

Chapter 11

Maxima and Minima in One Variable

Finding a maximum or a minimum clearly is important in everyday experience. A manufacturer wants to maximize her profits, a contractor wants to minimize his costs subject to doing a good job, and a physicist wants to find the wavelength that produces the maximum intensity of radiation.

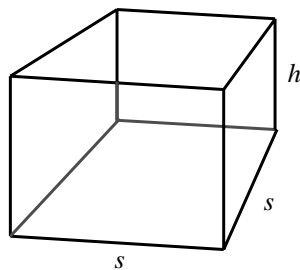
Even a manufacturer with a monopoly cannot maximize her profits by charging a very high price because at some point consumers will stop buying. She seeks an optimal balance between supply and demand. In everyday experience, these optima are sought in intuitive ways, probably incorrectly in many cases. The problems are not usually simple, and often they are not even clearly formulated. Calculus can help. It can solve closed-form problems and offer guidance when the mathematical models are incomplete. Much of the success of science and engineering is based on finding symbolic optima for accurate models, but no one pretends to know closed-form models for the national health profile and similar interesting, but complicated facts of everyday life.

This chapter starts with some basic mathematical theory and then looks at some ‘simple’ applications. They are simple in terms of life’s big questions but still offer some challenge mathematically. They may not be major “real world” revelations but are a start and have some practical significance. We are not retreating from our basic philosophy in this course, which is to show you what good calculus is. We think you know that optimization is important, so we will begin with the easy part: the mathematical theory. (Ugh, you say, but it really is the easy part.) Read the theory quickly, try the applications, and come back to the theory as needed.

11.1 A Graphical Minimum

We begin with a graphical maximization.

We want to make a box with a square base and no top. We need the box to hold 100 cubic inches and want to make it out of the least possible amount of cardboard. What dimensions should we use?



$$A = \frac{400}{s} + s^2$$

The graph of area as a function of the length of the side is shown in Figure 11.1:1.

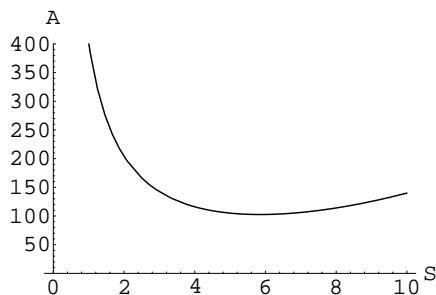


Figure 11.1:1: Area as a function of the side

It is clear on the graph of Figure 11.1:1 that the function decreases as s increases from 0 to about 5.8 and then the function increases as s increases beyond this. Hence, the minimum occurs when $s \approx 5.8$ and $h \approx 2.9$. The exercise below asks you to work through this reasoning.

This example is fine as far as it goes, but we will see that calculus can tell us more. The minimizing s and A are readily seen on the graph but values can only be read approximately. As you work through the exercise, think about how calculus could help you determine the shape and especially the point at which the graph changes from going down to going up.

Exercise Set 11.1 Graphical Minimum Box

1. (a) Show that the area of a box with an open top and a square base of volume 100 cubic inches is given by $A = \frac{400}{s} + s^2$, where s is the length of the edge of the base. (HINTS: The volume of 100 cubic inches can be expressed in terms of length, width and height, $V = lwh$, but $l = w = s$ and $V = 100$, so $100 = hs^2$. This makes $h = \frac{100}{s^2}$. The area of each side piece is $hs = \frac{100}{s}$, substituting $h = \frac{100}{s^2}$.)
- (b) Sketch the graph of $A = A[s]$. One approach is to sketch $y = \frac{400}{s}$ and $z = s^2$, then sketch the graph of the sum $y + z$ on the same graph. Use the same scales $0 < s < 10$. You may use the computer to check your graphs if you wish. (With or without the computer, how do we know which scales to use?)
- (c) Locate the side s_m that minimizes area on the graph of A vs. s .
- (d) What is the slope of the graph and the value of $\frac{dA}{ds}[s_m]$ at s_m ?

11.2 Critical Points

You should think of the theory of max-min as a lazy approach to graphing. If you had a “complete” graph, you could look and see where the maximum and minimum occurred (assuming all features occur on the same scale).

Even with a properly scaled graph, you might want to compute the symbolic formula for the maximum or minimum. Many scientific results amount to the symbolic way a max or a min depends on some parameter. We will see examples at the end of this chapter and several projects use this idea. (See Planck’s derivation of Wien’s Law or Resonant Frequency or the Notch Filter in the accompanying book of projects.)

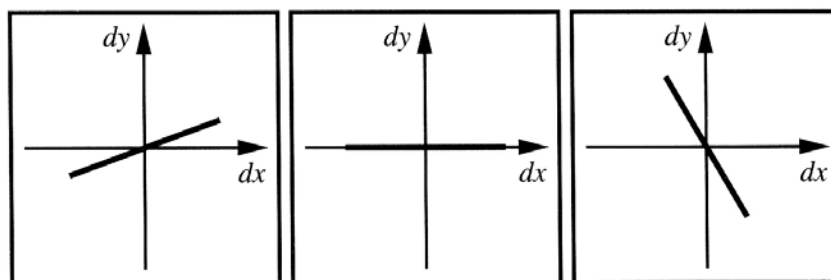


Figure 11.2:2: Up, over, down

Usually you do not need the whole graph (which is frequently hard to compute) to find a max or a min. For example, if you knew that a function $f[x]$ had a microscopic slope table starting at $x = 1$ with an upward slope that continued until $x = 2$ and then had a downward slope until $x = 3$, you would know that the maximum of $f[x]$ for $1 \leq x \leq 3$ occurs at $x = 2$. Use the microscope table to sketch such a curve.

Where does the minimum of this function occur? You need more information but only a little. It could be at either $x = 1$ or $x = 3$, but it could not be at any other x . Why? This is obvious. Think about it and sketch two graphs with the microscope table up-over-down, one with a min at $x = 1$ and the other with a min at $x = 3$. Endpoints or the lack of endpoints play an important role in max - min theory. You will learn to look for them in applications, because they simplify such problems.

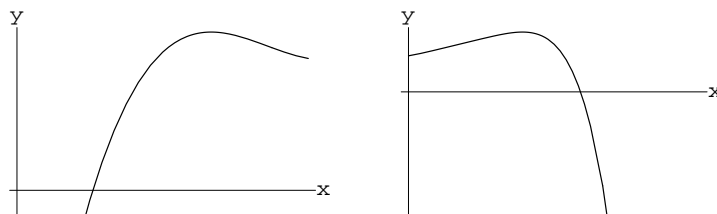


Figure 11.2:3: Minimum at different endpoints

Mathematicians are fussy about the exact meaning of technical terms. When we say the function $f[x]$ attains its maