $\delta < \delta+\delta$ and 2δ is still infinitesimal, while if ε is in B, $\varepsilon/2 < \varepsilon$ must also be in B.

Since the real numbers must have a number at the "gap," there cannot be any positive infinitesimal reals. Zero is at the gap in the reals and every positive real number is in B. This is what the theorem asserts, so it is proved. Notice that we have also proved that the hyperreals are not Dedekind complete, because the cut in the hyperreals must have a gap.

Two ordinary real numbers, a and b, satisfy $a \approx b$ only if $a = b$, since the ordinary real numbers do not contain infinitesimals. Zero is the only real number that is infinitesimal.

If you prefer not to say 'infinitesimal,' just say 'δ is a tiny positive number' and think of $\approx$ as 'close enough for the computations at hand.' The computation rules above are still important intuitively and can be phrased in terms of limits of functions if you wish. The intuitive rules help you find the limit.

The next axiom about the new "hyperreal" numbers says that you can continue to do the algebraic computations you learned in high school.

Axiom 1.9. *The Algebra Axiom (Including $<$ rules.)*
The hyperreal numbers are an ordered field, that is, they obey the same rules of ordered algebra as the real numbers, Axiom 1.1 *and Axiom* 1.2.

The algebra of infinitesimals that you need can be learned by working the examples and exercises in this chapter.

Functional equations like the addition formulas for sine and cosine or the laws of logs and exponentials are very important. (The specific high school identities are reviewed in the main text CD Chapter 28 on High School Review.) The Function Extension Axiom 2.1 shows how to extend the non-algebraic parts of high school math to hyperreal numbers. This axiom is the key to Robinson's rigorous theory of infinitesimals and it took 300 years to discover. You will see by working with it that it is a perfectly natural idea, as hindsight often reveals. We postpone that to practice with the algebra of infinitesimals.

Example 1.2. *The Algebra of Small Quantities*

Let's re-calculate the increment of the basic cubic using the new numbers. Since the rules of algebra are the same, the same basic steps still work (see Example 1.1), except now we may take x any number and δx an infinitesimal.

Small Increment of $f[x] = x^3$

$$f[x + \delta x] = (x + \delta x)^3 = x^3 + 3x^2 \delta x + 3x\delta x^2 + \delta x^3$$
$$f[x + \delta x] = f[x] + 3x^2\ \delta x + (\delta x[3x + \delta x])\ \delta x$$
$$f[x + \delta x] = f[x] + f'[x]\ \delta x + \varepsilon\ \delta x$$

with $f'[x] = 3x^2$ and $\varepsilon = (\delta x[3x + \delta x])$. The intuitive rules above show that $\varepsilon \approx 0$ whenever x is finite. (See Theorem 1.12 and Example 1.8 following it for the precise rules.)

Example 1.3. *Finite Non-Real Numbers*

The hyperreal numbers obey the same rules of algebra as the familiar numbers from high school. We know that $r+\Delta > r$, whenever $\Delta > 0$ is an ordinary positive high school number. (See the addition property of Axiom 1.2.) Since hyperreals satisfy the same rules of algebra, we also have new finite numbers given by a high school number r plus an infinitesimal,

$$a = r + \delta > r$$

The number $a = r + \delta$ is different from r, even though it is infinitely close to r. Since δ is small, the difference between a and r is small

$$0 < a - r = \delta \approx 0 \qquad \text{or} \qquad a \approx r \quad \text{but} \quad a \neq r$$

Here is a technical definition of "finite" or "limited" hyperreal number.

Definition 1.10. *Limited and Unlimited Hyperreal Numbers*

A hyperreal number x is said to be finite (or limited) if there is an ordinary natural number $m = 1, 2, 3, \cdots$ so that

$$|x| < m.$$

If a number is not finite, we say it is infinitely large (or unlimited).

Ordinary real numbers are part of the hyperreal numbers and they are finite because they are smaller than the next integer after them. Moreover, every finite hyperreal number is near an ordinary real number (see Theorem 1.11 below), so the previous example is the most general kind of finite hyperreal number there is. The important thing is to learn to compute with approximate equalities.

Example 1.4. *A Magnified View of the Hyperreal Line*

Of course, infinitesimals are finite, since $\delta \approx 0$ implies that $|\delta| < 1$. The finite numbers are not just the ordinary real numbers and the infinitesimals clustered near zero. The rules of algebra say that if we add or subtract a nonzero number from another, the result is a different number. For example, $\pi - \delta < \pi < \pi + \delta$, when $0 < \delta \approx 0$. These are distinct finite hyperreal numbers but each of these numbers differ by only an infinitesimal, $\pi \approx \pi + \delta \approx \pi - \delta$. If we plotted the hyperreal number line at unit scale, we could only put one dot for all three. However, if we focus a microscope of power $1/\delta$ at π we see three points separated by unit distances.

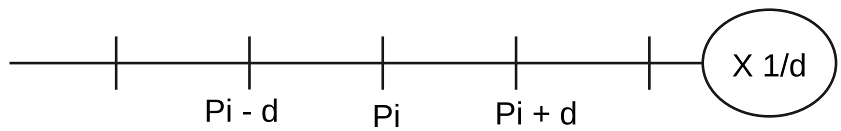


Figure 1.1: Magnification at Pi

The basic fact is that finite numbers only differ from reals by an infinitesimal. (This is equivalent to Dedekind's Completeness Axiom.)

Theorem 1.11. *Standard Parts of Finite Numbers*
Every finite hyperreal number x differs from some ordinary real number r by an infinitesimal amount, $x - r \approx 0$ or $x \approx r$. The ordinary real number infinitely near x is called the standard part of x, $r = \operatorname{st}(x)$.

PROOF:
Suppose x is a finite hyperreal. Define a cut in the real numbers by letting A be the set of all real numbers satisfying $a \leq x$ and letting B be the set of all real numbers b with $x < b$. Both A and B are nonempty because x is finite. Every a in A is below every b in B by transitivity of the order on the hyperreals. The completeness of the real numbers means that there is a real r at the gap between A and B. We must have $x \approx r$, because if $x - r > 1/m$, say, then $r + 1/(2m) < x$ and by the gap property would need to be in B.

A picture of the hyperreal number line looks like the ordinary real line at unit scale. We can't draw far enough to get to the infinitely large part and this theorem says each finite number is indistinguishably close to a real number. If we magnify or compress by new number amounts we can see new structure.

You still cannot divide by zero (that violates rules of algebra), but if δ is a positive infinitesimal, we can compute the following:

$$-\delta, \quad \delta^2, \quad \frac{1}{\delta} \qquad \text{What can we say about these quantities?}$$

The idealization of infinitesimals lets us have our cake and eat it too. Since $\delta \neq 0$, we can divide by δ. However, since δ is tiny, $1/\delta$ must be HUGE.

Example 1.5. *Negative infinitesimals*

In ordinary algebra, if $\Delta > 0$, then $-\Delta < 0$, so we can apply this rule to the infinitesimal number δ and conclude that $-\delta < 0$, since $\delta > 0$.

Example 1.6. *Orders of infinitesimals*

In ordinary algebra, if $0 < \Delta < 1$, then $0 < \Delta^2 < \Delta$, so $0 < \delta^2 < \delta$.

We want you to formulate this more exactly in the next exercise. Just assume δ is very small, but positive. Formulate what you want to draw algebraically. Try some small ordinary numbers as examples, like $\delta = 0.01$. Plot δ at unit scale and place δ^2 accurately on the figure.

Example 1.7. *Infinitely large numbers*

For real numbers if $0 < \Delta < 1/n$ then $n < 1/\Delta$. Since δ is infinitesimal, $0 < \delta < 1/n$ for every natural number $n = 1, 2, 3, \ldots$ Using ordinary rules of algebra, but substituting the infinitesimal δ, we see that $H = 1/\delta > n$ is larger than any natural number n (or is "infinitely large"), that is, $1 < 2 < 3 < \ldots < n < H$, for every natural number n. We can "see" infinitely large numbers by turning the microscope around and looking in the other end.

The new algebraic rules are the ones that tell us when quantities are infinitely close, $a \approx b$. Such rules, of course, do not follow from rules about ordinary high school numbers, but the rules are intuitive and simple. More important, they let us 'calculate limits' directly.

Theorem 1.12. *Computation Rules for Finite and Infinitesimal Numbers*

(a) *If p and q are finite, so are $p + q$ and $p \cdot q$.*
(b) *If ε and δ are infinitesimal, so is $\varepsilon + \delta$.*
(c) *If $\delta \approx 0$ and q is finite, then $q \cdot \delta \approx 0$. (finite x infsml = infsml)*
(d) $1/0$ *is still undefined and $1/x$ is infinitely large only when $x \approx 0$.*

To understand these rules, just think of p and q as "fixed," if large, and δ as being as small as you please (but not zero). It is not hard to give formal proofs from the definitions above, but this intuitive understanding is more important. The last rule can be "seen" on the graph of $y = 1/x$. Look at the graph and move down near the values $x \approx 0$.

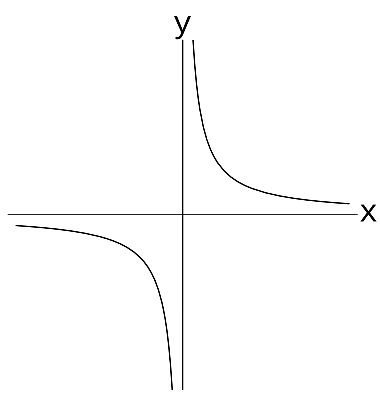


Figure 1.2: $y = 1/x$

PROOF:

We prove rule (c) and leave the others to the exercises. If q is finite, there is a natural number m so that $|q| < m$. We want to show that $|q \cdot \delta| < 1/n$ for any natural number n. Since δ is infinitesimal, we have $|\delta| < 1/(n \cdot m)$. By Exercise 1.2.5, $|q| \cdot |\delta| < m \cdot \frac{1}{n \cdot m} = \frac{1}{m}$.

Example 1.8. $y = x^3 \Rightarrow dy = 3x^2\, dx$*, for finite x*

The error term in the increment of $f[x] = x^3$, computed above is

$$\varepsilon = (\delta x[3x + \delta x])$$

If x is assumed finite, then $3x$ is also finite by the first rule above. Since $3x$ and δx are finite, so is the sum $3x + \delta x$ by that rule. The third rule, that says an infinitesimal times a finite number is infinitesimal, now gives $\delta x\times$ finite $= \delta x[3x + \delta x] =$ infinitesimal, $\varepsilon \approx 0$. This justifies the local linearity of x^3 at finite values of x, that is, we have used the approximation rules to show that

$$f[x + \delta x] = f[x] + f'[x]\ \delta x + \varepsilon\ \delta x$$

with $\varepsilon \approx 0$ whenever $\delta x \approx 0$ and x is finite, where $f[x] = x^3$ and $f'[x] = 3\,x^2$.

Exercise set 1.4

1. *Draw the view of the ideal number line when viewed under an infinitesimal microscope of power $1/\delta$. Which number appears unit size? How big does δ^2 appear at this scale? Where do the numbers δ and δ^3 appear on a plot of magnification $1/\delta^2$?*

2. Backwards microscopes or compression
Draw the view of the new number line when viewed under an infinitesimal microscope with its magnification reversed to power δ (not $1/\delta$). What size does the infinitely large number H (HUGE) appear to be? What size does the finite (ordinary) number $m = 10^9$ appear to be? Can you draw the number H^2 on the plot?

3. $y = x^p \Rightarrow dy = p\,x^{p-1}\ dx,\ p = 1, 2, 3, \ldots$
For each $f[x] = x^p$ below:

(a) *Compute $f[x + \delta x] - f[x]$ and simplify, writing the increment equation:*

$$\begin{aligned} f[x + \delta x] - f[x] &= f'[x] \cdot \delta x + \varepsilon \cdot \delta x \\ &= [\textit{term in } x \textit{ but not } \delta x]\delta x + [\textit{observed microscopic error}]\delta x \end{aligned}$$

Notice that we can solve the increment equation for $\varepsilon = \dfrac{f[x + \delta x] - f[x]}{\delta x} - f'[x]$

(b) *Show that $\varepsilon \approx 0$ if $\delta x \approx 0$ and x is finite. Does x need to be finite, or can it be any hyperreal number and still have $\varepsilon \approx 0$?*

(1) *If $f[x] = x^1$, then $f'[x] = 1x^0 = 1$ and $\varepsilon = 0$.*
(2) *If $f[x] = x^2$, then $f'[x] = 2x$ and $\varepsilon = \delta x$.*
(3) *If $f[x] = x^3$, then $f'[x] = 3x^2$ and $\varepsilon = (3x + \delta x)\delta x$.*
(4) *If $f[x] = x^4$, then $f'[x] = 4x^3$ and $\varepsilon = (6x^2 + 4x\delta x + \delta x^2)\delta x$.*
(5) *If $f[x] = x^5$, then $f'[x] = 5x^4$ and $\varepsilon = (10x^3 + 10x^2\delta x + 5x\delta x^2 + \delta x^3)\delta x$.*

4. Exceptional Numbers and the Derivative of $y = \dfrac{1}{x}$

(a) *Let $f[x] = 1/x$ and show that*

$$\frac{f[x + \delta x] - f[x]}{\delta x} = \frac{-1}{x(x + \delta x)}$$

(b) *Compute*

$$\varepsilon = \frac{-1}{x(x + \delta x)} + \frac{1}{x^2} = \delta x \cdot \frac{1}{x^2(x + \delta x)}$$

(c) *Show that this gives*

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

when $f'[x] = -1/x^2$.

(d) *Show that $\varepsilon \approx 0$ provided x is NOT infinitesimal (and in particular is not zero.)*

5. Exceptional Numbers and the Derivative of $y = \sqrt{x}$

(a) *Let* $f[x] = \sqrt{x}$ *and compute*

$$f[x + \delta x] - f[x] = \frac{1}{\sqrt{x + \delta x} + \sqrt{x}}$$

(b) *Compute*

$$\varepsilon = \frac{1}{\sqrt{x + \delta x} + \sqrt{x}} - \frac{1}{2\sqrt{x}} = \frac{-1}{2\sqrt{x}(\sqrt{x + \delta x} + \sqrt{x})^2} \cdot \delta x$$

(c) *Show that this gives*

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

when $f'[x] = \frac{1}{2\sqrt{x}}$.

(d) *Show that* $\varepsilon \approx 0$ *provided* x *is positive and NOT infinitesimal (and in particular is not zero.)*

6. *Prove the remaining parts of Theorem* 1.12.

CHAPTER 2 Functional Identities

In high school you learned that trig functions satisfy certain identities or that logarithms have certain "properties." This chapter extends the idea of functional identities from specific cases to a defining property of an unknown function.

The use of "unknown functions" is of fundamental importance in calculus, and other branches of mathematics and science. For example, differential equations can be viewed as identities for unknown functions.

One reason that students sometimes have difficulty understanding the meaning of derivatives or even the general rules for finding derivatives is that those things involve equations in unknown functions. The symbolic rules for differentiation and the increment approximation defining derivatives involve unknown functions. It is important for you to get used to this "higher type variable," an unknown function. This chapter can form a bridge between the specific identities of high school and the unknown function variables from rules of calculus and differential equations.

2.1 Specific Functional Identities

All the the identities you need to recall from high school are:

$(\text{Cos}[x])^2 + (\text{Sin}[x])^2 = 1$	CircleIden
$\text{Cos}[x+y] = \text{Cos}[x]\,\text{Cos}[y] - \text{Sin}[x]\,\text{Sin}[y]$	CosSum
$\text{Sin}[x+y] = \text{Sin}[x]\,\text{Cos}[y] + \text{Sin}[y]\,\text{Cos}[x]$	SinSum
$b^{x+y} = b^x\, b^y$	ExpSum
$(b^x)^y = b^{x\cdot y}$	RepeatedExp
$\text{Log}[x\cdot y] = \text{Log}[x] + \text{Log}[y]$	LogProd
$\text{Log}[x^p] = p\ \text{Log}[x]$	LogPower

but you must be able to *use* these identities. Some practice exercises using these familiar identities are given in main text CD Chapter 28.

2.2 General Functional Identities

A general functional identity is an equation which is satisfied by an unknown function (or a number of functions) over its domain.

The function

$$f[x] = 2^x$$

satisfies $f[x+y] = 2^{(x+y)} = 2^x 2^y = f[x]f[y]$, so eliminating the two middle terms, we see that the function $f[x] = 2^x$ satisfies the functional identity

(ExpSum) $$f[x+y] = f[x]\ f[y]$$

It is important to pay attention to the variable or variables in a functional identity. In order for an equation involving a function to be a functional identity, the equation must be valid for *all* values of the variables in question. Equation (ExpSum) above is satisfied by the function $f[x] = 2^x$ for all x and y. For the function $f[x] = x$, it is true that $f[2+2] = f[2]f[2]$, but $f[3+1] \neq f[3]f[1]$, so $= x$ does not satisfy functional identity (ExpSum).

Functional identities are a sort of 'higher laws of algebra.' Observe the notational similarity between the distributive law for multiplication over addition,

$$m \cdot (x+y) = m \cdot x + m \cdot y$$

and the additive functional identity

(Additive) $$f[x+y] = f[x] + f[y]$$

Most functions $f[x]$ do not satisfy the additive identity. For example,

$$\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y} \qquad \text{and} \qquad \sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$$

The fact that these are not identities means that for *some* choices of x and y in the domains of the respective functions $f[x] = 1/x$ and $f[x] = \sqrt{x}$, the two sides are not equal. You will show below that the only differentiable functions that do satisfy the additive functional identity are the functions $f[x] = m \cdot x$. In other words, the additive functional identity is nearly equivalent to the distributive law; the *only* unknown (differentiable) function that satisfies it *is* multiplication. Other functional identities such as the 7 given at the start of this chapter capture the most important features of the functions that satisfy the respective identities. For example, the pair of functions $f[x] = 1/x$ and $g[x] = \sqrt{x}$ do not satisfy the addition formula for the sine function, either.

Example 2.1. *The Microscope Equation*

The "microscope equation" defining the differentiability of a function $f[x]$ (see Chapter 5 of the text),

(Micro) $$f[x+\delta x] = f[x] + f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

with $\varepsilon \approx 0$ if $\delta x \approx 0$, is similar to a functional identity in that it involves an unknown function $f[x]$ and its related unknown derivative function $f'[x]$. It "relates" the function $f[x]$ to its derivative $\frac{df}{dx} = f'[x]$.

You should think of (Micro) as the definition of the derivative of $f[x]$ at a given x, but also keep in mind that (Micro) is the definition of the derivative of *any* function. If we let $f[x]$ vary over a number of different functions, we get different derivatives. The equation (Micro) can be viewed as an equation in which the function, $f[x]$, is the variable input, and the output is the derivative $\frac{df}{dx}$.

To make this idea clearer, we rewrite (Micro) by solving for $\frac{df}{dx}$:)

$$\frac{df}{dx} = \frac{f[x+\delta x] - f[x]}{\delta x} - \varepsilon$$

or

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f[x+\delta x] - f[x]}{\Delta x}$$

If we plug in the "input" function $f[x] = x^2$ into this equation, the output is $\frac{df}{dx} = 2x$. If we plug in the "input" function $f[x] = \mathrm{Log}[x]$, the output is $\frac{df}{dx} = \frac{1}{x}$. The microscope equation involves unknown functions, but strictly speaking, it is not a functional identity, because of the error term ε (or the limit which can be used to formalize the error). It is only an approximate identity.

Example 2.2. *Rules of Differentiation*

The various "differentiation rules," the Superposition Rule, the Product Rule and the Chain Rule (from Chapter 6 of the text) are functional identities relating functions and their derivatives. For example, the Product Rule states:

$$\frac{d(f[x]g[x])}{dx} = \frac{df}{dx}g[x] + f[x]\frac{dg}{dx}$$

We can think of $f[x]$ and $g[x]$ as "variables" which vary by simply choosing different actual functions for $f[x]$ and $g[x]$. Then the Product Rule yields an identity between the choices of $f[x]$ and $g[x]$, and their derivatives. For example, choosing $f[x] = x^2$ and $g[x] = \mathrm{Log}[x]$ and plugging into the Product Rule yields

$$\frac{d(x^2 \,\mathrm{Log}[x])}{dx} = 2x\,\mathrm{Log}[x] + x^2\frac{1}{x}$$

Choosing $f[x] = x^3$ and $g[x] = \mathrm{Exp}[x]$ and plugging into the Product Rule yields

$$\frac{d(x^3 \,\mathrm{Exp}[x])}{dx} = 3x^2\,\mathrm{Exp}[x] + x^3\,\mathrm{Exp}[x]$$

If we choose $f[x] = x^5$, but do not make a specific choice for $g[x]$, plugging into the Product Rule will yield

$$\frac{d(x^5 g[x])}{dx} = 5x^4 g[x] + x^5\frac{dg}{dx}$$

The goal of this chapter is to extend your thinking to identities in unknown functions.

Exercise set 2.2

1. (a) *Verify that for any positive number, b, the function $f[x] = b^x$ satisfies the functional identity (ExpSum) above for all x and y.*

(b) *Is (ExpSum) valid (for all x and y) for the function $f[x] = x^2$ or $f[x] = x^3$? Justify your answer.*

2. *Define $f[x] = \text{Log}[x]$ where x is any positive number. Why does this $f[x]$ satisfy the functional identities*

(LogProd) $$f[x \cdot y] = f[x] + f[y]$$

and

(LogPower) $$f[x^k] = k f[x]$$

where x, y, and k are variables. What restrictions should be placed on x and y for the above equations to be valid? What is the domain of the logarithm?

3. *Find values of x and y so that the left and right sides of each of the additive formulas for $1/x$ and $\sqrt{x}$ above are not equal.*

4. *Show that $1/x$ and $\sqrt{x}$ also do not satisfy the identity (SinSum), that is,*

$$\frac{1}{x+y} = \frac{1}{x}\sqrt{y} + \sqrt{x}\,\frac{1}{y}$$

is false for some choices of x and y in the domains of these functions.

5. (a) *Suppose that $f[x]$ is an unknown function which is known to satisfy (LogProd) (so $f[x]$ behaves "like" $\text{Log}[x]$, but we don't know if $f[x]$ is $\text{Log}[x]$), and suppose that $f[0]$ is a well-defined number (even though we don't specify exactly what $f[0]$ is). Show that this function $f[x]$ must be the zero function, that is show that $f[x] = 0$ for every x. (Hint: Use the fact that $0 * x = 0$).*

(b) *Suppose that $f[x]$ is an unknown function which is known to satisfy (LogPower) for all $x > 0$ and all k. Show that $f[1]$ must equal 0, $f[1] = 0$. (Hint: Fix $x = 1$, and try different values of k).*

6. (a) *Let m and b be fixed numbers and define*

$$f[x] = m\,x + b$$

Verify that if $b = 0$, the above function satisfies the functional identity

(Mult) $$f[x] = x\ f[1]$$

for all x and that if $b \neq 0$, $f[x]$ will not satisfy (Mult) for all x (that is, given a nonzero b, there will be at least one x for which (Mult) is not true).

(b) *Prove that any function satisfying (Mult) also automatically satisfies the two functional identities*

(Additive) $$f[x + y] = f[x] + f[y]$$

and

(Multiplicative) $$f[x\,y] = x\ f[y]$$

for all x and y.

(c) *Suppose $f[x]$ is a function which satisfies (Mult) (and for now that is the only thing you know about $f[x]$). Prove that $f[x]$ must be of the form $f[x] = m \cdot x$, for some fixed number m (this is almost obvious).*

(d) *Prove that a general power function, $f[x] = mx^k$ where k is a positve integer and m is a fixed number, will not satisfy (Mult) for all x if $k \neq 1$, (that is, if $k \neq 1$, there will be at least one x for which (Mult) is not true).*

(e) *Prove that $f[x] = \mathrm{Sin}[x]$ does not satisfy the additive identity.*

(f) *Prove that $f[x] = 2^x$ does not satisfy the additive identity.*

7. (a) *Let $f[x]$ and $g[x]$ be unknown functions which are known to satisfy $f[1] = 2$, $\frac{df}{dx}(1) = 3$, $g(1) = -3$, $\frac{dg}{dx}(1) = 4$. Let $h(x) = f[x]g[x]$. Compute $\frac{dh}{dx}(1)$.*

(b) *Differentiate the general Product Rule identity to get a formula for*

$$\frac{d^2(fg)}{dx^2}$$

Use your rule to compute $\frac{d^2(h)}{dx^2}(1)$ if $\frac{d^2(f)}{dx^2}(1) = 5$ and $\frac{d^2(g)}{dx^2}(1) = -2$, using other values from part 1 of this exercise.

2.3 The Function Extension Axiom

This section shows that all real functions have hyperreal extensions that are "natural" from the point of view of properties of the original function.

Roughly speaking, the Function Extension Axiom for hyperreal numbers says that the natural extension of any real function obeys the same functional identities and inequalities as the original function. In Example 2.7, we use the identity,

$$f[x + \delta x] = f[x] \cdot f[\delta x]$$

with x hyperreal and $\delta x \approx 0$ infinitesimal where $f[x]$ is a real function satisfying $f[x+y] = f[x] \cdot f[y]$. The reason this statement of the Function Extension Axiom is 'rough' is because we need to know precisely which values of the variables are permitted. Logically, we can express the axiom in a way that covers all cases at one time, but this is a little complicated so we will precede that statement with some important examples.

The Function Extension Axiom is stated so that we can apply it to the Log identity in the form of the implication

$$(x > 0 \;\&\; y > 0) \Rightarrow \mathrm{Log}[x] \text{ and } \mathrm{Log}[y] \text{ are defined and } \mathrm{Log}[x \cdot y] = \mathrm{Log}[x] + \mathrm{Log}[y]$$

The natural extension of $\mathrm{Log}[\cdot]$ is defined for all positive hyperreals and its identities hold for hyperreal numbers satisfying $x > 0$ and $y > 0$. The other identities hold for all hyperreal x and y. To make all such statements implications, we can state the exponential sum equation as

$$(x = x \quad \& \quad y = y) \quad \Rightarrow \quad e^{x+y} = e^x \cdot e^y$$

The differential

$$d(\text{Sin}[\theta]) = \text{Cos}[\theta]\ d\theta$$

is a notational summary of the valid approximation

$$\text{Sin}[\theta + \delta\theta] - \text{Sin}[\theta] = \text{Cos}[\theta]\delta\theta + \varepsilon \cdot \delta\theta$$

where $\varepsilon \approx 0$ when $\delta\theta \approx 0$. The derivation of this approximation based on magnifying a circle (given in a CD Section of Chapter 5 of the text) can be made precise by using the Function Extension Axiom in the place where it locates $(\text{Cos}[\theta+\delta\theta], \text{Sin}[\theta+\delta\theta])$ on the unit circle. This is simply using the extension of the (CircleIden) identity to hyperreal numbers, $(\text{Cos}[\theta + \delta\theta])^2 + (\text{Sin}[\theta + \delta\theta])^2 = 1$.

Logical Real Expressions, Formulas and Statements

Logical real expressions are built up from numbers and variables using functions. Here is the precise definition.

(a) A real number is a real expression.
(b) A variable standing alone is a real expression.
(c) If $E_1, E_2, \cdots, E_n$ are a real expressions and $f[x_1, x_2, \cdots, x_n]$ is a real function of n variables, then $f[E_1, E_2, \cdots, E_n]$ is a real expression.

A logical real formula is one of the following:

(a) An equation between real expressions, $E_1 = E_2$.
(b) An inequality between real expressions, $E_1 < E_2$, $E_1 \le E_2$, $E_1 > E_2$, $E_1 \ge E_2$, or $E_1 \ne E_2$.
(c) A statement of the form "*E is defined*" or of the form "*E is undefined*."

Let S and T be finite sets of real formulas. A logical real statement is an implication of the form,

$$S \Rightarrow T$$

or *"whenever every formula in S is true, then every formula in T is true."*

Logical real statements allow us to formalize statements like: "Every point in the square below lies in the circle below." Formalizing the statement does not make it true or false. Consider the figure below.

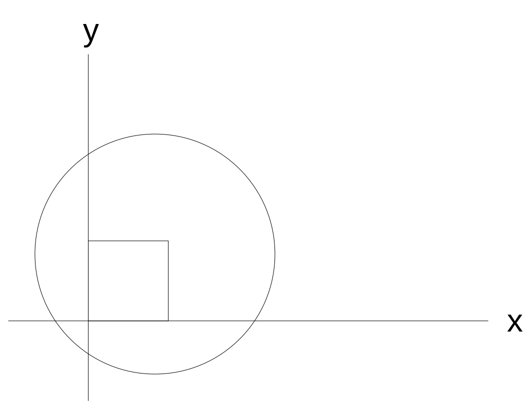


Figure 2.1: Square and Circle

The inside of the square shown can be described formally as the set of points satisfying the equations in the set $S = \{\ 0 \le x, 0 \le y, x \le 1.2, y \le 1.2\ \}$. The inside of the circle shown can be defined as the set of points satisfying the single equation $T = \{\ (x-1)^2+(y-1)^2 \le 1.6^2\ \}$. This is the circle of radius 1.6 centered at the point $(1,1)$. The logical real statement $S \Rightarrow T$ means that every point inside the square lies inside the circle. The statement is true for every real x and y. First of all, it is clear by visual inspection. Second, points (x, y) that make one or more of the formulas in S false produce a false premise, so no matter whether or not they lie in the circle, the implication is logically true (if uninteresting).

The logical real statement $T \Rightarrow S$ is a valid logical statement, but it is false since it says every point inside the circle lies inside the square. Naturally, only true logical real statements transfer to the hyperreal numbers.

Axiom 2.1. *The Function Extension Axiom*
Every logical real statement that holds for all real numbers also holds for all hyperreal numbers when the real functions in the statement are replaced by their natural extensions.

The Function Extension Axiom establishes the 5 identities for all hyperreal numbers, because $x = x$ and $y = y$ always holds. Here is an example.

Example 2.3. *The Extended Addition Formula for Sine*

$$\begin{aligned} S = \{x = x, y = y\} \Rightarrow T = \{\ & \text{Sin}[x] \text{ is defined },\\ & \text{Sin}[y] \text{ is defined },\\ & \text{Cos}[x] \text{ is defined },\\ & \text{Cos}[y] \text{ is defined },\\ & \text{Sin}[x+y] = \text{Sin}[x]\ \text{Cos}[y] + \text{Sin}[y]\ \text{Cos}[x]\} \end{aligned}$$

The informal interpretation of the extended identity is that the addition formula for sine holds for all hyperreals.

Example 2.4. *The Extended Formulas for Log*

We may take S to be formulas $x > 0$, $y > 0$ and $p = p$ and T to be the functional identities for the natural log plus the statements "Log[ ] is defined," etc. The Function Extension Axiom establishes that log is defined for positive hyperreals and satisfies the two basic log identities for positive hyperreals.

Example 2.5. *Abstract Uses of Function Extension*

There are two general uses of the Function Extension Axiom that underlie most of the theoretical problems in calculus. These involve extension of the discrete maximum and extension of finite summation. The proof of the Extreme Value Theorem 4.4 below uses a hyperfinite maximum, while the proof of the Fundamental Theorem of Integral Calculus 5.1 uses hyperfinite summation.

Equivalence of infinitesimal conditions for derivatives or limits and the "epsilon - delta" real number conditions are usually proved by using an auxiliary real function as in the proof of the limit equivalence Theorem 3.2.

Example 2.6. *The Increment Approximation*

NOTE: The increment approximation

$$f[x+\delta x] = f[x] + f'[x]\cdot\delta x + \varepsilon\cdot\delta x$$

with $\varepsilon \approx 0$ for $\delta x \approx 0$ and the simpler statement

$$\delta x \approx 0 \quad \Rightarrow \quad f'[x] \approx \frac{f[x]+\delta x) - f[x]}{\delta x}$$

are not **real** logical expressions, because they contain the relation $\approx$, which is not included in the formation rules for logical real statements. (The relation $\approx$ does not apply to ordinary real numbers, except in the trivial case $x = y$.)

For example, if θ is any hyperreal and $\delta\theta \approx 0$, then

$$\mathrm{Sin}[\theta+\delta\theta] = \mathrm{Sin}[\theta]\ \mathrm{Cos}[\delta\theta] + \mathrm{Sin}[\delta\theta]\ \mathrm{Cos}[\theta]$$

by the natural extension of the addition formula for sine above. Notice that the natural extension does NOT tell us the interesting and important estimate

$$\mathrm{Sin}[\theta+\delta\theta] = \mathrm{Sin}[\theta] + \delta\theta\cdot\mathrm{Cos}[\theta] + \varepsilon\cdot\delta\theta$$

with $\varepsilon \approx 0$ when $\delta\theta \approx 0$. (I.e., $\mathrm{Cos}[\delta\theta] = 1 + \iota\delta\theta$ and $\mathrm{Sin}[\delta\theta]/\delta\theta \approx 1$ are true, but not real logical statements we can derive just from natural extensions.)

Exercise set 2.3

1. *Write a formal logical real statement $S \Rightarrow T$ that says, "Every point inside the circle of radius 2, centered at $(-1, 3)$ lies outside the square with sides $x = 0$, $y = 0$, $x = 1$, $y = -1$. Draw a figure and decide whether or not this is a true statement for all real values of the variables.*
2. *Write a formal logical real statement $S \Rightarrow T$ that is equivalent to each of the functional identities on the first page of the chapter and interpret the extended identities in the hyperreals.*

2.4 Additive Functions

An identity for an unknown function together with the increment approximation combine to give a specific kind of function. The two ideas combine to give a differential equation. After you have learned about the calculus of the natural exponential function in Chapter 8 of the text, you will easily understand the exact solution of the problem of this section.

In the early 1800s, Cauchy asked the question: Must a function satisfying

(Additive) $$f[x+y] = f[x] + f[y]$$

be of the form $f[x] = m \cdot x$? This was not solved until the late 1800s by Hamel. The answer is "No." There are some very strange functions satisfying the additive identity that are not simple linear functions. However, these strange functions are not differentiable. We will solve a variant of Cauchy's problem for differentiable functions.

Example 2.7. *A Variation on Cauchy's Problem*

Suppose an unknown differentiable function $f[x]$ satisfies the (ExpSum) identity for all x and y,

$$f[x+y] = f[x] \cdot f[y]$$

Does the function have to be $f[x] = b^x$ for some positive b?

Since our unknown function $f[x]$ satisfies the (ExpSum) identity and is differentiable, both of the following equations must hold:

$$f[x+y] = f[x] \cdot f[y]$$
$$f[x+\delta x] = f[x] + f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

We let $y = \delta x$ in the first identity to compare it with the increment approximation,

$$f[x+\delta x] = f[x] \cdot f[\delta x]$$
$$f[x+\delta x] = f[x] + f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

so

$$f[x] \cdot f[\delta x] = f[x] + f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$
$$f[x][f[\delta x] - 1] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$
$$f'[x] = f[x]\frac{f[\delta x] - 1}{\delta x} - \varepsilon$$

or

$$\frac{f'[x]}{f[x]} = \frac{f[\delta x] - 1}{\delta x} - \varepsilon$$

with $\varepsilon \approx 0$ when $\delta x \approx 0$. The identity still holds with hyperreal inputs by the Function Extension Axiom. Since the left side of the last equation depends only on x and the right hand side does not depend on x at all, we must have $\frac{f[\delta x]-1}{\delta x} \approx k$, a constant, or $\frac{f[\Delta x]-1}{\Delta x} \to k$ as $\Delta x \to 0$. In other words, a differentiable function that satisfies the (ExpSum) identity satisfies the differential equation

$$\frac{df}{dx} = k\, f$$

What is the value of our unknown function at zero, $f[0]$? For any x and $y = 0$, we have

$$f[x] = f[x+0] = f[x] \cdot f[0]$$

so unless $f[x] = 0$ for all x, we must have $f[0] = 1$.

One of the main morals of this course is that if you know:

(1) where a quantity starts,

and

(2) how a quantity changes,

then you can compute subsequent values of the quantity. In this problem we have found (1) $f[0] = 1$ and (2) $\frac{df}{dx} = k\,f$. We can use this information with the computer to calculate values of our unknown function $f[x]$. The unique symbolic solution to

$$f[0] = 1$$
$$\frac{df}{dx} = k\,f$$

is

$$f[x] = e^{k\,x}$$

The identity (Repeated Exp) allows us to write this as

$$f[x] = e^{k\,x} = (e^k)^x = b^x$$

where $b = e^k$. In other words, we have shown that the only differentiable functions that satisfy the (ExpSum) identity are the ones you know from high school, b^x.

Problem 2.1. Smooth Additive Functions ARE Linear

Suppose an unknown function is additive and differentiable, so it satisfies both

(Additive) $$f[x + \delta x] = f[x] + f[\delta x]$$

and

(Micro) $$f[x + \delta x] = f[x] + f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

Solve these two equations for $f'[x]$ and argue that since the right side of the equation does not depend on x, $f'[x]$ must be constant. (Or $\frac{f[\Delta x]}{\Delta x} \to f'[x_1]$ and $\frac{f[\Delta x]}{\Delta x} \to f'[x_2]$, but since the left hand side is the same, $f'[x_1] = f'[x_2]$.)

What is the value of $f[0]$ if $f[x]$ satisfies the additive identity?

The derivative of an unknown function $f[x]$ is constant and $f[0] = 0$, what can we say about the function? (Hint: Sketch its graph.)

A project explores this symbolic kind of 'linearity' and the microscope equation from another angle.

2.5 The Motion of a Pendulum

Differential equations are the most common functional identities which arise in applications of mathematics to solving "real world" problems. One of the very important points in this regard is that you can often obtain significant information about a function if you know a differential equation the function satisfies, even if you do not know an exact formula for the function.

For example, suppose you know a function $\theta[t]$ satisfies the differential equation

$$\frac{d^2\theta}{dt^2} = \mathrm{Sin}[\theta[t]]$$

This equation arises in the study of the motion of a pendulum and $\theta[t]$ does not have a closed form expression. (There *is* no formula for $\theta[t]$.) Suppose you know $\theta[0] = \frac{\pi}{2}$. Then the differential equation forces

$$\frac{d^2\theta}{dt^2}[0] = \mathrm{Sin}[\theta[0]] = \mathrm{Sin}[\frac{\pi}{2}] = 1$$

We can also use the differential equation for θ to get information about the higher derivatives of $\theta[t]$. Say we know that $\frac{d\theta}{dt}[0] = 2$. Differentiating both sides of the differential equation yields

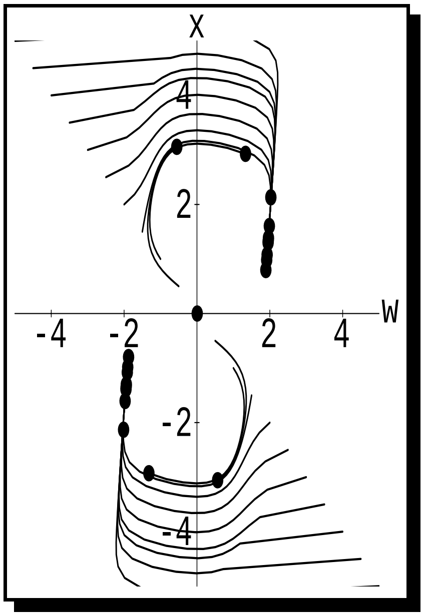
$$\frac{d^3\theta}{dt^3} = \mathrm{Cos}[\theta[t]]\frac{d\theta}{dt}$$

by the Chain Rule. Using the above information, we conclude that

$$\frac{d^3\theta}{dt^3}[0] = \mathrm{Cos}[\theta[0]]\frac{d\theta}{dt}[0] = \mathrm{Cos}[\frac{\pi}{2}]2 = 0$$

Problem 2.2.

Derive a formula for $\frac{d^4\theta}{dt^4}$ *and prove that* $\frac{d^4\theta}{dt^4}[0] = 1$.



Part 2

Limits

CHAPTER 3

The Theory of Limits

The intuitive notion of limit is that a quantity gets close to a "limiting" value as another quantity approaches a value. This chapter defines two important kinds of limits both with real numbers and with hyperreal numbers. The chapter also gives many computations of limits.

A basic fact about the sine function is

$$\lim_{x \to 0} \frac{\text{Sin}[x]}{x} = 1$$

Notice that the limiting expression $\frac{\text{Sin}[x]}{x}$ is defined for $0 < |x - 0| < 1$, but not if $x = 0$. The sine limit above is a difficult and interesting one. The important topic of this chapter is, "What does the limit expression mean?" Rather than the more "practical" question, "How do I compute a limit?"

Here is a simpler limit where we can see what is being approached.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

While this limit expression is also only defined for $0 < |x - 1|$, or $x \neq 1$, the mystery is easily resolved with a little algebra,

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{(x - 1)} = x + 1$$

So,

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2$$

The limit $\lim_{x \to 1}(x + 1) = 2$ is so obvious as to not need a technical definition. If x is nearly 1, then $x + 1$ is nearly $1 + 1 = 2$. So, while this simple example illustrates that the original expression does get closer and closer to 2 as x gets closer and closer to 1, it skirts the issue of "how close?"

3.1 Plain Limits

Technically, there are two equivalent ways to define the simple continuous variable limit as follows.

Definition 3.1. *Limit*

Let $f[x]$ be a real valued function defined for $0 < |x-a| < \Delta$ with Δ a fixed positive real number. We say

$$\lim_{x \to a} f[x] = b$$

when either of the equivalent the conditions of Theorem 3.2 hold.

Theorem 3.2. *Limit of a Real Variable*

Let $f[x]$ be a real valued function defined for $0 < |x-a| < \Delta$ with Δ a fixed positive real number. Let b be a real number. Then the following are equivalent:

(a) *Whenever the hyperreal number x satisfies $0 < |x - a| \approx 0$, the natural extension function satisfies*

$$f[x] \approx b$$

(b) *For every accuracy tolerance θ there is a sufficiently small positive real number γ such that if the real number x satisfies $0 < |x - a| < \gamma$, then*

$$|f[x] - b| < \theta$$

Proof:

We show that (a) $\Rightarrow$ (b) by proving that not (b) implies not (a), the contrapositive. Assume (b) fails. Then there is a real $\theta > 0$ such that for every real $\gamma > 0$ there is a real x satisfying $0 < |x - a| < \gamma$ and $|f[x] - b| \geq \theta$. Let $X[\gamma] = x$ be a real function that chooses such an x for a particular γ. Then we have the equivalence

$$\{\gamma > 0\} \Leftrightarrow \{X[\gamma] \text{ is defined }, 0 < |X[\gamma] - a| < \gamma, |f[X[\gamma] - b| \geq \theta\}$$

By the Function Extension Axiom 2.1 this equivalence holds for hyperreal numbers and the natural extensions of the real functions $X[\cdot]$ and $f[\cdot]$. In particular, choose a positive infinitesimal γ and apply the equivalence. We have $0 < |X[\gamma] - a| < \gamma$ and $|f[X[\gamma] - b| > \theta$ and θ is a positive real number. Hence, $f[X[\gamma]]$ is not infinitely close to b, proving not (a) and completing the proof that (a) implies (b).

Conversely, suppose that (b) holds. Then for every positive real θ, there is a positive real γ such that $0 < |x - a| < \gamma$ implies $|f[x] - b| < \theta$. By the Function Extension Axiom 2.1, this implication holds for hyperreal numbers. If $\xi \approx a$, then $0 < |\xi - a| < \gamma$ for every real γ, so $|f[\xi] - b| < \theta$ for every real positive θ. In other words, $f[\xi] \approx b$, showing that (b) implies (a) and completing the proof of the theorem.

Example 3.1. *Condition (a) Helps Prove a Limit*

Suppose we wish to prove completely rigorously that

$$\lim_{\Delta x \to 0} \frac{1}{2(2+\Delta x)} = \frac{1}{4}$$

The intuitive limit computation of just setting $\Delta x = 0$ is one way to "see" the answer,

$$\lim_{\Delta x \to 0} \frac{1}{2(2+\Delta x)} = \frac{1}{2(2+0} = \frac{1}{4}$$

but this certainly does not demonstrate the "epsilon - delta" condition (b).

Condition (a) is almost as easy to establish as the intuitive limit computation. We wish to show that when $\delta x \approx 0$

$$\frac{1}{2(2+\delta x)} \approx \frac{1}{4}$$

Subtract and do some algebra,

$$\begin{aligned} \frac{1}{2(2+\delta x)} - \frac{1}{4} &= \frac{2}{4(2+\delta x)} - \frac{(2+\delta x)}{4(2+\delta x)} \\ &= \frac{-\delta x}{4(2+\delta x)} = \delta x \cdot \frac{-1}{4(2+\delta x)} \end{aligned}$$

We complete the proof using the computation rules of Theorem 1.12. The fraction $-1/(4(2+\delta x))$ is finite because $4(2+\delta x) \approx 8$ is not infinitesimal. The infinitesimal δx times a finite number is infinitesimal.

$$\begin{aligned} \frac{1}{2(2+\delta x)} - \frac{1}{4} &\approx 0 \\ \frac{1}{2(2+\delta x)} &\approx \frac{1}{4} \end{aligned}$$

This is a complete rigorous proof of the limit. Theorem 3.2 shows that the "epsilon - delta" condition (b) holds.

Exercise set 3.1

1. *Prove rigorously that the limit* $\lim_{\Delta x \to 0} \frac{1}{3(3+\Delta x)} = \frac{1}{9}$. *Use your choice of condition (a) or condition (b) from Theorem* 3.2.
2. *Prove rigorously that the limit* $\lim_{\Delta x \to 0} \frac{1}{\sqrt{4+\Delta x}+\sqrt{4}} = \frac{1}{4}$. *Use your choice of condition (a) or condition (b) from Theorem* 3.2.
3. *The limit* $\lim_{x \to 0} \frac{\mathrm{Sin}[x]}{x} = 1$ *means that sine of a small value is nearly equal to the value, and near in a strong sense. Suppose the natural extension of a function* $f[x]$ *satisfies* $f[\xi] \approx 0$ *whenever* $\xi \approx 0$. *Does this mean that* $\lim_{x \to 0} \frac{f[x]}{x}$ *exists? (HINT: What is* $\lim_{x \to 0} \sqrt{x}$*? What is* $\sqrt{\xi}/\xi$*?)*
4. *Assume that the derivative of sine is cosine and use the increment approximation*

$$f[x+\delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

with $\varepsilon \approx 0$ *when* $\delta x \approx 0$*, to prove the limit* $\lim_{x\to 0} \frac{\text{Sin}[x]}{x} = 1$*. (It means essentially the same thing as the derivative of sine at zero is 1. HINT: Take* $x = 0$ *and* $\delta x = x$ *in the increment approximation.)*

3.2 Function Limits

Many limits in calculus are limits of functions. For example, the derivative is a limit and the derivative of x^3 *is the limit function* $3\,x^2$*. This section defines the function limits used in differentiation theory.*

Example 3.2. *A Function Limit*

The derivative of x^3 is $3\,x^2$, a function. When we compute the derivative we use the limit

$$\lim_{\Delta x\to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}$$

Again, the limiting expression is undefined at $\Delta x = 0$. Algebra makes the limit intuitively clear,

$$\frac{(x+\Delta x)^3 - x^3}{\Delta x} = \frac{(x^3 + 3\,x^2\,\Delta x + 3\,x\,\Delta x^2 + \Delta x^3) - x^3}{\Delta x} = 3\,x^2 + 3\,x\,\Delta x + \Delta x^2$$

The terms with Δx tend to zero as Δx tends to zero.

$$\lim_{\Delta x\to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x\to 0} (3\,x^2 + 3\,x\,\Delta x + \Delta x^2) = 3\,x^2$$

This is clear without a lot of elaborate estimation, but there is an important point that might be missed if you don't remember that you are taking the limit of a function. The graph of the approximating function approaches the graph of the derivative function. This more powerful approximation (than that just a particular value of x) makes much of the theory of calculus clearer and more intuitive than a fixed x approach. Intuitively, it is no harder than the fixed x approach and infinitesimals give us a way to establish the "uniform" tolerances with computations almost like the intuitive approach.

Definition 3.3. *Locally Uniform Function Limit*

Let $f[x]$ *and* $F[x, \Delta x]$ *be real valued functions defined when* x *is in a real interval* (a, b) *and* $0 < \Delta x < \Delta$ *with* Δ *a fixed positive real number. We say*

$$\lim_{\Delta x\to 0} F[x, \Delta x] = f[x]$$

uniformly on compact subintervals of (a, b)*, or "locally uniformly" when one of the equivalent the conditions of Theorem* 3.4 *holds.*

Theorem 3.4. *Limit of a Real Function*

Let $f[x]$ and $F[x, \Delta x]$ be real valued functions defined when x is in a real interval (a, b) and $0 < \Delta x < \Delta$ with Δ a fixed positive real number. Then the following are equivalent:

(a) *Whenever the hyperreal numbers δx and x satisfy $0 < |\delta x| \approx 0$, x is finite, and $a < x < b$ with neither $x \approx a$ nor $x \approx b$, the natural extension functions satisfy*

$$F[x, \delta x] \approx f[x]$$

(b) *For every accuracy tolerance θ and every real α and β in (a, b), there is a sufficiently small positive real number γ such that if the real number Δx satisfies $0 < |\Delta x| < \gamma$ and the real number x satisfies $\alpha \le x \le \beta$, then*

$$|F[x, \Delta x] - f[x]| < \theta$$

PROOF:

First, we prove not (b) implies not (a). If (b) fails, there are real α and β, $a < \alpha < \beta < b$, and real positive θ such that for every real positive γ there are x and Δx satisfying

$$0 < \Delta x < \gamma, \quad \alpha \le x \le \beta, \quad |F[x, \Delta x] - f[x]| \ge \theta$$

Define real functions $X[\gamma]$ and $DX[\gamma]$ that select such values of x and Δx,

$$0 < DX[\gamma] < \gamma, \quad \alpha \le X[\gamma] \le \beta, \quad |F[X[\gamma], DX[\gamma]] - f[X[\gamma]]| \ge \theta$$

Now apply the Function Extension Axiom 2.1 to the equivalent sets of inequalities,

$$\{\gamma > 0\} \Leftrightarrow \{0 < DX[\gamma] < \gamma, \quad \alpha \le X[\gamma] \le \beta, \quad |F[X[\gamma], DX[\gamma]] - f[X[\gamma]]| \ge \theta\}$$

Choose an infinitesimal $\gamma \approx 0$ and let $x = X[\gamma]$ and $\delta x = DX[\gamma]$. Then

$$0 < \delta x < \gamma \approx 0, \quad \alpha \le x \le \beta, \quad |F[x, \delta x] - f[x]| \ge \theta$$

so $F[x, \delta x] - f[x]$ is not infinitesimal showing not (a) holds and proving (a) implies (b).

Now we prove that (b) implies (a). Let δx be a non zero infinitesimal and let x satisfy the conditions of (a). We show that $F[x, \delta x] \approx f[x]$ by showing that for any positive real θ, $|F[x, \delta x] - f[x]| < \theta$. Fix any one such value of θ.

Since x is finite and not infinitely near a nor b, there are real values α and β satisfying $a < \alpha < \beta < b$. Apply condition (b) to these α and β together with θ fixed above. Then there is a positive real γ so that for every real ξ and Δx satisfying $0 < |\Delta x| < \gamma$ and $\alpha \le \xi \le \beta$, we have $|F[\xi, \Delta x] - f[\xi]| < \theta$. In other words, the following implication holds in the real numbers,

$$\{0 < |\Delta x| < \gamma, \alpha \le \xi \le \beta\} \Rightarrow \{|F[\xi, \Delta x] - f[\xi]| < \theta\}$$

Apply the Function Extension Axiom 2.1 to see that the same implication holds in the hyperreals. Moreover, $x = \xi$ and nonzero $\Delta x = \delta x \approx 0$ satisfy the left hand side of the implication, so the right side holds. Since θ was arbitrary, condition (a) is proved.

Example 3.3. *Computing Locally Uniform Limits*

The following limit is uniform on compact subintervals of $(-\infty, \infty)$.

$$\lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} (3\,x^2 + 3\,x\,\Delta x + \Delta x^2) = 3\,x^2$$

A complete rigorous proof based on condition (a) can be obtained with the computation rules of Theorem 1.12. The difference is infinitesimal

$$(3\,x^2 + 3\,x\,\delta x + \delta x^2) - 3\,x^2 = (3\,x + \delta x)\delta x$$

when δx is infinitesimal. First, $3\,x + \delta x$ is finite because a sum of finite numbers is finite. Second, infinitesimal times finite is infinitesimal. This is a complete proof and by Theorem 3.4 shows that both conditions (b) and (c) also hold.

Exercise set 3.2

1. *Prove rigorously that the locally uniform function limit* $\lim_{\Delta x \to 0} \frac{1}{x(x+\Delta x)} = \frac{1}{x^2}$. *Use your choice of condition (a) or condition (b) from Theorem* 3.4.
2. *Prove rigorously that the locally uniform function limit* $\lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$. *Use your choice of condition (a), condition (b), or condition (c) from Theorem* 3.4.
3. *Prove the following:*

Theorem 3.5. *Locally Uniform Derivatives*

Let $f[x]$ and $f'[x]$ be real valued functions defined when x is in a real interval (a, b). Then the following are equivalent:

(a) *Whenever the hyperreal numbers δx and x satisfy $\delta x \approx 0$, x is finite, and $a < x < b$ with neither $x \approx a$ nor $x \approx b$, the natural extension functions satisfy*

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

for $\varepsilon \approx 0$.

(b) *For every accuracy tolerance θ and every real α and β in (a, b), there is a sufficiently small positive real number γ such that if the real number Δx satisfies $0 < |\Delta x| < \gamma$ and the real number x satisfies $\alpha \le x \le \beta$, then*

$$\left| \frac{f[x + \Delta x] - f[x]}{\Delta x} - f'[x] \right| < \theta$$

(c) *For every real c in (a, b),*

$$\lim_{x \to c, \Delta x \to 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} = f'[c]$$

That is, for every real c with $a < c < b$ and every real positive θ, there is a real positive γ such that if the real numbers x and Δx satisfy $0 < \Delta x < \gamma$ and $0 < |x - c| < \gamma$, then $|\frac{f[x+\Delta x]-f[x]}{\Delta x} - f'[c]| < \theta$.

3.3 Computation of Limits

Limits can be computed in a way that rigorously establishes them as results by using the rules of Theorem 1.12.

Suppose we want to compute a limit like

$$\lim_{d\to 0} \frac{(x+d)^2 - x^2}{d}$$

First we observe that it does no good to substitute $d = 0$ into the expression, because we get $0/0$. We do some algebra,

$$\begin{aligned}\frac{(x+d)^2 - x^2}{d} &= \frac{x^2 + 2\,x\,d + d^2 - x^2}{d} \\ &= \frac{2\,x\,d + d^2}{d} \\ &= 2\,x + d\end{aligned}$$

Now,

$$\lim_{d\to 0} \frac{(x+d)^2 - x^2}{d} = \lim_{d\to 0} 2\,x + d = 2\,x$$

because making d smaller and smaller makes the expression closer and closer to $2\,x$. The rules of small, medium and large numbers given in Theorem 1.12 just formalize the kinds of operations that work in such calculations. Theorem 3.2 and Theorem 3.4 show that these rules establish the limits as rigorously proven facts.

Exercise 3.3.1 below contains a long list of drill questions that may be viewed as limit computations. For example,

$$\lim_{d\to 0} \frac{1}{d}\left(\frac{1}{b+d} - \frac{1}{b}\right) =?$$

is just asking what happens as d becomes small. Another way to ask the question is

$$\frac{1}{\delta}\left(\frac{1}{b+\delta} - \frac{1}{b}\right) \approx ? \qquad \text{when} \quad \delta \approx 0$$

The latter approach is well suited to direct computations and can be solved with the rules of Theorem 1.12 that formalize our intuitive notions of small, medium and large numbers.

Following are some sample calculations with parameters $a, b, c, \delta, \varepsilon, H, K$ from Exercise 3.3.1.

Example 3.4. *Infinitesimal, Finite and Infinite Computations*

We are told that $a \approx 2$ and $b \approx 5$, so we may write $a = 2 + \iota$ and $b = 5 + \theta$ with $\iota \approx 0$ and $\theta \approx 0$. Now we compute $b - a = 5 + \theta - 2 - \iota = 5 - 2 + (\theta - \iota) = 3 + (\theta - \iota)$ by rules of algebra. The negative of an infinitesimal is infinitesimal and the sum of a positive and negative infinitesimal is infinitesimal, hence $\theta - \iota \approx 0$. This makes

$$b - a \approx 3$$

Another correct way to do this computation is the following

$$a \approx 2$$
$$b \approx 5$$
$$b - a \approx 5 - 2 = 3$$

However, this is NOT use of ordinary rules of algebra, because ordinary rules of algebra do not refer to the infinitely close relation $\approx$. This form of the computation incorporates the fact that negatives of infinitesimals are infinitesimal and the fact that sums of infinitesimals are infinitesimal.

Example 3.5. *Small, Medium and Large as Limits*

The approximate computations can be re-phrased in terms of limits. We can replace the occurrences of $\delta \approx 0$ and $\varepsilon \approx 0$ by variables approaching zero and so forth. Let's just do this by change of alphabet, $\lim_{d \to 0}$ for $\delta \approx 0$ and $\lim_{\alpha \to 2}$ for $a \approx 2$.

The computation $b - a \approx 3$ can be viewed as the limit

$$\lim_{\alpha \to 2, \beta \to 5} \beta - \alpha = 3$$

The computation $(2 - \delta)/a \approx 1$ becomes

$$\lim_{d \to 0 \& \alpha \to 2} \frac{2 - d}{\alpha} = 1$$

The computation $\frac{\sqrt{a+\delta} - \sqrt{a}}{\delta} \approx \frac{1}{2\sqrt{2}}$ becomes

$$\lim_{d \to 0 \& \alpha \to 2} \frac{\sqrt{\alpha + d} - \sqrt{\alpha}}{d} = \frac{1}{2\sqrt{2}}$$

Example 3.6. *Hyperreal Roots*

When $a \approx 2$ and $c \approx -7$, the Function Extension Axiom guarantees in particular that $\sqrt{a}$ is defined and that $\sqrt{c}$ is undefined, since $a > 0$ is positive and $c < 0$ is negative. The computation rules may be used to show that $\sqrt{a} \approx \sqrt{2}$, that is, we do not need any more rules to show this. First, $\sqrt{a}$ is finite, because $a < 3$ implies

$$\sqrt{a} < \sqrt{3} < 2$$

by the Function Extension Axiom. Next,

$$\begin{aligned} \sqrt{a} - \sqrt{2} &= \frac{(\sqrt{a} - \sqrt{2})(\sqrt{a} + \sqrt{2})}{\sqrt{a} + \sqrt{2}} \\ &= \frac{a - 2}{\sqrt{a} + \sqrt{2}} \\ &= \iota \cdot \frac{1}{\sqrt{a} + \sqrt{2}} \end{aligned}$$

an infinitesimal times a finite number, by approximation rule (4). Finally, approximation rule (3) shows

$$\sqrt{a} - \sqrt{2} \approx 0 \qquad \text{or} \qquad \sqrt{a} \approx \sqrt{2}$$

Example 3.7. *A Limit of Square Root*

The "epsilon - delta" proof (condition (b) of Theorem 3.2) of

$$\lim_{x\to 0} \sqrt{|x|} = 0$$

is somewhat difficult to prove. We establish the equivalent condition (a) as follows.

Let $0 < \xi \approx 0$ be a positive infinitesimal. Since $0 \geq x$ implies $\sqrt{x}$ is defined and positive, The Function Extension Axiom 2.1 guarantees that $\sqrt{\xi}$ is defined and positive.

Suppose $\sqrt{\xi}$ is not infinitesimal. Then there is a positive real number $0 < a$ with $a < \sqrt{\xi}$. Squaring and using the Function Extension Axiom on the property $0 < b < c$ implies $0 < \sqrt{b} < \sqrt{c}$, we see that $0 < a^2 < \xi$ contradicting the assumption that $\xi \approx 0$ is infinitesimal.

Example 3.8. *Infinite Limits*

We know that $c+7 \neq 0$ because we are given that $c \neq -7$ and $c \approx -7$ or $c = -7+\iota$ with $\iota \approx 0$, but $\iota \neq 0$. This means that $c+7 = \iota \neq 0$ and so

$$c+7 \approx 0$$

This, together with what we know about reciprocals of infinitesimals tells us that

$$\frac{1}{c+7} \quad \text{is infinite}$$

We do not know if it is positive or negative; we simply weren't told whether $c < 7$ or $c > 7$, but only that $c \approx 7$.

In this example, the limit formulation has the result

$$\lim_{\gamma\to 7} \frac{1}{\gamma - 7} \text{ does not exist or } \lim_{\gamma\to 7} \left|\frac{1}{\gamma-7}\right| = +\infty$$

The precise meaning of these symbols is as follows.

Definition 3.6. *Infinite Limits*

Let $f[x]$ be a real function defined for a neighborhood of a real number a, $0 < |x-a| < \Delta$. We say

$$\lim_{x\to a} f[x] = \infty$$

provided that for every large positive real number B, there is a sufficiently small real tolerance, τ such that if $0 < |x-a| < \tau$ then $f[x] > B$.

The symbol ∞ means "gets bigger and bigger." This is equivalent to the following hyperreal condition.

Theorem 3.7. *A Hyperreal Condition for Infinite Limits*

Let $f[x]$ be a real function defined for a neighborhood of a real number a, $0 < |x-a| < \Delta$. The definition of $\lim_{x\to a} f[x] = \infty$ is equivalent to the following.

For every hyperreal x infinitely close to a, but distinct from it, the natural extension satisfies, $f[x]$ is a positive infinite hyperreal.

Proof: Left as an exercise below.

Example 3.9. *∞ is NOT Hyperreal*

The symbol ∞ cannot stand for a number because it does not obey the usual laws of algebra. Viewed as a numerical equation $\infty \cdot \infty = \infty$ says that we must have $\infty = 1$ or $\infty = 0$, by ordinary rules of algebra. Since we want to retain the rules of algebra, ∞ in the sense of 'very big' can not be a hyperreal number.

Example 3.10. *An Infinite Limit with Roots*

The limit

$$\lim_{x\to 0} \frac{\sqrt{|x|}}{|x|} = +\infty$$

Proof:

Let $0 < \xi \approx 0$. We know from the previous example that $\sqrt{\xi} \approx 0$. We know from algebra and the Function Extension Axiom that

$$\frac{\sqrt{\xi}}{\xi} = \frac{1}{\sqrt{\xi}}$$

Using Theorem 1.12, we see that this expression is infinitely large.

Example 3.11. *Indeterminate Forms*

Even though arguments similar to the ones we have just done show that $a+b+c \approx 0$ we can not conclude that $\frac{1}{a+b+c}$ is defined. For example, we might have $a = 2-\delta$, $b = 5+3\delta$ and $c = -7-2\delta$. Then $a \approx 2$, $b \approx 5$ and $c \approx -7$, but $a+b+c = 0$. (Notice that it is true that $a+b+c \approx 0$.) In this case $\frac{1}{a+b+c}$ is not defined. Other choices of the perturbations might make $a+b+c \neq 0$, so $\frac{1}{a+b+c}$ is defined (and positive or negative infinite) in some cases, but not in others. This means that the value of

$$\frac{1}{a+b+c}$$

can not be determined knowing only that the sum is infinitesimal.

In Webster's unabridged dictionary, the term "indeterminate" has the following symbolic characters along with the verbal definition

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad \infty\cdot 0,\quad 1^{\infty},\quad 0^0,\quad \infty^0,\quad \infty-\infty$$

In the first place, Webster's definition pre-dates the discovery of hyperreal numbers. The symbol ∞ does NOT represent a real or hyperreal number, because things like $\infty\cdot\infty = \infty$ only denote 'limit of big times limit of big is big.' The limit forms above do not have a definite outcome.

Each of the symbolic short-cuts above has a hyperreal number calculation with indeterminate outcomes in the sense that they may be infinitesimal, finite or infinite depending on the particular infinitesimal, finite or infinite numbers in the computation. In this sense, the older infinities are compatible with infinitely large hyperreal numbers.

Example 3.12. *The Indeterminate Form $\infty - \infty$*

Consider $\infty - \infty$. The numbers H and $L = H + \delta$ are both infinite numbers, but $H - L = -\delta$ is infinitesimal. The numbers K and $M = K + b$ are both infinite and $K - M \approx -5$. The numbers H and $N = H^2$ are both infinite and $H - N = (1-H)\cdot H$ is a negative infinite number.

We may view the symbolic expression "$\infty - \infty$ is indeterminate" as a short-hand for the fact that the difference between two infinite hyperreal numbers can be positive or negative infinite, positive or negative and finite or even positive or negative infinitesimal. Of course, it can also be zero.

Example 3.13. *The Indeterminate Form* $0 \cdot \infty$

The short-hand symbolic expression "$0 \cdot \infty$ is indeterminate" corresponds to the following kinds of choices. Suppose that $H = 1/\delta$. Then $\delta \cdot H = 1$. An infinitesimal times an infinite number could be finite. Suppose $K = H^3$, so K is infinite. Now $\delta \cdot K = H$ is infinite. An infinitesimal times an infinite number could be infinite. Finally, suppose $\varepsilon = \delta^5$. Then $\varepsilon \cdot K = \delta^5/\delta^2 = \delta^3$ is infinitesimal.

The following is just for practice at using the computation rules for infinitesimal, finite, and infinite numbers from Theorem 1.12 to compute limits rigorously. These computations prove that the "epsilon - delta" conditions hold.

Exercise set 3.3

1. Drill with Rules of Infinitesimal, Finite and Infinite Numbers
In the following formulas,

$$0 < \varepsilon \approx 0 \quad \textit{and} \quad 0 < \delta \approx 0, \qquad H \quad \textit{and} \quad K \quad \textit{are infinite and positive.}$$

$$a \approx 2, \quad b \approx 5, \quad c \approx -7, \quad \textit{but} \quad a \neq 2, \quad b \neq 5, \quad c \neq -7$$

Say whether each expression is infinitesimal, finite (and which real number it is near), infinite, or indeterminate (that is, could be in different categories depending on the particular values of the parameters above.)

1 $y = \varepsilon \times \delta$	*2* $y = \varepsilon - \delta$	*3* $y = \varepsilon/b$
4 $y = \varepsilon/\delta$	*5* $y = \frac{a+7\varepsilon}{b-4\delta}$	*6* $y = b/\varepsilon$
7 $y = a + b - c$	*8* $y = a + \delta$	*9* $y = c - \varepsilon$
10 $y = a - 2$	*11* $y = \frac{1}{a-2}$	*12* $y = \frac{1}{a-b}$
13 $y = \frac{c}{a-b}$	*14* $y = \frac{2-\delta}{a}$	*15* $y = \frac{5\delta^4-3\delta^2+2\delta}{\delta}$
16 $y = \frac{1}{H}$	*17* $y = \frac{2-\delta}{a-K}$	*18* $y = \frac{5\delta^4-3\delta^2+2\delta}{4\delta+\delta^2}$
19 $y = \frac{H^2+3H}{H}$	*20* $y = \frac{H^2+3H}{H^2}$	*21* $y = \frac{3\delta^2}{\delta+8\delta^2}$
22 $y = \frac{H-K}{H}$	*23* $y = \frac{H-K}{H\,K}$	*24* $y = \frac{H-K}{H+K}$
25 $y = \sqrt{H}$	*26* $y = \sqrt{\delta}$	*27* $y = \frac{H+K}{H-K}$
28 $y = \frac{\sqrt{H}}{H+a}$	*29* $y = \sqrt{a+\delta} - \sqrt{a}$	*30* $y = \frac{1}{b+\delta} - \frac{1}{b}$

31 $y = \frac{3a\delta^2+\delta^3}{\delta}$ *32* $y = \frac{\sqrt{a+\delta}-\sqrt{a}}{\delta}$ *33* $y = \frac{1}{\delta}\left(\frac{1}{b+\delta} - \frac{1}{b}\right)$

2. *Re-write the problems of the previous exercise as limits.*

3. *Prove Theorem* 3.7.

CHAPTER 4 Continuous Functions

A function $f[x]$ is continuous if a small change in its input only produces a small change in its output. This chapter gives some fundamental consequences of this property.

Definition 4.1. *Continuous Function*
Suppose a real function $f[x]$ is defined in a neighborhood of a, $|x - a| < \Delta$. We say $f[x]$ is continuous at a if whenever $x \approx a$ in the hyperreal numbers, the natural extension satisfies $f[x] \approx f[a]$.

Notice that continuity assumes that $f[a]$ is defined. The function

$$f[x] = \frac{\text{Sin}[x]}{x}$$

is technically not continuous at $x = 0$, but since $\lim_{x \to 0} f[x] = 1$ we could extend the definition to include $f[0] = 1$ and then the function would be continuous.

Theorem 4.2. *Continuity as Limit*
Suppose a real function $f[x]$ is defined in a neighborhood of a, $|x - a| < \Delta$. Then $f[x]$ is continuous at a if and only if $\lim_{x \to a} f[x] = f[a]$.

Proof:
Apply Theorem 3.2.
We show in Section **??** that differentiable functions are continuous, so rules of calculus give us an easy way to verify that a function is continuous.

4.1 Uniform Continuity

A function is uniformly continuous if given an "epsilon," the same "delta" works "uniformly" for all x.

The simplest intervals are the ones of finite length that include their endpoints, $[a, b]$, for numbers a and b. These intervals are sometimes described as 'closed and bounded,' because they have the endpoints and have bounded length. A shorter name is 'compact' intervals.

Every hyperreal number satisfying $a \le x \le b$ is near a real number $x \approx c$ with $a \le c \le b$. First, the hyperreal x has a standard part since it is finite. Second, c must lie in the interval because real numbers r outside the interval are a noninfinitesimsl distance from the endpoints. We cannot have $x \approx r$ and r a noninfinitesimal distance from the interval.

The fact that every hyperreal point of a set is near a standard point of the set is equivalent to the "finite covering property" of general topologically compact spaces. The hyperreal condition is easy to apply directly. The following theorem illustrates this (although we do not need the theorem later in the course.)

Theorem 4.3. *Continuous on a Compact Interval*
Suppose that a real function $f[x]$ is defined and continuous on the compact real interval $[a, b] = \{x : a \le x \le b\}$. Then for every real positive θ there is a real positive γ such that if $|x_1 - x_2| < \gamma$ in $[a, b]$, then $|f[x_1] - f[x_2]| < \theta$.

Proof:

Since $f[x]$ is continuous at every point of $[a, b]$, if $\xi \approx c$ for $a \le c \le b$, then $f[\xi] \approx f[c]$.

Further, since the interval $[a, b]$ includes its real endpoints, if a hyperreal number x satisfies $a \le x \le b$ then its standard part from Theorem 1.11 c lies in the interval and $x \approx c$.

Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 \approx x_2$. Both of these numbers have the same standard part (since the real standard parts have to be infinitely close and real, hence equal.) We have

$$f[x_1] \approx f[c] \approx f[x_2]$$

so for any numbers $x_1 \approx x_2$ in $[a, b]$, $f[x_1] \approx f[x_2]$.

Suppose the conclusion of the theorem is false. Then there is a real $\theta > 0$ such that for every $\gamma > 0$ there exist x_1 and x_2 in $[a, b]$ with $|x_1 - x_2| < \gamma$ and $|f[x_1] - f[x_2]| \ge \theta$. Define real functions $X_1[\gamma]$ and $X_2[\gamma]$ that select such values and give us the real logical statement

$$\{\gamma > 0\} \Rightarrow \{a \le X_1[\gamma] \le b, a \le X_2[\gamma] \le b, |X_1[\gamma] - X_2[\gamma]| < \gamma, |f[X_1[\gamma]] - f[X_2[\gamma]]| \ge \theta\}$$

Now apply the Function Extension Axiom 2.1 to this implication and select a positive infinitesimal $\gamma \approx 0$. Let $x_1 = X_1[\gamma]$, $x_2 = X_2[\gamma]$ and notice that they are in the interval, $x_1 \approx x_2$, but $f[x_1]$ is not infinitely close to $f[x_2]$. This contradiction shows that the theorem is true.

4.2 The Extreme Value Theorem

Continuous functions attain their max and min on compact intervals.

Theorem 4.4. *The Extreme Value Theorem*
If $f[x]$ is a continuous real function on the real compact interval $[a, b]$, then f attains its maximum and minimum, that is, there are real numbers x_m and x_M such that $a \le x_m \le b$, $a \le x_M \le b$, and for all x with $a \le x \le b$

$$f[x_m] \le f[x] \le f[x_M]$$

Intuitive Proof:

We will show how to locate the maximum, you can find the minimum. Partition the interval into steps of size Δx,

$$a < a + \Delta x < a + 2\Delta x < \cdots < b$$

and define a real function

$$M[\Delta x] = \text{the x of the form } x_1 = a + k\Delta x$$

so that

$$f[M[\Delta x] = f[x_1] = max[f[x] : x = a + h\Delta x,\ h = 0, 1, \cdots , n]$$

This function is the discrete maximum from among a finite number of possibilities, so that $M[\Delta x]$ has two properties: (1) $M[\Delta x]$ is one of the partition points and (2) all other partition points $x = a + h\Delta x$ satisfy $f[x] \le f[M[\Delta x]]$.

Next, we partition the interval into infinitesimal steps,

$$a < a + \delta x < a + 2\delta x < \cdots < b$$

and consider the natural extension of the discrete maximizing function $M[\delta x]$. By the Function Extension Axiom 2.1 we know that (1) $x_1 = M[\delta x]$ is one of the points in the infinitesimal partition and (2) $f[x] \le f[x_1]$ for all other partition points x.

Since the hyperreal interval $[a, b]$ only contains finite numbers, there is a real number $x_M \approx x_1$ (standard part) and every other real number x_2 in $[a, b]$ is within δx of some partition point, $x_2 \approx x$.

Continuity of f means that $f[x] \approx f[x_2]$ and $f[x_M] \approx f[x_1]$. The numbers x_2 and x_M are real, so $f[x_2]$ and $f[x_M]$ are also real and we have

$$f[x_2] \approx f[x] \le f[x_1] \approx f[x_M]$$

Thus, for any real x_2, $f[x_2] \le f[x_M]$, which says f attains its maximum at x_M. This completes the proof.

Partition Details of the Proof:

Let a and b be real numbers and suppose a real function $f[x]$ is defined for $a \le x \le b$. Let Δx be a positive number smaller than $b - a$. There are finitely many numbers of the form $a + k\,\Delta x$ between a and b; $a = a + 0\,\Delta x$, $a + \Delta x$, $a + 2\,\Delta x, \cdots , a + n\,\Delta x \le b$. The corresponding function values, $f[a]$, $f[a + \Delta x]$, $f[a + 2\,\Delta x], \cdots , f[a + n\,\Delta x]$ have a largest amongst them, say $f[a + m\,\Delta x] \ge f[a + k\,\Delta x]$ for all other k. We can express this with a function $M[\Delta x] = a + m\,\Delta x$ "is the place amongst the points $a < a + \Delta x < a + 2\,\Delta x < \cdots < a + n\,\Delta x \le b$ such that $f[M[\Delta x]] \ge f[a + k\,\Delta x]$." (There could be more than one, but $M[\Delta x]$ chooses one of them.)

A better way to formulate this logically is to say, 'if x is of the form $a + k\,\Delta x$, then $f[x] \le f[M[\Delta x]]$.' This can also be formulated with functions. Let $I[x]$ be the 'indicator function of integers,' that is $I[x] = 1$ if $x = 0, \pm 1, \pm 2, \pm 3, \cdots$ and $I[x] = 0$ otherwise. Then the maximizing property of $M[\Delta x] = a + m\,\Delta x$ can be summarized by

$$\left[a \le x \le b \quad \text{and} \quad I\left(\frac{x-a}{\Delta x}\right) = 1\right] \quad \Rightarrow \quad f[x] \le f[M[\Delta x]]$$

The rigorous formulation of the Function Extension Axiom covers this case. We take S to be the set of formulas, $\Delta x > 0$, $a \le x$, $x \le b$ and $I[[x - a]/\Delta x] = 1$ and take T to be the inequality $f[x] \le f[M[\Delta x]]$. The Function Extension Axiom shows that $M[\delta x]$ is the place where $f[x]$ is largest among points of the form $a + k\,\delta x$, even when $\delta x \approx 0$ is infinitesimal, but it says this as follows:

$$\left[a \le x \le b \quad \text{and} \quad I\left(\frac{x-a}{\delta x}\right) = 1\right] \quad \Rightarrow \quad f[x] \le f[M[\delta x]]$$

We interpret this as meaning that, 'among the hyperreal numbers of the form $a + k\,\delta x$, $f[x]$ is largest when $x = M[\delta x]$,' even when δx is a positive infinitesimal.

4.3 Bolzano's Intermediate Value Theorem

The graphs of continuous functions have no "jumps."

Theorem 4.5. *Bolzano's Intermediate Value Theorem*
If $y = f[x]$ is continuous on the interval $a \le x \le b$, then $f[x]$ attains every value intermediate between the values $f[a]$ and $f[b]$. In particular, if $f[a] < 0$ and $f[b] > 0$, then there is an x_0, $a < x_0 < b$, such that $f[x_0] = 0$.

Proof:

The following idea makes a technically simple general proof. Suppose we want to hit a real value γ between the values of $f[a] = \alpha$ and $f[b] = \beta$. Divide the interval $[a, b]$ up into small steps each Δx long, a, $a + \Delta x$, $a + 2\Delta x$, $a + 3\Delta x$, $\cdots$, b. Suppose $\alpha < \gamma < \beta$. The function $f[x]$ starts at $x = a$ with $f[a] = \alpha < \gamma$. At the next step, it may still be below γ, $f[a + \Delta x] < \gamma$, but there is a first step, $a + k\Delta x$ where $f[a + k\Delta x] > \gamma$ and $f[x] < \gamma$ for all x of the form $x = a + h\Delta x$ with $h < k$.

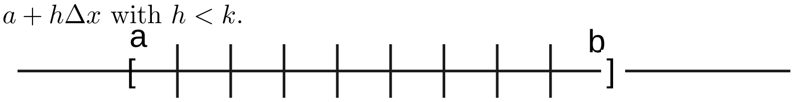


Figure 4.1: $[a, b]$ in steps of size Δx

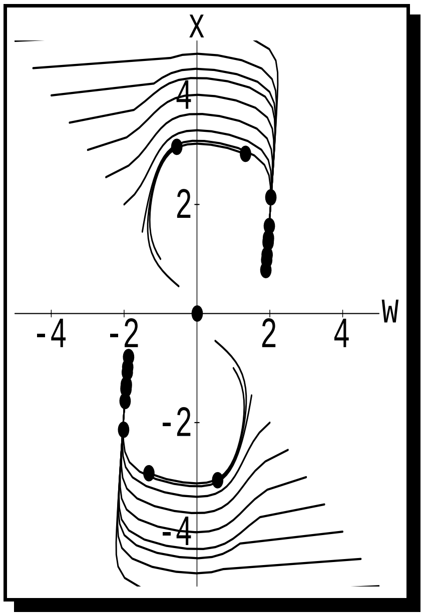
We need a general function for this. Let the function

$$\begin{aligned} M[\Delta x] &= \text{Min}[x : f[x] > \gamma, \quad a < x < b, \quad x = a + \Delta x, a + 2\,\Delta x, a + 3\,\Delta x, \cdots] \\ &= a + k\Delta x \end{aligned}$$

give this minimal x as a function of the step size Δx.

The natural extension of this Min function has the property that even when we compute at an infinitesimal step size, $\xi = M[\delta x]$ satisfies, $f[\xi] > \gamma$, and $f[x] < \gamma$ for $x = a + h\delta x < \xi$, in particular $f[\xi - \delta x] < \gamma$. Infinitesimals let continuity enter the picture.

Continuity of $f[x]$ means that if $c \approx x$, then $f[c] \approx f[x]$. We take c to be the standard real number such that $c \approx \xi = M[\delta x]$. We know $f[c] \approx f[\xi] > \gamma$ and $f[c] \approx f[\xi - \delta x] < \gamma$. Since $f[c]$ must be a real value for a real function at a real input, and since we have just shown that $f[c] \approx \gamma$, it follows that $f[c] = \gamma$, because ordinary reals can only be infinitely close if they are equal.



Part 3

1 Variable Differentiation

CHAPTER 5 The Theory of Derivatives

This chapter shows how the traditional "epsilon-delta" theory, rigorous infinitesimal analysis, and the intuitive approximations of the main text are related.

The chapter shows that

$$\lim_{\Delta x \to 0} \frac{f[x+\Delta x]-f[x]}{\Delta x} = f'[x] \qquad \text{locally uniformly}$$

$$\Leftrightarrow$$

$$f[x+\delta x] = f[x] + f'[x]\,\delta x + \varepsilon \cdot \delta x \quad \text{with} \quad \varepsilon \approx 0 \quad \text{for} \quad \delta x \approx 0$$

with all the provisos needed to make both of these exactly formally correct. Then this approximation is used to prove some of the basic facts about derivatives.

5.1 The Fundamental Theorem: Part 1

We begin with an overview that illustrates the two main approximations of calculus, how they fit together, and how the fine details are added if you wish to make formal arguments based on an intuitive approximation.

We re-write the traditional limit for a derivative as an approximation for the differential. Then we plug this differential approximation into an approximation of an integral to see why the Fundamental Theorem of Integral Calculus is true. The two main approximations interact to let us compute integrals by finding antiderivatives. For now, we simply treat the symbol "wiggly equals," $\approx$, as an intuitive "approximately equals." Subsections below justify the use both in terms of hyperreal infinitesimals and in terms of uniform limits. The point of the section is that the intuitive arguments are correct because we can fill in all the details if we wish.

The Intuitive Derivative Approximation

The traditional approach to derivatives is the approximation of secant lines approaching the slope of the tangent. Symbolically, this is

$$\lim_{\Delta x \to 0} \frac{f[x+\Delta x]-f[x]}{\Delta x} = f'[x]$$

The intuitive meaning of this formula is that $(f[x+\Delta x]-f[x])/\Delta x$ is approximately equal to $f'[x]$ when the difference, Δx, is small, $\Delta x = \delta x \approx 0$. We write this with an explicit error,

$$\frac{f[x+\delta x]-f[x]}{\delta x} = f'[x] + \varepsilon$$

where the error given by the Greek letter epsilon, ε, is small, provided that δx is small. We use lower case or small delta to indicate that the approximation is valid for a small difference in the value of x. [δ is lower case Δ. Both stand for "difference" because the difference in x-input is $\delta x = (x + \delta x) - x$.]

This approximation can be rewritten using algebra and expressed in the form

$$f[x+\delta x] - f[x] = f'[x]\,\delta x + \varepsilon \cdot \delta x \quad \text{with} \quad \varepsilon \approx 0 \quad \text{for} \quad \delta x \approx 0$$

where now the wiggly equals $\approx$ only means "approximately equals" in an intuitive sense. This expresses the change in a function $f[x+\delta x] - f[x]$ in moving from x to $x + \delta x$ as approximately given by a change $f'[x]\,\delta x$, linear in δx, with an error $\varepsilon \cdot \delta x$ that is small compared to δx, $(\varepsilon \cdot \delta x)/\delta x = \varepsilon \approx 0$.

This is a powerful intuitive formulation of the approximation of derivatives. (It is often called 'Landau's small oh formula.') This also has a direct geometric counterpart in terms of microscopes given in the main text, but here we use it symbolically.

An Antiderivative

Suppose that we begin with a function $f[x]$ and know an antiderivative $F[x]$, that is,

$$\frac{dF}{dx}[x] = f[x] \qquad \text{for} \quad a \le x \le b$$

The approximation above becomes

$$F[x+\delta x] - F[x] = f[x]\,\delta x + \varepsilon \cdot \delta x \quad \text{with} \quad \varepsilon \approx 0 \quad \text{for} \quad \delta x \approx 0$$

Flip this around to tell us

$$f[x]\,\delta x = (F[x+\delta x] - F[x]) - \varepsilon \cdot \delta x \quad \text{with} \quad \varepsilon \approx 0 \quad \text{for} \quad \delta x \approx 0$$

provided that $a \le x \le b$.

Integrals are Sums of Slices

The main idea of integral calculus is that integrals are 'sums of slices.' One way to express this is

$$\int_a^b f[x]\,dx = \lim_{\Delta x \to 0} (f[a]\Delta x + f[a+\Delta x]\Delta x + f[a+2\Delta x]\Delta x + f[a+3\Delta x]\Delta x + \cdots + f[b-2\Delta x]\Delta x + f[b-\Delta x]\Delta x)$$

where the sum is over values of $f[x]\,\Delta x$ where x starts at a and goes in steps of size Δx until we get to the slice ending at b.

The limiting quantity is approximately equal to the integral when the step size is "small enough."

$$\int_a^b f[x]\,dx \approx f[a]\delta x + f[a+\delta x]\delta x + \cdots + f[b-\delta x]\delta x \quad \text{for} \quad \delta x \approx 0$$

where $\approx$ temporarily only means the intuitive "approximately equals."

Now we incorporate the differential approximation above at each of the x points, $x = a$, $x = a + \delta x$, $\cdots$, $x = b - \delta x$, in our sum approximation, obtaining

$$\begin{aligned}\int_a^b f[x]\,dx \approx&\\ &([F(a+\delta x) - F[a]) + (F[a+2\delta x] - F[a+\delta x]) + \cdots + (F[b] - F[b-\delta x])\\ &\quad - (\varepsilon[a,\delta x]\,\delta x + \varepsilon[a+\delta x,\delta x])\,\delta x + \cdots + \varepsilon[b,\delta x]\,\delta x)\end{aligned}$$

The first sum 'telescopes,' that is, positive leading terms in one summand cancel negative second terms in the next, all except for the first and last terms,

$$\begin{aligned}&(F[a+\delta x] - F[a]) + (F[a+2\delta x] - F[a+\delta x]) + \cdots + (F[b] - F[b-\delta x])\\ &\qquad = -F[a] + F[b]\end{aligned}$$

The second sum and can be estimated as follows,

$$\begin{aligned}&|\varepsilon[a,\delta x]\,\delta x + \varepsilon[a+\delta x,\delta x]\,\delta x + \cdots + \varepsilon[b-\delta x,\delta x]\,\delta x|\\ &\qquad \le |\varepsilon[a,\delta x]|\,\delta x + |\varepsilon[a+\delta x,\delta x]|\,\delta x + \cdots + |\varepsilon[b-\delta x,\delta x]|\,\delta x\\ &\qquad \le |\varepsilon_{Max}|(\delta x + \delta x + \cdots + \delta x)\\ &\qquad \le |\varepsilon_{Max}|(b-a)\end{aligned}$$

where $|\varepsilon_{Max}|$ is the largest of the small errors, $\varepsilon[x,\delta x]$, coming from the differential approximation. The sum of δx enough times to move from a to b is $b-a$, the distance moved.

As long as we make the largest error small enough, the summed error less than $|\varepsilon_{Max}|(b-a)$ will also be small, so

$$\int_a^b f[x]\,dx \approx F[b] - F[a]$$

But these are both fixed and do not depend on how small we take δx, hence

$$\int_a^b f[x]\,dx = F[b] - F[a]$$

This intuitive estimation illustrates the Fundamental Theorem of Integral Calculus.

Theorem 5.1. *Fundamental Theorem of Integral Calculus: Part 1*

Suppose the real function $f[x]$ has an antiderivative, that is, a real function $F[x]$ so that the derivative of $F[x]$ satisfies

$$\frac{dF}{dx}[x] = f[x] \qquad \text{for all } x \text{ with } a \le x \le b$$

Then

$$\int_a^b f[x]\,dx = F[b] - F[a]$$

The above is not a formal proof, because we have not kept track of the "approximately equal" errors. This can be completed either from the "$\varepsilon - \delta$" theory of limits or by using hyperreal infinitesimals. Both justifications follow in separate subsections.

5.1.1 Rigorous Infinitesimal Justification

First, we take as our definition of derivative, $\frac{dF}{dx} = f[x]$, condition (a) of Theorem 3.5: for every hyperreal x with $a \le x \le b$ and every δx infinitesimal, there is an infinitesimal ε so that the extended functions satisfy

$$F[x + \delta x] - F[x] = f[x]\,\delta x + \varepsilon \cdot \delta x$$

The only thing we need to know from the theory of infinitesimals is that

$$\varepsilon_{Max}$$

exists in the sense that

$$\begin{aligned}
|\varepsilon[a, \delta x]\,\delta x + &\varepsilon[a + \delta x, \delta x]\,\delta x + \cdots + \varepsilon[b - \delta x, \delta x]\,\delta x| \\
&\le |\varepsilon[a, \delta x]|\,\delta x + |\varepsilon[a + \delta x, \delta x]|\,\delta x + \cdots + |\varepsilon[b - \delta x, \delta x]|\,\delta x \\
&\le |\varepsilon_{Max}|(\delta x + \delta x + \cdots + \delta x) \\
&\le |\varepsilon_{Max}|(b - a)
\end{aligned}$$

still holds when δx is infinitesimal. This follows from the Function Extension Axiom. Let $\varepsilon[x, \Delta x]$ be the real function of the real variables x and Δx,

$$\varepsilon[x, \Delta x] = \frac{F[x + \Delta x] - F[x]}{\Delta x} - f[x]$$

For each ordinary real Δx, there is an x of the form $x_m = a + m\,\Delta x \quad (m = 1, 2, 3, \cdots)$ so that the inequalities above hold with $\varepsilon_{Max} = \varepsilon[x_m, \Delta x]$. This is just a finite maximum being attained at one of the values. Define a real function $m[\Delta x] = x_m$.

Define a real function

$$S[\Delta x] = |\varepsilon[a, \Delta x]\,\Delta x + \varepsilon[a + \Delta x, \Delta x]\,\Delta x + \cdots + \varepsilon[\varepsilon[b - \Delta x, \Delta x]\,\Delta x|$$

The inequalities above say that for real Δx

$$S[\Delta x] \le |\varepsilon[m[\Delta x], \Delta x]|(b - a)$$

The Function Extension Axiom says

$$S[\delta x] \le |\varepsilon[m[\delta x], \delta x]|(b - a)$$

and the definition of derivative says that $\varepsilon[m[\delta x], \delta x]$ is infinitesimal, provided δx is infinitesimal. Since an infinitesimal times the finite number $(b - a)$ is also infinitesimal, we have shown that the difference between the real integral and the real answer

$$S[\delta x] = \int_a^b f[x]\,dx - (F[b] - F[a])$$

is infinitesimal. This means that they must be equal, since ordinary numbers can not differ by an infinitesimal unless they are equal.

5.1.2 Rigorous Limit Justification

We need our total error to be small. This total error, $\text{Error}_{Integral}$, is the difference between the quantity $F[b] - F[a]$ and the integral, so by the calculation above,

$$\text{Error}_{Integral} = \varepsilon[a, \Delta x]\,\Delta x + \varepsilon[a + \Delta x, \Delta x]\,\Delta x + \cdots + \varepsilon[b - \Delta x, \Delta x]\,\Delta x$$

We know from the calculation above that $|\text{Error}_{Integral}| \leq |\varepsilon_{Max}|(b - a)$. If we choose an arbitrary error tolerance of θ, then it is sufficient to have $|\varepsilon_{Max}| \leq \theta/(b - a)$, because then we will have $|\text{Error}_{Integral}| \leq \theta$. This means that we must show that the differential approximation

$$f[x]\Delta x = (F[x + \Delta x] - F[x]) - \varepsilon \cdot \Delta x$$

holds with $|\varepsilon| \leq \theta/(b - a)$ for every x in $[a, b]$. Using the algebra above in reverse, this is the same as showing that

$$\frac{F[x + \Delta x] - F[x]}{\Delta x} - f[x] = \varepsilon[x, \Delta x]$$

is never more than $\theta/(b - a)$, provided that Δx is small enough. The traditional way to say this is

$$\lim_{\Delta x \to 0} \frac{F[x + \Delta x] - F[x]}{\Delta x} = f[x] \qquad \text{uniformly for} \quad a \leq x \leq b$$

The rigorous definition of the limit in question is: for every tolerance η and every x in $[a, b]$, there exists a μ such that if $|\Delta x| < \mu$, then

$$|\frac{F[x + \Delta x] - F[x]}{\Delta x} - f[x]| < \eta$$

This is the formal definition of derivative condition (b) of Theorem 3.5. Our proof of the Fundamental Theorem is complete (letting $\eta = \theta/(b - a)$). The hypothesis says that if we can find a function $f[x]$ so that

$$\lim_{\Delta x \to 0} \frac{F[x + \Delta x] - F[x]}{\Delta x} = f[x] \qquad \text{uniformly for} \quad a \leq x \leq b$$

then the conclusion is

$$\int_a^b f[x]\,dx = F[b] - F[a]$$

(Notice that the existence of the limit defining the integral is part of our proof. The function $f[x]$ is continuous, because of our strong definition of derivative.)

5.2 Derivatives, Epsilons and Deltas

The fundamental approximation defining the derivative of a real valued function can be formulated with or without infinitesimals as follows.

Definition 5.2. *The Rigorous Derivative*

In Theorem 3.5 we saw that the following are equivalent definitions of, "The real function $f[x]$ is smooth with derivative $f'[x]$ on the interval (a, b)."

(a) *Whenever a hyperreal x satisfies $a < x < b$ and x is not infinitely near a or b, then an infinitesimal increment of the naturally extended dependent variable is approximately linear, that is, whenever $\delta x \approx 0$*

$$f[x + \delta x] - f[x] = f'[x]\,\delta x + \varepsilon \cdot \delta x$$

for some $\varepsilon \approx 0$.

(b) *For every compact subinterval $[\alpha, \beta] \subset (a, b)$,*

$$\lim_{\Delta x \to 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} = f'[x] \qquad \textit{uniformly for} \quad \alpha \le x \le \beta$$

in other words, for every accuracy tolerance θ and every real α and β in (a, b), there is a sufficiently small positive real number γ such that if the real number Δx satisfies $0 < |\Delta x| < \gamma$ and the real number x satisfies $\alpha \le x \le \beta$, then

$$\left| \frac{f[x + \Delta x] - f[x]}{\Delta x} - f'[x] \right| < \theta$$

(c) *For every real c in (a, b),*

$$\lim_{x \to c, \Delta x \to 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} = f'[c]$$

That is, for every real c with $a < c < b$ and every real positive θ, there is a real positive γ such that if the real numbers x and Δx satisfy $0 < \Delta x < \gamma$ and $0 < |x - c| < \gamma$, then $|\frac{f[x+\Delta x]-f[x]}{\Delta x} - f'[c]| < \theta$.

All derivatives computed by rules satisfy this strong approximation provided the formulas are valid on the interval. This is proved in Theorem 5.5 below.

5.3 Smoothness $\Rightarrow$ Continuity of Function and Derivative

This section shows that differentiability in the sense of Definition 5.2 implies that the function and derivative are continuous.

One difficult thing about learning new material is putting new facts together. Bolzano's Theorem and Darboux's Theorem have hypotheses that certain functions are continuous. This means you must show that the function you are working with is continuous. How do you tell if a function is continuous? You can't 'look' at a graph if you haven't drawn one and are using calculus to do so. What does continuity mean? Intuitively, it just means that small changes in the independent variable produce only small changes in the dependent variable.

In Theorem **??** we showed that the following are equivalent definitions

Definition 5.3. *Continuity of $f[x]$*
Suppose a real function $f[x]$ is defined for at least a small neighborhood of a real number a, $|x - a| < \Delta$, for some positive real Δ.
(a) *$f[x]$ is continuous at a if whenever a hyperreal x satisfies $x \approx a$, the natural extension satisfies $f[x] \approx f[a]$.*
(b) *$f[x]$ is continuous at a if $\lim_{x\to a} f[x] = f[a]$.*

Intuitively, this just means that $f[x]$ is close to $f[a]$ when x is close to a, for every $x \approx a$, $f[x]$ is defined and

$$f[x] \approx f[a]$$

The rules of calculus (together with Theorem 5.5) make it easy to verify that functions given by formulas are continuous: Simply calculate the derivative.

Theorem 5.4. *Continuity of $f[x]$ and $f'[x]$*
Suppose the real function $f[x]$ is smooth on the real interval (a, b) (see Definition 5.2). Then both $f[x]$ and $f'[x]$ are continuous at every real point c in (a, b).

Proof for $f[x]$:
Proof of continuity of f is easy algebraically but is obvious geometrically: A graph that is indistinguishable from linear in a microscope clearly only moves a small amount in a small x-step. Draw the picture on a small scale.

Algebraically, we want to show that if $x_1 \approx x_2$ then $f[x_1] \approx f[x_2]$, condition (a) above. Let c be any real point in (a, b) and $x = x_1 = c$. Let $\delta x = x_2 - x_1$ be any infinitesimal and use the approximation $f[x_2] = f[x+\delta x] = f[x_1]+f'[x_1]\delta x+\varepsilon\,\delta x$. The quantity $[f'[x_1]+\varepsilon]\delta x$ is medium times small = small, so $f[x_1] \approx f[x_2]$, by Theorem 1.12 (c). That is the algebraic proof.

Proof for $f'[x]$:
Proof of continuity of $f'[x]$ requires us to view the increment from both ends. First take any real c in (a, b), $x = x_1 = c$, and $\delta x = x_2 - x_1$ any nonzero infinitesimal. Use the approximation

$$f[x_2] = f[x + \Delta x] = f[x_1] + f'[x_1]\delta x + \varepsilon_1 \delta x.$$

Next let $x = x_2$, $\delta x = x_1 - x_2$ and use the approximation

$$f[x_1] = f[x + \Delta x] = f[x_2] + f'[x_2]\delta x + \varepsilon_2 \delta x.$$

The different x-increments are negatives, so we have

$$f[x_1] - f[x_2] = f'[x_2](x_1 - x_2) + \varepsilon_2(x_1 - x_2)$$

and

$$f[x_2] - f[x_1] = f'[x_1](x_2 - x_1) + \varepsilon_1(x_2 - x_1)$$

Adding, we obtain

$$0 = ((f'[x_2] - f'[x_1]) + (\varepsilon_2 - \varepsilon_1))\,(x_1 - x_2)$$

Dividing by the non-zero $(x_1 - x_2)$, we see that

$$f'[x_2] = f'[x_1] + (\varepsilon_1 - \varepsilon_2), \quad \text{so} \quad f'[x_2] \approx f'[x_1]$$

Note:

The derivative defined in many calculus books is a weaker pointwise notion than the notion of smoothness we have defined. The weak derivative function need not be continuous. (The same approximation does not apply at both ends with the weak definition.) This is explained in Chapter 6 on Pointwise Approximations.

Exercise set 5.3

1. (a) *Consider the real function $f[x] = 1/x$, which is undefined at $x = 0$. We could extend the definition by simply assigning $f[0] = 0$. Show that this function is not continuous at $x = 0$ but is continuous at every other real x.*
   (b) *Give an intuitive graphical description of the definition of continuity in terms of powerful microscopes and explain why it follows that smooth functions must be continuous.*
   (c) *The function $f[x] = \sqrt{x}$ is defined for $x \geq 0$; there is nothing wrong with $f[0]$. However, our increment computation for $\sqrt{x}$ above was not valid at $x = 0$ because a microscopic view of the graph focused at $x = 0$ looks like a vertical ray (or half-line). Explain why this is so, but show that $f[x]$ is still continuous "from the right;" that is, if $0 < x \approx 0$, then $\sqrt{x} \approx 0$ but $\frac{\sqrt{x}}{x}$ is very large.*

5.4 Rules $\Rightarrow$ Smoothness

This section shows that when we can compute a derivative by rules, then the smoothness Definition 5.2 is automatically satisfied on intervals where both formulas are defined.

Theorem 5.5. *Rules $\Rightarrow$ Smooth*
Suppose a function $y = f[x]$ is given by a formula to which we can apply the rules of Chapter 6 of the main text, obtaining a formula for $f'[x]$. If both $f[x]$ and $f'[x]$ are defined on the real interval (a, b), then then satisfy Definition 5.2 and, by Theorem 5.4, are continuous on (a, b).

Proof:
This is a special case of Theorem 10.1.

Exercise set 5.4

1. *What is the simple way to tell if a function is continuous?*
2. *Suppose that $y = f[x]$ is given by a formula that you can differentiate by the rules of calculus from Chapter 6 of the main text. As you know, you can differentiate many*

many formulas. What property does the function $\frac{dy}{dx} = f'[x]$ *have to have so that you can conclude* $f[x]$ *is continuous at all points of the interval* $a \le x \le b$*? (What about* $f[x] = 1/x$ *at* $x = 0$*?)*

Give examples of:

(a) *One* $y = f[x]$ *which you can differentiate by rules and an interval* $[a, b]$ *where* $f'[x]$ *is defined and* $f[x]$ *is continuous on the whole interval.*

(b) *Another* $y = f[x]$ *which you can differentiate by rules, but where* $f'[x]$ *fails to be defined on all of* $[a, b]$ *and where* $f[x]$ *is not continuous at some point* $a \le c \le b$*. (Hint: Read Theorem* 5.4*. What about* $y = 1/x$*?)*

What properties does the function $\frac{dy}{dx} = f'[x]$ *have to have so that you can conclude* $f'[x]$ *is continuous at all points of the interval* $a \le x \le b$*?*

Give Examples of:

(a) *One* $y = f[x]$ *which you can differentiate by rules and an interval* $[a, b]$ *where* $f'[x]$ *is defined and* $f'[x]$ *is continuous on the whole interval.*

(b) *Another* $y = f[x]$ *which you can differentiate by rules, but where* $f'[x]$ *fails to be defined on all of* $[a, b]$ *and where* $f'[x]$ *is not continuous at some point* $a \le c \le b$*. (Note: If* $f'[x]$ *is undefined at* $x = c$*, it cannot be considered continuous at* c*. Well, there is a sticky point here. Perhaps* $f'[x]$ *could be extended at an undefined point so that it would become continuous with the extension. It is actually fairly easy to rule this out with one of the functions you have worked with in previous homework problems.)*

5.5 The Increment and Increasing

A positive derivative means a function is increasing near the point. We prove this algebraically in this section.

It is 'clear' that if we view a graph in an infinitesimal microscope at a point x_0 and see the graph as indistinguishable from an upward sloping line, then the function must be 'increasing' near x_0. Certainly, the graph need not be increasing everywhere - draw $y = x^2$ and consider the point $x_0 = 1$ with $f'[1] = 2$. Exactly how should we formulate this? Even if you don't care about the symbolic proof of the algebraic formulation, the formulation itself may be useful in cases where you don't have graphs.

One way to say $f[x]$ increases near x_0 would be to say that if $x_1 < x_0 < x_2$ (and these points are not 'too far' from x_0), then $f[x_1] < f[x_0] < f[x_2]$. Another way to formulate the problem is to say that if $x_1 < x_2$ (and these points are not 'too far' from x_0), then $f[x_1] < f[x_2]$. Surprisingly, the second formulation is more difficult to prove (and even fails for pointwise differentiable functions). The second formulation essentially requires that we can move the microscope from x_0 to x_1 and continue to see an upward sloping line. We know from Theorem 5.4 that if $x_1 \approx x_0$ the slope of the microscopic line only changes an small amount, so we actually see the same straight line.

Theorem 5.6. *Local Monotony*

Suppose the function $f[x]$ is differentiable on the real interval $a < x < b$ and x_0 is a real point in this interval.

(a) *If $f'[x_0] > 0$, then there is a real interval $[\alpha, \beta]$, $a < \alpha < x_0 < \beta < b$, such that $f[x]$ is increasing on $[\alpha, \beta]$, that is,*

$$\alpha \le x_1 < x_2 \le \beta \qquad \Rightarrow \qquad f[x_1] < f[x_2]$$

(b) *If $f'[x_0] < 0$, then there is a real interval $[\alpha, \beta]$, $a < \alpha < x_0 < \beta < b$, such that $f[x]$ is decreasing on $[\alpha, \beta]$, that is,*

$$\alpha \le x_1 < x_2 \le \beta \qquad \Rightarrow \qquad f[x_1] > f[x_2]$$

Proof

We will only prove case (a), since the second case is proved similarly. First we verify that $f[x]$ is increasing on a microscopic scale. The idea is simple: Compute the change in $f[x]$ using the positive slope straight line and keep track of the error.

Take x_1 and x_2 so that $x_0 \approx x_1 < x_2 \approx x_0$. Since $f'[x_1] \approx f'[x_0]$ by Theorem 5.4 we may write $f'[x_1] = m + \varepsilon_1$ where $m = f'[x_0]$ and $\varepsilon_1 \approx 0$. Let $\delta x = x_2 - x_1$ so

$$\begin{aligned} f[x_2] = f[x_1 + \delta x] &= f[x_1] + f'[x_1] \cdot \delta x + \varepsilon_2 \cdot \delta x \\ &= f[x_1] + m \cdot \delta x + (\varepsilon_1 + \varepsilon_2) \cdot \delta x \end{aligned}$$

The number m is a real positive number, so $m + \varepsilon_1 + \varepsilon_2 > 0$ and , since $\delta x > 0$, $(m + \varepsilon_1 + \varepsilon_2) \cdot \delta x > 0$. This means $f[x_2] - f[x_1] > 0$ and $f[x_2] > f[x_1]$. This proves that for any infinitesimal interval $[\alpha, \beta]$ with $\alpha < x_0 < \beta$, the function satisfies

$$\alpha \le x_1 < x_2 \le \beta \qquad \Rightarrow \qquad f[x_1] < f[x_2]$$

The Function Extension Axiom guarantees that real numbers α and β exist satisfying the inequalities above, since if the equation fails for all real α and β, it would fail for infinitely close ones. That completes the proof.

Example 5.1. *A Non-Increasing Function with Pointwise Derivative 1*

The function

$$f[x] = \begin{cases} 0, & \text{if } x = 0 \\ x + x^2 \operatorname{Sin}[\frac{\pi}{x}], & \text{if } x \ne 0 \end{cases}$$

has a pointwise derivative at every point and $D_x f[0] = 1$ (but is not differentiable in the usual sense of Definition 5.2). This function is not increasing in any neighborhood of zero (so it shows why the pointwise derivative is not strong enough to capture the intuitive idea of the microscope). See Example 6.3.1 for more details.

5.6 Inverse Functions and Derivatives

If a function has a nonzero derivative at a point, then it has a local inverse. The project on Inverse Functions expands this section with more details.

The inverse of a function $y = f[x]$ is the function $x = g[y]$ whose 'rule un-does what the rule for f does.'

$$g[f[x]] = x$$

If we have a formula for $f[x]$, we can try to solve the equation $y = f[x]$ for x. If we are successful, this gives us a formula for $g[y]$,

$$y = f[x] \qquad \Leftrightarrow \qquad x = g[y]$$

Example 5.2. *$y = x^2$ and the Partial Inverse $x = \sqrt{y}$*

For example, if $y = f[x] = x^2$, then $x = g[y] = \sqrt{y}$, at least when $x \geq 0$. These two functions have the same graph if we plot g with its independent variable where the y axis normally goes, rather than plotting the input variable of $g[y]$ on the horizontal scale.

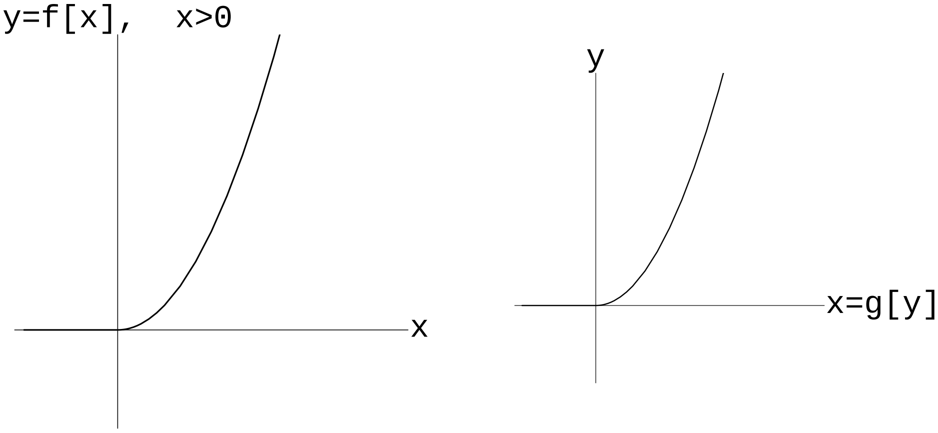


Figure 5.1: $y = f[x]$ and its inverse $x = g[y]$

The graph of $x = g[y]$ operationally gives the function g by choosing a y value on the y axis, moving horizontally to the graph, and then moving vertically to the x output on the x axis. This makes it clear graphically that the rule for g 'un-does' what the rule for f does. If we first compute $f[x]$ and then substitute that answer into $g[y]$, we end up with the original x.

Example 5.3. *$y = f[x] = x^9 + x^7 + x^5 + x^3 + x$ its Inverse*

The graph of the function $y = f[x] = x^9 + x^7 + x^5 + x^3 + x$ is always increasing because $f'[x] = 9\,x^8+7\,x^6+5\,x^4+3\,x^2+1 > 0$ is positive for all x. Since we know $\lim_{x\to-\infty} f[x] = -\infty$ and $\lim_{x\to+\infty} f[x] = +\infty$, Bolzano's Intermediate Value Theorem 4.5 says that $f[x]$ attains every real value y. By Theorem 5.6, $f[x]$ can attain each value only once. This means that for every real y, there is an $x = g[y]$ so that $f[x] = y$. In other words, we see abstractly that $f[x]$ has an inverse without actually solving the equation $y = x^9 + x^7 + x^5 + x^3 + x$ for x as a function of y.

Example 5.4. $\text{ArcTan}[y]$

The function $y = \text{Tan}[x]$ has derivative $\frac{dy}{dx} = \frac{1}{(\text{Cos}[x])^2}$. When $-\pi/2 < x < \pi/2$, cosine is not zero and therefore the tangent is increasing for $-\pi/2 < x < \pi/2$. How do we solve for

x in the equation $y = \mathrm{Tan}[x]$?

$$\begin{aligned} y &= \mathrm{Tan}[x] \\ \mathrm{ArcTan}[y] &= \mathrm{ArcTan}[\mathrm{Tan}[x]] = x \\ x &= \mathrm{ArcTan}[y] \end{aligned}$$

But what is the arctangent? By definition, the inverse of tangent on $(-\pi/2, \pi/2)$. So how would we compute it? The Inverse Function project answers this question.

Example 5.5. *A Non-Elementary Inverse*

Some functions are do not have classical expressions for their inverses. Let

$$y = f[x] = x^x$$

This may be written using $x = e^{\mathrm{Log}[x]}$, so $x^x = (e^{\mathrm{Log}[x]})^x = e^{x\ \mathrm{Log}[x]}$, and $f[x]$ has derivative

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(e^{x\ \mathrm{Log}[x]})}{dx} \\ &= (\mathrm{Log}[x] + x\,\frac{1}{x})e^{x\ \mathrm{Log}[x]} \\ &= (1 + \mathrm{Log}[x])\,x^x \end{aligned}$$

It is clear graphically that $y = f[x]$ has an inverse on either the interval $(0, 1/e)$ or $(1/e, \infty)$. We find where the derivative is positive, negative and zero as follows. First, $x^x = e^{x\ \mathrm{Log}[x]}$ is always positive, never zero, so

$$\begin{aligned} 0 &= (1 + \mathrm{Log}[x])\,x^x \\ 0 &= (1 + \mathrm{Log}[x]) \\ -1 &= \mathrm{Log}[x] \\ e^{-1} &= e^{\mathrm{Log}[x]} \\ \frac{1}{e} &= x \end{aligned}$$

If $x < 1/e$, say $x = 1/e^2$, then $\frac{dy}{dx} = (1 + Log[e^{-2}])(+) = (1-2)(+) = (-) < 0$. If $x = e > 1/e$, $\frac{dy}{dx} = (1 + \mathrm{Log}[e])(+) = (2)(+) = (+) > 0$. So $\frac{dy}{dx} < 0$ for $0 < x < 1/e$ and $\frac{dy}{dx} > 0$ for $1/e < x < \infty$. (Note our use of Darboux's Theorem 7.2.) This means that $f[x] = x^x$ has an inverse for $x > 1/e$.

It turns out that the inverse function $x = g[y]$ can not be expressed in terms of any of the classical functions. In other words, there is no formula for $g[y]$. (This is similar to the non-elementary integrals in the *Mathematica* NoteBook **SymbolicIntegr**. Computer algebra systems have a non-elementary function $\omega[x]$ which can be used to express the inverse.) The Inverse Function project has you compute the inverse approximately.

Example 5.6. *A Microscopic View of the Graph*

We view the graph $x = g[y]$ for the inverse as the graph $y = f[x]$ with the roles of the horizontal and vertical axes reversed. In other words, both functions have the same graph, but we view y as the input to the inverse function $g[y]$. A microscopic view of the graph can likewise be viewed as that of either $y = f[x]$ or $x = g[y]$.

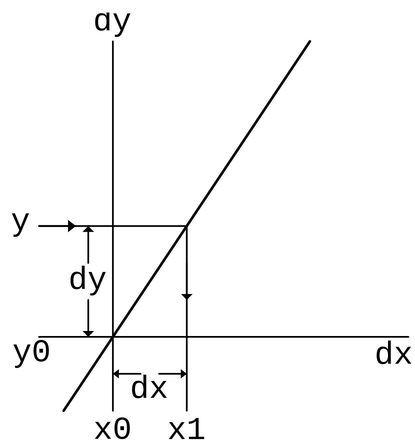


Figure 5.2: Small View of $x = g[y]$ and $y = f[x]$ at (x_0, y_0)

The ratio of a change in g-output dx to a change in g-input dy for the linear graph is the reciprocal of the change in f-output dy to the change in f-input dx for the function. In other words, if the inverse function really exists and is differentiable, we see from the microscopic view of the graph that we should have

$$\frac{dy}{dx} = f'[x_0] \qquad \text{and} \qquad \frac{dx}{dy} = \frac{1}{dy/dx} = g'[y_0]$$

The picture is right, of course, and the Inverse Function Theorem 5.7 justifies the needed steps algebraically (in case you don't trust the picture.)

Example 5.7. *The Symbolic Inverse Function Derivative*

Assume for the moment that $f[x]$ and $g[y]$ are smooth inverse functions. Apply the Chain Rule (in function notation) as follows.

$$\begin{aligned} x &= g[f[x]] \\ \frac{dx}{dx} &= g'[f[x]] \cdot f'[x] \\ 1 &= g'[f[x]] \cdot f'[x] \\ g'[f[x]] &= \frac{1}{f'[x]} \end{aligned}$$

At a point (x_0, y_0) on the graph of both functions, we have

$$g'[y_0] = \frac{1}{f'[x_0]}$$

In differential notation, this reads like ordinary fractions,

$$
\begin{aligned}
x = g[y] \quad &\Leftrightarrow \quad y = f[x] \\
\frac{dx}{dy} = g'[y] \quad &\Leftrightarrow \quad \frac{dy}{dx} = f'[x] \\
\frac{dx}{dy} &= 1/\frac{dy}{dx}
\end{aligned}
$$

A concrete example may help at this point.

Example 5.8. *Derivative of ArcTangent*

$$
\begin{aligned}
x = \mathrm{ArcTan}[y] \quad &\Leftrightarrow \quad y = \mathrm{Tan}[x] \\
\frac{dx}{dy} = \mathrm{ArcTan}'[y] \quad &\Leftrightarrow \quad \frac{dy}{dx} = \frac{1}{(\mathrm{Cos}[x])^2} \\
\mathrm{ArcTan}'[y] = \frac{dx}{dy} &= 1/\frac{dy}{dx} = (\mathrm{Cos}[x])^2
\end{aligned}
$$

Correct, but not in the form of a function of the same input variable y. We know that $\mathrm{Tan}^2[x] = \frac{\mathrm{Sin}^2[x]}{\mathrm{Cos}^2[x]} = y^2$ and $\mathrm{Sin}^2[x] + \mathrm{Cos}^2[x] = 1$, so we can express $\mathrm{Cos}^2[x]$ in terms of y,

$$
\begin{aligned}
\mathrm{Sin}^2[x] + \mathrm{Cos}^2[x] &= 1 \\
1 + \frac{\mathrm{Sin}^2[x]}{\mathrm{Cos}^2[x]} &= \frac{1}{\mathrm{Cos}^2[x]} \\
1 + y^2 &= \frac{1}{\mathrm{Cos}^2[x]} \\
\mathrm{Cos}^2[x] &= \frac{1}{1+y^2}
\end{aligned}
$$

So we can write

$$
\mathrm{ArcTan}'[y] = (\mathrm{Cos}[x])^2 = \frac{1}{1+y^2}
$$

The point of this concrete example is that we can compute the derivative of the arctangent even though we don't have a way (yet) to compute the arctangent. In general, the derivative of an inverse function at a point is the reciprocal of the derivative of the function. In this case a trig trick lets us find a general expression in terms of y as well.

Example 5.9. *Another Inverse Derivative*

It is sometimes easier to compute the derivative of the inverse function and invert for the derivative of the function itself – even if it is possible to differentiate the inverse function. For example, if $y = x^2 + 1$ and $x = \sqrt{y-1}$ when $y \geq 1$, then $\frac{dy}{dx} = 2x$. The inverse function rule says

$$\begin{aligned} \frac{dx}{dy} &= \\ 1/\frac{dy}{dx} &= \frac{1}{2x} \\ &= \frac{1}{2\sqrt{y-1}} \end{aligned}$$

The point of the last two examples is that computing derivatives by reciprocals is sometimes helpful. The next result justifies the method.

Theorem 5.7. *The Inverse Function Theorem*
Suppose $y = f[x]$ is a real function that is smooth on an interval containing a real point x_0 where $f'[x_0] \neq 0$. Then
(a) *There is a smooth real function $g[y]$ defined when $|y - y_0| < \Delta$, for some real $\Delta > 0$.*
(b) *There is a real $\varepsilon > 0$ such that if $|x - x_0| < \varepsilon$, then $|f[x] - y_0| < \Delta$ and $g[f[x]] = x$.*
$g[y]$ is a "local" inverse for $f[x]$.

PROOF:

Suppose we have a function $y = f[x]$ and know that $f'[x]$ exists on an interval around a point $x = x_0$ and we know the values $y_0 = f[x_0]$ and $m = f'[x_0] \neq 0$. In a microscope we would see the graph

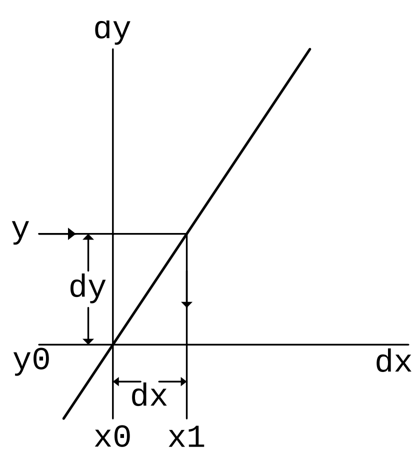


Figure 5.3: Small View of $y = f[x]$ at (x_0, y_0)

The point $(dx, dy) = (0, 0)$ in local coordinates is at (x_0, y_0) in regular coordinates.

Suppose we are given y near y_0, $y \approx y_0$. In the microscope, this appears on the dy axis at the local coordinate $dy = y - y_0$. The corresponding dx value is easily computed by inverting the linear approximation

$$\begin{aligned} dy &= m\,dx \\ m\,dx &= dy \\ dx &= dy/m \end{aligned}$$

The value x_1 that corresponds to $dx = dy/m$ is $dx = x_1 - x_0$. Solve for x_1,

$$\begin{aligned} x_1 &= x_0 + dx \\ &= x_0 + dy/m \\ &= x_0 + (y - y_0)/m \end{aligned}$$

Does this value of $x = x_1$ satisfy $y = f[x_1]$ for the value of y we started with? We wouldn't think it would be exact because we computed linearly and we only know the approximation

$$\begin{aligned} f[x_0 + dx] &= f[x_0] + f'[x_0]\,dx + \varepsilon \cdot dx \\ f[x_1] &= f[x_0] + m \cdot (x_1 - x_0) + \varepsilon \cdot (x_1 - x_0) \end{aligned}$$

We know that the error $\varepsilon \approx 0$ is small when $dx \approx 0$ is small, so we would have to move the microscope to see the error. Moving along the tangent line until we are centered over x_1, we might see

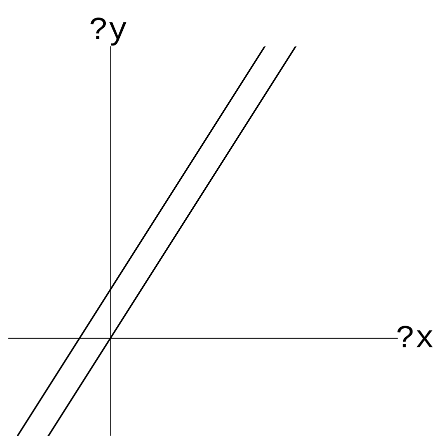


Figure 5.4: Small View at (x_1, y)

The graph of $y = f[x]$ appears to be the parallel line above the tangent, because we have only moved x a small amount and $f'[x]$ is continuous by Theorem 5.4. We don't know how to compute $x = g[y]$ necessarily, but we do know how to compute $y_1 = f[x_1]$. Suppose we have computed this and focus our microscope at (x_1, y_1) seeing

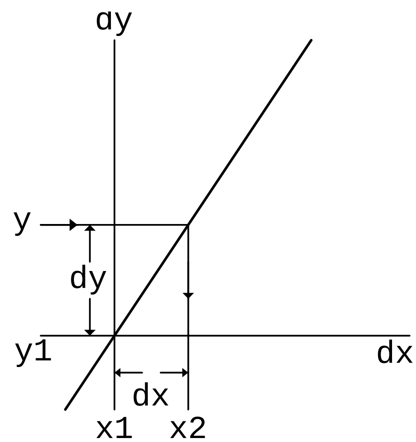


Figure 5.5: Small View at (x_1, y_1)

We still have the original $y \approx y_1$ and thus y still appears on the new view at $dy = y - y_1$. The corresponding dx value is easily computed by inverting the linear approximation

$$\begin{aligned} dy &= m\,dx \\ m\,dx &= dy \\ dx &= dy/m \end{aligned}$$

The x value, x_2, that corresponds to $dx = dy/m$ is $dx = x_2 - x_1$ with x_2 unknown. Solve for the unknown,

$$\begin{aligned} x_2 &= x_1 + dx \\ &= x_1 + dy/m \\ &= x_1 + (y - y_1)/m \end{aligned}$$

This is the same computation we did above to go from x_0 to x_1, in fact, this gives a discrete dynamical system of successive approximations

$$\begin{aligned} x_0 &= \text{given} \\ x_{n+1} &= x_n + (y - f[x_n])/m \\ x_{n+1} &= G[x_n], \qquad \text{with} \qquad G[x] = x + (y - f[x])/m \end{aligned}$$

The sequence of approximations is given by the general function iteration explained in Chapter 20 of the main text,

$$x_1 = G[x_0], \quad x_2 = G[x_1] = G[G[x_0]], \quad x_3 = G[G[G[x_0]]], \cdots$$

and we want to know how far we move,

$$\lim_{n\to\infty} x_n = ?$$

The iteration function $G[x]$ is smooth, in fact,

$$G'[x] = 1 - f'[x]/m$$

and in particular, $G'[x_0] = 1 - f'[x_0]/m = 1 - m/m = 0$. By Theorem 5.4, whenever $x \approx x_0$, $G'[x] \approx 0$. Differentiability of $G[x]$ means that if $x_i \approx x_j \approx x_0$,

$$\begin{aligned} G[x_i] - G[x_j] &= G'[x_j] \cdot (x_i - x_j) + \iota \cdot (x_i - x_j) \\ |G[x_i] - G[x_j]| &= |G'[x_j] + \iota| \cdot |x_i - x_j| \\ |G[x_i] - G[x_j]| &\le r \cdot |x_i - x_j| \end{aligned}$$

for any real positive r, since $G'[x_j] \approx G'[x_0] = 0$ and some $\iota \approx 0$.

If $y \approx y_0$, then $x_1 = x_0 - (y - y_0)/m \approx x_0$. Similarly, if $x_n \approx x_0$, then $f[x_n] \approx y$ and $x_{n+1} \approx x_0$, so $|G[x_{n+1}] - G[x_n]| \le r \cdot |x_{n+1} - x_n|$. Hence,

$$\begin{aligned} |x_2 - x_1| &= |G[x_1] - G[x_0]| \le r|x_1 - x_0| \\ |x_3 - x_2| &= |G[x_2] - G[x_1]| \le r|x_2 - x_1| \le r(r|x_1 - x_0|) = r^2|x_1 - x_0| \\ |x_4 - x_3| &= |G[x_3] - G[x_2]| \le r|x_3 - x_2| \le r(r^2|x_1 - x_0|) = r^3|x_1 - x_0| \end{aligned}$$

and in general

$$|x_{n+1} - x_n| \le r^n |x_1 - x_0|$$

The total distance moved in x is estimated as follows.

$$\begin{aligned} |x_{n+1} - x_0| &= |(x_{n+1} - x_n) + (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_1 - x_0)| \\ &\le |x_{n+1} - x_n| + |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_1 - x_0| \\ &\le r^n|x_1 - x_0| + r^{n-1}|x_1 - x_0| + \ldots + |x_1 - x_0| \\ &\le (r^n + r^{n-1} + \ldots + r + 1)|x_1 - x_0| \\ &\le \frac{1 - r^{n+1}}{1 - r}|x_1 - x_0| \end{aligned}$$

The sum $1 + r + r^2 + r^3 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$ is a geometric series as studied in the main text Chapter 25. Since $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$, $1 + r + r^2 + r^3 + \cdots + r^n \to \dfrac{1}{1-r}$ for $|r| < 1$. Thus, for any $y \approx y_0$ and any real r with $0 < r < 1$,

$$|x_n - x_0| \le \frac{1}{1-r}|x_1 - x_0|$$

for all $n = 1, 2, 3, \ldots$.

Similar reasoning shows that when $y \approx y_0$

$$|x_{k+n} - x_k| \le r^k \frac{1}{1-r}|x_1 - x_0|$$

because

$$\begin{aligned} |x_{k+1} - x_k| &\le r^k|x_1 - x_0| \\ |x_{k+2} - x_{k+1}| &= |G[x_{k+1}] - G[x_k]| \le r|x_{k+1} - x_k| \le r(r^k|x_1 - x_0|) \\ |x_{k+3} - x_{k+2}| &= |G[x_{k+2}] - G[x_{k+1}]| \le r|x_{k+2} - x_{k+1}| \le r^2(r^k|x_1 - x_0|) \\ &\vdots \end{aligned}$$

Take the particular case $r = 2$. We have shown in particular that whenever $0 < \delta \approx 0$ and $|y - y_0| < \delta$, then for all k and n,

$$|x_n - x_0| \leq 2|x_1 - x_0| \quad \text{and} \quad |x_{k+n} - x_k| \leq \frac{2}{2^k}|x_1 - x_0|$$

and $f[x]$ is defined and $|f'[x] - f'[x_0]| < |m|/2$ for $|x - x_0| < 3\delta/|m|$. By the Function Extension Axiom 2.1, there must be a positive real Δ such that if $|y - y_0| < \Delta$, then for all k and n,

$$|x_n - x_0| \leq 2|x_1 - x_0| \quad \text{and} \quad |x_{k+n} - x_k| \leq \frac{2}{2^k}|x_1 - x_0|$$

and $f[x]$ is defined and $|f'[x] - f'[x_0]| < |m|/2$ for $|x - x_0| < 3\Delta/|m|$. Fix this positive real Δ.

Also, if $|x - x_0| < E \approx 0$, then $|f[x] - y_0| < \Delta$ with Δ as above. By the Function Extension Axiom 2.1, there is a real positive ε so that if $|x - x_0| < \varepsilon < 3\Delta/|m|$, then $|f[x] - y_0| < \Delta$ and $|f'[x] - f'[x_0]| < |m|/2$.

Now, take any real y with $|y - y_0| < \Delta$ and consider the sequence

$$x_1 = G[x_0], \quad x_2 = G[x_1] = G[G[x_0]], \quad x_3 = G[G[G[x_0]]], \cdots$$

This converges because once we have gotten to the approximation x_k, we never move beyond that approximation by more than

$$|x_{k+n} - x_k| \leq \frac{2}{2^k}|x_1 - x_0|$$

In other words, if we want an approximation good to one one millionth, we need only take k large enough so that

$$\begin{aligned} \frac{2}{2^k}|x_1 - x_0| &\leq 10^{-6} \\ (1-k)\,\mathrm{Log}[2] + \mathrm{Log}[|x_1 - x_0|] &\leq -6\log[10] \\ k &\geq \frac{\mathrm{Log}[2] + \mathrm{Log}[|x_1 - x_0|] + 6\,\mathrm{Log}[10]}{\mathrm{Log}[2]} \end{aligned}$$

This shows by an explicit error formula that the sequence x_n converges. We define a real function $g[y] = \lim_{n\to\infty} x_n$ whenever $|y - y_0| < \Delta$. We can approximate $g[y] = x_\infty$ using the recursive sequence. (This is a variant of "Newton's method" for uniform derivatives.)

Moreover, if $|y - y_0| < \Delta$, then $f[x_\infty]$ is defined and $|f'[x_\infty] - f'[x_0]| < |m|/2$ because $|x_\infty - x_0| \leq 2|x_1 - x_0| = 2|y - y_0|/|m| < 3\Delta/|m|$.

Consider the limit

$$\begin{aligned} x_\infty &= \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} G[x_{n-1}] \\ &= G[\lim_{n\to\infty} x_{n-1}] \\ &= G[x_\infty] \\ x_\infty &= x_\infty + (y - f[x_\infty])/m \\ 0 &= (y - f[x_\infty])/m \\ y &= f[x_\infty] \end{aligned}$$

so $g[y] = x_\infty$ is the value of the inverse function and proves that the inverse function exists in our real interval $[x_0 - \varepsilon, x_0 + \varepsilon]$.

We conclude by showing that $g[y]$ is differentiable. Take $y_1 \approx y_2$ in the interval $(y_0 - \Delta, y_0 + \Delta)$, not near the endpoints. We know that $f[x_i]$ is defined and $|f'[x_i]| > |m|/2$ where $x_i = g[y_i]$ for $i = 1, 2$.

We have $x_2 \approx x_1$ because the defining sequences stay close. Let $x_{i1} = x_0 + (y_i - f[x_0])/m$ and $x_{i(n+1)} = x_{in} + (y_i + f[x_{in}])/m$, for $i = 1, 2$. Then $|x_{21} - x_{11}| = |y_2 - y_1|/|m| \approx 0$ and since $|x_i - x_0| < 3\Delta/|m|$, $f[x]$ is differentiable at x_{in}. We can show inductively that $|x_{2(n+1)} - x_{1(n+1)}| < B \cdot \left|\frac{y_2 - y_1}{m}\right|$ for some finite multiplier B when n is finite, because

$$\begin{aligned} x_{2(n+1)} - x_{1(n+1)} &= (x_{2n} - x_{1n}) + \frac{y_2 - y_1}{m} + \frac{f[x_{2n}] - f[x_{1n}]}{m} \\ &= (x_{2n} - x_{1n}) + \frac{y_2 - y_1}{m} + \frac{(f'[x_{1n}] + \iota) \cdot (x_{2n} - x_{1n})}{m} \\ &= \frac{y_2 - y_1}{m} + \left(1 + \frac{f'[x_{1n}] + \iota}{m}\right) \cdot (x_{2n} - x_{1n}) \end{aligned}$$

For any real positive θ, choose k large enough so that $|x_i - x_{ik}| < \frac{2}{2^k}|x_{i1} - x_0| < \theta/3$. Then $|x_2 - x_1| < |x_2 - x_{2k}| + |x_1 - x_{1k}| + |x_{2k} - x_1 1k| < \theta$. Since θ is arbitrary, we must have $x_2 \approx x_1$.

Differentiability of $f[x]$ means that

$$f[x_2] - f[x_1] = f'[x_1](x_2 - x_1) + \iota \cdot (x_2 - x_1)$$

with $\iota \approx 0$. Solving for $(x_2 - x_1)$ gives

$$\begin{aligned} (x_2 - x_1) &= \frac{y_2 - y_1}{f'[x_1] + \iota} \\ g[y_2] - g[y_1] &= \frac{1}{f'[x_1] + \iota} \cdot (y_2 - y_1) \end{aligned}$$

Since $|f'[x_1]| > |m|/2$, $\frac{1}{f'[x_1]+\iota} = \frac{1}{f'[x_1]} + \eta$ with $\eta \approx 0$. Hence,

$$g[y_2] - g[y_1] = \frac{1}{f'[x_1]} \cdot (y_2 - y_1) + \eta \cdot (y_2 - y_1)$$

and $g[y]$ is differentiable with $g'[y_1] = 1/f'[x_1]$.

CHAPTER 6 Pointwise Derivatives

This chapter explores the pathological consequences of weakening the definition of the derivative to only requiring the limit of difference quotients to converge pointwise.

Could a function have a derivative of 1 at $x = a$ and not be increasing near $x = a$? Could we have $F'[x] = f[x]$ on $[a, b]$ and yet not have $\int_a^b f[x]\, dx = F[b] - F[a]$? The strange answer is "yes" if you weaken the definition of derivative to allow only "pointwise" derivatives. We chose Definition 5.2 because ordinary functions given by formulas do not exhibit these pathologies, as shown in Theorem 5.5. We make the theory simple and natural with Definition 5.2 and lose nothing more than strange curiosities. If $f[x]$ is smooth on an interval around $x = a$ and $f'[a] = 1$, then $f[x]$ is increasing. (See Theorem 5.6.) The Fundamental Theorem of Integral Calculus above holds for smooth functions, as our proof shows, whereas a pointwise derivative need not even be integrable.

It is an unfortunate custom in many calculus texts to use the pointwise derivative. (As Peter Lax said in his lecture at a conference in San Jose, 'No self-respecting analyst would study the class of only pointwise differentiable functions.') This chapter explores the pathologies of the pointwise derivative and concludes with the connection with Definition 5.2 in Theorem 7.4: If pointwise derivatives are continuous, they satisfy Definition 5.2. The contrast of the straightforward proofs by approximation in this book with the round-about proofs of things like the Fundamental Theorem in many "traditional" books is then clear. The Mean Value Theorem 7.1 is used to make an approximation uniform. The traditional approach obscures the approximation concepts, makes the Mean Value Theorem seem more central than it actually is, and contributes no interesting new theorems other than the Mean Value Theorem whose main role is to recover from the over-generalization using Theorem 7.4.

6.1 Pointwise Limits

This section reviews the idea of a limit both from the point of view of "epsilons and deltas" and infinitesimals.

Suppose $g[\Delta x]$ is a function that only depends on one real variable, Δx, and is defined when $0 < |\Delta x| < b$. (The function may or may not be defined when $\Delta x = 0$.) Let g_0 be a number. The intuitive meaning of

$$\lim_{\Delta x \to 0} g[\Delta x] = g_0$$

is that $g[\delta x]$ is close to the value g_0, when δx is small or close to zero, but not necessarily equal to zero. (We exclude $\delta x = 0$, because we often have an expression which is not defined at the limiting value, such as $g[\Delta x] = \mathrm{Sin}[\Delta x]/\Delta x$ and want to know that $\lim_{\Delta x \to 0} \mathrm{Sin}[\Delta x]/\Delta x = 1$.) Technically, the limit is g_0 if the natural extension function satisfies $g[\delta x] \approx g_0$, whenever $0 \neq \delta x \approx 0$.

We proved the following in Theorem 3.2.

Theorem 6.1. *Let $g[\Delta x]$ be a function of one real variable defined for $0 < |\Delta x| < b$ and let g_0 be a number. Then the following are equivalent definitions of*

$$\lim_{\Delta x \to 0} g[\Delta x] = g_0$$

(a) *For every nonzero infinitesimal δx, the natural extension function satisfies*

$$g[\delta x] \approx g_0$$

(b) *For every real positive error tolerance, ε, there is a corresponding input allowance $D[\varepsilon]$, such that whenever the real value satisfies $0 < |\Delta x| < D[\varepsilon]$, then*

$$|g[\Delta x] - g_0| < \varepsilon$$

Example 6.1. $\lim_{\Delta x \to 0} \mathrm{Sin}[\pi/\Delta x]$ *does not exist*

We want to see why a limit need not exist in the case

$$g[\Delta x] = \mathrm{Sin}[\frac{\pi}{\Delta x}]$$

Notice that $g[\Delta x]$ is defined for all real Δx except $\Delta x = 0$. The fact that it is not defined is not the reason that there is no limit. We will show below that

$$\lim_{\Delta x \to 0} \Delta x \ \mathrm{Sin}[\frac{\pi}{\Delta x}] = 0$$

even though this second function is also undefined at $\Delta x = 0$.

We know from the 2π periodicity of sine that

$$\mathrm{Sin}[\theta] = +1 \quad \text{if} \quad \theta = 2k\pi + \frac{\pi}{2} \qquad \text{and} \qquad \mathrm{Sin}[\theta] = -1 \quad \text{if} \quad \theta = 2k\pi - \frac{\pi}{2}$$

whenever $k = \pm 1, \pm 2, \cdots$ is an integer. Hence we see that

$$g[\Delta x_1] = \mathrm{Sin}[\frac{\pi}{\Delta x_1}] = +1 \qquad \text{if} \qquad \Delta x_1 = \frac{1}{2k + \frac{1}{2}}$$

$$g[\Delta x_2] = \mathrm{Sin}[\frac{\pi}{\Delta x_2}] = -1 \qquad \text{if} \qquad \Delta x_2 = \frac{1}{2k - \frac{1}{2}}$$

INTUITIVE REASON FOR NO LIMIT

We can take k as large as we please, making Δx_1 and Δx_2 both close to zero, yet $g[\Delta x_1] = +1$ and $g[\Delta x_2] = -1$, so there is no single limiting value g_0.

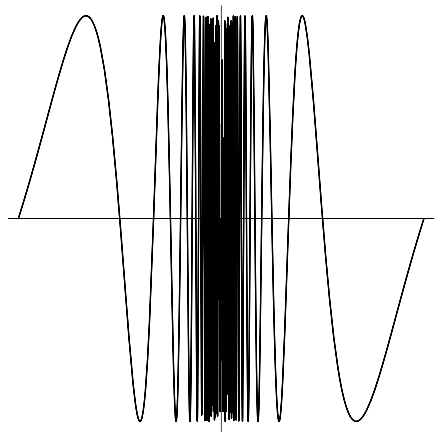


Figure 6.1: $y = \mathrm{Sin}[\frac{\pi}{\Delta x}]$

Rigorous Infinitesimal Reason for No Limit

We want an infinite integer K, so we can let $\delta x_1 = 1/(2\,K + \frac{1}{2})$ and $\delta x_2 = 1/(2\,K - \frac{1}{2})$ and have $\delta x_1 \approx \delta x_2 \approx 0$, $g[\delta x_1] = +1$ and $g[\delta x_2] = -1$. We can rigorously define infinite integers by function extension.

Let $N[x]$ be the function that indicates whether or not a number is an integer,

$$N[x] = \begin{cases} 0, & \text{if } x \text{ is not an integer} \\ 1, & \text{if } x \text{ is an integer} \end{cases}$$

Then we know that if $N[k] = 1$, then $\mathrm{Sin}[(2\,K \pm \frac{1}{2})\pi] = \pm 1$ (respectively). We also know that every real number is within $\frac{1}{2}$ unit of an integer; for every x, there is a k with $|x - k| \leq \frac{1}{2}$ and $N[k] = 1$. The natural extension of $N[x]$ also has these properties, so given any infinite hyperreal H, there is another K satisfying $|H - K| \leq \frac{1}{2}$, $N[K] = 1$ and $\mathrm{Sin}[(2\,K \pm \frac{1}{2})\pi] = \pm 1$. This technically completes the intuitive argument above, since we have two infinitesimals $\delta x_1 = \dfrac{1}{2K + \frac{1}{2}}$ and $\delta x_2 = \dfrac{1}{2K - \frac{1}{2}}$ with $g[\delta x_1]$ a distance of 2 units from $g[\delta x_2]$.

Rigorous $\varepsilon - D$ Reason for No Limit

No limit means that the negation of the $\varepsilon - D$ statement holds for every real value g_0. Negation of quantifiers is tricky, but the correct negation is that for every g_0, there is some real $\varepsilon > 0$ so that for every real $D > 0$ there is a real Δx such that $|\Delta x| < D$ and $|g[\Delta x] - g_0| \geq \varepsilon$.

Let $\varepsilon = \frac{1}{3}$ and let g_0 and $D > 0$ be arbitrary real numbers. We know that we can take an integer k large enough so that $0 < |\Delta x_1| < |\Delta x_2| < D$, $g[\Delta x_1] = +1$ and $g[\Delta x_2] = -1$. At most one of the two values can be within $\frac{1}{3}$ of g_0, because if $|g[\Delta x_1] - g_0| < \frac{1}{3}$, then $|g[\Delta x_2] - g_0| \geq \frac{5}{3}$ or vice versa. This shows that the negation of the $\varepsilon - D$ statement holds.

Example 6.2. $\lim_{\Delta x \to 0} \Delta x\ \mathrm{Sin}[\pi/\Delta x] = 0$

Since $|\mathrm{Sin}[\theta]| \leq 1$, $|\Delta x\ \mathrm{Sin}[\text{anything}]| \leq |\Delta x|$, so if $|\Delta x|$ is small, $|\Delta x\ \mathrm{Sin}[\pi/\Delta x]|$ is small. The rigorous justification with infinitesimals is obvious and the rigorous $\varepsilon - D$ argument follows simply by taking $D[\varepsilon] = \varepsilon$.

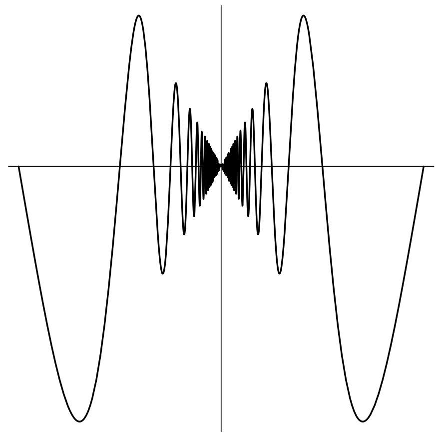


Figure 6.2: $y = \Delta x \ \mathrm{Sin}[\frac{\pi}{\Delta x}]$

Exercise set 6.1

1. *Show that* $\lim_{\Delta x \to 0} \Delta x^2 \ \mathrm{Sin}[\pi/\Delta x] = 0$.
2. *Show that* $\lim_{\Delta x \to 0} \Delta \, \mathrm{Cos}[\pi/\Delta x^2]$ *does not exist.*

6.2 Pointwise Derivatives

What happens if we apply the pointwise limit idea to $g[\Delta x] = (f[x+\Delta x] - f[x])/\Delta x$, "holding x fixed"? In fact, many books use this to define the following weak notion of derivative.

Definition 6.2. *Pointwise Derivative*
We say that the function $f[x]$ has pointwise derivative $D_x f[x_0]$ at a point x_0 if

$$\lim_{\Delta x \to 0} \frac{f[x_0 + \Delta x] - f[x_0]}{\Delta x} = D_x f[x_0]$$

What is the difference between this definition and Definition 5.2? We can explain this either in terms of the $\varepsilon - D$ definition, or in terms of infinitesimals. In terms of $\varepsilon - D$ limits, the input allowance $D[\varepsilon]$ can depend on the point x_0 in the pointwise definition. In the following example, $f[x] = x^2 \ \mathrm{Sin}[\pi/x]$, a $D[\varepsilon]$ that works at $x = 0$ does not work at $x = \sqrt{\varepsilon}$.

In terms of infinitesimals, the increment approximation

$$f[x_0 + \delta x] - f[x_0] = f'[x_0] \cdot \delta x + \varepsilon \cdot \delta x$$

only holds at fixed real values. In the following example, this approximation fails at hyperreal values like $x = 1/(2\,K)$.

Before we proceed with the example, we repeat an important observation of Theorem 5.5. Derivatives computed by rules automatically satisfy the stronger approximation, provided the formulas are valid on intervals. If you compute derivatives by rules, you know that you will see straight lines in microscopic views of the graph. The next example shows that this is false for the weaker approximation of pointwise derivatives.

Example 6.3. *A Non-Smooth, Pointwise Differentiable Function*

From the exercise above, the following definition makes $f[x]$ a continuous function,

$$f[x] = \begin{cases} 0, & \text{if } x = 0 \\ x^2 \ \mathrm{Sin}[\frac{\pi}{x}], & \text{if } x \neq 0 \end{cases}$$

That is, $\lim_{x\to 0} f[x] = f[0] = 0$. Differentiation of $f[x]$ by rules does not apply at $x = 0$, since we obtain

$$f'[x] = 2\,x\ \mathrm{Sin}[\frac{\pi}{x}] - \pi\ \mathrm{Cos}[\frac{\pi}{x}]$$

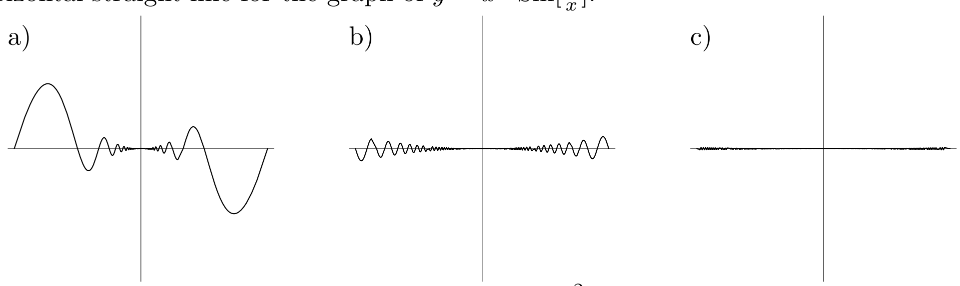
which is undefined at $x = 0$.

We can apply the pointwise definition of derivative to this function at $x_0 = 0$,

$$\begin{aligned} \lim_{\Delta x\to 0} \frac{f[x_0+\Delta x] - f[x_0]}{\Delta x} &= \lim_{\Delta x\to 0} \frac{f[0+\Delta x] - f[0]}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\Delta x^2\ \mathrm{Sin}[\frac{\pi}{\Delta x}] - 0}{\Delta x} \\ &= \lim_{\Delta x\to 0} \Delta x\ \mathrm{Sin}[\frac{\pi}{\Delta x}] = 0 \end{aligned}$$

So the pointwise derivative at $x = 0$ is$D_x f[0] = 0$.

If we focus a microscope at $(x, y) = (0, 0)$ and magnify enough, we will see only a horizontal straight line for the graph of $y = x^2\ \mathrm{Sin}[\frac{\pi}{x}]$.



Three Views of $y = x^2\ \mathrm{Sin}[\frac{\pi}{x}]$

However, if we first magnify this much and then move the focus point to

$$x = 1/\sqrt{\text{magnification}}$$

we will no longer see a straight line, but rather a pure sinusoid!

Example 6.4. *Why we see a sinusoid*

Suppose we focus our microscope at the point $x = 1/(2\,K)$, for a very large K. We know that $\mathrm{Sin}[\frac{\pi}{x}] = \pm 1$ for $x = 1/(2\,K \pm \frac{1}{2})$. This makes the function values $x^2\ \mathrm{Sin}[\frac{\pi}{x}]$ differ by

$$\frac{1}{(2\,K-\frac{1}{2})^2} - \frac{-1}{(2\,K+\frac{1}{2})^2} = \frac{1}{(2\,K)^2}\left[\frac{1}{(1-\frac{1}{4\,K})^2} + \frac{1}{(1+\frac{1}{4\,K})^2}\right]$$

so that we see 2 after magnification by $(2\,K)^2$,

$$\frac{1}{(1-\frac{1}{4\,K})^2} + \frac{1}{(1+\frac{1}{4\,K})^2} \approx 2$$

We compute the distance between these x points,

$$\begin{aligned}\frac{1}{2\,K-\frac{1}{2}} - \frac{1}{2\,K+\frac{1}{2}} &= \frac{2\,K+\frac{1}{2}-2\,K+\frac{1}{2}}{(2\,K-\frac{1}{2})(2\,K+\frac{1}{2})}\\ &= \frac{1}{4\,K^2-\frac{1}{4}}\\ &= \frac{1}{(2\,K)^2}\frac{1}{1-\frac{1}{16\,K^2}}\end{aligned}$$

These points are one unit apart on the $1/(2\,K)^2$ scale,

$$\frac{1}{1-\frac{1}{16\,K^2}} \approx 1$$

We will see a difference of two units in function values at magnification $(2\,K)^2$ and the differing points lie one unit apart at this magnification.

We can say more. If we magnify by $4\,K^2$ and observe the function $f[x+\delta x]$ with the microscope centered at $(x,0) = (1/(2\,K),0)$, we see the magnified values

$$4\,K^2(x+\delta x)^2\ \mathrm{Sin}[\frac{\pi}{x+\delta x}], \qquad x \text{ fixed}, \quad \delta x \text{ varying}.$$

But we also see magnified values on the dx axis. Let $\delta x = dx/(4\,K^2)$, for dx finite and let

$$F[dx] = 4\,K^2(x+\delta x)^2\ \mathrm{Sin}[\frac{\pi}{x+\delta x}]$$

with this relationship between the true δx and the dx we see in the microscope. Our microscopic view is the same as $F[dx]$ at unit scale. The coefficient in front of the sine above is actually constant on the scale of our microscope,

$$\begin{aligned}4\,K^2(x+\delta x)^2 &= 4\,K^2(\frac{1}{2\,K} + \frac{dx}{4\,K^2})^2\\ &= (\frac{2\,K}{2\,K} + \frac{dx\,2\,K}{4\,K^2})^2\\ &= (1+\frac{dx}{2\,K})^2\\ &\approx 1\end{aligned}$$

for dx finite, so $F[dx] \approx \text{Sin}[\frac{\pi}{x+\delta x}]$ on this scale. By algebra (as in Chapters 5 and 28 of the main text)

$$\begin{aligned}\frac{1}{x+\delta x} &= \frac{1}{x} + \frac{-\delta x}{x(x+\delta x)} \\ &= 2\,K - \frac{dx}{(2\,K)^2(\frac{1}{2\,K})(\frac{1}{2\,K}} + \frac{dx}{(2\,K)^2} \\ &= 2\,K - \frac{dx}{1+\frac{dx}{2\,K}} \\ &\approx 2\,K - dx\end{aligned}$$

This means that $\text{Sin}[\pi/(x+\delta x)] \approx \text{Sin}[2\,K\,\pi - \pi\,dx] = -\,\text{Sin}[\pi\,dx]$. At the point $x = 1/(2\,K)$ with magnification $(2\,K)^2$, we see the function

$$dy = -\,\text{Sin}[\pi\,dx]$$

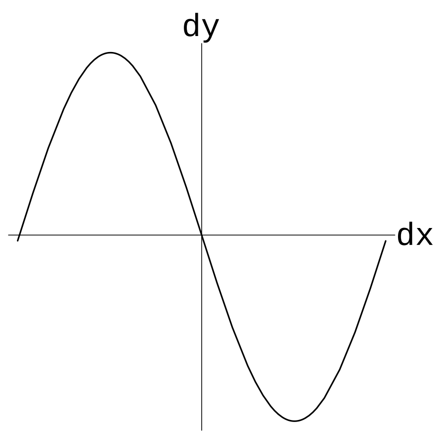


Figure 6.3: $y = f[x]$ at $x = 1/(2\,K)$

Example 6.5. *More Trouble with Pointwise Derivatives*

The sinusoidal view we see in the microscope is just a hint of what can go wrong with derivatives that are only given by pointwise limits. A pointwise derivative can be 1 and yet the function need not be increasing near the point. The Fundamental Theorem of Integral Calculus is false if we only assume that $D_x F(x) = f[x]$, because then $\int_a^b f[x]\,dx$ need not exist. The section below on the Mean Value Theorem unravels the mystery. A pointwise derivative $D_x f[x]$ is a continuous function on an interval if and only if it is actually an ordinary derivative, $D_x f[x] = f'[x]$.

Exercise set 6.2

1. *Show that* $\lim_{x\to 0} f'[x] = \lim_{x\to 0}\left(2\,x\,\text{Sin}[\frac{\pi}{x}] - \pi\,\text{Cos}[\frac{\pi}{x}]\right)$ *does not exist.*

2. *Use Mathematica to Plot* $y = x^2 \operatorname{Sin}[\frac{\pi}{x}]$ *from -0.0001 to + 0.0001. Use AspectRatio -> 1 and PlotRange -> {-0.0001,0.0001}. You should see a straight line, but if you do not control the PlotRange, you won't. (Try the plot without setting equal scales.)*
Now move the focus point of your microscope to $x = 0.01$. *Plot from 0.0099 to 0.0101 with PlotRange -> {-0.0001,0.0001}. You will see a sinusoid. (If you use equal scales.)*
3. *Show that the function* $f[x] = x \operatorname{Sin}[\frac{\pi}{x}]$ *is continuous if we extend its definition to* $f[0] = 0$. *Show that the extended function does not even have a pointwise derivative at* $x = 0$. *What do you see if you Plot this function at a very small scale at zero?*
Show that the function $f[x] = x^3 \operatorname{Sin}[\frac{\pi}{x}]$ *is continuous if we extend its definition to* $f[0] = 0$. *Show that the extended function has a pointwise derivative at* $x = 0$. *What do you see if you Plot this function at a very small scale near zero? Do wiggles appear?*

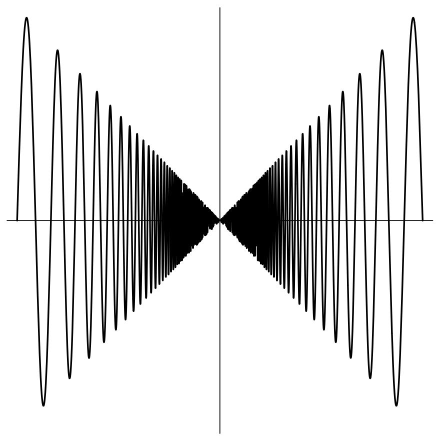


Figure 6.4: $y = x \operatorname{Sin}[\frac{\pi}{x}]$

6.3 Pointwise Derivatives Aren't Enough for Inverses

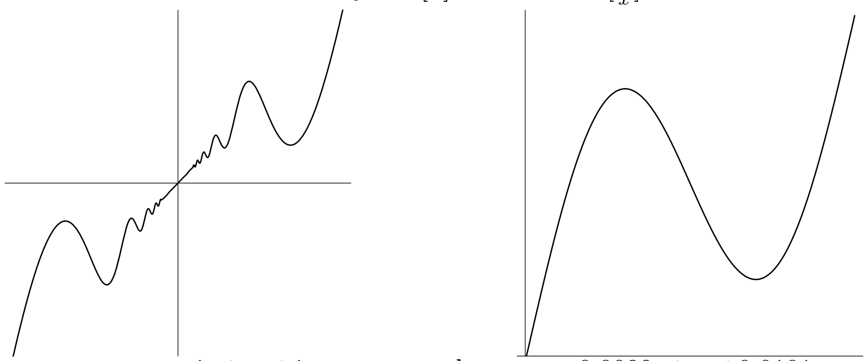
A function can have a pointwise derivative at every point, $Df_x[x_0] = 1$, *but not be increasing in any neighborhood of* x_0.

The function

$$w[x] = \begin{cases} 0, & \text{if } x = 0 \\ x + x^2 \operatorname{Sin}[\frac{\pi}{x}], & \text{if } x \neq 0 \end{cases}$$

has pointwise derivative $D_x w[0] = 1$. However, this function does not have an inverse in any neighborhood of zero. It is NOT increasing in any neighborhood of zero. You can verify this yourself. Here are plots of $w[x]$ on two scales:

$$y = w[x] = x + x^2 \ \mathrm{Sin}[\tfrac{\pi}{x}]$$



$-1 \le x \le 1$ and $0.0099 \le x \le 0.0101$

Exercise set 6.3

1. (a) *Show that the function $w[x]$ above has an ordinary derivative at every $x \neq 0$.*

(b) *Show that the function $w[x]$ above has a pointwise derivative at every x, in particular, the pointwise derivative $D_x w[0] = 1$. (HINT: Write the definition and estimate.)*

(c) *Verify the plots shown using Mathematica with equal scales. One plot is from -1 to 1 and the other is from 0.0099 to 0.0101 both with AspectRatio -> 1 and PlotRanges equal to x ranges.*

(d) *Prove that for every real $\theta > 0$, there are numbers $x_1 < x_2$ with $w[x_1] > w[x_2]$ as shown on the decreasing portion of the small scale graph above.*

CHAPTER 7 The Mean Value Theorem

The following situation illustrates the main result of this chapter.

You travel a total distance of 100 miles in an elapsed time of 2 hours for an average or "mean" speed of 50 mph. However, your speed varies. For example, you start from rest, drive through city streets, stop at stop signs, then enter the Interstate and travel most of the way at 65 mph. Were you ever going exactly 50 mph? Of course, but how can we show this mathematically?

Exercise set 7.0

1. *Sketch the graph of a trip beginning at 2 pm, 35 miles from a reference point, ending at 4 pm 135 miles from the point and having the features of stopping at stop signs, etc., as described above.*
Sketch the line connecting the end points of the graph $(a, f[a])$ *and* $(b, f[b])$*. What is the slope of this line?*
Find a point on your sketch where the speed is 50 mph and sketch the tangent line at that point. Call the point c. Why does this satisfy $f'[c] = \frac{f[b]-f[a]}{b-a}$ *?*

7.1 The Mean Value Theorem

The Mean Value Theorem asserts that there is a place where the value of the instantaneous speed equals the average speed. This theorem is true even if the derivative is only defined pointwise.

We want to formulate the speed problem above in a general way for a function $y = f[x]$ on an interval $[a, b]$. You may think of x as the time variable with $x = a$ at the start of the trip and $x = b$ at the end. The elapsed time traveled is $b - a$, or 2 hours in the example. (Perhaps you start at 2 and end at 4, 4 - 2 = 2.) You may think of $y = f[x]$ as a distance from a reference point, so we start at $f[a]$, end at $f[b]$ and travel a total of $f[b] - f[a]$. The average speed is $(f[b] - f[a])/(b - a)$.

We state the Mean Value Theorem in its ultimate generality, only assuming weakly approximating pointwise derivatives and those only at interior points. This complicates the proof, but will be the key to seeing why regular derivatives and pointwise derivatives are the same when the pointwise derivative is continuous.

Theorem 7.1. *The Mean Value for Pointwise Derivatives*
Let $f[x]$ be a function which is pointwise differentiable at every point of the open interval (a, b) and is continuous on the closed interval $[a, b]$. There is a point c in the open interval $a < c < b$ such that

$$D_x f[c] = \frac{f[b] - f[a]}{b - a}$$

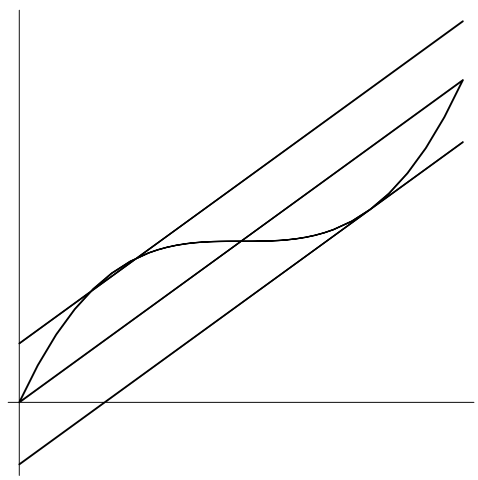


Figure 7.1: Mean Slope and Tangents

There may be more than one point where $D_x f[c]$ equals the mean speed or slope.

Proof

The average speed over a sub-interval of length Δx is

$$g[x] = \frac{f[x + \Delta x] - f[x]}{\Delta x}$$

and this new function is defined and continuous on $[a, b - \Delta x]$.

Suppose we let $\Delta x_1 = (b - a)/3$ compute the average of 3 averages, the speeds on $[a, a + \Delta x_1]$, $[a + \Delta x_1, a + 2\Delta x_1]$ and $[a + 2\Delta x_1, a + 3\Delta x_1]$. This ought to be the same as the overall average and the telescoping sum below shows that it is:

$$\begin{aligned}
&\frac{1}{3}\left(g[a]+g[a+\Delta x_1]+g[a+2\Delta x_1]\right)= \\
&\frac{1}{3}\left(\frac{f[a+\Delta x_1]-f[a]}{\Delta x_1}+\frac{f[a+2\Delta x_1]-f[a+\Delta x_1]}{\Delta x_1}+\frac{f[a+3\Delta x_1]-f[a+2\Delta x_1]}{\Delta x_1}\right) \\
&\quad=\frac{f[a+\Delta x_1]-f[a]+f[a+2\Delta x_1]-f[a+\Delta x_1]+f[a+3\Delta x_1]-f[a+2\Delta x_1]}{3\Delta x_1} \\
&\quad=\frac{-f[a]+f[b]}{3\frac{b-a}{3}}=\frac{f[b]-f[a]}{b-a}
\end{aligned}$$

This implies that there is an adjacent pair of sub-intervals with

$$g[x_{lo}]=\frac{f[x_{lo}+\Delta x_1]-f[x_{lo}]}{\Delta x_1}\le\frac{f[b]-f[a]}{b-a}\le\frac{f[x_{hi}+\Delta x_1]-f[x_{hi}]}{\Delta x_1}=g[x_{hi}]$$

because the average of the three sub-interval speeds equals the overall average and so either all three also equal the overall average, or one is below and another is above the mean slope. (We know that x_{lo} and x_{hi} differ by Δx_1, but we do not care in which order they occur $x_{lo}<x_{hi}$ or $x_{hi}<x_{lo}$.)

Since $g[x]$ is continuous, Bolzano's Intermediate Value Theorem 4.5 says that there is an x_1 between x_{lo} and x_{hi} with $g[x_1]=(f[x_1+\Delta x_1]-f[x_1])/\Delta x_1=(f[b]-f[a])/(b-a)$. The subinterval $[x_1,x_1+\Delta x_1]$ lies inside (a,b), has length $(b-a)/3$ and $f[x]$ has the same mean slope over the subinterval as over the whole interval. (So far we have only used continuity of $f[x]$.)

Let $\Delta x_2=(b-a)/3^2$, one third of the length of $[x_1,x_1+\Delta x_1]$. We can repeat the average of averages procedure above on the interval $[x_1,x_1+\Delta x_1]$ and obtain a new sub-interval $[x_2,x_2+\Delta x_2]$ inside the old sub-interval such that $(f[x_2+\Delta x_2]-f[x_2])/\Delta x_2=(f[b]-f[a])/(b-a)$.

Continuing recursively, we can find x_n in $(x_{n-1},x_{n-1}+\Delta x_{n-1})$ with $\Delta x_n=(b-a)/3^n$ and $(f[x_n+\Delta x_n]-f[x_n])/\Delta x_n=(f[b]-f[a])/(b-a)$.

The sequence of numbers $a_n=x_n$ increases to a limit c in (a,b), and the sequence $b_n=x_n+\Delta x_n$ decreases to c. In addition, we have

$$\begin{aligned}
\frac{f[b]-f[a]}{b-a}&=\frac{f[x_n+\Delta x_n]-f[x_n]}{\Delta x_n} \\
&=\frac{f[b_n]-f[a_n]}{b_n-a_n}=\frac{f[b_n]-f[c]+f[c]-f[a_n]}{b_n-a_n} \\
&=\frac{b_n-c}{b_n-a_n}\frac{f[b_n]-f[c]}{b_n-c}+\frac{c-a_n}{b_n-a_n}\frac{f[c]-f[a_n]}{c-a_n}
\end{aligned}$$

Notice that coefficients are positive and satisfy

$$\frac{b_n-c}{b_n-a_n}+\frac{c-a_n}{b_n-a_n}=1$$

Also notice that

$$\lim_{n\to\infty}\frac{f[b_n]-f[c]}{b_n-c}=\lim_{n\to\infty}\frac{f[c]-f[a_n]}{c-a_n}=D_xf[c]$$

Hence

$$\lim_{n\to\infty} \frac{b_n - c}{b_n - a_n}\frac{f[b_n]-f[c]}{b_n - c} + \frac{c - a_n}{b_n - a_n}\frac{f[c]-f[a_n]}{c - a_n} = D_x f[c]$$

and we have proved the general result of the graphically 'obvious' Mean Value Theorem, by finding a sequence of shorter and shorter sub-intervals with the same mean slope and 'taking the limit.'

Exercise set 7.1

1. *Suppose $f[x]$ satisfies Definition 5.2. Show that the step in the proof of the Mean Value Theorem where we write*

$$\frac{b_n - c}{b_n - a_n}\frac{f[b_n]-f[c]}{b_n - c} + \frac{c - a_n}{b_n - a_n}\frac{f[c]-f[a_n]}{c - a_n}$$

can be skipped. If we take an infinite n, we must automatically have

$$\frac{f[b_n]-f[a_n]}{b_n - a_n} \approx f'[a_n] \approx f'[b_n] \approx f'[c]$$

when $f[x]$ satisfies Definition 5.2. Why?

2. *This exercise seeks to explain why we call the fraction*

$$\frac{f[b]-f[a]}{b-a}$$

the average speed in the case of the ordinary derivative, Definition 5.2.
The average of a continuous function $g[x]$ over the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b g[x]\,dx$$

If $f[x]$ satisfies Definition 5.2, show that the average of the speed is

$$\frac{1}{b-a}\int_a^b f'[x]\,dx = \frac{f[b]-f[a]}{b-a}$$

What theorem do you use to make this general calculation? Why do you need Definition 5.2 rather than only a pointwise derivative?
Write an approximating sum for the integral and substitute the microscope approximation $f'[x]\,\delta x = f[x+\delta x] - f[x] - \varepsilon\,\delta x$ as the summand. The latter sum telescopes to $f[b] - f[a]$ with your adjusting constants.
Write the average of small interval speeds, $(f[x+\delta x] - f[x])/(\delta x)$ for enough terms to move from a to b. How many terms are there in the sum? Why is this sum

$$\frac{1}{(b-a)/\delta x}\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x}\left[\frac{f[x+\delta x]-f[x]}{\delta x}\right]$$

approximately the integral above?

3. Alternate Proof to Averaging Averages
Let $f[x]$ *satisfy the hypotheses of the Mean Value Theorem for Pointwise Derivatives. Let the constant* m *denote the mean slope,*

$$m = \frac{f[b] - f[a]}{b - a}$$

Define a function

$$h[x] = f[x] - m \cdot (x - a)$$

Show that $h[x]$ *has the following properties:*

(a) $h[x]$ *is continuous on* $[a, b]$.
(b) $h[x]$ *is pointwise differentiable on* (a, b).
(c) $h[a] = f[a]$
(d) $h[b] = f[a] = h[a]$, *so that the mean slope of* $h[x]$ *is zero,*

$$\frac{h[b] - h[a]}{b - a} = 0$$

(e) *For any* x, *we have* $D_x h[x] = 0 \quad \Leftrightarrow \quad D_x f[x] = m$
The function $h[x]$ *has a horizontal mean cord. We want you to show that there is a point* c *in* (a, b) *where* $D_x h[c] = 0$.
(f) *Show that* $h[x]$ *satisfies the hypotheses of the Extreme Value Theorem* 4.4 *on* $[a, b]$, *hence must have both a max and a min.*
(g) *Show that either* $h[x]$ *is constant or not both the max and min occur at endpoints. In other words, there is a c in the open interval* (a, b) *where either* $h[c]$ *is a max or min for* $[a, b]$.
(h) *Prove a pointwise version of the the Interior Critical Points Theorem 10.2 from the main text and show that* $D_x h[c] = 0$.
(i) *Show that* $D_x f[c] = (f[b] - f[a])/(b - a)$.

7.2 Darboux's Theorem

Suppose that $f[x]$ *is pointwise differentiable, but* $D_x f[x]$ *is not necessarily continuous. The derivative function still has the intermediate value property. In other words, a derivative cannot be defined and take a jump in values. (Pointwise derivatives can oscillate to a discontinuity, be defined, and NOT be continuous. Ordinary derivatives are continuous by Theorem* 5.4*)*

How do we know that it is sufficient to just check one point between the zeros of $f'[x]$ in the graphing procedure of the main text Chapter 9? If $f'[x]$ is not zero in an interval $a < x < b$ and if $f'[x]$ cannot change sign without being zero, then the sign of any one point determines the sign of all the others in the interval. Derivatives have the property that they cannot change sign without being zero, but not every function has this property.

It was 5°C when I woke up this morning, but has warmed up to a comfortable 16° now (61° F). Was it ever 10° this morning? Most people would say, 'Yes.' They implicitly reason that temperature 'moves continuously' through values and hence hits all intermediate values. This idea is a precise mathematical theorem and its most difficult part is in correctly formulating what we mean by 'continuous' function. See Theorem 4.5.

Darboux's Theorem even holds for the discontinuous weak pointwise derivatives defined above. We showed in Theorem 5.4 that our definition of $f'[x]$ makes it a continuous function. This means we can apply Bolzano's Theorem to $f'[x]$ to prove the case of Darboux's theorem for the ordinary derivatives we have defined. This is the result:

Theorem 7.2. *Darboux's Intermediate Value Theorem*
If $f'[x]$ exists on the interval $a \le x \le b$, then $f'[x]$ attains every value intermediate between the values $f'[a]$ and $f'[b]$. In particular, if $f'[a] < 0$ and $f'[b] > 0$, then there is an x_0, $a < x_0 < b$, such that $f'[x_0] = 0$.

Theorem 7.3. *Intermediate Values for Pointwise Derivatives*
Suppose that $f[x]$ is pointwise differentiable at every point of $[a, b]$. Then the derivative function $D_x f[x]$ attains every value between $D_x f[a]$ and $D_x f[b]$, even though it can be discontinuous.

Proof:
The functions

$$g[x] = \begin{cases} D_x f[a], & \text{if } x = a \\ \frac{f[x]-f[a]}{x-a}, & \text{if } x \neq a \end{cases} \qquad \text{and} \qquad h[x] = \begin{cases} \frac{f[b]-f[x]}{b-x}, & \text{if } x \neq b \\ D_x f[b], & \text{if } x = b \end{cases}$$

are continuous on $[a, b]$. The function $g[x]$ attains every value between $D_x f[a]$ and $[f[b] - f[a]]/[b-a]$, while $h[x]$ attains every value between $[f[b] - f[a]]/[b-a]$ and $D_x f[b]$. Consequently, one or the other attains every value between $D_x f[a]$ and $D_x f[b]$ by Bolzano's Intermediate Value Theorem 4.5. In either case, an intermediate value v satisfies

$$v = \frac{f[\beta] - f[\alpha]}{\beta - \alpha}$$

so the Mean Value Theorem for Derivatives above asserts that there is a γ with $D_x f[\gamma] = v$. This proves the theorem.

Exercise set 7.2

1. *Show that the function $y = j[x] = \frac{\sqrt{x^2+2x+1}}{x+1}$ equals -1 when $x = -2$, equals $+1$ when $x = +3$, but never takes the value $y = \frac{1}{2}$ for any value of x. Why doesn't $j[x]$ violate Bolzano's Theorem 4.5?*

2. *1) Show that the function $y = k[x] = \sqrt{x^2 + 2x + 1}$ has $k'[x] = -1$ when $x = -2$, has $k'[x] = +1$ when $x = +3$, but $k'[x]$ never takes the value $y' = \frac{1}{2}$ for any value of x. Why doesn't $k[x]$ violate Darboux's Theorem above?*
*2) In the graphing procedures using the first and second derivatives, you must compute **all** values where the derivative is zero or fails to exist. Why is this a crucial part of*

making the shape table? In particular, suppose you missed one x value where $f'[x]$ failed to exist or was zero. How could this lead you to make an incorrect graph?

7.3 Continuous Pointwise Derivatives are Uniform

Pointwise derivatives are peculiar because they do not arise from computations with rules of calculus. This section explores the question of when a pointwise derivative is actually the stronger uniform kind. The answer is simple.

Theorem 7.4. *Continuous Pointwise Derivatives are Uniform*

Let $f[x]$ be defined on the open interval (a,b). The following are equivalent:

(a) *The function $f[x]$ is smooth with derivative $f'[x]$ on (a,b) as defined Definition 5.2.*

(b) *The pointwise derivative exists at every point of (a,b) and defines a continuous function, $D_x f[x] = g[x]$.*

(c) *The double limit*

$$\lim_{x_1 \to x, x_2 \to x} \frac{f[x_2] - f[x_1]}{x_2 - x_1} = h[x]$$

exists at every point x in (a,b).

Proof

(1) $\Rightarrow$ (3) : Assume that (1) holds. Let $x_2 \approx x_1 \approx x_0$, a real value in (a,b). Then $x_2 = x_1 + \delta x$ with $\delta x = x_2 - x_1 \approx 0$ and x_1 is not infinitely near the ends of the interval (a,b). By smoothness at x_1,

$$f[x_2] = f[x_1] + f'[x_1] \cdot (x_2 - x_1) + \varepsilon \cdot (x_2 - x_1)$$

so $[f[x_2] - f[x_1]]/[x_2 - x_1] \approx f'[x_1]$. We know from Theorem 5.39 of the main text that $f'[x]$ is continuous, so $f'[x_1] \approx f'[x_0]$ and we have shown that for any real value x_0 in (a,b) and any pair of nearby values,

$$\frac{f[x_2] - f[x_1]}{x_2 - x_1} \approx f'[x_0]$$

which is equivalent to (3).

(3) $\Rightarrow$ (2): Now assume (3). As a special case of the double limit, we may let $x_1 = x_0$ and take $\lim_{x_2 \to x_0} [f[x_2] - f[x_0]]/[x_2 - x_0] = h[x_0] = D_x f[x_0]$, showing that the pointwise derivative exists. It remains to show that $h[x] = D_x f[x]$ is continuous.

If $x_1 \approx x_0$, we need to show that $h[x_1] \approx h[x_0]$. We may apply the Function Extension Axiom to show that given an infinitesimal ε, for 'sufficiently small' differences between x_2

and x_1,

$$\frac{f[x_2]-f[x_1]}{x_2-x_1} = D_x f[x_1] + \varepsilon_1$$

with $|\varepsilon_1| < \varepsilon$. We know that $D_x f[x] = h[x]$ at all real points, hence by Extension, at all hyperreal points and $h[x_1] \approx [f[x_2]-f[x_1]]/[x_2-x_1]$.

The double limit means that whenever $x_2 \approx x_1 \approx x_0$,

$$\frac{f[x_2]-f[x_1]}{x_2-x_1} \approx h[x_0]$$

Hence, $h[x_1] \approx h[x_0]$ and $h[x] = D_x f[x]$ is continuous.

(2) $\Rightarrow$ (1): Finally, assume that the pointwise derivative exists at every point of (a,b) and that $g[x] = D_x f[x]$ is continuous. This means that for any real x_0 in (a,b) and $x_1 \approx x_0$, we have $g[x_0] \approx g[x_1]$. We must show that for any finite x in (a,b), not infinitely near an endpoint, and any infinitesimal δx,

$$f[x+\delta x] = f[x] + D_x f[x]\cdot\delta x + \varepsilon\cdot\delta x$$

for an infinitesimal ε.

By the Mean Value Theorem on $[x, x+\delta x]$, there is an x_1 in $(x, x+\delta x)$ such that

$$\frac{f[x+\delta x]-f[x]}{\delta x} = D_x f[x_1]$$

By the continuity hypothesis, since $x \approx x_1$, we have $D_x f[x] \approx D_x f[x_1]$, so $D_x f[x] = D_x f[x_1] + \varepsilon$, with $\varepsilon \approx 0$. This means

$$\frac{f[x+\delta x]-f[x]}{\delta x} = D_x f[x] + \varepsilon$$

so by algebra we have shown (1) with $f'[x] = D_x f[x]$:

$$f[x+\delta x] = f[x] + D_x f[x]\cdot\delta x + \varepsilon\cdot\delta x$$

We have shown (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1), so all three conditions are equivalent.

CHAPTER

8 Higher Order Derivatives

This chapter relates behavior of a function to its successive derivatives.

The derivative is a function; its derivative is the second derivative of the original function. Taylor's formula is a more accurate local formula than the "microscope approximation" based on a number of derivatives. It has versions for each order of derivative and has many uses. Taylor's formula of order n is equivalent to having n successive derivatives.

Higher order derivatives can also be defined directly in terms of local properties of a function. The first derivative arises from a local fit by a linear function. We can successfully fit a quadratic locally if and only if the function has two derivatives. We give a general result for nth order fit and n derivatives.

8.1 Taylor's Formula and Bending

The first derivative tells the slope of a graph and the second derivative says which way the graph bends. This section gives algebraic forms of the graphical "smile" and "frown" icons that say which way a graph bends.

When the second derivative is positive, a curve bends upward like part of a smile. When the second derivative is negative, the curve bends downward like part of a frown. The smile and frown icons are based on a simple intuitive mathematical idea: when the slope of the tangent increases, the curve bends up. We have two questions. (1) How can we formulate bending symbolically? (2) How do we prove that the formulation is true? First things first.

If a curve bends up, it lies above its tangent line. Draw the picture. The tangent line at x_0 has the formula $y = b + m(x - x_0)$ with $b = f[x_0]$ and $m = f'[x_0]$. If the graph lies above the tangent, $f[x_1]$ should be greater than $b + m(x_1 - x_0) = f[x_0] + f'[x_0](x_1 - x_0)$ or

$$f[x_1] > f[x_0] + f'[x_0](x_1 - x_0)$$

This is the answer to question (1), but now we are faced with question (2): How do we prove it? The increment approximation says

$$f[x_1] = f[x_0] + f'[x_0](x_1 - x_0) + \varepsilon(x_1 - x_0)$$

so this direct formulation of 'bending up' requires that we show that the whole error $\varepsilon(x_1 - x_0)$ stays positive for $x_1 \approx x_0$. All we have to work with is the increment approximation for $f'[x]$ and the fact that $f''[x_0] > 0$. A direct proof is not very easy to give - at least we don't know a simple one.

We have formulated the result as follows.

Theorem 8.1. *Local Bending*
Suppose the function $f[x]$ is twice differentiable on the real interval $a < x < b$ and x_0 is a real point in this interval.

(a) *If $f''[x_0] > 0$, then there is a real interval $[\alpha, \beta]$, $a < \alpha < x_0 < \beta < b$, such that $y = f[x]$ lies above its tangent over $[\alpha, \beta]$, that is,*

$$\alpha \le x_1 \le \beta \qquad \Rightarrow \qquad f[x_1] > f[x_0] + f'[x_0](x_1 - x_0)$$

(b) *If $f''[x_0] < 0$, then there is a real interval $[\alpha, \beta]$, $a < \alpha < x_0 < \beta < b$, such that $y = f[x]$ lies below its tangent over $[\alpha, \beta]$, that is,*

$$\alpha \le x_1 \le \beta \qquad \Rightarrow \qquad f[x_1] < f[x_0] + f'[x_0](x_1 - x_0)$$

Proof:

This algebraic formulation of convexity (or bending) follows easily from the second order Taylor formula. This formula approximates by a quadratic function in the change variable δx (where x is considered fixed), not just a linear function in δx. A general higher order Taylor Formula is proved later in this chapter. We want to use the second order case as follows to show the algebraic form of the smile icon.

Theorem 8.2. *The Second Order Taylor Small Oh Formula*
If $f[x]$ is twice differentiable on a real interval (a, b), $a < x < b$ and x is not infinitely near a or b, then for any infinitesimal δx

$$f[x + \delta x] = f[x] + f'[x]\,\delta x + \frac{1}{2} f''[x](\delta x)^2 + \varepsilon \cdot \delta x^2$$

with $\varepsilon \approx 0$.

Suppose that $f''[x_0] > 0$ at the real value x_0. If $x_1 \approx x_0$, substitute $x = x_0$ and $\delta x = x_1 - x_0$ into Taylor's Second Order Formula to show:

$$f[x + \delta x] = f[x] + f'[x]\,\delta x + \frac{1}{2} f''[x](\delta x)^2 + \varepsilon \cdot \delta x^2$$
$$f[x_1] = f[x_0] + f'[x_0]\,(x_1 - x_0) + \frac{1}{2} f''[x_0]\,(x_1 - x_0) + \varepsilon\,(x_1 - x_0)^2$$

The infinitesimal smile formula follows from using the fact that $\frac{1}{2}(f''[x_0] + \varepsilon)(x_1 - x_0)^2 > 0$

$$f[x_1] > f[x_0] + f'[x_0]\,(x_1 - x_0)$$

The Function Extension Axiom 2.1 says that since $f[x_1] > f[x_0] + f'[x_0]\,(x_1 - x_0)$ for all $|x_1 - x_0| < d$, for infinitesimal d, there must be a non-infinitesimal D such that $f[x_1] > f[x_0] + f'[x_0]\,(x_1 - x_0)$ whenever $|x_1 - x_0| < D$. This proves the Local Bending Theorem.

Exercise set 8.1

1. *Give an algebraic condition that says a curve bends downward. One way to do this is to "say" the curve lies below its tangent line. Prove that your condition holds for a small real interval containing x_0 provided $f''[x_0] < 0$.*

8.2 Symmetric Differences and Taylor's Formula

The symmetric difference

$$\frac{f[x+\delta x/2]-f[x-\delta x/2]}{\delta x} \approx f'[x]$$

gives a more accurate approximation to first derivative than the formula $(f[x+\delta x]-f[x])/\delta x \approx f'[x]$.

In the computations for Galileo's Law of Gravity in Chapter 10 of the main text, we used symmetric differences to approximate the derivative. There is an obvious geometric reason to suspect this is a better approximation. Look at the figure below. Graphically, the approximation of slope given by the symmetric difference is "clearly" better on a "typical" graph as illustrated in Figure 8.1 below.

A line through the points $(x, f[x])$ and $(x+\delta x, f[x+\delta x])$ is drawn with the tangent at x in the left view , while a line through $(x-\delta x, f[x-\delta x])$ and $(x+\delta x, f[x+\delta x])$ is drawn with the tangent at x in the right view. The second slope is closer to the slope of the tangent, even though the line does not go through the point of tangency.

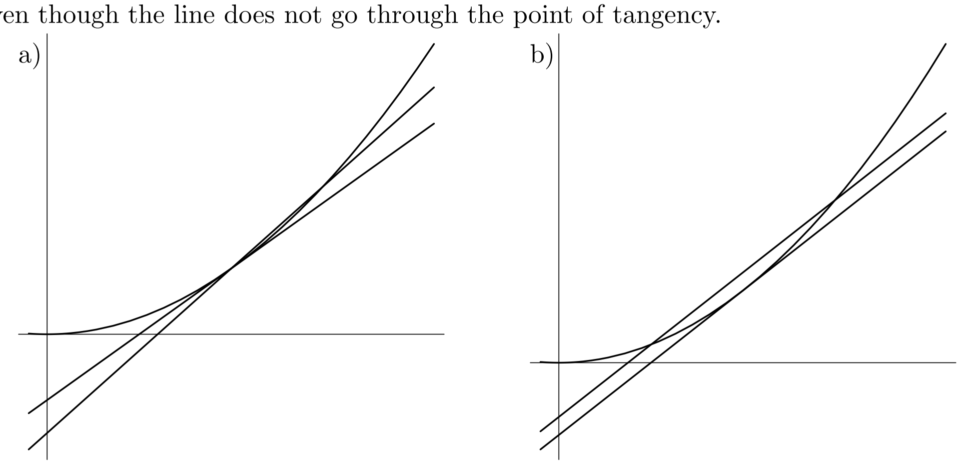


Figure 8.1: $(f[x+\delta x]-f[x])/\delta x$ and $(f[x+\delta x]-f[x-\delta x])/\delta x$

Now we use the second order Taylor formula to prove the algebraic form of this geometric condition. Substitute $\delta x/2$ and $-\delta x/2$ into Taylor's Second Order Formula to obtain

$$f[x+\delta x/2] = f[x] + f'[x]\delta x/2 + \frac{1}{8}f''[x]\,(\delta x)^2 + \varepsilon_1 \delta x^2$$
$$f[x-\delta x/2] = f[x] - f'[x]\delta x/2 + \frac{1}{8}f''[x]\,(\delta x)^2 + \varepsilon_2 \delta x^2$$

subtract the two to obtain

$$f[x+\delta x/2] - f[x-\delta x/2] = f'[x]\delta x + (\varepsilon_1 + \varepsilon_2)\,\delta x^2$$

Solve for $f'[x]$ obtaining

$$f'[x] = \frac{f[x+\delta x/2] - f[x-\delta x/2]}{\delta x} - \varepsilon_4\,\delta x$$

with $\varepsilon_4 \approx 0$. This formula algebraically a better approximation for $f'[x]$ than the ordinary increment approximation $f[x+\delta x] = f[x] + f'[x]\delta x + \varepsilon_3 \delta x$ which gives

$$f'[x] = \frac{f[x+\delta x] - f[x]}{\delta x} - \varepsilon_3$$

Note the importance of δx being small: $\varepsilon_4 \cdot \delta x$ is a product of two small quantities.

Exercise set 8.2

1. (a) *Sketch the line through the points* $(x-\delta x, f[x-\delta x])$ *and* $(x, f[x])$ *on the left view of Figure* 8.1.

(b) *Substitute* $x \pm \delta x$ *into Taylor's second order formula and do some algebra to obtain the approximation*

$$f'[x] = \frac{f[x+\delta x] - f[x-\delta x]}{2\,\delta x} + \varepsilon_5 \cdot \delta x$$

(c) *Show that the average of the slopes of the two secant lines on the left figure is* $(f[x+\delta x] - f[x-\delta x])/(2\delta x)$*, the same as the slope of the symmetric secant line in the second view.*

(d) *A quadratic function* $q[dx]$ *in the local variable* dx *that matches the graph* $y = f[x]$ *at the three* x *values,* $x-\delta x$*,* x*, and* $x+\delta x$ *is given by*

$$q[dx] = y_1 + \frac{y_2 - y_1}{\delta x}\,dx + \frac{y_3 - 2y_1 + y_2}{2\delta x^2}[dx(dx - \delta x)]$$

where $y_1 = f[x]$*,* $y_2 = f[x+\delta x]$ *and* $y_3 = f[x-\delta x]$*. Verify that the values agree at these points by substituting the values* $dx = 0$*,* $dx = \delta x$ *and* $dx = -\delta x$ *into* $q[dx]$.

(e) *Show that the derivative* $q'[0] = (f[x+\delta x] - f[x-\delta x])/(2\delta x)$*, the same as the symmetric secant line slope.*

A quadratic fit gives the same slope approximation as the symmetric one, which is also the same as the average of a left and right approximation. All these approximations are "second order."

It is interesting to compare numerical approximations to the derivative in a difficult, but known case. The experiments give a concrete form to the error estimates of the previous exercise. When we only have data (such as in the law of gravity in Chapter 9 of the main text or in the air resistance project in the Scientific Projects), we must use an approximation. In that case the symmetric formula is best.

2. Numerical Difference Experiments
In the Project on Exponentials, you compute the derivative of $y = b^t$ directly from the increment approximation. Type and enter the following Mathematica program and compare the two methods of finding $y'[0]$

```
b = 2;
y[t_] := b∧t;
t = 0;
Do[
dt = 0.5∧n;
Print[dt,(y[t + dt] - 1)/dt,(y[t + dt] - y[t - dt])/(2 dt)]
,{n,0,16}]
N[Log[b],10]
```

8.3 Approximation of Second Derivatives

$$f''[x] = \frac{f[x+\delta x] - 2f[x] + f[x-\delta x]}{\delta x^2} + \varepsilon$$

Substitute δx and $-\delta x$ into Taylor's Second Order Formula to obtain

$$f[x+\delta x] = f[x] + f'[x]\delta x + \frac{1}{2}f''[x]\,(\delta x)^2 + \varepsilon_1 \delta x^2$$
$$f[x-\delta x] = f[x] - f'[x]\delta x + \frac{1}{2}f''[x]\,(\delta x)^2 + \varepsilon_2 \delta x^2$$

add the two to obtain

$$f[x+\delta x] - 2f[x] + f[x-\delta x] = f''[x]\delta x^2 + (\varepsilon_1 + \varepsilon_2)\,\delta x^2$$

Solve for $f''[x]$ to give a formula approximating the second derivative with values of the function:

$$f''[x] = \frac{f[x+\delta x] - 2f[x] + f[x-\delta x]}{\delta x^2} + \varepsilon$$

Exercise set 8.3

1. Second Differences for Second Derivatives

(a) *The acceleration data in the electronic homework for the Chapter on velocity and acceleration is obtained by taking differences of differences. Suppose three* x *values are* $x - \delta x$, x *and* $x + \delta x$. *Two velocities correspond to the difference quotients*

$$\frac{f[x] - f[x - \delta x]}{\delta x} \quad \& \quad \frac{f[x + \delta x] - f[x]}{\delta x}$$

Compute the difference of these two differences and divide by the correct x *step size. What formula do you obtain?*

(b) *Compare the approximation for* $f''[x]$ *preceding the Exercise Set to the answer from part (a) of this exercise. What does the comparison tell you?*

(c) *Make a program like the one in Exercise* 8.2.2 *above to compute this direct numerical approximation to the second derivative and compare it with the exact symbolically calculated derivative of* b^t.

8.4 The General Taylor Small Oh Formula

The general higher order Taylor formula is the following approximation of the change function $g[\delta x] = f[x+\delta x]$ *by a polynomial in the change variable* δx *(or sometimes* dx*) when* x *is held fixed.*

Continuity of all the derivatives is equivalent to the fact that the approximation works for all the values of x strictly inside the interval. The converse result is given below.

Theorem 8.3. *Taylor's Small Oh Formula*

Suppose that the real function $f[x]$ *has* n *real function continuous derivatives on the interval* (a, b). *If* x *is not infinitely near* a *or* b *and* $\delta x \approx 0$ *is infinitesimal, then the natural extension functions satisfy*

$$f[x+\delta x] = f[x]+f'[x]\cdot\delta x+\frac{1}{2}f''[x]\cdot\delta x^2+\frac{1}{3\cdot 2}f^{(3)}[x]\cdot\delta x^3+\cdots+\frac{1}{n!}f^{(n)}[x]\cdot\delta x^n+\varepsilon\cdot\delta x^n$$

for $\varepsilon \approx 0$.

Equivalently,for every compact subinterval $[\alpha, \beta] \subset (a, b)$,

$$\lim_{\Delta x\to 0}\frac{f[x+\Delta x] - \left(f[x] + f'[x]\cdot\delta x + \frac{1}{2}f''[x]\cdot\Delta x^2 + \cdots + \frac{1}{n!}f^{(n)}[x]\cdot\Delta x^n\right)}{\Delta x^n} = 0$$

uniformly in $[\alpha, \beta]$.

Before we give the proof of this approximation formula, we would like you to see for yourself how it looks. The claim of the theorem is "local," that is, the approximations are better than δx^n, but only for small δx, or only 'in the limit.' (Notice that if $\delta x = 0.01$ and $n = 3$, this means that the error is small compared to $\delta x^3 = 0.000001$.)

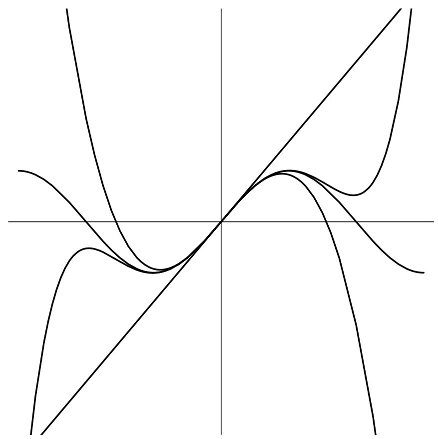


Figure 8.2: Sine and Taylor

You will need to review Integration by Parts from Chapter 12 of the main text,

$$\int_a^b F[u]\ dG[u] = F[x]\,G[x]|_a^b - \int_a^b G[u]\ dF[u]$$

in order to follow the proof of Taylor's formula.

Taylor's Remainder Formula using Integration by Parts

When $n = 1$, Taylor's approximation is the increment equation of Definition 5.2. However, we want to derive a general formula for the error ε using integration. In the case where $n = 1$, this is just uses the Fundamental Theorem of Integral Calculus 5.1.

$$\begin{aligned}
\int_0^{\delta x} [f'[x+u] - f'[x]]\ du &= \int_0^{\delta x} f'[x+u]\ du - \int_0^{\delta x} f'[x]\ du \\
&= \int_0^{\delta x} f'[x+u]\ du - f'[x] \int_0^{\delta x} du \\
&= \int_0^{\delta x} f'[x+u]\ du - f'[x] \cdot \delta x \\
&= f[x+\delta x] - f[x] - f'[x] \cdot \delta x
\end{aligned}$$

because if we take $F[u] = f[x+u]$, then $dF[u] = f'[x+u]\ du$ and the Fundamental Theorem of Integral Calculus says $\int_a^b dF[u] = F[b] - F[a]$. Rearranging the calculation we have the first order formula

$$f[x+\delta x] = f[x] + f'[x] \cdot \delta x + \int_0^{\delta x} [f'[x+u] - f'[x]]\ du$$

Integration by Parts shows

$$f[x+\delta x] = f[x] + f'[x]\cdot\delta x + \frac{1}{2}f''[x]\cdot\delta x^2 + \cdots$$
$$\cdots + \frac{1}{n!}f^{(n)}[x]\cdot\delta x^n + \frac{\delta x^{(n-1)}}{(n-1)!}\cdot\int_0^{\delta x}(1-u/\delta x)^{(n-1)}\ [f^{(n)}[x+u]-f^{(n)}[x]]\ du$$

because

$$\frac{\delta x^{(n-1)}}{(n-1)!}\cdot f^{(n)}[x]\cdot\int_0^{\delta x}(1-u/\delta x)^{(n-1)}\ du = \frac{\delta x^{(n-1)}}{(n-1)!}\cdot f^{(n)}[x]\cdot\frac{\delta x}{n} = \frac{\delta x^n}{n!}\cdot f^{(n)}[x]$$

and Integration by Parts with $F[u]=(1-u/\delta x)^{(n-1)}$ and $dG[u]=f^{(n)}[x+u]\ du$ gives

$$\frac{\delta x^{(n-1)}}{(n-1)!}\cdot\int_0^{\delta x}(1-u/\delta x)^{(n-1)}\ f^{(n)}[x+u]\ du =$$
$$-\frac{\delta x^{(n-1)}}{(n-1)!}\cdot f^{(n-1)}(x) + \frac{\delta x^{(n-2)}}{(n-2)!}\cdot\int_0^{\delta x}(1-u/\delta x)^{(n-2)}\ f^{(n-1)}[x+u]\ du$$

which could be further reduced (or used as an inductive hypothesis),

$$\frac{\delta x^{(n-2)}}{(n-2)!}\cdot\int_0^{\delta x}(1-u/\delta x)^{(n-2)}\ [f^{(n-1)}[x+u]-f^{(n)}[x]]\ du$$
$$= \frac{1}{(n-2)!}f^{(n-2)}[x]\cdot\delta x^{(n-2)} + \frac{\delta x^{(n-3)}}{(n-3)!}\cdot\int_0^{\delta x}(1-u/\delta x)^{(n-3)}\ f^{(n-2)}[x+u]\ du$$
$$\vdots$$

The Error Formula: We have shown that

$$f[x+\delta x] = f[x] + f'[x]\cdot\delta x + \frac{1}{2}f''[x]\cdot\delta x^2 + \frac{1}{3\cdot 2}f^{(3)}[x]\cdot\delta x^3 + \cdots + \frac{1}{n!}f^{(n)}[x]\cdot\delta x^n + \varepsilon\cdot\delta x^n$$

with the explicit formula

$$\varepsilon = \frac{1}{\delta x}\int_0^{\delta x}\frac{(1-u/\delta x)^{(n-1)}}{(n-1)!}\ [f^{(n)}[x+u]-f^{(n)}[x]]\ du$$

Now we will show that ε is small when δx is small.

Proof of Taylor's Small Oh Formula

To show that $\varepsilon\approx 0$, it is sufficient to notice that continuity makes $f^{(n)}[x+u]-f^{(n)}[x]\approx 0$ for $0\le u\le\delta x$, so the maximum

$$m = \text{Max}[|f^{(n)}[x+u]-f^{(n)}[x]|\ :\ 0\le u\le\delta x]\approx 0$$

and

$$|\varepsilon| \le \frac{m}{\delta x}\int_0^{\delta x}\frac{(1-u/\delta x)^{(n-1)}}{(n-1)!}\ du$$
$$= \frac{m}{\delta x}\cdot\frac{\delta x}{n!} = \frac{m}{n!}\approx 0$$

This completes the proof. The equivalent "epsilon - delta" condition follows as in the proof of Theorem 3.4.

8.4.1 The Converse of Taylor's Theorem

Theorem 8.4. *The Converse of Taylor's Theorem*

Let $f[x]$ be a real function defined on an interval (α, ω). Suppose there are real functions $a_h[x]$, $h = 0, \cdots, k$, defined on (α, ω) such that whenever a hyperreal x is in (α, ω) and not infinitely near the endpoints, and $\delta x \approx 0$, the natural extensions satisfy

$$f[x + \delta x] = \sum_{h=0}^{k} \frac{1}{h!} a_h[x] \, \delta x^h + \varepsilon \cdot \delta x^k$$

with $\varepsilon \approx 0$. Then $f[x]$ is k-times differentiable with derivatives $f^{(h)}[x] = a_h[x]$.

Proof:

First, we show that the coefficient functions are continuous. Consider $k = 0$. Take $\delta x = 0$ to see that $f[x + 0] = a_0[x] + 0$. Take $\delta x \approx 0$ to see $f[x]$ is continuous,

$$f[x + \delta x] = f[x] + \varepsilon$$

with $\varepsilon \approx 0$.

Consider $k = 1$ and take two infinitesimals δx_0 and δx_1 of comparable size, $\delta x_0/\delta x_1$ and $\delta x_1/\delta x_0$ both finite. Expand at $\xi = x + \delta x_0$ and at x,

$$\begin{aligned}
&f[x + \delta x_0 + \delta x_1] - f[x + \delta x_0] = a_1[x + \delta x_0] \, \delta x_1 + \varepsilon_1 \, \delta x_1 \\
&f[x + \delta x_0 + \delta x_1] - f[x] + f[x] - f[x + \delta x_0] = a_1[x] \, (\delta x_0 + \delta x_1) - a_1[x] \, \delta x_0 + \varepsilon_2 \, \delta x_1 \\
&f[x + \delta x_0 + \delta x_1] - f[x + \delta x_0] = a_1[x] \, \delta x_1 + \varepsilon_2 \, \delta x_1
\end{aligned}$$

Solving, $a_1[x + \delta x_0] = a_1[x] + (\varepsilon_2 - \varepsilon_1)$ proves continuity,

$$a_1[x + \delta x_0] \approx a_1[x]$$

The case $k = 2$ is Exercise 8.4.3 below. The general case follows by expanding the deleted differences ($\delta x_i^{'}$ indicates it is deleted from the expression) for $k + 1$ comparable infinitesimals, δx_0, $\delta x_1, \cdots$, δx_k,

$$\begin{aligned}
&f[x + \delta x_0 + \delta x_1 + \cdots + \delta x_k] \\
&\qquad - \sum_{j=1}^{k} f[x + \delta x_0 + \delta x_1 + \cdots + \delta x_j^{'} + \cdots + \delta x_k] \\
&\qquad + \sum_{1 \le i < j \le k} f[x + \delta x_0 + \delta x_1 + \cdots + \delta x_i^{'} + \cdots + \cdots + \delta x_j^{'} + \cdots + \delta x_k] \\
&\qquad + \cdots + \\
&\qquad (-1)^{k-1} \sum_{j=1}^{k} f[x + \delta x_0 + \delta x_j] \\
&\qquad + (-1)^k \, f[x + \delta x_0]
\end{aligned}$$

about $\xi = x + \delta x_0$ and about x, obtaining this expression equal to both

$$a_k[x + \delta x_0] \, \delta x_1 \cdot \delta x_2 \cdots \delta x_k + \varepsilon_1 \cdot \delta x_0^{k+1} = a_k[x] \, \delta x_1 \cdot \delta x_2 \cdots \delta x_k + \varepsilon_2 \cdot \delta x_0^k$$

Proof that $f^{(h)}[x] = a_h[x]$ is by induction on k. The case $k = 1$ is the Definition 5.2. Suppose that we know $f^{(h)}[x] = a_h[x]$, for $h = 0, 1, \cdots, k$. Let δx_1 and δx_2 be comparable infinitesimals and expand two ways:

$$
\begin{aligned}
&f[x + \delta x_1 + \delta x_2] \\
&\qquad = \sum_{h=0}^{k} \frac{1}{h!}\ f^{(h)}[x + \delta x_1]\delta x_2^h + \frac{1}{(k+1)!} a_{k+1}[x + \delta x_1]\, \delta x_2^{k+1} + \varepsilon_1\, \delta x_2^{k+1} \\
&\qquad = \sum_{h=0}^{k} \frac{1}{h!}\ f^{(h)}[x](\delta x_1 + \delta x_2)^h + \frac{1}{(k+1)!} a_{k+1}[x]\, (\delta x_1 + \delta x_2)^{k+1} + \varepsilon_2\, \delta x_2^{k+1}
\end{aligned}
$$

Now form the difference, and write it as a polynomial in δx_2,

$$
\begin{aligned}
\varepsilon_3 \cdot \delta x_2^{k+1} &= \sum_{h=0}^{k} \left(f^{(h)}[x + \delta x_1]\, \delta x_2^h - f^{(h)}[x]\, (\delta x_1 + \delta x_2)^h \right) \\
&\quad + \frac{1}{(k+1)!} \left(a_{k+1}[x + \delta x_1]\, \delta x_2^{k+1} - a_{k+1}[x]\, (\delta x_1 + \delta x_2)^{(k+1)} \right) \\
&= \sum_{h=0}^{k+1} b_h[x, \delta x_1]\, \delta x_2^h
\end{aligned}
$$

Expansion in δx_2 gives the terms

$$
b_{k+1}[x, \delta x_1] = \frac{1}{(k+1)!} \left(a_{k+1}[x + \delta x_1] - a_{k+1}[x] \right)
$$

and

$$
b_k[x, \delta x_1] = \frac{1}{k!} \left(f^{(k)}[x + \delta x_1] - f^{(k)}[x] - a_{k+1}[x]\, \delta x_1 \right)
$$

Since $a_{k+1}[x]$ is continuous, we have

$$
\varepsilon \cdot \delta x_2^{k+1} = \frac{1}{k!} \left(f^{(k)}[x + \delta x_1] - f^{(k)}[x] - a_{k+1}[x]\, \delta x_1 \right) \delta x_2^k + \sum_{h=0}^{k-1} b_h[x, \delta x]\, \delta x_2^h
$$

For any $\lambda_0, \cdots, \lambda_k$ distinct nonzero real numbers, we have the invertible Vandermonde system of equations

$$
\begin{bmatrix} 1, \lambda_0, \cdots, \lambda_0^k \\ 1, \lambda_1, \cdots, \lambda_1^k \\ \vdots \\ 1, \lambda_k, \cdots, \lambda_k^k \end{bmatrix}
\begin{bmatrix} b_0 \\ b_1\, \delta x_2 \\ \vdots \\ b_k\, \delta x_2^k \end{bmatrix}
=
\begin{bmatrix} \varepsilon_0 (\lambda_0 \delta x_2)^{k+1} \\ \varepsilon_1 (\lambda_1 \delta x_2)^{k+1} \\ \vdots \\ \varepsilon_k (\lambda_k \delta x_2)^{k+1} \end{bmatrix}
= \delta x_2^{k+1}
\begin{bmatrix} \iota_0 \\ \iota_1 \\ \vdots \\ \iota_k \end{bmatrix}
$$

with $\iota_h \approx 0$ for $h = 0, \cdots, k$. Applying the real inverse matrix to both sides, we obtain

$$
\begin{bmatrix} b_0[x, \delta x_1] \\ b_1[x, \delta x_1]\, \delta x_2 \\ \vdots \\ b_k[x, \delta x_2]\, \delta x_2^k \end{bmatrix}
= \delta x_2^{k+1}
\begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_k \end{bmatrix}
$$

with $\eta_h \approx 0$ for $h = 0, \cdots, k$. In particular,

$$b_k[x, \delta x_1] = \frac{1}{k!}\left(f^{(k)}[x + \delta x_1] - f^{(k)}[x] - a_{k+1}[x]\,\delta x_1\right) = \zeta\,\delta x_1$$

with $\zeta \approx 0$. This proves that $f^{(k)}[x]$ satisfies Definition 5.2.

Exercise set 8.4

1. *Show that the Taylor polynomials for sine at $x = 0$ satisfy*

$$\text{Sin}[0 + dx] = dx - \frac{1}{6}\cdot dx^3 + \frac{1}{5!}\cdot dx^5 + \cdots$$

and use Mathematica to compare the plots as follows
Plot[{Sin[0 + dx],dx, dx - dx∧3/6, dx - dx∧3/6 + dx∧5/5!}
, {dx,-3 Pi/2,3 Pi/2}]
Make similar graphical comparisons for $\text{Cos}[0 + dx]$ *and* $Exp[0 + dx] = e^{0+dx}$.

2. *We to work through the steps in finding the second order formula.*
Calculate the following second order integral by breaking it into two pieces,

$$\begin{aligned}\delta x \cdot \int_0^{\delta x} &(1 - u/\delta x)[f''[x + u] - f''[x]]\,du \\ &= \delta x \cdot \int_0^{\delta x}(1 - u/\delta x)f''[x + u]\,du - \delta x \int_0^{\delta x}(1 - u/\delta x)f''[x]\,du \\ &= \delta x \cdot \int_0^{\delta x}(1 - u/\delta x)f''[x + u]\,du - \delta x \cdot f''[x]\int_0^{\delta x}(1 - u/\delta x)du\end{aligned}$$

First, compute the integral $\int_0^{\delta x}(1 - u/\delta x)du = \delta x/2$, *by symbolic means or by noticing that it is the area of a triangle of height 1 and base δx. Second, use Integration by Parts with $F[u] = (1 - u/\delta x)$ and $dG[u] = f''[x + u]\,du$ to show*

$$\begin{aligned}\delta x \cdot \int_0^{\delta x}(1 - u/\delta x)f''[x + u]\,du &= -\delta x \cdot f'[x] + \int_0^{\delta x} f'[x + u]\,du \\ &= -\delta x \cdot f'[x] + f[x + \delta x] - f[x]\end{aligned}$$

Finally, combine your second order results to show that

$$f[x + \delta x] = f[x] + f'[x]\cdot\delta x + \frac{1}{2}f''[x]\cdot\delta x^2 + \delta x \cdot \int_0^{\delta x}(1 - u/\delta x)[f''[x + u] - f''[x]]\,du$$

3. *Suppose that*

$$f[x + \delta x] = f[x] + a_1[x]\,\delta x + \frac{1}{2}\,a_2[x]\,\delta x^2 + \varepsilon\delta x^2$$

Take three comparable infinitesimals δx_0, δx_1, δx_2, and expand the following:

$$\begin{aligned} f[\xi + \delta x_1 + \delta x_2] - f[\xi] + f[\xi] - f[\xi + \delta x_1] - f[\xi + \delta x_2] + f[\xi] \\ = a_2[\xi]\, \delta x_1 \delta x_2 + \varepsilon_1 \delta x_0^2, \quad \xi = x + \delta x_0 \\ f[x + \delta x_0 + \delta x_1 + \delta x_2] - f[x] + f[x] - f[x + \delta x_0 + \delta x_1] - f[x + \delta x_0 + \delta x_2] \\ + f[x] - f[x] + f[x + \delta x_0] \\ = a_2[x]\, \delta x_1 \delta x_2 + \varepsilon_2 \delta x_0^2 \end{aligned}$$

to show that $a_2[x + \delta x_0] \approx a_2[x]$, $a_2[x]$ is continuous.

8.5 Direct Interpretation of Higher Order Derivatives

We know that the first derivative tells us the slope and the second derivative tells us the concavity or convexity (frown or smile), but what do the third, fourth, and higher derivatives tell us?

The symmetric limit interpretation of derivative arose from fitting the curve $y = f[x]$ at the points $x - \delta x$ and $x + \delta x$ and then taking the limit of the quadratic fit. A more detailed approach to studying higher order properties of the graph is to fit a polynomial to several points and take a limit. To determine a quadratic fit to a curve, we would need three points, say $x - \delta x$, x, and $x + \delta x$. We would then have three values of the function, $f[x - \delta x]$, $f[x]$, and $f[x + \delta x]$ to use to determine unknown coefficients in the interpolation polynomial $p[dx] = a_0 + a_1 dx + a_2 dx^2$. We could solve for these coefficients in order to make $f[x - \delta x] = p[-\delta x]$, $f[x] = p[0]$, and $f[x + \delta x] = p[\delta x]$. This solution can be easily done with *Mathematica* commands given in the next exercise. The limit of this fit tends to the second order Taylor polynomial,

$$\lim_{\delta x \to 0} p[dx] = f[x] + f'[x]\, dx + \frac{1}{2} f''[x]\; dx^2$$

This approach extends to as many derivatives as we wish. If we fit to $n + 1$ points, we can determine the $n + 1$ coefficients in the polynomial

$$p[dx] = a_0 + a_1\; dx + \cdots + a_n\; dx^n$$

so that $p[\delta x_i] = f[x + \delta x_i]$ for $i = 0, 1, \cdots, n$. If the function $f[x]$ is n times continuously differentiable,

$$\lim_{\delta x \to 0} p[dx] = f[x] + f'[x]\, dx + \frac{1}{2} f''[x]\; dx^2 + \cdots + \frac{1}{n!} f^{(n)}[x]\; dx^n$$

specifically, if $p[dx] = a_0[x,\delta x] + a_1[x,\delta x]\ dx + \cdots + a_n[x,\delta x]\ dx^n$, then

$$\lim_{\delta x \to 0} a_k[x,\delta x] = \frac{1}{k!}\ f^{(k)}[x], \qquad \text{for all } k = 0, 1, \cdots, n$$

uniformly for x in compact intervals. The higher derivatives mean no more or less than the coefficients of a local polynomial fit to the function. In other words, once we understand the geometric meaning of the dx^3 coefficient in a cubic polynomial, we can apply that knowledge locally to a thrice differentiable function. Before we prove this amazing fact, we would like you to "see" how it works by using *Mathematica* to fit the polynomials in Exercise 8.5.1

8.5.1 Basic Theory of Interpolation

Let $f[x]$ be a real-valued function defined on (α, ω), and let $X = \{x_0, x_1, \ldots, x_n\}$ be $n+1$ distinct points in the interval. The "Lagrange form" of the polynomial of degree n that has the same values at the x_i is

$$p_X[x] = \sum_{i=0}^{n} f[x_i] \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

We say "$p_X[x]$ interpolates f on X." For example, when $n = 2$,

$$p_X[x] =$$
$$= f[x_0]\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f[x_1]\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f[x_2]\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

By substitution we see that $p_X[x_i] = f[x_i]$ for $i = 0, 1, \ldots, n$. A polynomial of order n with this interpolation property is unique, because the difference of two such polynomials has $n+1$ zeros and thus is identically zero.

The interpolation polynomial may also be written in "Newton form" with ascending terms depending only on successive values of x_i:

$$p_X[x] = a_0[x_0] + a_1[x_0, x_1](x - x_0) + \cdots + a_n[x_0, \ldots, x_n] \prod_{i=0}^{n-1} (x - x_i)$$

Substitution of x_0 in the Newton form shows

$$a_0[x_0] = f[x_0]$$

Equating the nth order terms in both the Newton and Lagrange form, we obtain

(NewtLa)
$$a_n[x_0, \ldots, x_n] = \sum_{i=0}^{n} \frac{f[x_i]}{\prod\limits_{j=0, j\neq i}^{n} (x_j - x_i)}$$

This formula (NewtLa) for $a_n[x_0, \ldots, x_n]$ shows the symmetry of the coefficients. That is, if k_i is any permutation of $\{0, 1, \ldots, n\}$, then

(Symm)
$$a_n[x_{k_0}, \ldots, x_{k_n}] = a_n[x_0, \ldots, x_n]\ .$$

Applying the formula (NewtLa) to the right hand side of the next equation and putting the resulting expression on a common denominator justifies the divided difference recursion

$$\text{(DiffQ)} \qquad a_n[x_0,\dots,x_n] = \frac{a_{n-1}[x_1,\dots,x_n] - a_{n-1}[x_0,\dots,x_{n-1}]}{x_n - x_0}$$

Successive substitution into (DiffQ) shows

$$\begin{aligned} a_0[x_0] &= f[x_0] \ , \\ a_1[x_0,x_1] &= \frac{f[x_1]-f[x_0]}{x_1-x_0} \\ a_2[x_0,x_1,x_2] &= \frac{1}{x_2-x_0}\left(\frac{f[x_2]-f[x_1]}{x_2-x_1} - \frac{f[x_1]-f[x_0]}{x_1-x_0}\right) \\ &\vdots \end{aligned}$$

Because of the relation of the Newton coefficients to divided differences of $f[x]$, we denote them by

$$\delta^n f[x_0,\dots,x_n] = a_n[x_0,\dots,x_n]$$

If $f[x]$ is differentiable at a point, we may extend the definition of $\delta^n f$ to include a repetition. This extension continues to satisfy the functional identity (DiffQ).

Theorem 8.5. *Limiting Differences*

(a) *If $f'[x_n]$ exists (as pointwise limit) then the following limit exists,*

$$\lim_{t\to 0} \delta^{n+1} f[x_0,\dots,x_i,x_i+t,\dots,x_n]$$

We denote this limit $\delta^{n+1} f[x_0,\dots,x_{i-1},x_i,x_i,x_{i+1},\dots,x_n]$.

(b) *If $f[x]$ is pointwise differentiable on (α,ω) and $\{x_0,\dots,x_{n+1}\}$ has one repetition amongst $x_0,\dots,x_n$, then the functional identity (DiffQ) still holds*

$$\delta^{n+1} f[x_0,\dots,x_{n+1}] = \frac{1}{x_{n+1}-x_0}\Big\{\delta^n f[x_1,\dots,x_{n+1}] - \delta^n f[x_0,\dots,x_n]\Big\}$$

with the extended definition of part (a) as needed.

Proof(a):

If $n = 0$, the limit

$$\lim_{t\to 0} \frac{f[x_0+t]-f[x_0]}{t} = f'[x_0]$$

exists and we may define $\delta^1 f[x_0,x_0] = f'[x_0]$.

If $n \geq 0$, Newton's interpolation formula at distinct points $x_0, x_0+t, x_1, x_2, \dots, x_n$ gives

$$\begin{aligned} f[x_0+t] = f[x_0] &+ \sum_{i=1}^{n} t\delta^i f[x_0,\dots,x_i] \prod_{j=1}^{i-1}(x_0+t-x_j) \\ &+ t\delta^{n+1} f[x_0,\dots,x_n,x_0+t] \prod_{j=1}^{n}(x_0+t-x_j) \end{aligned}$$

so that

$$\begin{aligned}&\delta^{n+1} f[x_0, x_1, \dots, x_n, x_0 + t]\\ &\quad = \frac{\frac{f[x_0+t]-f[x_0]}{t} - \delta^1 f[x_0, x_1] - \sum_{i=2}^{n} \delta^i f[x_0, \dots, x_i] \prod_{j=1}^{i-1}(x_0 + t - x_j)}{\prod_{j=1}^{n}(x_0 + t - x_j)}\\ &\quad \longrightarrow \frac{f'[x_0] - \delta^1 f[x_0, x_1] - \sum_{i=2}^{n} \delta^i f[x_0, \dots, x_i] \prod_{j=1}^{i-1}(x_0 - x_j)}{\prod_{j=1}^{n}(x_0 - x_j)}\end{aligned}$$

This proves (a).

PROOF (b):

To prove (b), we may use symmetry and assume $x_0 = x_1$, so by definition of the extended formula and identity (DiffQ),

$$\begin{aligned}&\delta^{n+1} f[x_0, x_1, \dots, x_{n+1}] = \lim_{t\to 0} \delta^{n+1} f[x_0, x_0 + t, x_2, \dots, x_{n+1}]\\ &= \lim_{t\to 0} \frac{1}{x_{n+1} - x_0} \left\{ \delta^n f[x_0 + t, x_2, \dots, x_{n+1}] - \delta^n f[x_0, x_0 + t, \dots, x_n] \right\}\\ &= \frac{1}{x_{n+1} - x_0} \left\{ \delta^n f[x_1, x_2, \dots, x_{n+1}] - \delta^n f[x_0, x_0, x_2, \dots, x_n] \right\}\end{aligned}$$

8.5.2 Interpolation where f is Smooth

If we know that $f[x]$ has n ordinary continuous derivatives (Definition 5.2), then we have the following elegant formula.

Theorem 8.6. *Hermite-Gennocci's Formula*

Suppose $f[x]$ has n derivatives in (α, ω), for $n \geq 1$. Choose distinct points $x_0, \dots, x_n$ in (α, ω). Then

$$\begin{aligned}\delta^n f[x_0, \dots, x_n] &= \int_{t_i \geq 0, \sum_{i=0}^{n} t_i = 1} \cdots \int f^{(n)}[t_0 x_0 + \cdots + t_n x_n] dt_1 \dots dt_n\\ &= \int_{t_i \geq 0, \sum_{i=1}^{n} t_i \leq 1} \cdots \int f^{(n)}[x_0 + t_1(x_1 - x_0) + \cdots + t_n(x_n - x_0)] dt_1 \dots dt_n\end{aligned}$$

PROOF:

First, the two integrals are equivalent because

$$\sum_{i=1}^{n} t_i = 1 - t_0$$

Second, if $n = 1$, the Fundamental Theorem of Integral Calculus, with $G[t] = f[x_0 + t(x_1 - x_0)]/(x_1 - x_0)$ and $\frac{dG}{dt} = f'[x_0 + t(x_1 - x_0)]$, shows the Hermite-Gennoci Formula,

$$\delta^1 f[x_1, x_0] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \int_0^1 f'[x_0 + t(x_1 - x_0)] dt$$

Third, for $n > 1$, use successive integration and The Fundamental Theorem to show that the the Hermite-Gennoci integrals satisfy the recursion (DiffQ):

$$\int_{t_i \geq 0, \sum_{i=1}^{n} t_i \leq 1} \cdots \int f^{(n)}(x_0 + t_1(x_1 - x_0) + \cdots + t_n(x_n - x_0) dt_1 \ldots dt_n$$

$$= \int \cdots \int \left[\int_{t_n=0}^{1-\sum_{i=1}^{n-1} t_i} f^{(n)}(x_0 + t_1(x_1 - x_0) + \cdots + t_n(x_n - x_0))dt_n \right] dt_1 \ldots dt_{n-1}$$

$$= \int \cdots \int \frac{1}{x_n - x_0} f^{(n-1)}[x_0 + t_1(x_1 - x_0) + \cdots + t_n(x_n - x_0)] \Big|_{t_n=0}^{1-\sum_{i=1}^{n-1} t_i} dt_1 \ldots dt_{n-1}$$

$$= \frac{1}{x_n - x_0} \{ \int \cdots \int f^{(n-1)}(x_n + t_1(x_1 - x_n) + \cdots + t_{n-1}(x_{n-1} - x_n))dt_1 \ldots dt_{n-1}$$

$$- \int \cdots \int f^{(n-1)}(x_0 + t_1(x_1 - x_0) + \cdots + t_{n-1}(x_{n-1} - x_0))dt_1 \ldots dt_{n-1} \}$$

Since both $\delta^n f$ and the integrals agree when $n = 1$ and since both satisfy the recursion (DiffQ), the two are equal and we have proved the theorem.

If $f[x] = x^n$, then $f^{(n)}[x] = n!$ and $\delta^n f[x_0, \ldots, x_n] = 1$ by equating coefficients of x^n, so

$$1 = \int_{t_i \geq 0, \sum_{i=0}^{n} t_i = 1} \cdots \int n! \;\; dt_1 \ldots dt_n$$

8.5.3 Smoothness From Differences

We say $\delta^n f$ is S-continuous on (α, ω) if whenever we choose nearby infinitesimal sequences, we obtain nearly the same finite results. That is, suppose $x_0 \approx x_1 \approx \cdots \approx x_n$ are distinct infinitely close points near a real b in (α, ω) and $\xi_0 \approx \xi_1 \approx \cdots \approx \xi_n$ are also near b. (The x_i points are distinct and the ξ_i points are distinct, but the sets $\{x_0, \ldots, x_n\}$ and $\{\xi_0, \ldots, \xi_n\}$ may overlap.) Then

$$\delta^n f[x_0, \ldots, x_n] \approx \delta^n f[\xi_0, \ldots, \xi_n]$$

and both are finite numbers.

Theorem 8.7. *Theorem on Higher Order Smoothness*

Let $f[x]$ be a real function defined on a real open interval (α, ω). Then $f[x]$ is n-times continuously differentiable on (α, ω) if and only if the nth-order differences $\delta^n f$ are S-continuous on (α, ω).

Proof that Smooth Implies Stable Differences:

The implication $\Rightarrow$ follows from the Hermite-Gennoci Formula, Theorem 8.6, and shows

$$\delta^n f[x_0, \ldots, x_n] \approx \frac{1}{n!} f^{(n)}[b]$$

whenever $x_0 \approx \cdots \approx x_n \approx b$.

If $x_0 \approx x_1 \approx \cdots \approx x_n$, then for all $0 \le t_i \le 1$, $f^{(n)}[t_0 x_0 + \cdots + t_n x_n] \approx f^{(n)}[x_0]$, so

$$\begin{aligned}\delta f^{(n)}[x_0, \cdots, x_n] &= \int_{t_i \ge 0, \sum_{i=0}^{n} t_i = 1} \cdots \int f^{(n)}[t_0 x_0 + \cdots + t_n x_n] dt_1 \dots dt_n \\ &\approx f^{(n)}[x_0] \int_{t_i \ge 0, \sum_{i=0}^{n} t_i = 1} \cdots \int \quad dt_1 \dots dt_n\end{aligned}$$

We prove the converse by induction and need some technical lemmas. The case $n = 0$ is trivial and the case $n = 1$ follows from Theorem 3.5.

Theorem 8.8. *Technical Lemma 1*
If $f[x]$ is pointwise differentiable with derivative $f'[x]$ on (α, ω) and $x_0, x_1, \dots x_n$ are distinct points in (α, ω), then

$$(\delta') \qquad \delta^n f'[x_0, \dots, x_n] = \sum_{i=0}^{n} \delta^{n+1} f[x_0, \dots, x_n, x_i]$$

Proof of Lemma 1:

If $n = 0$,

$$\delta^0 f'[x_0] = f'[x_0] = \lim_{t \to 0} \delta^1 f[x_0, x_0 + t] = \delta^1 f[x_0, x_0]$$

Assume that the formula holds for n and that $x_0, \dots, x_{n+1}$ are distinct points in (α, ω). Use the recurrence formula (DiffQ),

$$\delta^{n+1} f'[x_0, \dots, x_{n+1}] = \frac{1}{x_{n+1} - x_0} \left\{ \delta^n f'[x_1, \dots x_{n+1}] - \delta^n f'[x_0, \dots, x_n] \right\}$$

Next, use the induction hypothesis,

$$\begin{aligned}&\delta^{n+1} f'[x_0, \dots, x_{n+1}] \\ &= \frac{1}{x_{n+1} - x_0} \left\{ \sum_{i=1}^{n+1} \delta^{n+1} f[x_1, \dots, x_{n+1}, x_i] - \sum_{i=0}^{n} \delta^{n+1} f[x_0, \dots, x_n, x_i] \right\}\end{aligned}$$

Finally, use part (b) of Theorem 8.5

$$
\begin{aligned}
&\delta^{n+1} f'[x_0, \dots, x_{n+1}] \\
&= \frac{1}{x_{n+1} - x_0} \Bigg\{ \delta^{n+1} f[x_1, \dots, x_{n+1}, x_{n+1}] \\
&\qquad + \sum_{i=1}^{n} \Big(\delta^{n+1} f[x_1, \dots, x_{n+1}, x_i] - \delta^{n+1} f[x_0, \dots, x_n, x_i] \Big) \\
&\qquad\qquad - \delta^{n+1} f[x_0, \dots, x_n, x_0] \Bigg\} \\
&= \sum_{i=1}^{n} \delta^{n+2} f[x_0, \dots, x_{n+1}, x_i] \\
&\qquad + \frac{1}{x_{n+1} - x_0} \Bigg\{ \delta^{n+1} f[x_1, \dots, x_{n+1}, x_{n+1}] - \delta^{n+1} f[x_1, \dots, x_{n+1}, x_0] \Bigg\} \\
&\qquad + \frac{1}{x_{n+1} - x_0} \Bigg\{ \delta^{n+1} f[x_0, x_1, \dots, x_{n+1}] - \delta^{n+1} f[x_0, x_1, \dots, x_n, x_0] \Bigg\} \\
&= \sum_{i=0}^{n+1} \delta^{n+2} f[x_0, \dots, x_{n+1}, x_i]
\end{aligned}
$$

This proves the lemma.

Theorem 8.9. *Technical Lemma 2*
If $\delta^{n+1} f$ is S-continuous on (α, ω), then $\delta^k f$ is also S-continuous for all $k = 0, 1, \dots, n$.

Proof of Lemma 2:

It is sufficient to prove this for $k = n$, by reduction. Suppose $x_0 \approx x_1 \approx \dots \approx x_n$ and $\xi_0 \approx \cdots \approx \xi_n$ are near a real b. We wish to show $\delta^n f[x_0, \dots, x_n] \approx \delta^n f[\xi_0, \dots, \xi_n]$ and both are finite. We may assume that $\{x_0, \dots, x_n\} \neq \{\xi_0, \dots, \xi_n\}$ and if there is an overlap between the sets let $x_m, x_{m+1}, \dots, x_n$ be the overlapping points. Take $\xi_0 = x_m$, $\xi_1 = x_{m+1}, \dots, \xi_{n-m} = x_n$. Now we have $x_i \neq \xi_i$ and $\{x_0, x_1, \dots, x_j, \xi_j, \xi_{j+1}, \dots, \xi_n\}$ a set of $n+1$ distinct infinitely close points for each j.

To show that $\delta^n f[x_0, \dots, x_n] \approx \delta^n f[\xi_0, \dots, \xi_n]$, we form a telescoping sum and apply identity (DiffQ):

$$
\begin{aligned}
&\delta^n f[x_0, \dots, x_n] - \delta^n f[\xi_0, \dots, \xi_n] \\
&\qquad = \sum_{j=0}^{n} \delta^n f[x_0, \dots, x_j, \xi_{j+1}, \dots, \xi_n] - \delta^n f[x_0, \dots, x_{j-1}, \xi_j, \dots, \xi_n] \\
&\qquad = \sum_{j=0}^{n} (x_j - \xi_j) \delta^{n+1} f[x_0, \dots, x_j, \xi_j, \dots, \xi_n]
\end{aligned}
$$

By hypothesis all the $n+1$ order differences are near the same finite number so $\delta^n f[x] \approx \delta^n f[\xi]$, since $x_j \approx \xi_j$.

We can also use this identity to show that the nth order differences are finite. Since the identity holds for all infinitely close points, The Function Extension Axiom 2.1 shows that it must almost hold for sufficiently close differences (say within real θ for x's within real η). A real difference is finite, so the nearby infinitesimal one is too.

PROOF THAT STABLE DIFFERENCES IMPLIES SMOOTH:

We need to show that the differences with repetition are S-continuous, not just defined as hyperreals. It follows from The Function Extension Axiom 2.1 (as in the proof of Theorem 3.4) that for sufficiently small infinitesimal t,

$$\delta^{n+1} f[x_0, x_1, \dots, x_n, x_0 + t] \approx \delta^{n+1} f[x_0, \dots, x_n, x_0]$$

If $\xi_0 \approx \xi_1 \approx \cdots \approx \xi_n \approx x_0 \approx \cdots \approx x_n$, then

$$\begin{aligned} \delta^{n+1} f[x_0, \dots, x_n, x_0] &\approx \delta^{n+1} f[x_0, \dots, x_n, x_0 + t] \\ &\approx \delta^{n+1} f[\xi_0, \dots, \xi_n, \xi_0 + s] \\ &\approx \delta^{n+1} f[\xi_0, \dots, \xi_n, \xi_0] \end{aligned}$$

So S-continuity of $\delta^{n+1} f$ implies S-continuity of $\delta^{n+1} f$ with one repetition. (This is the theorem on first order smoothness if $n = 0$.)

Our induction hypothesis is applied at n to the function $g[x] = f'[x]$, that is, we assume that it is given that when $\delta^n g$ is S-continuous, g is n-times continuously differentiable. Now apply (δ'):

$$\begin{aligned} \delta^n f'[x_0, \dots, x_n] &= \sum_{i=0}^{n} \delta^{n+1} f[x_0, \dots, x_n, x_i] \\ &\approx \sum_{i=0}^{n} \delta^{n+1} f[\xi_0, \dots, \xi_n, \xi_i] \\ &= \delta^n f'[\xi_0, \dots, \xi_n] \end{aligned}$$

so $\delta^n f'$ is S-continuous and f' is n-times continuously differentiable. This proves that f is $n+1$ times continuously differentiable as claimed.

Exercise set 8.5

1. Local Higher Order Fit

First, make a table of values to fit:

```
n = 1;
x = 0.0;
dx = 0.5;
f[x_] := Exp[x]
values = Table[{x + k dx,f[x+ k dx]},{k,-n,n}] < Enter >
```

Next, make a list of basic functions:

```
Clear[dx];
polys = Table[dx∧i,{i,0,2 n}] < Enter >
```

Now Fit[] the data:
p = Fit[values,polys, dx];
***p**< Enter >*
And finally, Plot[] for comparison:
Plot[{ Exp[x + dx],1 + dx + dx∧2/2, p},{dx,-2,2}]

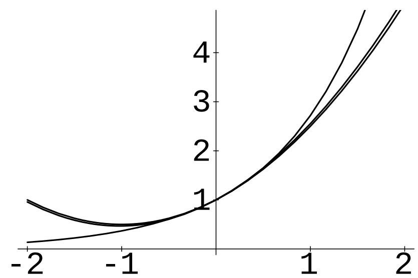
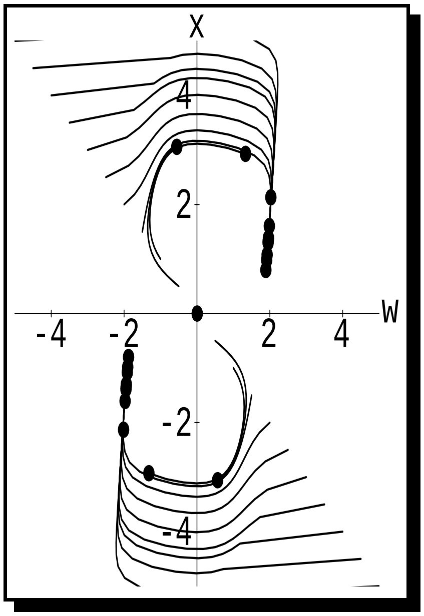


Figure 8.3: *2nd Comparison*

Take $dx = 0.25$*,* $dx = 0.125$ *and compare the coefficients of your Fit[]* $p[dx]$ *to the Taylor polynomial.*
Extend your program to fit a polynomial of degree 4 *and use the program to compare to the Taylor coefficients in the limit as* $\delta x \to 0$*.*



Part 4

Integration

CHAPTER

9 Basic Theory of the Definite Integral

The second half of the Fundamental Theorem technically requires that we prove existence of the integral before we make the estimates given in the proof.

We are assuming that we have a function $f[x]$ and do not know whether or not there is an expression for its antiderivative. We want to take the following limit and be assured that it converges.

$$\lim_{\Delta x \downarrow 0}(f[a]\Delta x+f[a+\Delta x]\Delta x+f[a+2\Delta x]\Delta x+f[a+3\Delta x]\Delta x+\cdots+f[b-2\Delta x]\Delta x+f[b-\Delta x]\Delta x)$$

If the limit exists it equals $\int_a^b f[x]\ dx$,

$$\int_a^b f[x]\,dx = \lim_{\Delta x \downarrow 0}(f[a]\Delta x + f[a+\Delta x]\Delta x + f[a+2\Delta x]\Delta x + f[a+3\Delta x]\Delta x + \cdots + f[b-2\Delta x]\Delta x + f[b-\Delta x]\Delta x)$$

or

$$\int_a^b f[x]\,dx \approx f[a]\delta x + f[a+\delta x]\delta x + \cdots + f[b-\delta x]\delta x \quad \text{for every} \quad 0 < \delta x \approx 0$$

When $f[x]$ is continuous and $[a, b]$ is a compact interval, of course this works as we prove next. When $f[x]$ is not continuous or if we want to integrate over an interval like $[1, \infty)$, then the theory of integration is more complicated. Later sections in this chapter show you why. (Even the first half of the Fundamental Theorem does not work if we only have a pointwise differentiable antiderivative, because then $f[x]$ can be discontinuous.)

The proof of the first half of the Fundamental Theorem of Integral Calculus 5.1 does not require a separate proof of the convergence of the sum approximations to the integral. The fact that the limit converges in the case where we know an antiderivative $F[x]$, where $dF[x] = f[x]\ dx$ as an ordinary derivative, follows directly from the increment approximation.

When we cannot find an antiderivative $F[x]$ for a given $f[x]$, we sometimes still want to work directly with the definition of the integral. A number of important functions are given by integrals of simple functions. The logarithm and arctangent are elementary examples. Some other functions like the probability function of the "bell-shaped curve" of probability and statistics do not have elementary formulas but do have integral formulas. The second half of the Fundamental Theorem justifies the integral formulas, which are useful as approximations. The **NumIntAprx** computer program shows us efficient ways to estimate the limit of sums directly.

9.1 Existence of the Integral

The next result states the approximation result we need without writing a symbol for $\int_a^b f[x]\,dx$. Once we show that I exists, we are justified in writing $I = \int_a^b f[x]\,dx$. Continuity of $f[x]$ is needed to show that the limit actually "converges."

Theorem 9.1. *Existence of the Definite Integral*

Let $f[x]$ be a continuous function on the interval $[a, b]$. Then there is a real number I such that

$$\lim_{\Delta x \downarrow 0} \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [f[x]\Delta x] = I$$

or, equivalently, for any $0 < \delta x \approx 0$ the natural extension of the sum function satisfies

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x]\,\delta x] \approx I$$

Proof:

First, by the Extreme Value Theorem 4.4, $f[x]$ has a min, m, and a Max, M, on the interval $[a, b]$. Monotony of summation tells us

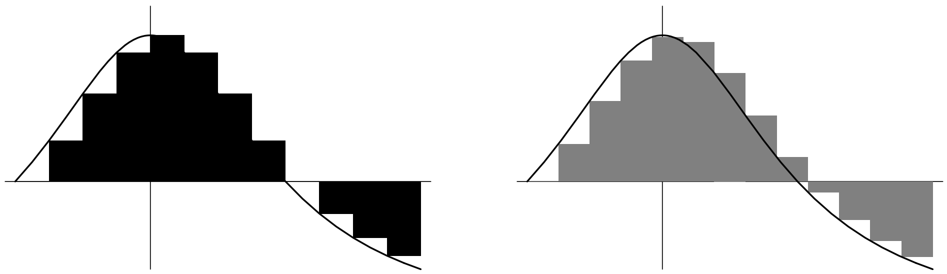
$$m \times (b-a) \le \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x]\,\delta x] \le M \times (b-a)$$

So that $\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x]\,\delta x]$ is a finite number and thus near some real value $I[\delta x] \approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x]\,\delta x]$. What we need to show is that if we choose a different infinitesimal, say δu, then $I[\delta x] = I[\delta u]$ or

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x]\,\delta x] \approx \sum_{\substack{u=a \\ \text{step } \delta u}}^{b-\delta u} [f[u]\,\delta u]$$

In other words, we need to compare "rectangular" approximations to the "area" with different step sizes. These step sizes may not be multiples of one another. This creates a messy technical complication as illustrated next with several finite step sizes. (You can experiment further using the program **GraphIntApprox**.)

The next two graphs show a function over an interval with 12 and 13 subdivisions.



If we superimpose these two subdivisions, we see

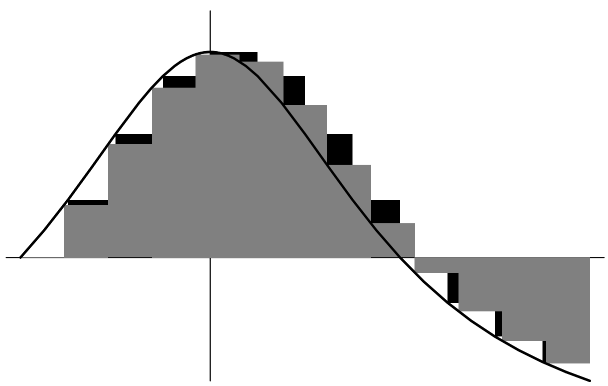
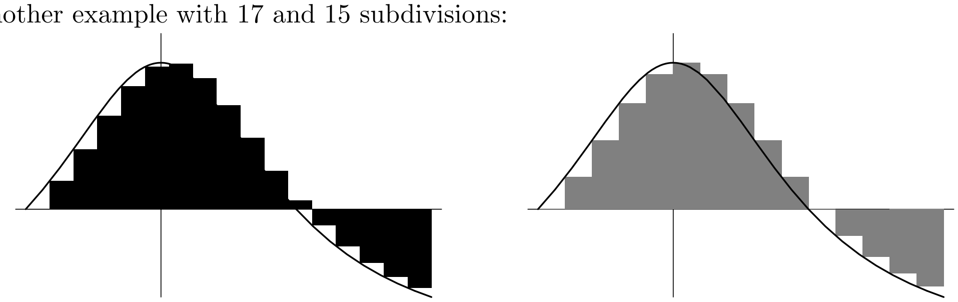


Figure 9.1: Both 12 and 13 Subdivisions Together

Notice that the overlaps between the various rectangles are not equal sizes. Here is another example with 17 and 15 subdivisions:



Again, the overlapping portions of the rectangles are unequal in size.

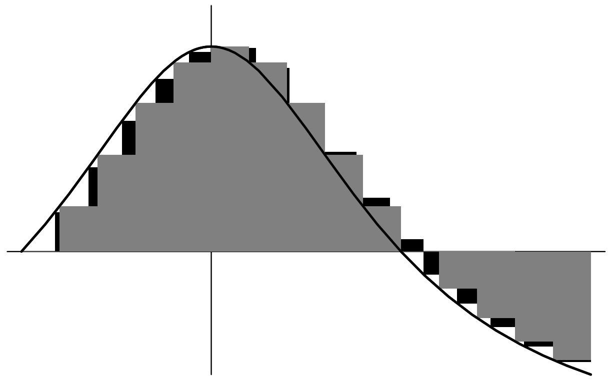


Figure 9.2: Both 17 and 15 Subdivisions Together

As a first step, we will make upper and lower estimates for the sum with a fixed step size. We know by the Extreme Value Theorem 4.4 that for each Δx and each $x = a + k\ \Delta x < b$, the function $f[x]$ has a max and a min on the small subinterval $[x, x + \Delta x]$,

$$f[x_m] \le f[\xi] \le f[x_M], \qquad \text{for any } \xi \text{ satisfying } x \le \xi \le x + \Delta x$$

We may 'code' this fact by letting $x_m = x_m[x, \Delta x]$ and $x_M = x_M[x, \Delta x]$ be functions that give these values. We extend and sum these to see that

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x_m]\,\delta x] \le \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x]\,\delta x]$$

and

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x]\,\delta x] \le \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x_M]\,\delta x]$$

When $\delta x \approx 0$ is infinitesimal, we also know that $f[x_m] \approx f[x_M]$ and in fact that the largest difference along the partition is also infinitesimal,

$$0 \le \text{Max}[(f[x_M[x, \delta x]] - f[x_m[x, \delta x]]) : x = a, a + \delta x, a + 2\,\delta x, \cdots < b] = \theta \approx 0$$

This shows that our upper and lower estimates are close,

$$\begin{aligned} \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x_M]\,\delta x] - \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x_m]\,\delta x] &= \\ &= \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [(f[x_M] - f[x_m])\,\delta x] \\ &\le \theta \cdot \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\delta x] = \theta \cdot (b - a) \approx 0 \end{aligned}$$

We have shown that for any equal size infinitesimal partition, the upper and lower estimate sums are infinitely close. Now consider the unequal partition given by overlapping a

δx partition and a δu partition for two different infinitesimals. A lower estimate with the min of $f[x]$ over the refined partition is LARGER than the min estimate of either equal size partition, because the subinterval mins are taken over smaller subintervals. An upper estimate with the max of $f[x]$ over the refined partition is SMALLER than the max estimate over either equal size partition. The difference between the refined upper and lower estimates is infinitesimal by the same kind of computation as above. We therefore have

$$I[\delta x] \approx \text{Min } \delta x\text{-Sum} \approx \text{Max } \delta x\text{-Sum} \quad \text{and} \quad \text{Min } \delta u\text{-Sum} \approx \text{Max } \delta u\text{-Sum} \approx I[\delta u]$$
$$\text{Min } \delta x\text{-Sum} \leq \text{Refined Min Sum} \leq \text{Refined Max Sum} \leq \text{Max } \delta x\text{-Sum}$$
$$\text{Min } \delta u\text{-Sum} \leq \text{Refined Min Sum} \leq \text{Refined Max Sum} \leq \text{Max } \delta u\text{-Sum}$$

so $I[\delta x] \approx I[\delta u]$ and since these are real numbers $I[\delta x] = I[\delta u]$. This proves that the integral of a continuous function over a compact interval exists.

Exercise set 9.1

1. Keisler's Proof of Existence
Let $f[x]$ be continuous on $[a, b]$. We want to show that the integral exists. This is equivalent to showing that for every two positive infinitesimals δx and δu we have

$$\sum_{\substack{x=a \\ step\ \delta x}}^{b-\delta x} [f[x]\,\delta x] \approx \sum_{\substack{u=a \\ step\ \delta u}}^{b-\delta u} [f[u]\,\delta u]$$

(a) *First, show that we can reduce the problem to the case where $f[x] \geq 0$ on $[a, b]$. (HINT: Consider $f[x] = F[x] + m$ where $m = Min[F[x] : a \leq x \leq b]$.)*

(b) *Given two positive infinitesimals δx and δu, show that we can reduce the problem to showing that for any positive real number r we have*

$$\sum_{\substack{x=a \\ step\ \delta x}}^{b-\delta x} [f[x]\,\delta x] \leq r + \sum_{\substack{u=a \\ step\ \delta u}}^{b-\delta u} [f[u]\,\delta u]$$

(c) *Let r be a positive real number and take $c = (b - a)/r$. Below you will show that*

$$\sum_{\substack{x=a \\ step\ \delta x}}^{b-\delta x} [f[x]\,\delta x] \leq \sum_{\substack{u=a \\ step\ \delta u}}^{b-\delta u} [(f[u] + c)\,\delta u]$$

Why does this establish (b)?

(d) *To prove that the inequality in (c) holds, suppose to the contrary that*

$$\sum_{\substack{x=a \\ step\ \delta x}}^{b-\delta x} [f[x]\,\delta x] > \sum_{\substack{u=a \\ step\ \delta u}}^{b-\delta u} [(f[u] + c)\,\delta u]$$

Show that then there must be a pair of points x and u in $[a, b]$ so that

$$x - \delta u \leq u \leq x + \delta x \quad \textit{and} \quad f[x] > f[u] + c$$

(e) *When* $f[x]$ *is continuous and* $\delta x \approx 0$*, we cannot have* $x - \delta u \le u \le x + \delta x$ *and* $f[x] > c + f[u]$*. Why?*

(f) *Why does the contradiction of (e) prove that the integral exists?*

(HINTS: Step (d) is the hard one. The sums are areas of regions bounded by $x = a$*,* $x = b$*, the x-axis, and step functions* $s_{\delta x}[x]$ *and* $s_{\delta u}[x]$*. When the* δx*-sum is larger than the* δu*-sum, the region below* $s_{\delta x}[x]$ *cannot lie completely inside the region below* $s_{\delta u}[x]$*. Let* v *satisfy* $s_{\delta x}[v] > s_{\delta u}[v]$*. The value* $s_{\delta x}[v] = f[x]$ *for a* δx*-partition point,* $x \le v < x + \delta x$ *and the value* $s_{\delta u}[v] = f[u] + c$ *for a* δu*-partition point,* $u \le v < u + \delta u$*. Show that* $x - \delta u \le u \le x + \delta x$*.*

When Δx *and* Δu *are real, concoct functions that give the needed values of* $x = X[\Delta x]$ *and* $u = U[\Delta u]$*, then apply the implication when* δx *and* δu *are infinitesimal.)*

9.2 You Can't Always Integrate Discontinuous Functions

With discontinuous integrands, it is possible to make the sums oscillate as $\Delta x \to 0$*. In these cases, numerical integration by computer will likely give the wrong answer.*

Exercises 12.7.3 and 12.7.4 in the main text show you examples of false attempts to integrate discontinuous functions by antidifferentiation. The idea in those exercises is very important - it is a main text topic. You can not use the Fundamental Theorem without verifying the hypotheses.

More bizarre examples are possible if you permit the use of weakly approximating pointwise derivatives. Examples 6.3.1 and 6.3 show how a single oscillatory discontinuity in a pointwise derivative can lead to unexpected results. It is possible to tie a bunch of these oscillations together in such a way that the resulting function has oscillatory discontinuities on 'a set of positive measure.' In this case we have the pointwise derivative $D_x F[x] = f[x]$, but the limit of sums trying to define the integral do not converge to $F[b] - F[a]$.

There are two different kinds of discontinuity preventing convergence of the approximating sums for integrals. Isolated infinite discontinuities like the ones cited above from the main text are easiest to understand and we discuss them below in a section on "improper" integrals. There is also a project on improper integrals.

Oscillatory (even bounded) discontinuities are much more difficult to understand. B. Riemann discovered the best condition that would allow convergence of the sums. The integral is often called the "Riemann integral" as a result. This is peculiar, because the notation for integrals originated in 1675 in a paper of Leibniz, while Riemann's integrability result appears over 150 years later in his paper on Fourier series. It took a very long time for mathematicians to understand integration of discontinuous functions. (You too can progress very far by only integrating continuous functions.)

Riemann was interested in passing a limit under the integral,

$$\lim_{n\to\infty}\int_a^b f_n[x]\,dx = \int_a^b \lim_{n\to\infty} f_n[x]\,dx$$

In his particular case the functions $f_n[x]$ were Fourier series approximations. This idea seems harmless enough, but the appearance is deceiving. If we can do this once, we should be able to do it twice. We want you to see just how awful limits of functions can be.

Example 9.1. *A Very Discontinuous Limit*

The following limit exists for each x, but the limit function is discontinuous at every point.

$$\lim_{n\to\infty}\lim_{m\to\infty}(\mathrm{Cos}[\pi\, n!\; x])^m = I_Q[x]$$

If $x = p/q$ is rational, then when $n \ge q$, $\mathrm{Cos}[\pi\, n!\; \frac{p}{q}] = 1$, so

$$\lim_{m\to\infty}\left(\mathrm{Cos}[\pi\, n!\; \frac{p}{q}]\right)^m = 1$$

Also

$$\lim_{n\to\infty} 1 = \lim_{n\to\infty}\lim_{m\to\infty}\left(\mathrm{Cos}[\pi\, n!\; \frac{p}{q}]\right)^m = 1$$

If x is not rational, then no matter how large we take a fixed n,

$$\lim_{m\to\infty}(\mathrm{Cos}[\pi\, n!\; x])^m = 0$$

since the fixed quantity $|\,\mathrm{Cos}[\pi\, n!\; x]| < 1$. The limit of zero is zero, so when x is fixed and irrational,

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty}\lim_{m\to\infty}(\mathrm{Cos}[\pi\, n!\; x])^m = 0$$

Together, these two parts show that the limit exists and equals the indicator function of the rational numbers,

$$I_Q[x] = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

This function is discontinuous at every point since every x has both rational and irrational points arbitrarily nearby. In other words, we can approximate $\xi \approx x \approx \eta$ with $I_Q[\xi] = 1$ and $I_Q[\eta] = 0$, so

$$\xi \approx \eta \qquad \text{but} \qquad I_Q[\xi] = 1 \quad \text{and} \quad I_Q[\eta] = 0$$

It is even difficult to make a useful plot of low approximations to the limit.

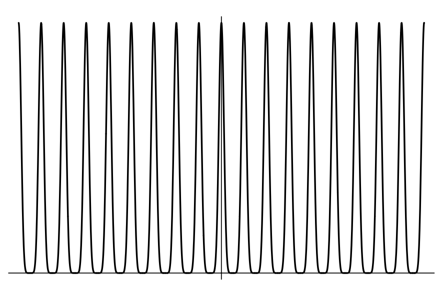


Figure 9.3: $(\mathrm{Cos}[\pi\, 3!\; x])^8$

We can show that

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{-\pi}^{\pi}(\text{Cos}[\pi\, n!\; x])^m \;\; dx = 0$$

and the limits defining the ordinary integral

$$\int_{-\pi}^{\pi} I_Q[x]\; dx = \int_{-\pi}^{\pi}\left(\lim_{n\to\infty}\lim_{m\to\infty}\text{Cos}^m[\pi\, n!\; x]\right)\; dx \quad \text{do not converge.}$$

You cannot interchange these limits and the ordinary integral.

After Riemann, the study of Fourier series motivated Lebesgue's work on integration that ultimately led to a more powerful kind of integral now called the Lebesgue integral. Lebesgue integrals of continuous functions are the same as the integrals we have been studying, but Lebesgue integrals are defined for more discontinuous functions and satisfy more general and flexible interchange of limit and integral theorems. When you really need to integrate wild discontinuities, study Lebesgue integrals.

9.3 Fundamental Theorem: Part 2

The second part of the Fundamental Theorem of Integral Calculus says that the derivative of an integral of a continuous function is the integrand,

$$\frac{d}{dX}\int_a^X f[x]\,dx = f\,[X]$$

The function $A[X] = \int_a^X f[x]\,dx$ can be thought of as the "accumulated area" under the curve $y = f[x]$ from a to X shown in Figure 9.4. The "accumulation function" can also be thought of as the reading of your odometer at time X for a given speed function $f[x]$.

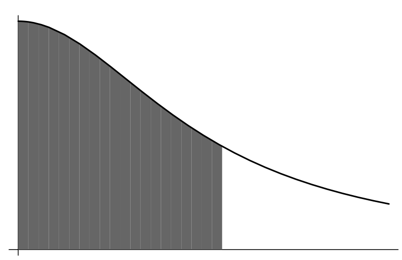


Figure 9.4: $A[X] = \int_a^X f[x]\,dx$

Theorem 9.2. *Second Half of the Fundamental Theorem*

Suppose that $f[x]$ is a continuous function on an interval containing a and we define a new function by "accumulation,"

$$A[X] = \int_a^X f[x]\,dx$$

Then $A[X]$ is smooth and $\frac{dA}{dX}[X] = f[X]$; in other words,

$$\frac{d}{dX}\int_a^X f[x]\,dx = f[X]$$

PROOF:

We show that $A[X]$ satisfies the differential approximation $A[X+\delta X]-A[X] = f[X]\,\delta X + \varepsilon\cdot\delta X$ with $\varepsilon \approx 0$ when $\delta X \approx 0$. This proves that $f[X]$ is the derivative of $A[X]$.

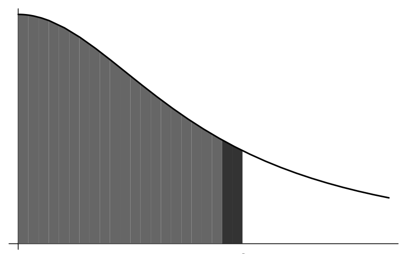


Figure 9.5: $\int_a^{X+\delta X} f[x]\,dx$

By definition of A and the additivity property of integrals, for every real ΔX (small enough so $X+\Delta X$ is still in the interval where $f[x]$ is continuous) we have

$$\begin{aligned} A[X+\Delta X] &= \int_a^{X+\Delta X} f[x]\,dx = \int_a^X f[x]\,dx + \int_X^{X+\Delta X} f[x]\,dx \\ &= A[X] + \int_X^{X+\Delta X} f[x]\,dx \\ A[X+\Delta X] - A[X] &= \int_X^{X+\Delta X} f[x]\,dx \end{aligned}$$

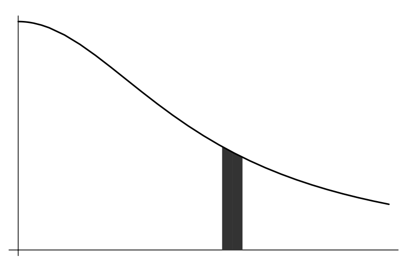


Figure 9.6: $\int_X^{X+\Delta X} f[x]\,dx$

The Extreme Value Theorem for Continuous Functions 4.4 says that $f[x]$ has a max and a min on the interval $[X, X+\Delta X]$, $m = f[X_m] \le f[x] \le M = f[X_M]$ for all $X \le x \le$

$X + \Delta X$. Monotony of the integral and simple algebra gives us the estimates

$$m \cdot \Delta X = \int_X^{X+\Delta X} m \, dx \le \int_X^{X+\Delta X} f[x]\, dx \le \int_X^{X+\Delta X} M \, dx = M \cdot \Delta X$$
$$m \cdot \Delta X \le A[X + \Delta X] - A[X] \le M \cdot \Delta X$$
$$m \le \frac{A[X + \Delta X] - A[X]}{\Delta X} \le M$$
$$m = f[X_m] \le f[X] \le f[X_M] = M$$
$$\left| \frac{A[X + \Delta X] - A[X]}{\Delta X} - f[X] \right| \le f[X_M] - f[X_m]$$

with both X_m and X_M in the interval $[X, X + \Delta X]$.

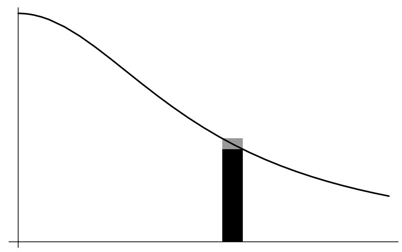


Figure 9.7: Upper and lower estimates

The Function Extension Axiom 2.1 means that this inequality also hold for positive infinitesimal $\delta X = \Delta X$,

$$\left| \frac{A[X + \delta X] - A[X]}{\delta X} - f[X] \right| \le f[X_M] - f[X_m]$$

We know that $X_m \approx X_M$ when $\delta X \approx 0$. Continuity of $f[x]$ means that $f[X_m] \approx f[X_M] \approx f[X]$ in this case, so

$$\frac{A[X + \delta X] - A[X]}{\delta X} = f[X] + \varepsilon$$

with $\varepsilon \approx 0$. So,

$$A[X + \delta X] - A[X] = f[X] \cdot \delta X + \varepsilon \cdot \delta X$$

with $\varepsilon \approx 0$, when $\delta X \approx 0$. This proves the theorem because we have verified the microscope equation from Definition 5.2

$$A[X + \delta X] = A[X] + A'[X]\,\delta X + \varepsilon \cdot \delta X$$

with $A'[X] = f[X]$.

Exercise set 9.3

1. A Formula for ArcTangent
Prove that

$$\text{ArcTan}[x] = \int_0^x \frac{1}{1+\xi^2}\, d\xi$$

(HINT: See Example 5.4.*)*

9.4 Improper Integrals

The ordinary integral does not cover the case where the integrand function is discontinuous or the case where the interval of integration is unbounded. This section extends the integral to these cases.

There are two main kinds of improper integrals that we can study simply. One kind has an isolated discontinuity in the integrand like

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx \qquad \text{and} \qquad \int_0^1 \frac{1}{x^2}\, dx$$

The other kind of improper integral is over an unbounded interval like the probability distribution

$$\frac{2}{\sqrt{\pi}} \int_{-\infty}^{X} e^{-x^2}\, dx \qquad \text{or} \qquad \int_1^{\infty} \frac{1}{x^2}\, dx$$

In both of these cases, we can calculate $\int_a^c f[x]\, dx$ and take a limit as $c \to b$. The theory of these kinds of limiting integrals is similar to the theory of infinite series. We begin with a very basic pair of examples.

Example 9.2. $\int_0^1 1/x^p\, dx$

The function $\frac{1}{x^p}$ has a discontinuity at $x = 0$ when $p > 0$, but we can compute

$$\lim_{b \downarrow 0} \int_b^1 \frac{1}{x^p}\, dx$$

The Fundamental Theorem applies to the continuous function $1/x^p$ for $0 < b \le x \le 1$, so

$$\begin{aligned}
\int_b^1 \frac{1}{x^p}\, dx &= \int_b^1 x^{-p}\, dx \\
&= \begin{cases} \frac{1}{1-p} x^{1-p}|_b^1, & \text{if } p \neq 1 \\ \text{Log}[x]|_b^1, & \text{if } p = 1 \end{cases} \\
&= \begin{cases} \frac{1-b^{1-p}}{1-p}, & \text{if } p < 1 \\ -\text{Log}[b], & \text{if } p = 1 \\ \frac{1/b^{p-1}-1}{p-1}, & \text{if } p > 1 \end{cases}
\end{aligned}$$

The limits in these cases are

$$\lim_{b \downarrow 0} \int_b^1 \frac{1}{x^p}\, dx = \begin{cases} \lim_{b\downarrow 0} \frac{1-b^{1-p}}{1-p} = \frac{1}{1-p}, & \text{if } p < 1 \\ \lim_{b\downarrow 0} -\text{Log}[b] = \lim_{b\downarrow 0} \text{Log}[1/b] = \lim_{c\to\infty} \text{Log}[c] = \infty, & \text{if } p = 1 \\ \lim_{b\downarrow 0} \frac{1/b^{p-1}-1}{p-1} = \infty, & \text{if } p > 1 \end{cases}$$

To summarize, we have

$$\int_0^1 x^{-p}\, dx = \begin{cases} \frac{1}{1-p}, & \text{if } p < 1 \\ \infty, & \text{if } p \geq 1 \end{cases}$$

Now we consider the other difficulty, an unbounded interval of integration.

Example 9.3. $\int_1^\infty 1/x^p\, dx$

The integrand $1/x^p$ is continuous on the interval $[1,\infty)$, so we can compute

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^p}\, dx$$

Again we have cases,

$$\begin{aligned} \int_1^b \frac{1}{x^p}\, dx &= \int_1^b x^{-p}\, dx \\ &= \begin{cases} \frac{1}{1-p} x^{1-p}|_1^b, & \text{if } p \neq 1 \\ \mathrm{Log}[x]|_1^b, & \text{if } p = 1 \end{cases} \\ &= \begin{cases} \frac{b^{1-p}-1}{1-p}, & \text{if } p < 1 \\ \mathrm{Log}[b], & \text{if } p = 1 \\ \frac{1-1/b^{p-1}}{p-1}, & \text{if } p > 1 \end{cases} \end{aligned}$$

The limits in these cases are

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^p}\, dx = \begin{cases} \lim_{b\to\infty} \frac{b^{1-p}-1}{1-p} = \infty, & \text{if } p < 1 \\ \lim_{b\to\infty} \mathrm{Log}[b] = \infty, & \text{if } p = 1 \\ \lim_{b\to\infty} \frac{1-1/b^{p-1}}{p-1} = \frac{1}{p-1}, & \text{if } p > 1 \end{cases}$$

To summarize, we have the opposite cases of convergence in the infinite interval case that we had in the $(0,1]$ case above,

$$\int_1^\infty x^{-p}\, dx = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \infty, & \text{if } p \leq 1 \end{cases}$$

Infinite intervals of integration arise often in probability. One such place is in exponential waiting times. This can be explained intuitively as follows. Suppose you have light bulbs that fail in some random way, but if one of them is still working, it is as good as new, that is, the time you expect to wait for it to fail is the same as if it was new. If we write $P[t]$ for the probability of the bulb lasting at least until time t, then the 'good as new' statement becomes

$$\text{Probability of lasting to } t + \Delta t \text{ given that it lasted to } t \;=\; \frac{P[t+\Delta t]}{P[t]} = P[\Delta t]$$

If we re-write this algebraically, this probabilistic statement is the same as the exponential functional identity (see Chapter 2 above for the background on functional identities),

$$P[t+\Delta t] = P[t] \times P[\Delta t]$$

If we assume that $P[t]$ is a smooth function, we saw in Example 2.7 above that

$$P[t] = e^{-\lambda\ t}$$

for some $\lambda > 0$. Example 2.7 shows that $P[t] = e^{k\,t}$ and the reason that we have a negative constant $k = -\lambda$ in our exponential is because the light bulb can not last forever, $\lim_{t\to\infty} P[t] = \lim_{t\to\infty} e^{-\lambda\,t} = 0$.

Notice that $P[0] = 1$ says we start with a good bulb.

We can think of the expression $\lambda\ e^{-\lambda\ t}\ dt$ as the probability that the bulb burns out during the time interval $[t, t+dt)$. (See the next exercise.)

Example 9.4. *Expected Life of the Bulb*

The average or expected lifetime of a bulb that fails by an exponential waiting time is given by the improper integral

$$\int_0^\infty t\ \lambda\ e^{-\lambda\ t}\ dt$$

This can be computed with integration by parts,

$$u = t \qquad dv = \lambda\ e^{-\lambda\ t}\ dt$$
$$du = dt \qquad v = -e^{-\lambda\ t}$$

so

$$\begin{aligned}\int_0^b t\ \lambda e^{-\lambda\ t}\ dt &= -t\ e^{-\lambda\ t}|_0^b + \int_0^b e^{-\lambda\ t}\ dt \\ &= b\ e^{-\lambda\ b} - \frac{1}{\lambda}\ e^{-\lambda\ t}|_0^b \\ &= \frac{1}{\lambda} + b\ e^{-\lambda\ b} - \frac{1}{\lambda}\ e^{-\lambda\ b}\end{aligned}$$

We know (from Chapter 7 of the main text) that exponentials beat powers to infinity, so

$$\lim_{b\to\infty} b\ e^{-\lambda\ b} - \frac{1}{\lambda}\ e^{-\lambda\ b} = \lim_{b\to\infty} \frac{b}{e^{\lambda\ b}} - \frac{1}{\lambda\ e^{\lambda\ b}} = 0$$

We have shown that the expected life of the bulb is

$$\int_0^\infty t\ \lambda\ e^{-\lambda\ t}\ dt = \frac{1}{\lambda}$$

9.4.1 Comparison of Improper Integrals

The most important integral in probability is the Gaussian or "normal" probability related to the integral

$$\mathrm{Erf}[X] = \frac{2}{\sqrt{\pi}} \int_{-\infty}^X e^{-x^2}\ dx$$

You saw in the NoteBook **SymbolicIntegr** that this does not have an antiderivative that can be expressed in terms of elementary functions. *Mathematica* calculates this integral in terms of "Erf[ ]," its built in Gaussian error function. We often want to be certain that an integral converges, but without calculating the limit explicitly (since this is sometimes impossible). We may do this by comparing an integral to a known one. This is similar to the comparison tests for convergence of series in the main text Chapter 18 on Series.

In the case of the integral above, $e^{-x^2} < e^{-|x|}$ for $|x| > 1$, so

$$\int_{-\infty}^{-1} e^{-x^2}\,dx < \int_{-\infty}^{-1} e^{x}\,dx \qquad \text{and} \qquad \int_{1}^{\infty} e^{-x^2}\,dx < \int_{1}^{\infty} e^{-x}\,dx$$

The estimating integral converges,

$$\int_{1}^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\int_{1}^{b} e^{-x}\,dx = \lim_{b\to\infty}\left(e - e^{-b}\right) = e$$

so the tails of the e^{-x^2} integral converges and

$$\int_{-\infty}^{-1} e^{-x^2}\,dx + \int_{-1}^{1} e^{-x^2}\,dx + \int_{1}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

Exercise set 9.4

1. Improper Drill

1 $\int_0^1 1/\sqrt{x}\,dx =$	*2* $\int_1^\infty 1/\sqrt{x}\,dx =$	*3* $\int_0^\infty 1/\sqrt{x}\,dx =$
4 $\int_0^1 1/x^2\,dx =$	*5* $\int_1^\infty 1/x^2\,dx =$	*6* $\int_0^\infty 1/x^2\,dx =$
7 $\int_0^1 1/\sqrt[3]{x}\,dx =$	*8* $\int_1^\infty 1/\sqrt[3]{x}\,dx =$	*9* $\int_0^\infty 1/\sqrt[3]{x}\,dx =$
10 $\int_0^1 1/x^3\,dx =$	*11* $\int_1^\infty 1/x^3\,dx =$	*12* $\int_0^\infty 1/x^3\,dx =$

2. *Show that*

$$P[t] = \int_{t}^{\infty} \lambda\, e^{-\lambda\, t}\,dt = \lim_{b\to\infty}\int_{t}^{b} \lambda\, e^{-\lambda\, t}\,dt = e^{-\lambda\, t}$$

and

$$\int_{0}^{t} \lambda\, e^{-\lambda\, t}\,dt = 1 - e^{-\lambda\, t}$$

3. *Calculate the integral*

$$\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} x\, e^{-x^2}\,dx$$

symbolically using a change of variables. Explain your answer geometrically. Calculate the integral

$$\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^2\, e^{-x^2}\,dx$$

4. *Prove that the integral*

$$\int_{0}^{1} \frac{\mathrm{Sin}[x]}{x}\,dx$$

converges. (Is $\mathrm{Sin}[x]/x$ *really discontinuous at zero? Consider its power series.)*

Do the integrals

$$\int_0^\infty \frac{\text{Sin}[x]}{x}\, dx \qquad \text{and} \qquad \int_0^\infty \frac{|\,\text{Sin}[x]|}{x}\, dx$$

converge? This is a tough question, but perhaps you can at least use Mathematica to make conjectures.

The previous exercise is related to conditionally convergent series studied in the Mathematical Background Chapter on Series below. Lebesgue integrals are always absolutely convergent, but we can have conditionally convergent improper integrals when we define them by limits like

$$\lim_{b\to\infty} \int_0^b \frac{\text{Sin}[x]}{x}\, dx$$

5. The Gamma Function

The Gamma function is given by the improper integral

$$\Gamma(s) = \int_0^\infty t^{s-1}\, e^{-t}\, dt$$

This integral has both kinds of improperness. Why? Show that it converges anyway by breaking up the two cases

$$\int_0^\infty t^{s-1}\, e^{-t}\, dt = \int_0^1 t^{s-1}\, e^{-t}\, dt + \int_1^\infty t^{s-1}\, e^{-t}\, dt$$

Use Integration by Parts to show $\Gamma(s+1) = s\,\Gamma(s)$*.*

Use $s = n$*, a positive integer and induction on the functional identity above to show that* $\Gamma(s+1)$ *is an extension of the factorial function,*

$$\Gamma(n+1) = n!$$

9.4.2 A Finite Funnel with Infinite Area?

Suppose we imagine an infinite funnel obtained by rotating the curve $y = 1/x$ about the x-axis. We can compute the volume of the funnel by slicing it into disks,

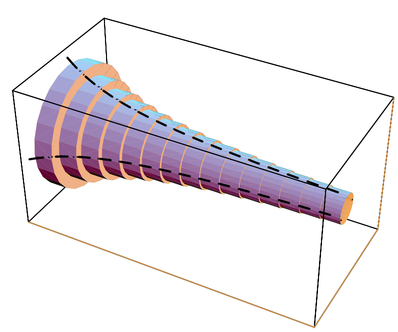


Figure 9.8: $y = 1/x$ Rotated

$$\text{Volume} = \int_1^{\infty} \pi r^2(x)\ dx = \pi \int_1^{\infty} x^{-2}\ dx$$

6. Finite Volume

Calculate the integral above and show that the volume is finite (π).

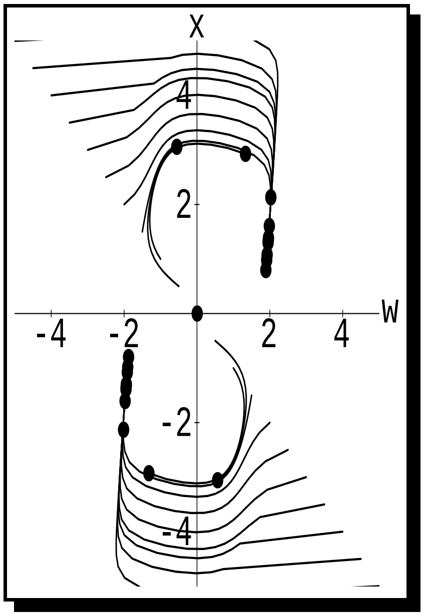
Paradoxically, the surface area of this infinite funnel is infinite. If you review the calculation of surface area integrals from Chapter 12 of the main text, you will obtain the formula

$$\text{Area} = \pi \int_1^{\infty} \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\ dx$$

7. Infinite Area

Show that the surface area integral above is infinite by comparing it to a smaller integral that you know diverges.

Perhaps it is a good thing that we can't build infinite objects. We would run out of paint to cover them, even though we could fill them with paint...



Part 5

Multivariable Differentiation

CHAPTER 10

Derivatives of Multivariable Functions

Functions of several variables whose partial derivatives can be computed by rules automatically are differentiable when the function and its partial derivative formulas are defined on a rectangle.

Theorem 10.1. *Defined Formulas Imply Approximation*
Suppose that $z = f[x,y]$ is given by formulas and that the partial derivatives $\frac{\partial f}{\partial x}[x,y]$ and $\frac{\partial f}{\partial y}[x,y]$ can be computed using the rules of Chapter 6 (Specific Functions, Superposition Rule, Product Rule, Chain Rule) holding one variable at a time fixed. If the resulting three formulas $f[x,y]$, $\frac{\partial f}{\partial x}[x,y]$, $\frac{\partial f}{\partial y}[x,y]$, are all defined in a compact box, $\alpha \le x \le \beta$, $\gamma \le y \le \eta$, then

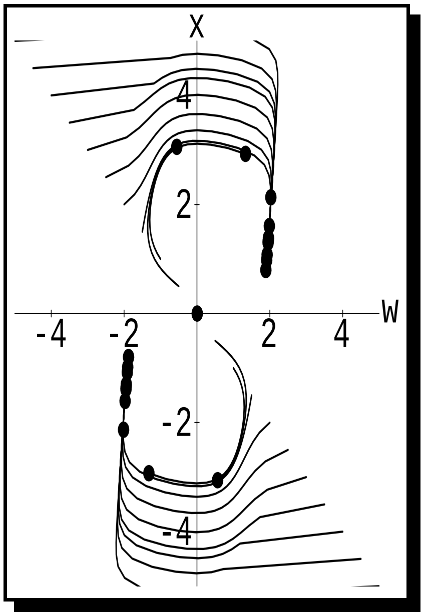
$$f[x+\delta x, y+\delta y] - f[x,y] = \frac{\partial f}{\partial x}[x,y]\cdot \delta x + \frac{\partial f}{\partial y}[x,y]\cdot \delta y + \varepsilon \cdot \sqrt{\delta x^2 + \delta y^2}$$

with ε uniformly small in the (x,y)-box for sufficiently small δx and δy.

The "high tech" reason this theorem is true is this. All the specific classical functions are complex analytic; having infinitely many derivatives and convergent power series expansions. Formulas are built up using these functions together with addition, multiplication and composition - the exact rules by which we differentiate. These formation rules only result in more complex analytic functions of several variables. The only thing that can "go wrong" is to have the functions undefined.

Despite this clear reason, it would be nice to have a more direct elementary proof. In fact, it would be nice to show that uniform differentiability is "closed" under the operations of basic calculus and specifically including solution of initial value problems and indefinite integration in particular. Try to prove this yourself or WATCH OUR WEB SITE!

http://www.math.uiowa.edu/ stroyan/



Part 6

Differential Equations

CHAPTER 11

Theory of Initial Value Problems

One of the main ideas of calculus is that if we know

(1) *where a quantity starts,* $x[t_0] = x_0$

and

(2) *how the quantity changes,* $\frac{dx}{dt} = f[t, x]$

then we can find out where the quantity goes. The basic start and continuous change information is called an initial value problem.

The solution of an initial value problem is an unknown function $x[t]$. This chapter shows that there is only one solution, but since the solutions may not be given by simple formulas, it also studies properties of the solutions.

11.1 Existence and Uniqueness of Solutions

If we know where to start and how to change, then we should be able to figure out where we go. This sounds simple, but it means that there is only one possible place for the quantity to go by a certain time.

The precise one dimensional theorem that captures this is:

Theorem 11.1. *Existence & Uniqueness for I. V. P.s*
Suppose that the functions $f[t, x]$ and $\frac{\partial f}{\partial x}[t, x]$ are continuous in a rectangle around (x_0, t_0). Then the initial value problem

$$x[t_0] = x_0$$
$$dx = f[t, x]\, dt$$

has a unique solution $x[t]$ defined for some small time interval $t_0 - \Delta < t < t_0 + \Delta$. Euler's Method converges to $x[t]$ on closed subintervals $[t_0, t_1]$, for $t_1 < t_0 + \Delta$.

The Idea of the Proof

The simplest proof of this theorem is based on a "functional." The unknown variable in the differential equation is a function $x[t]$. Suppose we had a solution and integrated the differential,

$$\int_{t_0}^{\tau} dx[t] = \int_{t_0}^{\tau} f[t, x[t]]\ dt$$
$$x[\tau] - x[t_0] = \int_{t_0}^{\tau} f[t, x[t]]\ dt$$

We may think of the integral as a function of functions (or "functional.") We make a translation of variables, $u[t] = x[t + t_0] - x_0,\ g[t, u] = f[t + t_0, u + x_0]$. Given an input function $v[t]$, we get an output function by computing

$$w = G[v], \qquad \text{where} \quad w[\tau] = \int_0^{\tau} g[t, v[t]]\ dt$$

An equivalent problem to our differential equation is: Find a "fixed point" of the functional $G[u]$, that is, a function $u[t]$ with $u[0] = 0$ so that $G[u] = u$. Notice that

$$u[\tau] = \int_0^{\tau} g[t, u[t]]\ dt \qquad \Leftrightarrow \qquad u = G[u]$$

See Exercise 11.1.1.

The proof of the theorem is very much like the computation of inverse functions in Theorem 5.7 above. We begin with $u_0[t] = 0$ and successively compute

$$\begin{aligned}
u_1 &= G[u_0] \\
u_2 &= G[u_1] = G[G[u_0]] \\
u_3 &= G[u_2] = G[G[u_1]] = G[G[G[u_0]]] \\
&\vdots
\end{aligned}$$

This iteration is an infinite dimensional discrete dynamical system or a discrete dynamical system on functions. If we choose a small enough time interval, $0 \le t \le \Delta$, we can show that the functional $G[\cdot]$ decreases the distance between functions as measured by

$$\|u - v\| = \text{Max}[|u[t] - v[t]| : 0 \le t \le \Delta]$$

That is, $\|G[u] - G[v]\| \le r\,\|u - v\|$, with a constant r satisfying $0 \le r < 1$. (We will show that we can take $r = 1/2$ below.) The proof of convergence of these approximating functions is just like the proof in Theorem 5.7 once we have this "contraction" property. The iteration scheme above reduces the distance between successive terms, and we can prove that $\|u_{n+1} - u_n\| \le r^n \cdot \|u_1 - u_0\|$ by recursively using each inequality in the next as follows:

$$\begin{aligned}
&\|u_2 - u_1\| = \|G[u_1] - G[u_0]\| \le r \cdot \|u_1 - u_0\| \\
&\|u_3 - u_2\| = \|G[u_2] - G[u_1]\| \le r \cdot \|G[u_1] - G[u_0]\| \le r^2 \cdot \|u_1 - u_0\| \\
&\|u_4 - u_3\| = \|G(u_3) - G[u_2]\| \le r \cdot \|G[u_2] - G[u_1]\| \le r^3 \cdot \|u_1 - u_0\| \\
&\qquad \vdots \\
&\|u_{n+1} - u_n\| \le r^n \cdot \|u_1 - u_0\|
\end{aligned}$$

The sequence of iterates tends to the actual solution, $u_n \to u$, the function satisfying $u = G[u]$. To see this we use the geometric series from Chapter 25 of the main text on the numbers $\|u_{n+1} - u_n\|$. Notice that the distance we move for any number m steps beyond the n^{th} approximation satisfies

$$\begin{aligned}
|u_{n+m}[t] - u_n[t]| &\le |u_{n+m}[t] - u_{n+m-1}[t] + u_{n+m-1}[t] - \cdots - u_n[t]| \\
&\le |u_{n+m}[t] - u_{n+m-1}[t]| + |u_{n+m-1}[t] - u_{n+m-2}[t]| + \cdots \\
&\qquad + |u_{n+1}[t] - u_n[t]| \\
&\le \|u_{n+m} - u_{n+m-1}\| + \|u_{n+m-1} - u_{n+m-2}\| + \cdots \\
&\qquad + \|u_{n+1} - u_n\| \\
&\le r^n \cdot \|u_1 - u_0\| \cdot (r^m + r^{m-1} + \cdots + 1) \\
&\le r^n \cdot \|u_1 - u_0\| \cdot \frac{1 - r^{m+1}}{1 - r} \le r^n \cdot \|u_1 - u_0\| \cdot \frac{1}{1 - r}
\end{aligned}$$

We use of the explicit formula $(r^m + r^{m-1} + \cdots + 1) = \frac{1-r^{m+1}}{1-r}$ for the finite geometric series. (We use $0 \le r < 1$ in the last step.)

Uniqueness follows from the same functional, because if both u and v satisfy the problem, $u = G[u]$ and $v = G[v]$, then

$$\|u - v\| = \|G[u] - G[v]\| \le r\,\|u - v\| \qquad \text{with} \quad r < 1$$

This shows that $\|u - v\| = 0$, or $u = v$ as functions, since the maximum difference is zero. This is all there is to the proof conceptually, but the details that show the integral functional is defined on the successive iterates are quite cumbersome. The details follow, if you are interested.

The Details of the Proof

The proof uses two maxima. In a sense, the maximum M below gives existence of solutions and the maximum L gives uniqueness. This is where the hypothesis about the differentiability of $f[t, x]$ enters the proof.

In Exercise 11.1.2 you show that if $f[t, x]$ and $\frac{\partial f}{\partial x}[t, x]$ are defined and continuous on the compact rectangle $|x - x_0| \le b_x$ and $|t - t_0| \le b_t$, then the maxima M and L below exist and

$$\begin{aligned}
M &= \text{Max}[|f[t, x]| : |x - x_0| \le b_x \& |t - t_0| \le b_t] \\
&= \text{Max}[|g[t, u]| : |u| \le b_x \& |t| \le b_t]
\end{aligned}$$

and

$$\begin{aligned}
L &= \text{Max}[\left|\frac{\partial f}{\partial x}[t, x]\right| : |x - x_0| \le b_x \& |t - t_0| \le b_t] \\
&= \text{Max}[\left|\frac{\partial g}{\partial u}[t, u]\right| : |u| \le b_x \& |t| \le b_t]
\end{aligned}$$

An important detail in the proof is a prior estimate of the time that a solution could last. (We know from Problem 21.7 of the main text that solutions can "explode" in finite time.) This is transferred technically to the problem of making $G[v]$ defined when $v = G[u]$.

As long as $g[t,u[t]]$ is defined for $0 \le t \le \tau$, we know our functional $G[\cdot]$ satisfies

$$|G[u](\tau)| = \left|\int_0^\tau g[t,u[t]]\;dt\right| \le M\;\left|\int_{t_0}^\tau \;dt\right| = M\cdot\tau$$

when $|g[t,u]| \le M$. This is a little circular, because we need to have $g[t,u[t]]$ defined in order to use the estimate. What we really need is that $g[t,u_{n+1}[t]]$ is defined on $0 \le t \le \tau$ provided that $g[t,u_n[t]]$ is defined on $0 \le t \le \tau$ and $g[t,u_0[t]]$ is also defined on $0 \le t \le \tau$.

Exercise11.1.3 shows the following successive definitions. Let $u_0[t] = 0$ for all t. Then $g[t,u_0[t]] = f[t+t_0,x_0]$ is defined for all t with $0 \le t \le \tau$ as long as $\tau \le b_t$ (the constant given above.) Let Δ_1 be a positive number satisfying $\Delta_1 < b_t$ and $\Delta_1 < b_x/M$ (for the maximum M above and in Exercise 11.1.2.) Let $u_1[t] = G[u_0][t]$. Then $|u_1[t]| \le b_x$, so that $g[t,u_1[t]]$ is defined for $0 \le t \le \Delta_1$. Continuing this procedure, the whole sequence $u_{n+1} = G[u_n]$ is defined for $0 \le t \le \Delta_1$.

The maximum partial derivative L is needed for the contraction property. For each fixed t and any numbers u and v with $|u| \le b_x$ and $|v| \le b_x$, we can use (integration or) the Mean Value Theorem 7.1 (on the function $F[u] = g[t,u]$) to show that

$$|g[t,v] - g[t,u]| \le L\cdot|v-u|$$

This is called a "Lipschitz estimate" for the change in g. See Exercise 11.1.4.

Let Δ be a positive number with $\Delta < \Delta_1$ and $\Delta < 1/(2\,L)$. For any two continuous functions $u[t]$ and $v[t]$ defined on $[0,\Delta]$ with maximum less than b_x,

$$\begin{aligned}|G[u]-G[v]|(\tau) &= \left|\int_0^\tau g[t,u[t]]\;dt - \int_0^\tau g[t,v[t]]\;dt\right|\\ &\le \int_0^\tau |g[t,u[t]] - g[t,v[t]]|\;dt\\ &\le \int_0^\tau L\cdot \mathrm{Max}[|u[t]-v[t]| : 0 \le t \le \Delta]\;dt\\ &\le \int_0^\tau L\|u-v\|\;dt \le \|u-v\|\cdot L\cdot\Delta \le \frac{1}{2}\;\|u-v\|\end{aligned}$$

This shows that the iteration idea above will produce a solution defined for $0 \le t \le \Delta$ and completes the details of the proof.

Once we know that there is an exact solution, the idea for a proof of convergence of Euler's method given in Section 21.2 of the core text applies and shows that the Euler approximations converge to the true solution. (The functional $G[\cdot]$, called the Picard approximation, is usually not a practical approximation method.)

Exercise set 11.1

1. *Define the functional $G[u]$ as in the proof of Theorem* 11.1 *Show that a function $u[t]$ satisfies $u = G[u]$ if and only if $x[t+t_0] = u[t]+x_0$ satisfies $x[t_0] = x_0$ and $\frac{dx}{dt} = f[t,x[t]]$.*

2. *If* $f[t,x]$ *and* $\frac{\partial f}{\partial x}[t,x]$ *are defined and continuous on the compact rectangle* $|x-x_0| \le b_x$ *and* $|t-t_0| \le b_t$*, show that the maxima* M *and* L *below exist and*

$$\begin{aligned} M &= Max[|f[t,x]| : |x-x_0| \le b_x \& |t-t_0| \le b_t] \\ &= Max[|g[t,u]| : |u| \le b_x \& |t| \le b_t] \end{aligned}$$

and

$$\begin{aligned} L &= Max[\left|\frac{\partial f}{\partial x}[t,x]\right| : |x-x_0| \le b_x \& |t-t_0| \le b_t] \\ &= Max[\left|\frac{\partial g}{\partial u}[t,u]\right| : |u| \le b_x \& |t| \le b_t] \end{aligned}$$

3. *Let* $u_0[t]=0$ *for all* t*. Prove that* $g[t,u_0[t]]$ *is defined for all* t *with* $0 \le t \le \tau$ *as long as* $\tau \le b_t$ *(the constant given in the Exercise* 11.1.2 *above.)*
Let Δ_1 *be a positive number satisfying* $\Delta_1 < b_t$ *and* $\Delta_1 < b_x/M$ *(for the maximum* M *in Exercise* 11.1.2*.) Let* $u_1[t] = G[u_0][t]$ *and show that* $|u_1[t]| \le b_x$*, so that* $g[t,u_1[t]]$ *is defined for* $0 \le t \le \Delta_1$*.*
Continue this procedure and show that the whole sequence $u_{n+1} = G[u_n]$ *is defined for* $0 \le t \le \Delta_1$*.*
4. *Calculate the integral*

$$\int_u^v \frac{\partial g}{\partial x}[t,x]\ dx$$

(where t *is fixed.)*
If $h[x]$ *is any continuous function for* $|x| \le b_x$*, and* $Max[|h[x]| : |x| \le b_x] \le L$ *(in particular, if* $h[x] = g[t,x]$*) show that*

$$\left|\int_u^v h[x]\ dx\right| \le L \cdot \left|\int_u^v\ dx\right| = L \cdot |v-u|$$

provided $|u| \le b_x$ *and* $|v| \le b_x$*. Which property of the integral do you use?*
Combine the two previous parts to show that for any t *with* $|t| \le b_t$ *and any numbers* u *and* v *with* $|u| \le b_x$ *and* $|v| \le b_x$*,*

$$|g[t,v] - g[t,u]| \le L \cdot |v-u|$$

11.2 Local Linearization of Dynamical Systems

Now we consider a microscopic view of a nonlinear equilibrium point.

Theorem 11.2. *Microscopic Equilibria*

Let $f[x,y]$ and $g[x,y]$ be smooth functions with $f[x_e,y_e]=g[x_e,y_e]=0$. The flow of

$$\frac{dx}{dt}=f[x,y]$$
$$\frac{dy}{dt}=g[x,y]$$

under infinite magnification at (x_e,y_e) appears the same as the flow of its linearization

$$\begin{bmatrix}\frac{du}{dt}\\ \frac{dv}{dt}\end{bmatrix}=\begin{bmatrix}a_x & a_y\\ b_x & b_y\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix} \qquad \text{where} \qquad \begin{bmatrix}a_x & a_y\\ b_x & b_y\end{bmatrix}=\begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}[x_e,y_e]$$

Specifically, if our magnification is $1/\delta$, for $\delta\approx 0$, and our solution starts in our view,

$$(x[0]-x_e,y[0]-y_e)=\delta\cdot(a,b)$$

for finite a and b and if $(u[t],v[t])$ satisfies the linear equation and starts at $(u[0],v[0])=(a,b)$, then

$$(x[t]-x_e,y[t]-y_e)=\delta\cdot(u[t],v[t])+\delta\cdot(\varepsilon_x[t],\varepsilon_y[t])$$

where $(\varepsilon_x[t],\varepsilon_y[t])\approx(0,0)$ for all finite t.

Equivalently, for every screen resolution θ and every bound β on the time of observation and observed scale of initial condition, there is a magnification large enough so that if $|a|\le\beta$ and $|b|\le\beta$, then the error observed at that magnification is less than θ for $0\le t\le\beta$, and in particular, the solution lasts until time β.

Proof:

We give the proof in the 1-D case, where the pictures are not very interesting, but the ideas of the approximation are the same except for the technical difficulty of estimating vectors rather than numbers.

We define functions

$$z[t]=\frac{1}{\delta}\,(x[t]-x_e) \qquad \text{and} \qquad F[z]=\frac{1}{\delta}\,f[z\cdot\delta+x_e]$$

when $x[t]$ is a solution of the original equation. $z[t]$ is what we observe when we focus a microscope of magnification $1/\delta$ at the equilibrium point x_e, with $f[x_e]=0$ and watch a solution of the original equation. We want to compare $z[t]$ starting with $z[0]=a$ to the solution of $\frac{du}{dt}=b\,u$ where $b=f'[x_e]$ and $u[0]=a$ as well. Exercise 11.2.1 shows:

(a) $z[t]$ satisfies $\frac{dz}{dt}=F[z]$, when $x[t]$ satisfies $\frac{dx}{dt}=f[x]$.
(b) If z is a finite number, $F[z]\approx b\cdot z$.
(c) Let $\rho>0$ be any real number so that the following max is defined and let

$$L=\text{Max}[|f'[x]|:|x-x_e|\le\rho]+1$$

Then if z_1 and z_2 are finite,

$$|F[z_2]-F[z_1]|\le L\,|z_2-z_1|$$

Our first lemma in the microscope proof is:

Theorem 11.3. *Lemma on Existence of the Infinitesimal Solution*
The problem $\frac{dz}{dt} = F[z]$ *and* $z[0] = a$ *has a solution defined for all finite time.*

Proof of the Lemma: See Problem 11.1.

Let $u[t]$ satisfy $u[0] = a$ and $\frac{du}{dt} = b \cdot u$. (Ignore the fact that we know a formula in the 1-D case.) Define $\varepsilon[t]$ by

$$z[t] = u[t] + \varepsilon[t]$$

Now use Taylor's formula for $f[x]$ at x_e, and use the fact that $f[z[t]] = \frac{dz}{dt}$,

$$\begin{aligned} F[z] &= \frac{1}{\delta} f[x_e + \delta z] \\ &= f[x_e] + f'[x_e] \cdot (u[t] + \varepsilon[t]) + \int_0^1 (f'[x_e + s\,\delta z] - f'[x_e])\, z\ ds \\ \frac{dz}{dt} = \frac{du}{dt} + \frac{d\varepsilon}{dt} &= b \cdot (u[t] + \varepsilon[t]) + \int_0^1 (f'[x_e + s\,\delta z] - f'[x_e])\, z\ ds \end{aligned}$$

so

$$\frac{d\varepsilon}{dt} = b\,\varepsilon[t] + \eta[t]$$

with $\eta[t] \approx 0$ for all finite t.

This differential equation is equivalent to

$$\varepsilon[t] = \int_0^1 \eta[s]\ ds + b \int_0^1 \varepsilon[s]\ ds$$

so for any positive real a, no matter how small (but not infinitesimal) and any finite t,

$$|\varepsilon[t]| \le a + b \int_0^1 |\varepsilon[s]|\ ds$$

Which implies that

$$|\varepsilon[t]| \le a\, e^{b\,t}$$

and since a is arbitrarily small, $\varepsilon[t] \approx 0$ for all finite t.

To see this last implication, let

$$H[t] = a + b \int_0^1 |\varepsilon[s]|\ ds$$

so $|\varepsilon[t]| \le H[t]$. We know $H[0] = a$ and $H'[t] = b\,|\varepsilon[t]| \le b\,H[t]$ by the second half of the Fundamental Theorem of Integral Calculus and the previous estimate. Hence,

$$\begin{aligned} &\frac{H'[t]}{H[t]} \le b \\ &\int_0^s \frac{H'[t]}{H[t]}\ dt \le \int_0^s b\ dt \\ &\mathrm{Log}[\frac{H[s]}{a}] \le b\,s \\ &H[s] \le a\, e^{b\,t} \end{aligned}$$

This proves the infinitesimal microscope theorem for dynamical systems, but better than a proof, we offer some interesting experiments for you to try yourself in the exercises.

This is a finite time approximation result. In the limit as t tends to infinity, the nonlinear system can "look" different. Here is another way to say this. If we magnify a lot, but not by an infinite amount, then we may see a separation between the linear and nonlinear system after a very very long time. As a matter of fact, a solution beginning a small finite distance away from the equilibrium can 'escape to infinity' in large finite time.

Exercise set 11.2

1. *Show that* $z[t]$ *satisfies* $\frac{dz}{dt} = F[z]$, *when* $x[t]$ *satisfies* $\frac{dx}{dt} = f[x]$.
Show that if z *is a finite number,* $F[z] \approx b \cdot z$.
Let $\rho > 0$ *be any real number so that the following max is defined and let*

$$L = Max[|f'[x]| : |x - x_e| \le \rho] + 1$$

Show that when z_1 *and* z_2 *are finite,*

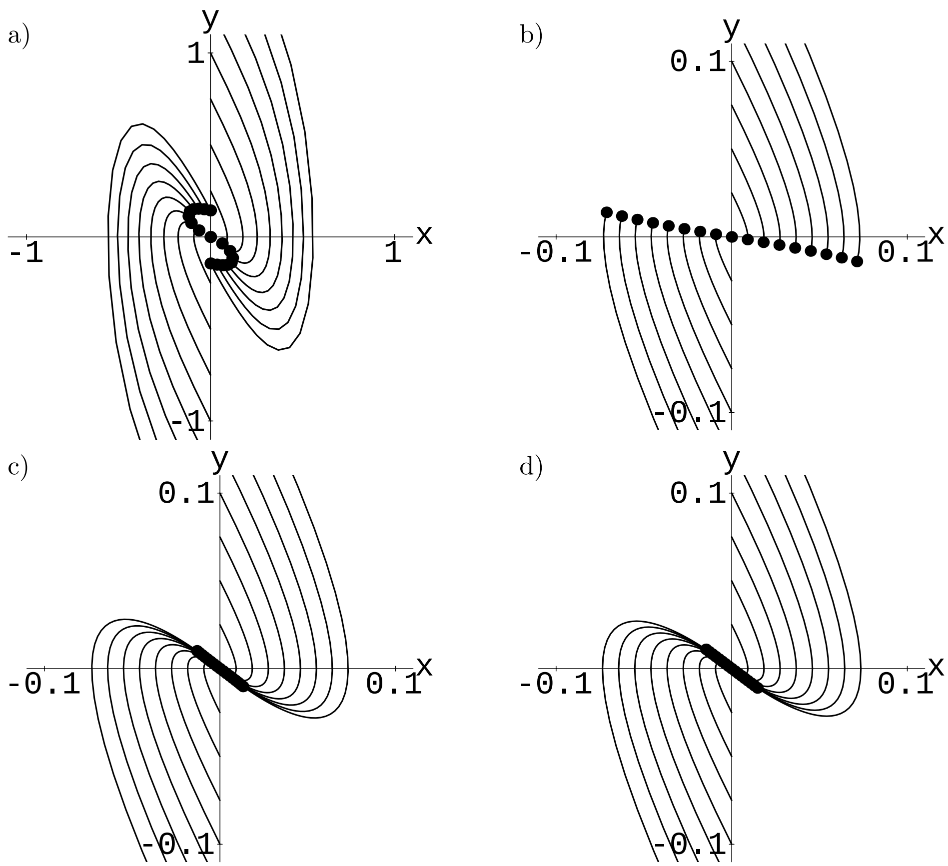
$$|F[z_2] - F[z_1]| \le L\,|z_2 - z_1|$$

The following experiments go beyond the main text examples of magnifying a flow near an equilibrium where the linearized problems are non-degenerate.

2. *Use the phase variable trick to write the differential equation* $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x + 15\,x^3 = 0$ *as a two dimensional first order system with* $f[x,y] = y$ *and* $g[x,y] = -x - 2\,y - 15\,x^3$.
Prove that the only equilibrium point is at $(x_e, y_e) = (0,0)$.
Prove that the linearization of the system is the system

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

and that it has only the single characteristic value $r = -1$.
Use the ***Flow2D.ma*** *NoteBook to solve the linear and nonlinear systems at various scales. A few experiments are shown next. Notice the different shape of the nonlinear system at large scale and that the difference gradually vanishes as the magnification increases. The first three figures are nonlinear, and the fourth is linear at same scale as the small nonlinear case.*

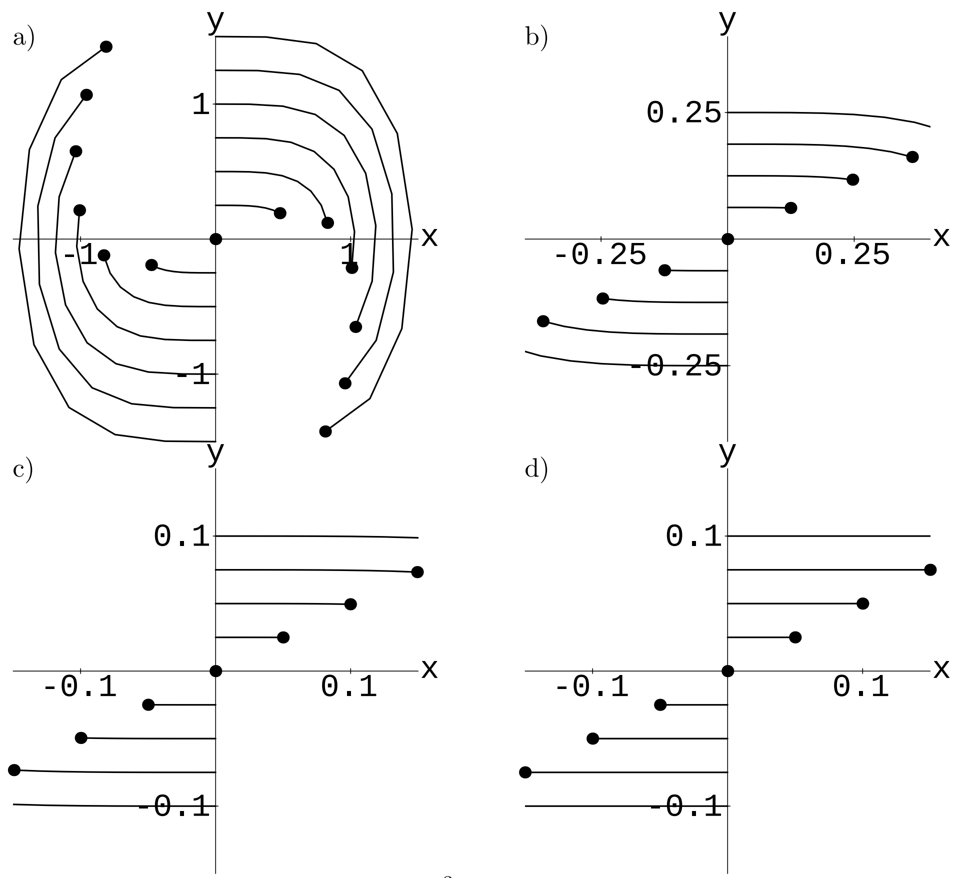


$x'' + 2x' + x + 15\,x^3 = 0$ vs. $x'' + 2x' + x = 0$

3. *Use the phase variable trick to write the differential equation* $\frac{d^2x}{dt^2} + x^3 = 0$ *as a two dimensional first order system with* $f[x,y] = y$ *and* $g[x,y] = -x^3$.
Prove that the only equilibrium point is at $(x_e, y_e) = (0,0)$.
Prove that the linearization of the system is the system

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

and that it has only the single characteristic value $r = 0$.
Use the ***Flow2D.ma*** *NoteBook to solve the linear and nonlinear systems at various scales. A few experiments are shown next. The first three figures are nonlinear, and the fourth is linear at same scale as the small nonlinear case. What is the analytical solution of the linear system?*



$x'' + x^3 = 0$ vs. $x'' = 0$

Problem 11.1. Existence of the Infinitesimal Solution

Prove that the problem $\frac{dz}{dt} = F[z]$ *and* $z[0] = a$ *has a solution defined for all finite time.*

HINTS:

Apply the idea in the proof of existence-uniqueness, Theorem 11.1 above. Define,

$$z_0[t] = a$$
$$z_{n+1}[t] = a + \int_0^t F[z_n(s)]\ ds$$

We have $|z_1[t] - z_0[t]| \le \int_0^t F[a]\ ds| \le t\,|F[a]| \approx b \cdot a \cdot t$ *is finite. Next,*

$$\begin{aligned}|z_2[t] - z_1[t]| &\le \int_0^t |F[z_2(s)] - F[z_1(s)]|\ ds \\ &\le \int_0^t L\,|z_1(s) - a|\ ds \\ &\le L\,\frac{t^2}{2}\end{aligned}$$

$$\begin{aligned}|z_2[t] - a| &\le |z_2[t] - z_1[t]| + |z_1[t] - a| \\ &\le |F[a]| \cdot |L\,t + \frac{1}{2}(L\,t)^2| \le |F[a]| \cdot |1 + L\,t + \frac{1}{2}(L\,t)^2|\end{aligned}$$

Continue by induction to show that

$$|z_{n+1}[t] - a| \le |F[a]|\,|1 + L\,t + \frac{1}{2}(L\,t)^2 + \cdots + \frac{1}{n!}(L\,t)^n| \le |F[a]|\,e^{L\,t}$$

This shows that $z_n[t]$ *is finite when* t *is. We can also show that* $z_n[t] \to z[t]$*, the solution to the initial value problem, as we did in the existence-uniqueness theorem.*

▲

11.3 Attraction and Repulsion

This section studies the cases where solutions stay in the microscope for infinite time.

The local stability of an equilibrium point for a dynamical system is formulated as the next result. Notice that stability is an "infinite time" result, whereas the localization of the previous theorem is a finite time result after magnification.

Theorem 11.4. *Local Stability*

Let $f[x,y]$ and $g[x,y]$ be smooth functions with $f[x_e,y_e] = g[x_e,y_e] = 0$. The coefficients given by the partial derivatives evaluated at the equilibrium

$$\begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} [x_e, y_e]$$

define the characteristic equation of the equilibrium,

$$\det \begin{vmatrix} a_x - r & a_y \\ b_x & b_y - r \end{vmatrix} = (a_x - r)(b_y - r) - a_y\, b_x = r^2 - (a_x + b_y)\, r + (a_x\, b_y - a_y\, b_x) = 0$$

Suppose that the real parts of both of the roots of this equation are negative. Then there is a real neighborhood of (x_e, y_e) or a non-infinitesimal $\varepsilon > 0$ such that when a solution satisfies

$$\frac{dx}{dt} = f[x,y]$$
$$\frac{dy}{dt} = g[x,y]$$

with initial condition in the neighborhood, $|x[0] - x_e| < \varepsilon$ and $|y[0] - y_e| < \varepsilon$, then

$$\lim_{t\to\infty} x[t] = x_e \qquad \text{and} \qquad \lim_{t\to\infty} y[t] = y_e$$

Proof

One way to prove this theorem is to 'keep on magnifying.' If we begin with any solution inside a square with an infinitesimal side $2\ \varepsilon$, then the previous magnification result says that the solution appears to be in a square of half the original side in the time that it takes the linearization to do this. It might be complicated to compute the maximum time for a linear solution starting on the square, but we could do so based on the characteristic roots in the linear solution terms of the form $e^{-r\ t}$. It is a fixed finite time τ. We could then start up again at the half size position and watch again for time τ. After each time interval of length τ, we would arrive nearly in a square of half the previous side.

If we want to formulate this with only reference to real quantities, we need to remove the fact that the true solution is only infinitely near the linear one on the scale of the magnification. Since it appears to be in a square of one half the side on that scale, the true solution must be inside a square of $2/3$ the side within time τ. Since this holds for every infinitesimal radius, the Function Extension Axiom guarantees that it also holds for some positive real ε. After time $n \times \tau$, the true solution is inside a square of side $\left(\frac{2}{3}\right)^n$ times the original length of the side. $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$, so our theorem is proved.

Continuous dynamical systems have a local repeller theorem, unlike discrete dynamical systems. Discrete solutions can "jump" inside the microscope, but continuous solutions 'move continuously.' You could formulate a local repeller theorem by 'zooming out' with the microscopic theorem above. How would a ring of initial conditions move when viewed inside a microscope if the characteristic values had only positive real parts?

11.4 Stable Limit Cycles

Solutions of a dynamical system do not necessarily tend to an attracting point or to infinity.

There are nonlinear oscillators which have stable oscillations in the sense that every solution (except zero) tends to the same oscillatory solution. One of the most famous examples is the Van der Pol equation:

$$\frac{d^2x}{dt^2} + a(x^2 - 1)\frac{dx}{dt} + x = 0$$

Your experiments in the first exercise below will reveal a certain sensitivity of this form of the Van der Pol equation. A more stable equivalent system may be obtained by a different change of variables, $w = -\int x\ dt$,

$$\begin{aligned}\frac{dw}{dt} &= -x \\ \frac{dx}{dt} &= w - a(\frac{x^3}{3} - x)\end{aligned}$$

Since

$$\begin{aligned}&\frac{d^2x}{dt^2} + a\frac{dx}{dt}(x^2 - 1) = \frac{d}{dt}(\frac{dx}{dt} + a(\frac{x^3}{3} - x)) = -x \\ &\int \frac{d}{dt}(\frac{dx}{dt} + a(\frac{x^3}{3} - x)\ dt = -\int x\ dt = w \\ &\frac{dx}{dt} + a(\frac{x^3}{3} - x) = w \\ &\frac{dx}{dt} = -a(\frac{x^3}{3} - x) + w\end{aligned}$$

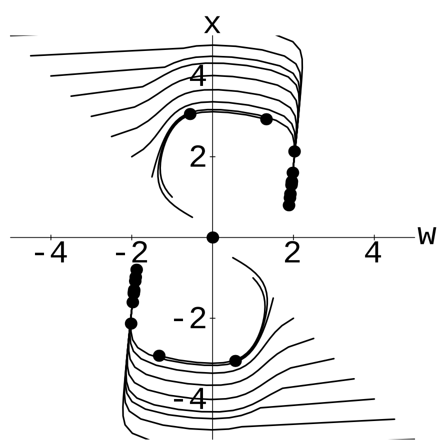


Figure 11.1: Van der Pol Flow

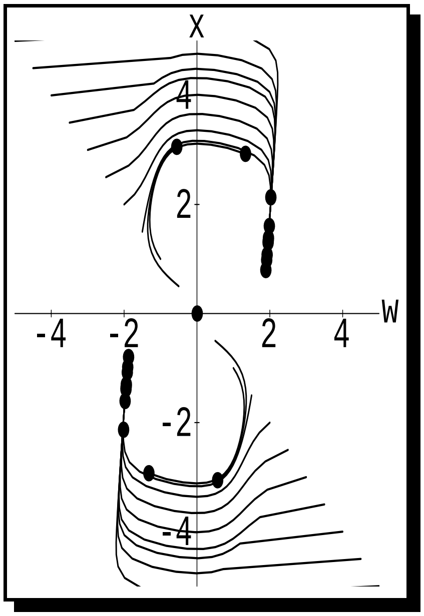
Exercise set 11.4

1. Van der Pol 1
Show that the following is an equivalent system to the first form of the Van der Pol equation via the phase variable trick, $y = \frac{dx}{dt}$*,*

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x - ay(x^2 - 1)$$

*Use **Flow2D.ma** to make a flow for this system and observe that every solution except zero tends to the same oscillation.*

2. Van der Pol 2
*Use **Flow2D.ma** to create an animation of the second version of the Van der Pol dynamics.*



Part 7

Infinite Series

CHAPTER

12

The Theory of Power Series

This chapter fills in a few details of the theory of series not covered in the main text Chapters 24 and 25.

What do we mean by an infinite sum

$$u_1[x] + u_2[x] + u_3[x] + \cdots$$

The traditional notation is

$$\sum_{k=1}^{\infty} u_k[x] = \lim_{m \to \infty} \sum_{k=1}^{m} u_k[x]$$

where the limit may be defined at least two different ways using real numbers. One way to define the limit allows the rate of convergence to be different at different values of x and the other has the whole graph of the partial sum functions approximate the limiting graph. The weaker kind of limit makes the "sum" $\sum_{k=1}^{\infty}$ have fewer "calculus" properties. The stronger limit makes the "infinite sum" behave more like an ordinary sum. The traditional notation $\sum_{k=1}^{\infty}$ makes no distinction between uniform and non-uniform convergence. Perhaps this is unfortunate since the equation

$$\int_a^b \sum_{k=1}^{\infty} u_k[x]\ dx = \sum_{k=1}^{\infty} \int_a^b u_k[x]\ dx$$

is true for uniform convergence (when the $u_k[x]$ are continuous) and may be false for pointwise convergence (even when the $u_k[x]$ are continuous).

The love knot symbol "∞" is NOT a hyperreal number, and is not an integer, because it does not satisfy the formal properties of arithmetic (as the Axioms in Chapter 1 require of hyperreals. For example, $\infty + \infty \neq 2\infty$.) Hyperreal integers will retain the arithmetic properties of an ordered field, so we always have

$$\int_a^b \sum_{k=1}^{n} u_k[x]\ dx = \sum_{k=1}^{n} \int_a^b u_k[x]\ dx$$

when n is a hyperreal integer, even an infinite one. This seems a little paradoxical, since we expect to have

$$\sum_{k=1}^{n} u_k[x] \approx \sum_{k=1}^{\infty} u_k[x]$$

It turns out that we can understand uniform and non-uniform convergence quite easily from this approximation. The secret lies in examining the approximation when x is hyperreal. If the convergence is non-uniform, then even when n is infinite the hyperreal sum will NOT be near the limit for some non-ordinary x's. In other words, non-uniform convergence is "infinitely slow" at hyperreal x's. (You might even wish to say the series is not converging at these hyperreal values.)

What do we mean by an infinite sum, $\sum_{k=1}^{n} u_k[x]$, when n is an infinite hyperreal integer? What do we even mean by an infinite integer n? On the ordinary real numbers we can define the indicator function of the integers

$$I[x] = \begin{cases} 1, & \text{if } x \text{is an integer} \\ 0, & \text{if } x \text{is not an integer} \end{cases}$$

The equation $I[m] = 1$ says "m is an integer". The formal statement

$$\{I[m] = 1, a \le x \le b\} \Rightarrow s[m, x] \text{ is defined}$$

is true when m and x are real. The function $I[x]$ has a natural extension and we take the equation $I[n] = 1$ to be the meaning of the statement "n is a hyperinteger." The natural extensions of these functions satisfy the same implication, so when n is an infinite hyperreal and $I[n] = 1$, we understand the "hyperreal infinite sum" to be the natural extension

$$\sum_{k=1}^{n} u_k[x] = s[n, x]$$

Next, we will show that hyperintegers are either ordinary integers or infinite as you might expect from a sketch of the hyperreal line. Every real number r is within a half unit of an integer. For example, we can define the nearest integer function

$$N[r] = n, \quad \text{the integer } n \text{ such that } |r - n| < \frac{1}{2} \text{ or } n = r - \frac{1}{2}$$

and then every real r satisfies

$$|r - N[r]| \le \frac{1}{2} \text{ and } I[N[r]] = 1$$

(As a formal logical statement we can write this $\{x = x\} \Rightarrow \{|x - N[x]| \le \frac{1}{2}, I[N[x]] = 1\}$.)

If $m = 1, 2, 3, \cdots$ is an ordinary natural number and $|x| \le m$ in the real numbers, then we know $N[x] = -m$ or $N[x] = -m + 1$ or $N[x] = -m + 2$ or $\cdots$ or $N[x] = m - 1$ or $N[x] = m$. By the Function Extension Axiom, if x is a finite hyperreal (so $|x| \le m$ for some m), then $N[x]$ must equal an ordinary integer in the finite list from $-m$ to m described in these equations.

If x is an infinite hyperreal, then $N[x]$ is still a "hyperinteger" in the sense $I[N[x]] = 1$. Since $|x - N[x]| \le 1/2$, $N[x]$ is infinite, yet $s[n, x] = \sum_{k=1}^{n} u_k[x]$ is defined.

Similarly, we can show that hyperreal infinite sums given by natural extension of sum function satisfy formal properties like

$$\sum_{k=1}^{m} u_k[x] + \sum_{k=m+1}^{n} u_k[x] = \sum_{k=1}^{n} u_k[x]$$

12.1 Uniformly Convergent Series

A series of continuous functions

$$u_1[x] + u_2[x] + u_3[x] + \cdots$$

can converge to a discontinuous limit function as in Example 13.18 *below. However, this can only happen when the rate of convergence of the series varies with x. A series whose convergence does not depend on x is said to converge "uniformly."*

Following is the real tolerance ("epsilon - delta") version of uniform convergence. We state the definition for a general sequence of functions, $f_m[x] = s[m, x]$, defined for all positive integers m and x in some interval.

Definition 12.1. *Uniformly Convergent Sequence*
A sequence of functions $s[m, x]$ all defined on an interval I is said to converge to the function $S[x]$ uniformly for x in I if for every small positive real tolerance, θ, there exists a sufficiently large real index N, so that all real functions beyond N are θ-close to $S[x]$, specifically, if $m > N$ and x is in I, then

$$|S[x] - s[m, x]| < \theta$$

The needed N for a given accuracy θ does not depend on x in A, N is "uniform in x" for θ.

If we have a series of functions

$$u_1[x] + u_2[x] + u_3[x] + \cdots$$

let the partial sum sequence be denoted

$$s[m, x] = \sum_{k=1}^{m} u_k[x]$$

Definition 12.2. *Uniformly Convergent Series*
A series of functions all defined on an interval I

$$u_1[x] + u_2[x] + u_3[x] + \cdots$$

is said to converge to the function $S[x] = \sum_{k=1}^{\infty} u_k[x]$ uniformly for x in I if the sequence of partial sums $s[m, x]$ converges to $S[x]$ uniformly on I.

The equivalent definition in terms of infinitesimals is given in Theorem 12.3. By "extended interval" we mean that the same defining inequalities hold. For example, if $I = [a, b]$ then a hyperreal x is in the extended interval if $a \le x \le b$. In the case $I = (a, \infty)$, a hyperreal x is in the extended interval if $a < x$.

Theorem 12.3. *A sequence of functions $s[m,x]$ converges uniformly on the interval to the real function $S[x]$ if and only if for every infinite n and every hyperreal x in the extended interval, the natural extension functions satisfy*

$$S[x] \approx s[n,x]$$

Proof:

The proof is similar to that of Theorem 3.4. Given the real tolerance condition, there is a function $N[\theta]$ such that the following implication holds in the real numbers.

$$\{a \le x \le b, \theta > 0, m > N[\theta]\} \Rightarrow |S[x] - s[m,x]| < \theta$$

By the Function Extension Axiom 2.1, this also holds for the hyperreals.

Fix any infinite integer n. Let $\theta > 0$ be an arbitrary positive real number. We will show that $|S[x] - s[n,x]| < \theta$. Since $N[\theta]$ is real and n is infinite, $n > N[\theta]$. Thus the extended implication above applies with this θ and $m = n$, so for any $a \le x \le b$ we have $|S[x] - s[n,x]| < \theta$. Since θ is an arbitrary real, $S[x] \approx s[n,x]$.

Conversely, suppose the real tolerance condition fails. Then there are real functions $M[\theta,N]$ and $X[\theta,N]$ so that the following holds in the reals and hyperreals.

$$\{\theta > 0, N > 0\} \Rightarrow \{a \le X[\theta,N] \le b, M[\theta,N] > N, |S[X[\theta,N]] - s[M[\theta,N],X[\theta,N]]| \ge \theta\}$$

Applying this to an infinite hyperinteger N, we see that $|S[X[\theta,N]] - s[M[\theta,N],X[\theta,N]]| \ge \theta$ and the infinitesimal condition also fails, that is, if $x = X[\theta,N]$ and $n = M[\theta,N]$, then $n > N$ is infinite and we do NOT have $S[x] \approx s[n,x]$. This proves the contrapositive of the statement and completes the proof.

Example 12.1. *Uniformly Convergent Series*

A series of functions $u_1[x] + u_2[x] + u_3[x] + \cdots$ all defined on an interval I with partial sums $s[m,x] = \sum_{k=1}^{m} u_k[x]$ converges uniformly to $S[x] = \sum_{k=1}^{\infty} u_k[x]$ on I if and only if, for every infinite hyperinteger n

$$\sum_{k=1}^{n} u_k[x] \approx \sum_{k=1}^{\infty} u_k[x]$$

for all hyperreal x, $a \le x \le b$. In other words if the whole graph of the any infinite sum is infinitely near the graph of the real limiting function.

If a series is NOT converging uniformly, then there is an x where even an infinite number of terms of the series fails to approximate the limit function. This can happen even though the point x where the approximation fails is not real. We can see this in Example 13.18 and Example 12.2.

We think it is intuitively easier to understand non-uniformity of convergence as "infinitely slow convergence" at hyperreal values.

Example 12.2. *x^n Convergence*

The sequence of functions

$$s[m,x] = x^m$$

converges to the function

$$S[x] = \begin{cases} 1, & x = 1 \\ 0, & -1 < x < 1 \end{cases}$$

In particular, $x^m \to 0$ for $-1 < x < 1$. However, this convergence is not uniform on all of the interval $-1 < x < 1$.

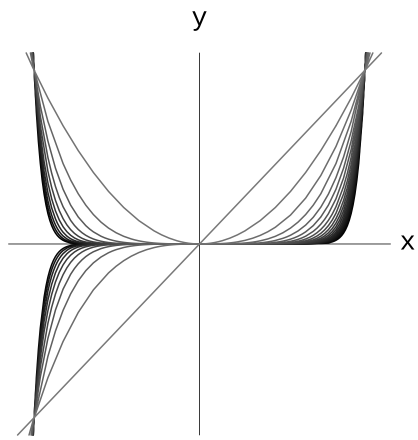


Figure 12.1: Limit of x^n

You can see from the graphs above that each function x^m climbs from near zero to $x^m = 1$ when $x = 1$. In particular, for every real m there is a ξ_m, $0 < \xi_m < 1$, such that $\xi_m^m = \frac{1}{2}$ – the graph $y = x^m$ crosses the line $y = 1/2$. The Function Extension Axiom 2.1 says that this holds with $m = n$ an infinite integer. (Extend the real function $\xi_m = \sqrt[m]{1/2} = f[m, x]$.) The infinite function x^n is 1/2 unit away from the limit function $S[x] = 0$ when $x = \xi_n$. Of course, $1 \approx \xi_n < 1$.

If we fix any $\xi \approx 1$, sufficiently large n will make $\xi^n \approx 1$, but *some* infinite powers will not be infinitesimal. We could say ξ^m converges to zero "infinitely slowly."

Example 12.3. *x^n Convergence is Uniform for $-r \leq x \leq r$ if $0 < r < 1$*

When $r < 1$ is a fixed real number, we know that $r^m \to 0$ and if $|x| \leq r$ then $|x^m| \leq r^m$, so the convergence of x^m to zero is uniform on $[-r, r]$.

12.2 Robinson's Sequential Lemma

Functionally defined sequences that are infinitesimal for finite indices continue to be infinitesimal for sufficiently small infinite indices.

Theorem 12.4. *Robinson's Sequential Lemma*

Suppose a hyperreal sequence ε_k is given by a real function of several variables specified at fixed hyperreal values, $\xi_1, \cdots, \xi_i$,

$$\varepsilon_k = f[k, \xi_1, \xi_2, \cdots, \xi_i] = f[k, \Xi], \quad k = 1, 2, 3, \cdots$$

If $\varepsilon_m \approx 0$ for finite m, then there is an infinite n so that $\varepsilon_k \approx 0$ for all $k \le n$.

Proof:

Let $X = (x_1, x_2, \cdots, x_i)$ be a real multivariable and suppose $f[k, X]$ is defined for all natural numbers. Assume $f[n, X]$ is non-negative, if necessary replacing it with $|f[n, X]|$. (By hypothesis, $\varepsilon_k = f[k, \Xi]$ is defined at $\Xi = (\xi_1, \cdots, \xi_i)$ for all hyperintegers k, $I[k] = 1$.) Define a real function

$$\mu[X] = \begin{cases} 1/\mathrm{Max}[n : m \le n \Rightarrow f[m, X] < 1/m] \text{ when } f[h, X] \ge 1/h \text{ for some } h \\ 0, \text{ if } f[m, X] < 1/m \text{ for all } m \end{cases}$$

In the reals we have either $\mu[X] = 0$ and $f[k, X] < 1/k$ for all k, or $\mu[X] > 0$ and

$$k \le 1/\mu[X] \Rightarrow f[k, X] < 1/k$$

Now, consider the value of $\mu[\Xi]$. Since $f[m, \Xi] \approx 0$ for all finite m, $f[m, \Xi] < 1/m$ for all finite m. Thus $\mu[\Xi] \approx 0$ and either $\mu[\Xi] = 0$ so all infinite n satisfy $f[n, \Xi] < 1/n$ or $1/\mu[\Xi] = n$ is an infinite hyperinteger and all $k \le n$ satisfy $f[k, \Xi] < 1/k$. When k is finite we already know $f[k, \Xi] \approx 0$ and when k is infinite, $1/k \approx 0$, so the Lemma is proved.

12.3 Integration of Series

We can interchange integration and infinite series summation of unifromly convergent series of continuous functions.

Theorem 12.5. *Integration of Series*

Suppose that the series of continuous functions $u_o[x] + u_1[x] + \cdots$ converges uniformly on the interval $[a, b]$ to a "sum"

$$S[x] = \lim_{n \to \infty} u_0[x] + \cdots + u_n[x]$$

Then the limit $S[x]$ is continuous and

$$\int_a^b \lim_{n \to \infty} u_0[x] + \cdots + u_n[x] \; dx = \lim_{n \to \infty} \int_a^b u_0[x] + \cdots + u_n[x] \; dx$$

Short notation for this result would simply be that

$$\int_a^b \sum_{k=0}^{\infty} u_k[x] \; dx = \sum_{k=0}^{\infty} \int_a^b u_k[x] \; dx$$

provided the series is uniformly convergent and the terms are continuous.

Proof:

Continuity means that if $x_1 \approx x_2$, then $S[x_1] \approx S[x_2]$. We need this in order to integrate $S[x]$. By continuity of the functions $u_k[x]$, if $x_1 \approx x_2$ and m is a finite integer, $\sum_{k=0}^{m} u_k[x_1] \approx \sum_{k=0}^{m} u_k[x_2]$. Robinson's Lemma 12.4 with the function $f[m, x_1, x_2] = \sum_{k=0}^{m}(u_k[x_1] - u_k[x_2])$ shows that there is an infinite n so that we still have $\sum_{k=0}^{n} u_k[x_1] \approx \sum_{k=0}^{n} u_k[x_2]$. Uniform convergence gives us the conclusion

$$\sum_{k=0}^{\infty} u_k[x_1] \approx \sum_{k=0}^{n} u_k[x_1] \approx \sum_{k=0}^{n} u_k[x_2] \approx \sum_{k=0}^{\infty} u_k[x_2]$$

The integral part is easy. Let $\theta > 0$ be any real number and n sufficiently large so that $|S[x] - (u_0[x] + \cdots + u_n[x])| < \theta/(2(b-a))$

$$\begin{aligned}\left|\int_a^b S[x]\ dx - \left(\int_a^b u_0[x]\ dx + \cdots + \int_a^b u_n[x]\ dx\right)\right| &= \\ \left|\int_a^b S[x] - (u_0[x] + \cdots + u_n[x])\ dx\right| &= \\ \leq \int_a^b |S[x] - (u_0[x] + \cdots + u_n[x])|\ dx & \\ \leq \int_a^b \frac{\theta}{2(b-a)}\ dx = (b-a)\frac{\theta}{2(b-a)} &< \theta\end{aligned}$$

This shows that the series of numbers $\int_a^b u_0[x]\ dx + \int_a^b u_1[x]\ dx + \int_a^b u_2[x]\ dx + \cdots$ converges to the number $\int_a^b S[x]\ dx$.

Exercise set 12.3

1. *When a series converges uniformly and* n *is infinite, we know* $\sum_{k=0}^{\infty} u_k[x] \approx \sum_{k=0}^{n} u_k[x]$. *Show that*

$$\int_a^b \sum_{k=0}^{\infty} u_k[x]\ dx \approx \int_a^b \sum_{k=0}^{n} u_k[x]\ dx$$

and

$$\int_a^b \left(\sum_{k=0}^{\infty} u_k[x] - \sum_{k=0}^{n} u_k[x]\right) dx \approx 0$$

What is the meaning of the equation

$$\int_a^b \sum_{k=0}^{n} u_k[x]\ dx = \sum_{k=0}^{n} \int_a^b u_k[x]\ dx$$

and why is it true?

Prove that

$$\int_a^b \sum_{k=0}^{\infty} u_k[x]\ dx = \sum_{k=0}^{\infty} \int_a^b u_k[x]\ dx$$

2. *Let* $f[m,x] = (m+1)\,2\,x\,\left(1-x^2\right)^m$

(a) *Show that*

$$\int_0^1 f[m,x]\;dx = 1 \quad \textit{for each } m$$

(b) *Show that for a fixed real value of* x

$$\lim_{m\to\infty} f[m,x] = 0$$

(c) *Show that*

$$\int_0^1 \left(\lim_{m\to\infty} f[m,x]\right)\,dx = 0$$

(d) *Show that*

$$\lim_{m\to\infty}\left(\int_0^1 f[m,x]\;dx\right) = 1$$

(e) *Explain why* $f[m,x]$ *does NOT tend to zero uniformly. In fact, for a given infinite* $m = n$ *show that there is an* $x = \xi \approx 0$ *where* $f[n,\xi]$ *is infinitely far from zero.*

(HINT: See Section 26.3 of the main text.)

12.4 Radius of Convergence

A power series $a_0 + a_1\,x + a_2\,x^2 + \cdots + a_n\,x^n + \cdots$ *does one of the following:*

(a) *Converges for all* x *and converges uniformly and absolutely for* $|x| \le \rho$ *for any constant* $\rho < \infty$*. In this case we say the series has radius of convergence* ∞*.*

(b) *There is a number* r *so that the series converges for all* x *in the open interval* $(-r, r)$*, uniformly and absolutely for* $|x| \le \rho < r$ *for any constant* ρ*, and diverges for all* x *with* $|x| > r$*. In this case we say the series has radius of convergence* r*. Such a series may converge or diverge at either* $x = r$ *or* $x = -r$*.*

(c) *The series does not converge for any nonzero* x*. In this case we say the series has radius of convergence* 0*.*

A general fact about power series is that if we can find a point of convergence, even conditional convergence, then we can use geometric comparison to prove convergence at smaller values. See Theorem 27.4 of the main text where the following is discussed in more detail - but where there is a typo in the proof, (:-().

Theorem 12.6. *If the power series*

$$a_0 + a_1\, x + a_2\, x^2 + \cdots + a_n\, x^n + \cdots$$

converges for a particular $x = x_1$, then the series converges uniformly and absolutely for $|x| \le \rho < |x_1|$, for any constant ρ.

PROOF:

Because the series converges at x_1, we must have $a_n\, x_1^n \to 0$. If $|x| \le \rho < |x_1|$, then

$$|a_n\, x^n| = |a_n\, x_1^n| \left|\frac{x}{x_1}\right|^n \le \left|\frac{\rho}{x_1}\right|^n = r^n$$

is a geometric majorant for the tail of the series. That is, eventually $|a_n x^n| \le r^n$ and $\sum_{n=m}^{\infty} r^n$ converges. This proves the theorem.

Example 12.4. *The Radius of Convergence*

Now consider the cases described in the section summary at the beginning of this section. If the series converges for all x we simply say the radius of convergence is ∞ and apply the theorem to see that convergence is uniform on any compact interval.

If the series diverges for all nonzero x there is nothing to show. We simply say the radius of convergence is zero.

If the series converges for some values of x and diverges for others, we need to show that it converges in $(-r, r)$, and diverges for $|x| > r$. Theorem 12.6 shows that if the series converges for x_1, then it converges for all real x satisfying $|x| < |x_1|$.

Consider the sets numbers

$$L = \{s : s < 0 \text{ or the series converges when } x = s\}$$
$$R = \{t : t > 0 \text{ and the series diverges when } x = t\}$$

The pair (L, R) is a Dedekind cut on the real numbers (see Definition 1.4.) First, both L and R are nonempty since there are positive values where the series converges and where it diverges. Second, if $s \in L$ and $t \in R$, then $s < t$ by Theorem 12.6. Let r be the real number at the gap of this cut. Then whenever $|x| < r$, $|x| \in L$ and the series converges, while when $r < |x|$, $|x| \in R$ and the series diverges at the positive $|x|$. It cannot converge at $-|x|$ because Theorem 12.6 would make the series converge at $(|x| + r)/2 > r$. Thus the series converges for $|x| < r$ and diverges for $|x| > r$.

Exercise set 12.4

1. (a) *Find a power series with finite radius of convergence r that converges when $x = r$, but diverges when $x = -r$.*

(b) *Find a power series with finite radius of convergence r that diverges when $x = r$ and diverges when $x = -r$.*

(c) *Find a power series with radius of convergence ∞.*

(d) *Find a power series with radius of convergence 0.*

(HINT: Try $\text{Log}[1 + x]$, $1/(1 - x)$, *make substitutions,* $\cdots$*.)*

12.5 Calculus of Power Series

We can differentiate and integrate power series inside their radius of convergence (defined in the preceding section).

Theorem 12.7. *Differentiation and Integration of Power Series*
Suppose that a power series

$$a_0 + a_1\, x + a_2\, x^2 + \cdots + a_n\, x^n + \cdots$$

converges to $S[x]$ *for* $|x| < r$*, its radius of convergence,*

$$S[x] = a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + \cdots = \sum_{k=0}^{\infty} a_k\ x^k$$

Then, the derivative of $S[x]$ *exists and the series obtained from term by term differentiation has the same radius of convergence and converges uniformly absolutely to it on* $|x| \le \rho < r$*,*

$$\frac{dS[x]}{dx} = a_1 + 2\, a_2\, x + 3\, a_3\, x^2 + \cdots + n\, a_n\, x^{n-1} + \cdots = \sum_{k=1}^{\infty} k\, a_k\ x^{k-1}$$

The integral of $S[x]$ *exists and the series obtained from term by term integration has the same radius of convergence and converges uniformly absolutely to it on* $|x| \le \rho < r$*,*

$$\int_0^x S[\xi]\ d\xi = a_0\ x + \frac{a_1}{2}\, x^2 + \frac{a_2}{3}\, x^3 + \cdots + \frac{a_n}{n+1}\ x^{n+1} + \cdots = \sum_{k=0}^{\infty} \frac{a_k}{k+1}\ x^{k+1}$$

PROOF:

First, we show that the series $\sum_{k=1}^{\infty} k\, a_k\ x^{k-1}$ and $\sum_{k=0}^{\infty} \frac{a_k}{k+1}\ x^{k+1}$ have the same radius of convergence as $\sum_{k=0}^{\infty} a_k\ x^k$.

For any x and ρ,

$$\left|k\, a_k\ x^{k-1}\right| = k \left|\frac{x}{\rho}\right|^k \cdot \left|\frac{1}{x}\right| \left|a_k\ \rho^k\right| \quad \text{and} \quad \left|\frac{a_k}{k+1}\ x^{k+1}\right| = \left|\frac{x}{k+1}\right| \left|\frac{x}{\rho}\right|^k \cdot \left|a_k\ \rho^k\right|$$

When we fix $|x| < \rho < r$

$$\left|k\, a_k\ x^{k-1}\right| < \left|a_k\ \rho^k\right| \quad \text{and} \quad \left|\frac{a_k}{k+1}\ x^{k+1}\right| < \left|a_k\ \rho^k\right|$$

for sufficiently large k because then $|x/\rho| < 1$ so

$$k\left|\frac{x}{\rho}\right|^k \cdot \left|\frac{1}{x}\right| \to 0 \quad \text{and} \quad \left|\frac{x}{k+1}\right|\left|\frac{x}{\rho}\right|^k \to 0 \qquad \text{as } k \to \infty$$

See Exercise 12.5.1. In this case since the series $\sum_{k=0}^{\infty} |a_k|\ \rho^k$ converges, the term-by-term derivative and integral series also converge. (See Exercise 12.5.5.)

When $|x| > \rho > r$

$$\left|k\, a_k\ x^{k-1}\right| > \left|a_k\ \rho^k\right| \quad \text{and} \quad \left|\frac{a_k}{k+1}\ x^{k+1}\right| > \left|a_k\ \rho^k\right|$$

for sufficiently large k because then $|x/\rho| > 1$ so

$$k\left|\frac{x}{\rho}\right|^k \cdot \left|\frac{1}{x}\right| \to \infty \quad \text{and} \quad \left|\frac{x}{k+1}\right|\left|\frac{x}{\rho}\right|^k \to \infty \qquad \text{as } k \to \infty$$

See Exercise 12.5.2. In this case since the series $\sum_{k=0}^{\infty} |a_k|\ \rho^k$ diverges, the term-by-term derivative and integral series also diverge. (See Exercise 12.5.5.)

The fact that the integral of the series equals the series of integrals now follows from Theorem 12.5 applied to and interval $[-\rho, \rho]$ with $\rho < r$, the radius of convergence.

To prove the derivative part, define a new function

$$T[x] = \sum_{k=1}^{\infty} k\, a_k\ x_{k-1}$$

on the interval of convergence, $(-r, r)$. $T[x]$ is continuous by Theorem 12.5. The integral

$$\int_0^x T[\xi]\ d\xi = \sum_{k=1}^{\infty} a_k \int_0^x k\, \xi_{k-1}\ d\xi = \sum_{k=1}^{\infty} a_k\ x^k = S[x]$$

The second half of the Fundamental Theorem 9.2 says

$$T[x] = \frac{d}{dx}\int_0^x T[\xi]\ d\xi = \frac{dS}{dx}[x]$$

This proves that the derivative of the series is the series of derivatives.

Exercise set 12.5

1. *Show that if* $0 \le \rho < 1$ *then* $\lim_{k\to\infty} k\, \rho^k = 0$
2. *Show that if* $\rho > 1$ *then* $\lim_{k\to\infty} \frac{\rho^k}{k} = \infty$
3. *Prove:*

> **Theorem 12.8.**
> *If the series* $a_0 + a_1 + a_2 + a_3 + \dots$ *converges (with terms of arbitrary sign), then* $\lim_{k\to\infty} a_k = 0$.

4. *Give a divergent series* $a_0 + a_1 + a_2 + a_3 + \dots$ *of positive terms with* $\lim_{k\to\infty} a_k = 0$. *(HINT: Harmonic series.)*

5. *Prove the following.*

> **Theorem 12.9.** *Comparison Suppose that a_k and b_k are sequences of positive numbers with $a_k \le b_k$ for all $k \ge n$. Then*
> (a) *If $\sum_{k=1}^{\infty} b_k < \infty$ converges, so does $\sum_{k=1}^{\infty} a_k < \infty$.*
> (b) *If $\sum_{k=1}^{\infty} a_k = \infty$ diverges, so does $\sum_{k=1}^{\infty} b_k = \infty$.*

6. Euler's Criterion for Convergence
Show that the series $\sum_{k=1}^{\infty} a_k$ converges if and only if whenever m and n are both infinite hyperintegers,

$$\sum_{k=m}^{n} a_k \approx 0$$

7. *Prove:*

> **Theorem 12.10.** *Limit Comparison*
> *Suppose two sequences a_k and b_k satisfy $\lim_{k\to\infty} \frac{a_k}{b_k} = L \neq 0$. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.*

HINT: If k is infinite, $a_k = (L + \varepsilon_k) b_k$ with $\varepsilon \approx 0$. How much is $\sum_{k=m}^{n} (a_k - L\, b_k)$?

CHAPTER 13 The Theory of Fourier Series

This chapter gives some examples of Fourier series and a basic convergence theorem.

Fourier series and general "orthogonal function expansions" are important in the study of heat flow and wave propagation as well as in pure mathematics. The reason that these series are important is that sines and cosines satisfy the 'heat equation' or 'wave equation' or 'Laplace's equation' for certain geometries of the domain. A general solution of these partial differential equations can sometimes be approximated by a series of the simple solutions by using superposition. We conclude the background on series with this topic, because Fourier series provide many interesting examples of delicately converging series where we still have a simple general result on convergence.

The project on Fourier series shows you how to compute some of your own examples. The method of computing Fourier series is quite different from the methods of computing power series.

The Fourier sine-cosine series associated with $f[x]$ for $-\pi < x \le \pi$ is:

$$f[x] \sim a_0 + \sum_{k=1}^{\infty} [a_k \operatorname{Cos}[kx] + b_k \operatorname{Sin}[kx]]$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f[x]\ dx = \text{Average of } f[x]$$

and for $k = 1, 2, 3, \cdots$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f[x] \cdot \operatorname{Cos}[kx]\ dx \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f[x] \cdot \operatorname{Sin}[kx]\ dx$$

Dirichlet's Theorem 13.4 says that if $f[x]$ and $f'[x]$ are 2π-periodic and continuous except for a finite number of jumps or kinks and if the value $f[x_j]$ is the midpoint of the jump if there is one at x_j, then the Fourier series converges to the function at each point. It may not converge uniformly, in fact, the approximating *graphs* may not converge to the *graph* of the function, as shown in Gibb's goalposts below. If the periodic function $f[x]$ has no jumps (but may have a finite number of kinks, or jumps in $f'[x]$), then the series converges uniformly to $f[x]$.

Convergence of Fourier series is typically weaker than the convergence of power series, as we shall see in the examples, but the weak convergence is still quite useful. Actually, the most important kind of convergence for Fourier series is "mean square convergence,"

$$\int_{-\pi}^{\pi} (f[x] - S_n[x])^2 \; dx \to 0$$

where $S_n[x]$ is the sum of n terms. This is only a kind of average convergence, since the integral is still small if the difference is big only on a small set of $x's$. We won't go into mean square convergence except to mention that it sometimes corresponds to the difference in energy between the 'wave' $f[x]$ and its approximation $S_n[x]$. Mean square convergence has important connections to "Hilbert spaces."

Convergence of Fourier series at 'almost every' point was a notorious problem in mathematics, with many famous mathematicians making errors about the convergence. Fourier's work was in the early 1800's and not until 1966 did L. Carleson prove that the Fourier series of any continuous function $f[x]$ converges to the function at almost every point. (Dirichlet's Theorem **??** uses continuity of $f'[x]$ which may not be true if $f[x]$ is only continuous. Mean square convergence is much easier to work with, and was well understood much earlier.)

13.1 Computation of Fourier Series

This section has some examples of specific Fourier series.

Three basic examples of Fourier sine - cosine series are animated in the computer program **FourierSeries**. These follow along with some more. "Calculus" is about calculating. The following examples indicate the many specific results that we can obtain by performing algebra and calculus on Fourier series. Of course, the computation of the basic coefficients also requires calculus.

Example 13.1. *Fourier Series for the Zig-Zag* $f[x] = |x|$ *for* $-\pi < x \leq \pi$

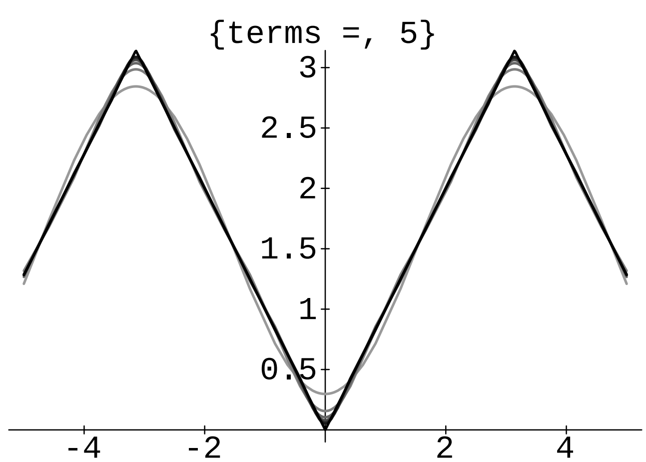


Figure 13.1: Fourier Series for $f[x] = |x|$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{4}{(2k+1)^2}\ \text{Cos}[(2k+1)x]$$

The series

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\text{Cos}[x]}{1} + \frac{\text{Cos}[3\,x]}{3^2} + \frac{\text{Cos}[5\,x]}{5^2} + \cdots + \frac{\text{Cos}[(2n+1)\,x]}{(2n+1)^2} + \cdots\right)$$

converges to the function that equals $|x|$ for $-\pi < x \leq \pi$ and is then repeated periodically.

The average value of $f[x]$ is clearly $\pi/2$ and can be computed as the integral

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f[x]\ dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\ dx \\ &= 2\,\frac{1}{2\pi}\int_{0}^{\pi} x\ dx \\ &= \frac{1}{\pi}\frac{1}{2}\ x^2|_0^\pi = \frac{\pi^2}{2\pi} \\ &= \frac{\pi}{2} \end{aligned}$$

Notice the step in the computation of the integral where we get rid of the absolute value. We must do this in order to apply the Fundamental Theorem of Integral Calculus. Absolute value does not have an antiderivative. We do the same thing in the computation of the other coefficients.

$$\begin{aligned} a_{2k} &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\ \text{Cos}[2k\,x]\ dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\ \text{Cos}[2k\,x]\ dx \\ &= \frac{2}{\pi}\left[x\ \frac{1}{2k}\,\text{Sin}[2kx]\ |_0^\pi - \int_0^\pi \frac{1}{2k}\ \text{Sin}[2k\,x]\ dx\right] \\ &= \frac{2}{\pi}\left[2k\pi\,\text{Sin}[2k\pi] - 0 - \frac{1}{(2k)^2}\,(\text{Cos}[2kx]\ |_0^\pi)\right] = 0 \end{aligned}$$

using integration by parts with

$$u = x \qquad dv = \text{Cos}[2kx]\ dx$$

$$du = dx \qquad v = \frac{1}{2k}\,\text{Sin}[2kx]$$

In the fourier Series project you show that the a_k terms of the Fourier series for $f[x] = |x|$ with odd k are

$$a_{2k+1} = -\frac{4}{\pi}\cdot\frac{1}{(2k+1)^2}$$

and all $b_k = 0$.

Example 13.2. *A Particular Case of the $|x|$ Series*

Set $x = 0$ in the series

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\text{Cos}[x]}{1} + \frac{\text{Cos}[3\,x]}{3^2} + \frac{\text{Cos}[5\,x]}{5^2} + \cdots + \frac{\text{Cos}[(2n+1)\,x]}{(2n+1)^2} + \cdots\right)$$
$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} + \cdots\right)$$
$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Example 13.3. *Fourier Series for* $f[x] = |\,\text{Sin}[x]|$ *for* $-\pi < x \le \pi$

$$|\,\text{Sin}[x]| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\ \text{Cos}[2kx]$$

In the project on Fourier Series you show that

$$|\,\text{Sin}[x]| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\text{Cos}[2x]}{3} + \frac{\text{Cos}[4\,x]}{15} + \frac{\text{Cos}[6\,x]}{35} + \cdots + \frac{\text{Cos}[2n\,x]}{(2n)^2-1} + \cdots\right)$$

converges to the function that equals $|x|$ for $-\pi < x \le \pi$ and is then repeated periodically.

Example 13.4. *A Particular Case of the* $|\,\text{Sin}[x]|$ *Series*

Set $x = 0$ in the series

$$|\,\text{Sin}[x]| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\text{Cos}[2x]}{3} + \frac{\text{Cos}[4\,x]}{15} + \frac{\text{Cos}[6\,x]}{35} + \cdots + \frac{\text{Cos}[2n\,x]}{(2n)^2-1} + \cdots\right)$$
$$0 = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots + \frac{1}{(2n)^2-1} + \cdots\right)$$
$$\frac{1}{2} = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots + \frac{1}{(2n)^2-1} + \cdots$$

Example 13.5. *Another Case of the* $|\,\text{Sin}[x]|$ *Series*

Set $x = \pi/2$ in the series

$$|\,\text{Sin}[x]| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\text{Cos}[2x]}{3} + \frac{\text{Cos}[4\,x]}{15} + \frac{\text{Cos}[6\,x]}{35} + \cdots + \frac{\text{Cos}[2n\,x]}{(2n)^2-1} + \cdots\right)$$
$$1 = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{-1}{3} + \frac{1}{15} + \frac{-1}{35} + \cdots + \frac{(-1)^n}{(2n)^2-1} + \cdots\right)$$
$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{3} - \frac{1}{15} + \frac{1}{35} + \cdots + \frac{(-1)^{n+1}}{(2n)^2-1} + \cdots$$
$$\frac{\pi}{8} = \frac{1}{3} + \frac{1}{35} + \cdots + \frac{1}{4(2k+1)^2-1} + \cdots$$

Example 13.6. *Fourier Series for* $f[x] = x^2$*, for* $-\pi < x \le \pi$

$$x^2 = \frac{\pi^2}{3} - 4\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}\ \mathrm{Cos}[kx]$$

In the project on Fourier Series you show that

$$x^2 = \frac{\pi^2}{3} - 4\left(\mathrm{Cos}[x] - \frac{\mathrm{Cos}[2x]}{2^2} + \frac{\mathrm{Cos}[3x]}{3^2} + \cdots + (-1)^{n+1}\frac{\mathrm{Cos}[nx]}{n^2} + \cdots\right)$$

for $-\pi < x \leq \pi$.

Example 13.7. *A Particular Case of the x^2 Series*

Set $x = 0$ in the series

$$\begin{aligned}
x^2 &= \frac{\pi^2}{3} - 4\left(\mathrm{Cos}[x] - \frac{\mathrm{Cos}[2x]}{2^2} + \frac{\mathrm{Cos}[3x]}{3^2} + \cdots + (-1)^{n+1}\frac{\mathrm{Cos}[nx]}{n^2} + \cdots\right) \\
0 &= \frac{\pi^2}{3} - 4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} + \cdots + (-1)^{n+1}\frac{1}{n^2} + \cdots\right) \\
\frac{\pi^2}{12} &= 1 - \frac{1}{2^2} + \frac{1}{3^2} + \cdots + (-1)^{n+1}\frac{1}{n^2} + \cdots
\end{aligned}$$

Example 13.8. *The Formal Derivative of the x^2 Series*

Notice that if we differentiate both sides and (without justification) interchange derivative and (infinite) sum, we obtain

$$\begin{aligned}
\frac{dx^2}{dx} &= \frac{d}{dx}\left(\frac{\pi^2}{3} - 4\left(\mathrm{Cos}[x] - \frac{\mathrm{Cos}[2x]}{2^2} + \frac{\mathrm{Cos}[3x]}{3^2} + \cdots + (-1)^{n+1}\frac{\mathrm{Cos}[nx]}{n^2} + \cdots\right)\right) \\
2\,x &= 4\left(\mathrm{Sin}[x] - \frac{\mathrm{Sin}[2x]}{2} + \frac{\mathrm{Sin}[3x]}{3} + \cdots + (-1)^{n+1}\frac{\mathrm{Sin}[nx]}{n} + \cdots\right)
\end{aligned}$$

Example 13.9. *Fourier Series for the Sawtooth Wave $f[x] = x$*

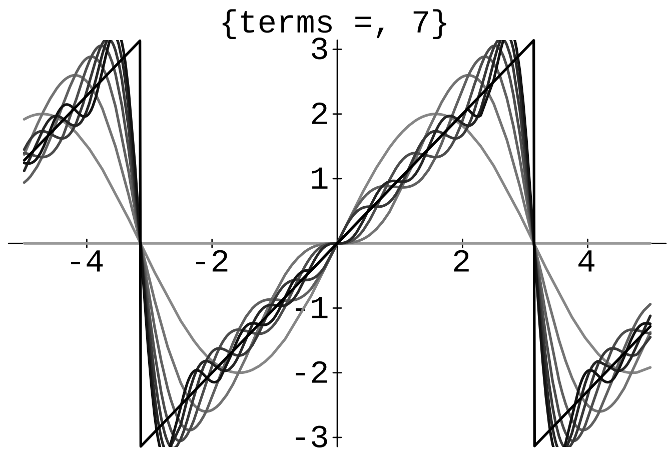


Figure 13.2: Fourier Series for $f[x] = x$

$$x = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \operatorname{Sin}[k\,x]$$

$$x = 2\left(\operatorname{Sin}[x] - \frac{\operatorname{Sin}[2x]}{2} + \frac{\operatorname{Sin}[3x]}{3} + \cdots + (-1)^{n+1}\frac{\operatorname{Sin}[nx]}{n} + \cdots\right)$$

Without the absolute value the integrals of the Fourier coefficients can be computed directly, without breaking them into pieces. The Fourier sine-cosine series for the "sawtooth wave,"

$$f[x] = \begin{cases} x, & \text{for } -\pi < x < \pi \\ 0, & \text{for } |x| = \pi \end{cases}$$

extended to be 2π periodic is easier to compute. Notice that the average $a_0 = 0$, by inspection of the graph or by computation of an integral. Moreover, $x\operatorname{Cos}[2kx]$ is an odd function, that is, $-x\operatorname{Cos}[2k\cdot(-x)] = -(x\operatorname{Cos}[2kx])$, so the up areas and down areas of the integral cancel, $a_k = 0$. Finally, you can show that

$$b_k = 2\frac{(-1)^{k+1}}{k}$$

Example 13.10. *A Particular Case of the x Series*

Set $x = \pi/2$ in the series

$$x = 2\left(\operatorname{Sin}[x] - \frac{\operatorname{Sin}[2x]}{2} + \frac{\operatorname{Sin}[3x]}{3} + \cdots + (-1)^{n+1}\frac{\operatorname{Sin}[nx]}{n} + \cdots\right)$$
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$$

Example 13.11. *Fourier Series for the Square Wave $f[x] = Sign[x]$*

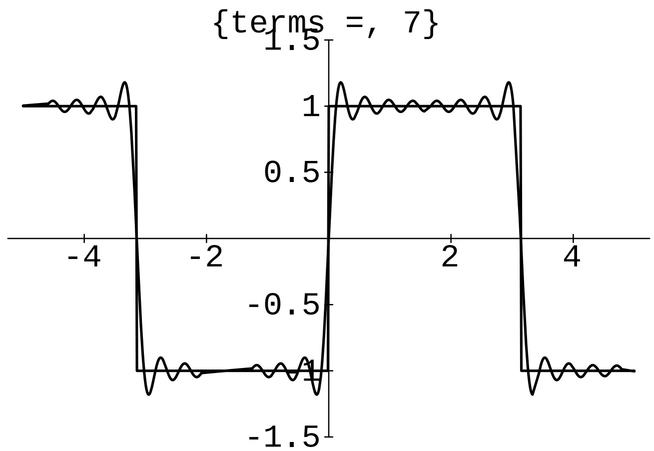


Figure 13.3: Gibbs Goalposts $f[x] = \text{Sign}[x]$

$$\text{Sign}[x] = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \ \text{Sin}[(2k+1)\,x]$$

$$\frac{4}{\pi}\left(\text{Sin}[x] + \frac{1}{3}\ \text{Sin}[3\,x] + \frac{1}{5}\ \text{Sin}[5\,x] + \cdots\right) = \text{Sign}[x] = \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

The coefficients for the Fourier series of

$$f[x] = \text{Sign}[x] = \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

must be computed by breaking the integrals into pieces where the Fundamental Theorem applies, for example,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f[x]\ dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -1\ dx + \int_{0}^{\pi} +1\ dx\right] = 0$$

In the Fourier Series project you show that

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{Sign}[x]\ \ \text{Cos}[kx]\ dx = 0$$

and

$$b_{2k} = 0$$

because each piece of the integral $\int_0^{\pi} \text{Sin}[2kx]\ dx = 0$, being the integral over whole periods of the sine function.

Also,

$$\begin{aligned} b_{2k+1} &= \frac{2}{\pi}\int_0^{\pi} \mathrm{Sin}[(2k+1)x]\ dx \\ &= \frac{2}{\pi}\cdot\frac{1}{2k+1}\int_0^{(2k+1)\pi} \mathrm{Sin}[u]\ du \\ &= \frac{4}{\pi}\cdot\frac{1}{2k+1} \end{aligned}$$

Example 13.12. *A Particular Case of the Sign[x] Series*

Substituting $x = \pi/2$ in the series

$$\mathrm{Sign}[x] = \frac{4}{\pi}\left(\mathrm{Sin}[x] + \frac{1}{3}\ \mathrm{Sin}[3\,x] + \frac{1}{5}\ \mathrm{Sin}[5\,x] + \cdots\right)$$
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$$

Example 13.13. *The Series for* Cos[ωx]*, for Non-Integer* ω

In the project on Fourier Series you show that when ω is not an integer,

$$\mathrm{Cos}[\omega x] = \frac{2\omega\ \mathrm{Sin}[\omega\pi]}{\pi}\left(\frac{1}{2\omega^2} - \frac{\mathrm{Cos}[x]}{\omega^2-1} + \frac{\mathrm{Cos}[2x]}{\omega^2-2^2} + \cdots + \frac{(-1)^n}{\omega^2-n^2}\ \mathrm{Cos}[n\,x] + \cdots\right)$$

Substituting $x = \pi$ and performing some algebra, we obtain,

$$\mathrm{Cos}[\omega x] = \frac{2\omega\ \mathrm{Sin}[\omega\pi]}{\pi}\left(\frac{1}{2\omega^2} - \frac{\mathrm{Cos}[x]}{\omega^2-1} + \frac{\mathrm{Cos}[2x]}{\omega^2-2^2} + \cdots + \frac{(-1)^n}{\omega^2-n^2}\ \mathrm{Cos}[n\,x] + \cdots\right)$$
$$\pi\,\frac{\mathrm{Cos}[\omega\pi]}{\mathrm{Sin}[\omega\pi]} = 2\,\omega\left(\frac{1}{2\omega^2} + \frac{1}{\omega^2-1} + \frac{1}{\omega^2-2^2} + \cdots + \frac{1}{\omega^2-n^2} + \cdots\right)$$
$$\pi\,\frac{\mathrm{Cos}[\omega\pi]}{\mathrm{Sin}[\omega\pi]} - \frac{1}{\omega} = 2\,\omega\left(\frac{1}{\omega^2-1} + \frac{1}{\omega^2-2^2} + \cdots + \frac{1}{\omega^2-n^2} + \cdots\right)$$
$$\pi\int_0^x \frac{\mathrm{Cos}[\omega\pi]}{\mathrm{Sin}[\omega\pi]} - \frac{1}{\pi\,\omega}\ d\omega = \left(\int_0^x \frac{2\,\omega}{\omega^2-1}\ d\omega + \int_0^x \frac{2\,\omega}{\omega^2-2^2}\ d\omega + \cdots + \int_0^x \frac{2\,\omega}{\omega^2-n^2}\ d\omega + \cdots\right)$$
$$\mathrm{Log}\left[\frac{\mathrm{Sin}[\pi\,x]}{\pi\,x}\right] = \mathrm{Log}[1 - \frac{x^2}{1}] + \mathrm{Log}[1 - \frac{x^2}{2^2}] + \cdots + \mathrm{Log}[1 - \frac{x^2}{n^2}] + \cdots$$
$$\mathrm{Sin}[\pi\,x] = \pi\,x\prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right)$$

Example 13.14. *The Derivative of The Series for* Cos[ωx]

Formally differentiating,

$$\frac{d\,\mathrm{Cos}[\omega x]}{dx} = \frac{2\omega\ \mathrm{Sin}[\omega\pi]}{\pi}\frac{d}{dx}\left(\frac{1}{2\omega^2} - \frac{\mathrm{Cos}[x]}{\omega^2-1} + \frac{\mathrm{Cos}[2x]}{\omega^2-2^2} + \cdots + \frac{(-1)^n}{\omega^2-n^2}\ \mathrm{Cos}[n\,x] + \cdots\right)$$

$$-\,\mathrm{Sin}[\omega\,x] = \frac{2\ \mathrm{Sin}[\omega\pi]}{\pi}\left(\frac{\mathrm{Sin}[x]}{\omega^2-1} - \frac{2\ \mathrm{Cos}[2\,x]}{\omega^2-2^2} + \cdots + (-1)^{n+1}\frac{n}{\omega^2-n^2}\ \mathrm{Sin}[n\,x] + \cdots\right)$$

Example 13.15. *The Series for* $\mathrm{Sin}[\omega x]$*, for Non-Integer* ω

In the project on Fourier Series you show that when ω is not an integer,

$$\mathrm{Sin}[\omega x] = \frac{2\,\mathrm{Sin}[\omega\pi]}{\pi}\left(-\frac{\mathrm{Sin}[x]}{\omega^2-1} + \frac{2\ \mathrm{Sin}[2x]}{\omega^2-2^2} + \cdots + (-1)^n\,\frac{n}{\omega^2-n^2}\ \mathrm{Sin}[n\,x] + \cdots\right)$$

Example 13.16. *Hyperbolic Functions Restricted to* $-\pi < x < \pi$

We can also compute

$$\cosh[\omega\,x] = \frac{2\,\omega\ \sinh[\omega\,\pi]}{\pi}\left(\frac{1}{2\omega^2} - \frac{\mathrm{Cos}[x]}{\omega^2+1^2} + \frac{\mathrm{Cos}[2\,x]}{\omega^2+2^2} + \cdots + (-1)^n\frac{1}{\omega^2+n^2}\ \mathrm{Cos}[n\,x] + \cdots\right)$$

$$\sinh[\omega\,x] = \frac{2\ \sinh[\omega\,\pi]}{\pi}\left(\frac{\mathrm{Sin}[x]}{\omega^2+1^2} - \frac{2\ \mathrm{Sin}[2\,x]}{\omega^2+2^2} + \cdots + (-1)^{n+1}\,\frac{n}{\omega^2+n^2}\,\mathrm{Sin}[n\,x] + \cdots\right)$$

Exercise set 13.1

1. *Use the computer to plot the Fourier series examples above.*

13.2 Convergence for Piecewise Smooth Functions

Fourier series of piecewise smooth functions converge.

A function $f[x]$ on $-\pi < x \le \pi$ is said to be piecewise continuous if it is continuous except for at most a finite number of jump discontinuities. That is, except for finitely many values of x, $\lim_{\xi\to x} f[\xi] = f[x]$, and at the finite number of other points x_j, $f[x_j]$ has a jump discontinuity, meaning $\lim_{x\uparrow x_j} f[x]$ exists and $\lim_{x\downarrow x_j} f[x]$ exists. ($f[x_j]$ can exist, but need not equal either one-sided limit. Fourier series will converge to the midpoint of a jump.) A function is said to be piecewise smooth if both $f[x]$ and $f'[x]$ are piecewise continuous.

Piecewise smooth functions can be continuous, like the periodic extension of $|x|$ for $-\pi < x \le \pi$. In this case the function has a kink or its derivative $f'[x]$ has a jump discontinuity because $f'[x] = -1$ for $x < 0$ and $f'[x] = +1$ for $x > 0$. (See Figure 13.1.)

Periodic extension can create jump discontinuities, like x for $-\pi < x < \pi$. In this case $\lim_{x\uparrow\pi} f[x] = \pi$ but $\lim_{x\downarrow\pi}$ "periodic x" $= \lim_{x\downarrow-\pi} x = -\pi$. (See Figure 13.2.)

Periodic extension can create jumps in the derivative of a smooth function like x^2. Sketch the graph from $-\pi$ to π, extend periodically, and observe that the graph is continuous, but not smooth at multiples of π.

We need the following

Theorem 13.1. *A Trig Identity and Integral*

$$\frac{\mathrm{Sin}[(n+\frac{1}{2})\,\phi]}{2\ \mathrm{Sin}[\frac{1}{2}\phi]} = \frac{1}{2} + \sum_{k=1}^{n} \mathrm{Cos}[k\,\phi]$$

and

$$\frac{1}{\pi}\int_0^{\pi} \frac{\mathrm{Sin}[(n+\frac{1}{2})\,\phi]}{\mathrm{Sin}[\frac{1}{2}\phi]}\ d\phi = 1$$

Proof:

By Euler's formula, $\mathrm{Cos}[\theta] = \frac{1}{2}(e^{\mathbf{i}\theta} + e^{-\mathbf{i}\theta})$,

$$\begin{aligned}
\frac{1}{2} + \sum_{k=1}^{n} \mathrm{Cos}[k\,\phi] &= \frac{1}{2}\sum_{k=-n}^{n} e^{\mathbf{i}\,k\,\phi} = e^{-\mathbf{i}\,n\,\phi}\sum_{h=0}^{2n} r^h, \text{ for } r = e^{\mathbf{i}\,\phi} \\
&= e^{-\mathbf{i}\,n\,\phi}\frac{1-r^{2n+1}}{1-r} = e^{-\mathbf{i}\,n\,\phi}\frac{1-e^{\mathbf{i}\,(2n+1)\,\phi}}{1-e^{\mathbf{i}\,\phi}} \\
&= \frac{e^{-\mathbf{i}\,n\,\phi} - e^{\mathbf{i}\,(n+1)\,\phi}}{2}\cdot\frac{e^{-\mathbf{i}\,\frac{1}{2}\,\phi}}{e^{-\mathbf{i}\,\frac{1}{2}\,\phi}}\cdot\frac{1}{1-e^{\mathbf{i}\,\phi}} \\
&= \frac{e^{\mathbf{i}\,(n+\frac{1}{2})\,\phi} - e^{-\mathbf{i}\,(n+\frac{1}{2})\,\phi}}{2\mathbf{i}}\cdot\frac{1}{2}\cdot\frac{2\mathbf{i}}{e^{\mathbf{i}\,\frac{1}{2}\,\phi} - e^{-\mathbf{i}\,\frac{1}{2}\,\phi}} \\
&= \frac{\mathrm{Sin}[(n+\frac{1}{2})\phi]}{2\ \mathrm{Sin}[\frac{1}{2}\,\phi]}
\end{aligned}$$

using Euler's formula, $\mathrm{Sin}[\theta] = \frac{1}{2\mathbf{i}}\left(e^{\mathbf{i}\theta} - e^{-\mathbf{i}\theta}\right)$.

For the integral notice that all the terms of the integral of the right hand side vanish except for the constant term.

Example 13.17. *The Dirichlet Kernel* $S_n[x] = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} f[x+\theta]\frac{\mathrm{Sin}[(n+\frac{1}{2})\theta]}{2\ \mathrm{Sin}[\frac{1}{2}\,\theta]}\ d\theta$

We use this identity to write a partial sum of a Fourier series as an integral. By the definition of a partial sum of the Fourier series, for any n,

$$\begin{aligned}
S_n[x] &= a_0 + \sum_{k=1}^{n} (a_k \ \mathrm{Cos}[k\,x] + b_k \ \mathrm{Sin}[k\,x]) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \frac{1}{2} f[\xi]\,d\xi + \sum_{k=1}^{n}\left(\int_{-\pi}^{\pi} f[\xi]\ \mathrm{Cos}[k\,\xi]\,d\xi\ \mathrm{Cos}[k\,x] + \int_{-\pi}^{\pi} f[\xi]\ \mathrm{Sin}[k\,\xi]\,d\xi\ \mathrm{Sin}[k\,x]\right)\right) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f[\xi]\left(\frac{1}{2} + \sum_{k=1}^{n} (\mathrm{Cos}[k\,\xi]\ \mathrm{Cos}[k\,x] + \mathrm{Sin}[k\,\xi]\ \mathrm{Sin}[k\,x])\right) d\xi \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f[\xi]\left(\frac{1}{2} + \sum_{k=1}^{n} \mathrm{Cos}[k\,(\xi - x)]\right) d\xi \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f[\xi]\cdot\frac{\mathrm{Sin}[(n+\frac{1}{2})(\xi - x)]}{2\ \mathrm{Sin}[\frac{1}{2}\,(\xi - x)]}\,d\xi, \text{ by the trig identity above.}
\end{aligned}$$

Now make a change of variable and differential, $\xi = x + \theta$, $d\xi = d\theta$ and use periodicity of $f[x]$ to see,

$$S_n[x] = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} f[x+\theta]\frac{\mathrm{Sin}[(n+\frac{1}{2})\theta]}{\mathrm{Sin}[\frac{1}{2}\,\theta]}\ d\theta$$

Notice that this integrand is well-behaved even near $\theta = 0$ where the denominator tends to zero. ($\mathrm{Sin}[x]/x$ extends smoothly to $x = 0$, as you can easily see from the power series for sine.)

The intuitive idea using this formula in the convergence theorem given next is to think of the sine fraction as giving a "measure" for each n.

$$\int_{-\pi}^{\pi} ()\ d\mu_n[\theta] = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} ()\ \frac{\mathrm{Sin}[(n+\frac{1}{2})\theta]}{\mathrm{Sin}[\frac{1}{2}\,\theta]}\ d\theta$$

Each of these measures has total "mass" 1,

$$\int_{-\pi}^{\pi} (1)\ d\mu_n[\theta] = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} (1)\ \frac{\mathrm{Sin}[(n+\frac{1}{2})\theta]}{\mathrm{Sin}[\frac{1}{2}\,\theta]}\ d\theta = 1$$

As n increases, more and more of this unit mass is concentrated near $\theta = 0$,

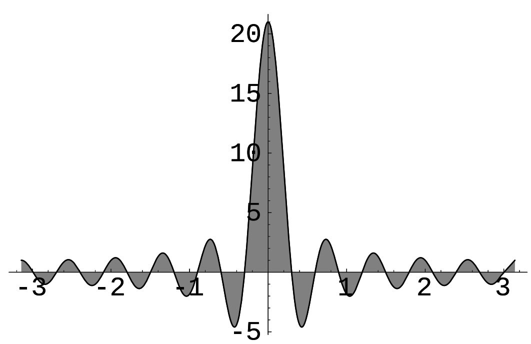


Figure 13.4: Dirichlet's Kernel $\frac{\mathrm{Sin}[(n+\frac{1}{2})\theta]}{\mathrm{Sin}[\frac{1}{2}\,\theta]}$ with $n = 10$

Since $\theta \approx 0$ implies $f[x-\theta] \approx f[x]$, we expect

$$\int_{-\pi}^{\pi} (f[x+\theta]) \; d\mu_n[\theta] \approx f[x]$$

when n is large. It is not quite this simple, but it is easy to show that the measure is less and less important away from zero.

Theorem 13.2. *Suppose $f[x]$ is 2π-periodic and piecewise smooth. Then for any fixed $\varepsilon > 0$,*

$$\lim_{n\to\infty} \int_{\varepsilon}^{\pi} f[x+\theta] \frac{\mathrm{Sin}[(n+\frac{1}{2})\theta]}{\mathrm{Sin}[\frac{1}{2}\,\theta]} \; d\theta + \int_{-\pi}^{-\varepsilon} f[x+\theta] \frac{\mathrm{Sin}[(n+\frac{1}{2})\theta]}{\mathrm{Sin}[\frac{1}{2}\,\theta]} \; d\theta = 0$$

Proof:
Define the function

$$g[\theta] = f[x+\theta]/\,\mathrm{Sin}[\theta/2]$$

Since $f[x]$ is piecewise smooth and $\mathrm{Sin}[\theta/2]$ is nonzero for $\varepsilon \le \theta \le \pi$,

$$\int_{\varepsilon}^{\pi} f[x+\theta] \frac{\mathrm{Sin}[(n+\frac{1}{2})\theta]}{\mathrm{Sin}[\frac{1}{2}\,\theta]} \; d\theta = \int_{\varepsilon}^{\pi} g[\theta] \; \mathrm{Sin}[(n+\frac{1}{2})\theta] \; d\theta$$

$g[\theta]$ is piecewise smooth and we may integrate by parts:

$$\int_{\varepsilon}^{\pi} g[\theta] \; \mathrm{Sin}[(n+\frac{1}{2})\theta] \; d\theta = \frac{1}{n+\frac{1}{2}}\Big(g[\varepsilon] \; \mathrm{Cos}[(n+\frac{1}{2})\varepsilon] - g[\pi] \; \mathrm{Cos}[(n+\frac{1}{2})\pi] + \int_{\varepsilon}^{\pi} g'[\theta] \; \mathrm{Cos}[(n+\frac{1}{2})\theta] \; d\theta \Big) \to 0, \text{ as } n \to \infty$$

In order to prove the strongest form of the convergence theorem, we need the following generalization of this result.

Theorem 13.3. *Suppose that $g[\theta]$ is piecewise continuous on the subinterval $[a,b]$ of $[-\pi,\pi]$. Then*

$$\int_a^b g[\theta] \; \mathrm{Sin}[\nu\theta] \; d\theta \to 0 \text{ as } \nu \to \infty$$

Proof:
Let ν be a large integer, the interval $a \le x \le b$ is divided into a sequence of adjacent subintervals $[x_{2j}, x_{2j+1}]$, $[x_{2j+1}, x_{2j+2}]$ of length π/ν where $\nu\, x_{2j}$ is an even multiple of π and $\nu\, x_{2j+1}$ is an odd multiple of π. These are simply the points that lie in the interval of the form $k\,\pi/\nu$, for integers k.

There may be as many as 2 exceptional unequal length subintervals at the ends and one additional non-matched subinterval of odd and even multiples of π/ν. Re-number the sequence beginning with $x_1 = a$ and ending with b.

The integral

$$\int_a^b g[x]\,\mathrm{Sin}[\nu\, x] \; dx = \sum_{j=1}^{n} \left(\int_{x_{2j}}^{x_{2j+1}} g[x] \; \mathrm{Sin}[\nu\, x] \; dx + \int_{x_{2j+1}}^{x_{2j+2}} g[x] \; \mathrm{Sin}[\nu\, x] \; dx \right)$$

Sine is positive on one subinterval of each pair and negative on the next with

$$\int_{x_{2j}}^{x_{2j+1}} \mathrm{Sin}[\nu\xi]\ d\xi = -\int_{x_{2j+1}}^{x_{2j+2}} \mathrm{Sin}[\nu\xi]\ d\xi$$

The same decomposition is true when ν is an infinite hyperreal. We may write

$$\begin{aligned}\int_a^b g[x]\,\mathrm{Sin}[\nu\,x]\ dx = &\int_a^{x_1} g[x]\ \mathrm{Sin}[\nu\,x]\ dx + \\ &\sum_{j=1}^{n}\left(\int_{x_{2j-1}}^{x_{2j}} g[x]\ \mathrm{Sin}[x]\ dx + \int_{x_{2j}}^{x_{2j+1}} g[x]\ \mathrm{Sin}[x]\ dx\right) \\ &+ \int_{x_{2n+2}}^{b} g[x]\ \mathrm{Sin}[\nu\,x]\ dx\end{aligned}$$

When ν is infinite, $g[\theta]$ changes very little on $[x_{2j-1}, x_{2j+1}]$, in fact, the maximum and minimum over a pair of subintervals differ by an infinitesimal: there is a sequence of infinitesimals η_j such that

$$|g[\xi] - g[\zeta]| \le \eta_j \approx 0, \text{ for } x_{2j-1} \le \xi \le \zeta \le x_{2j+1}$$

Sine is positive on one and negative on the other so that the adjacent subintegrals nearly cancel,

$$\begin{aligned}&\left|\int_{x_{2j-1}}^{x_{2j}} g[x]\ \mathrm{Sin}[\nu\,x]\ dx + \int_{x_{2j}}^{x_{2j+1}} g[x]\ \mathrm{Sin}[\nu\,x]\ dx\right| = \\ &\left|\int_{x_{2j-1}}^{x_{2j}} g[x]\ \mathrm{Sin}[\nu\,x] + g[x+\frac{\pi}{\nu}]\ \mathrm{Sin}[\nu\,x+\pi]\ dx\right| \le \\ &\left|\int_{x_{2j-1}}^{x_{2j}} (g[x] - g[x+\frac{\pi}{\nu}])\,\mathrm{Sin}[\nu\,x]\ dx\right| \le \\ &\mathrm{Max}[|g[x] - g[x+\frac{\pi}{\nu}]| : x_{2j-1} \le x \le x_{2j}] \int_{x_{2j-1}}^{x_{2j}} 1\ dx \le \eta_j\,\frac{\pi}{\nu}\end{aligned}$$

and

$$\begin{aligned}\Big|\sum_{j=1}^{n}\Big(\int_{x_{2j-1}}^{x_{2j}} g[x]\ \mathrm{Sin}[\nu\,x]\ dx + \int_{x_{2j}}^{x_{2j+1}} g[x]\ \mathrm{Sin}[\nu\,x]\ dx\Big)\Big| &\le \sum_{j=1}^{n}\eta_j\,\frac{\pi}{\nu} \\ &\le \mathrm{Max}[\eta_j : 1 \le j \le n]\sum_{j=1}^{n}\frac{\pi}{\nu} \\ &\le \eta_h \cdot (b-a)/2 \approx 0\end{aligned}$$

The few stray infinitesimal end subintervals contribute at most an infinitesimal since $g[x]$ is bounded, so

$$\int_a^b g[x]\ \mathrm{Sin}[\nu\,x]\ dx \approx 0$$

and the theorem is proved (by the characterization of limits with infinite indices.)

Theorem 13.4. *Dirichlet's Convergence Theorem*

If $f[x]$ is a piecewise smooth 2π-periodic function, then the Fourier series of $f[x]$ converges to the function at each point of continuity of $f[x]$ and converges to the midpoint of the jump at the finite number of jump discontinuities, for all x,

$$\frac{1}{2}\left(\lim_{\xi\uparrow x} f[\xi] + \lim_{\xi\downarrow x} f[\xi]\right) = a_0 + \sum_{k=0}^{\infty} a_k \operatorname{Cos}[k\,x] + \sum_{k=0}^{\infty} b_k \operatorname{Sin}[k\,x]$$

If $f[x]$ is continuous at x, $f[x] = \frac{1}{2}\left(\lim_{\xi\uparrow x} f[\xi] + \lim_{\xi\downarrow x} f[\xi]\right)$.

Proof:

Fix a value of x and let $F_x = \lim_{\xi\downarrow x} f[\xi]$. Then $\lim_{\xi\downarrow x} f[x] - F_x = 0$ and the piecewise derivative of $f[x]$ means $\lim_{\Delta x\downarrow 0} \frac{f[x+\Delta x]-F_x}{\Delta x} = F_x'$ exists.

Since $\lim_{\Delta x\to 0} \frac{\operatorname{Sin}[\Delta x]}{\Delta x} = \lim_{\Delta x\to 0} \frac{\operatorname{Sin}[\Delta x/2]}{\Delta x/2} = 1$, we also have

$$\lim_{\Delta x\downarrow 0} \frac{f[x+\Delta x] - F_x}{2\operatorname{Sin}[\Delta x/2]} = F_x'$$

This discussion means that the function

$$g[\theta] = \frac{f[x+\theta] - F_x}{\operatorname{Sin}[\theta/2]}$$

is piecewise continuous on $[0,\pi]$ and Theorem 13.3 says

$$\int_0^{\pi} g[\theta]\,\operatorname{Sin}[(n+\frac{1}{2})\theta]\,d\theta \to 0 \text{ as } n\to\infty$$

Thus, we see that

$$\frac{1}{\pi}\int_0^{\pi} \frac{f[x+\theta]-F_x}{\operatorname{Sin}[\theta/2]}\,\operatorname{Sin}[(n+\frac{1}{2})\theta]\,d\theta \to 0$$
$$\frac{1}{\pi}\int_0^{\pi} f[x+\theta]\frac{\operatorname{Sin}[(n+\frac{1}{2})\theta]}{\operatorname{Sin}[\theta/2]}\,d\theta \to F_x\frac{1}{\pi}\int_0^{\pi} \frac{\operatorname{Sin}[(n+\frac{1}{2})\theta]}{\operatorname{Sin}[\theta/2]}\,d\theta = \frac{\lim_{\xi\downarrow x} f[\xi]}{2}$$

Similarly,

$$\frac{1}{\pi}\int_{-\pi}^{0} f[x+\theta]\frac{\operatorname{Sin}[(n+\frac{1}{2})\theta]}{\operatorname{Sin}[\theta/2]}\,d\theta \to \frac{\lim_{\xi\uparrow x} f[\xi]}{2}$$

This proves the theorem.

Exercise set 13.2

Intuitively, many of the weakly convergent Fourier series are converging by cancelling oscillations. If this is true, we would expect averages to be even better approximations.

1. *Let $s_m[x] = \frac{1}{2} + \sum_{k=1}^{m}\left(\operatorname{Cos}[k\,x] + \operatorname{Sin}[k\,x]\right)$ be the partial Fourier-like sum. Define the average of the partial sums to be*

$$a_n[x] = \frac{1}{1+n}\sum_{m=0}^{n} s_m[x]$$

When $S_m[x] = a_0 + \sum_{k=1}^{m} (a_k \text{ Cos}[k\, x] + b_k \text{ Sin}[k\, x])$ *are the partial Fourier sums of a function* $f[x]$*, let*

$$A_n[x] = \frac{1}{1+n} \sum_{m=0}^{n} S_m[x]$$

denote the average of the first n *Fourier sums.*

(a) *Plot the average Fourier sums* $A_n[x]$ *for the examples of the previous section, especially those that converge weakly like* $f[x] = x$ *at* $\pm\pi$ *or* $f[x] = Sign[x]$*.*

(b) *Show that*

$$a_n[x] = \frac{1}{1+n} \left(\frac{\text{Sin}[(n+1)\, \frac{x}{2}]}{\text{Sin}[\frac{x}{2}]} \right)^2$$

(c) *Show that*

$$\frac{1}{\pi} \int_0^{\pi} \frac{1}{1+n} \left(\frac{\text{Sin}[(n+1)\, \frac{x}{2}]}{\text{Sin}[\frac{x}{2}]} \right)^2 \, dx = 1$$

(d) *Show that the average of the first* n *Fourier series of a function* $f[x]$ *are given by*

$$A_n[x] = \frac{1}{2\,\pi} \int_{-\pi}^{\pi} f[x+\theta]\, \frac{1}{1+n} \left(\frac{\text{Sin}[(n+1)\, \frac{x}{2}]}{\text{Sin}[\frac{\theta}{2}]} \right)^2 \, d\theta$$

13.3 Uniform Convergence for Continuous Piecewise Smooth Functions

Fourier series of continuous piecewise smooth functions converge uniformly.

Theorem 13.5. *Uniform Convergence of Fourier Series*
If $f[x]$ *is continuous and* $f'[x]$ *is piecewise continuous, then its Fourier series converges absolutely and uniformly to the function. Moreover, the Fourier series of any piecewise smooth function converges uniformly to the function on any closed subinterval where the function is continuous.*

Proof of this theorem requires some inequalities related to mean square convergence. In particular,

$$a_0^2 + \sum_{k=1}^{n} (a_k^2 + b_k^2) \le \frac{1}{\pi} \int_{-\pi}^{\pi} (f[x])^2 \, dx \text{ and for any sequences } \left(\sum_{k=1}^{n} u_k\, v_k \right)^2 \le \sum_{k=1}^{n} u_k^2 \cdot \sum_{k=1}^{n} v_k^2$$

We refer the reader to a book on fourier series.

We do want to compare the convergence of some of the continuous and discontinuous examples computed above to compare uniform and non-uniform convergence.

Example 13.18. *Infinitely Slowly Convergent Series with a Discontinuous Limit*

Fourier series can converge delicately. For example, the identity

$$x = 2\left(\mathrm{Sin}[x] - \frac{\mathrm{Sin}[2x]}{2} + \frac{\mathrm{Sin}[3x]}{3} + \cdots + (-1)^{n+1}\frac{\mathrm{Sin}[nx]}{n} + \cdots\right)$$

is a valid convergent series for $-\pi < x < \pi$. However, the Weierstrass majorization does not yield a simple convergence estimate, because

$$\left|(-1)^{n+1}\frac{\mathrm{Sin}[nx]}{n}\right| \leq \frac{1}{n}$$

is a useless upper estimate by a divergent series, $\sum_{k=1}^{\infty}\frac{1}{n} = \infty$. The Fourier series converges but not uniformly, and its limit function is discontinuous because repeating x periodically produces a jump at π as follows:

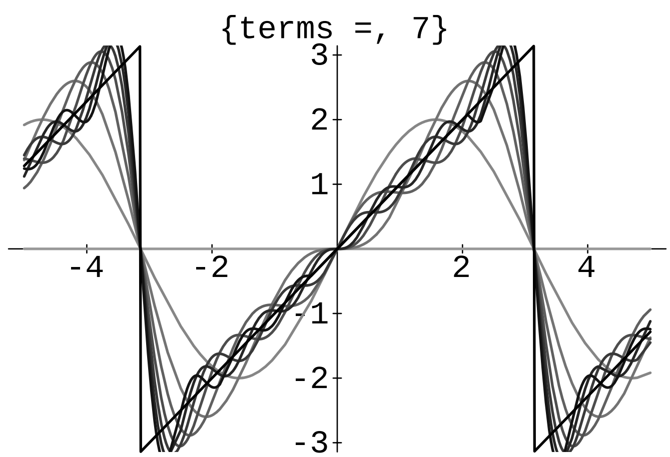


Figure 13.5: Fourier series for $f[x] = x$

The convergence of the Fourier series for $\mathrm{Sign}[x]$

$$\frac{4}{\pi}\left(\mathrm{Sin}[x] + \frac{1}{3}\,\mathrm{Sin}[3\,x] + \frac{1}{5}\,\mathrm{Sin}[5\,x] +\right) = \mathrm{Sign}[x]$$

holds at every fixed point, but the convergence is not uniform.

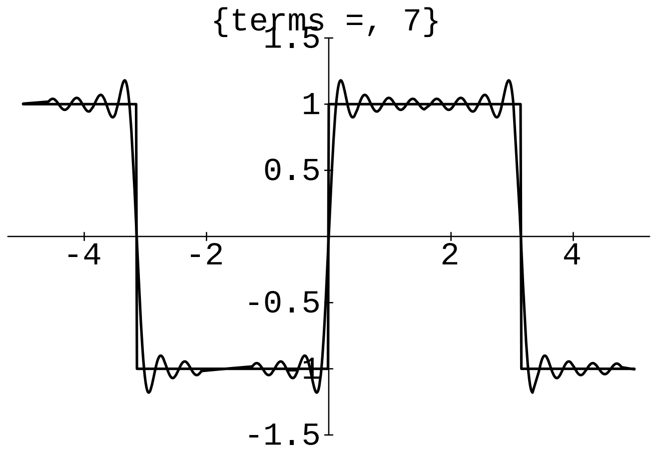


Figure 13.6: Gibbs Goalposts $f[x] = \text{Sign}[x]$

In fact, the graphs of the approximations do not converge to a "square wave," but rather to "goal posts." Each approximating term has an overshoot before the jump in Sign$[x]$ and these move to a straight line segment longer than the distance between ± 1. You can see this for yourself in the animation of the computer program **FourierSeries**. A book on Fourier series will show you how to estimate the height of the overshoot.

In both of these examples, no matter how many terms we take in the Fourier series, even a hyperreal infinite number, there will always be an x close to the jump where the partial sum is half way between the one-sided limit and the midpoint of the jump. In this sense the series is converging "infinitely slowly" near the jump.

13.4 Integration of Fourier Series

Fourier series of piecewise smooth functions can be integrated termwise, even if the series are not uniformly convergent.

Theorem 13.6. *Integration of Fourier Series*
Let $f[x]$ be a piecewise continuous 2π-periodic function with Fourier series

$$a_0 + \sum_{k=0}^{\infty} a_k \text{ Cos}[k\,x] + \sum_{k=0}^{\infty} b_k \text{ Sin}[k\,x]$$

(which we do not even assume is convergent.) The Fourier series can be integrated between any two limits $-\pi \le \alpha < \xi \le \pi$ and

$$\int_{\alpha}^{\xi} f[x]\;dx = a_0(\xi - \alpha) + \sum_{k=0}^{\infty} \int_{\alpha}^{\xi} a_k \text{ Cos}[k\,x]\;dx + \sum_{k=0}^{\infty} \int_{\alpha}^{\xi} b_k \text{ Sin}[k\,x]\;dx$$

Moreover, the series on the right converges uniformly in ξ.

PROOF:

Define the function

$$F[x] = \int_{-\pi}^{x} (f[\xi] - a_0)\, d\xi$$

Then $F[x]$ is continuous inside the interval $(-\pi, \pi)$ and piecewise smooth. Since $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f[\xi]\, d\xi$, $F[-\pi] = 0 = F[\pi]$, and $F[x]$ has a continuous periodic extension. Applying Theorem 13.5, the Fourier series for $F[x]$ converges uniformly. Denote this series by

$$F[x] = A_0 + \sum_{k=1}^{\infty} (A_k \operatorname{Cos}[k\,x] + B_k \operatorname{Sin}[k\,x])$$

Apply integration by parts to the definitions of the Fourier coefficients with $k > 0$,

$$\begin{aligned} A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} F[x] \operatorname{Cos}[k\,x]\, dx \\ &= \frac{1}{\pi}\left(F[\pi]\frac{\operatorname{Sin}[k\,\pi]}{k} - F[-\pi]\frac{\operatorname{Sin}[-k\,\pi]}{k} - \frac{1}{k}\int_{-\pi}^{\pi} f[x] \operatorname{Sin}[k\,x]\, dx\right) \\ &= -\frac{1}{k}\frac{1}{\pi}\int_{-\pi}^{\pi} f[x] \operatorname{Sin}[k\,x]\, dx = -\frac{b_k}{k} \end{aligned}$$

and similarly

$$B_k = \frac{a_k}{k}$$

Notice that the uniformly convergent series gives

$$\begin{aligned} F[x] - F[\xi] &= \sum_{k=1}^{\infty} (A_k(\operatorname{Cos}[k\,x] - \operatorname{Cos}[k\,\xi]) + B_k(\operatorname{Sin}[k\,x] - \operatorname{Sin}[k\,\xi])) \\ &= \sum_{k=1}^{\infty}\left(\frac{a_k}{k}(\operatorname{Sin}[k\,x] - \operatorname{Sin}[k\,\xi]) - \frac{b_k}{k}(\operatorname{Cos}[k\,x] - \operatorname{Cos}[k\,\xi])\right) \end{aligned}$$

Replace $F[x]$ by its definition and the differences by integrals,

$$-\frac{1}{k}(\operatorname{Cos}[k\,x] - \operatorname{Cos}[k\,\xi]) = \int_{\xi}^{x} \operatorname{Sin}[k\,\zeta]\, d\zeta \qquad \text{and} \qquad \frac{1}{k}(\operatorname{Sin}[k\,x] - \operatorname{Sin}[k\,\xi]) = \int_{\xi}^{x} \operatorname{Cos}[k\,\zeta]\, d\zeta$$

to see the uniformly convergent series

$$\int_{\xi}^{x} f[\zeta]\, d\zeta - a_0\int_{\xi}^{x} d\zeta = \sum_{k=1}^{\infty}\left(a_k\int_{\xi}^{x} \operatorname{Cos}[k\,\zeta]\, d\zeta + b_k\int_{\xi}^{x} \operatorname{Sin}[k\,\zeta]\, d\zeta\right)$$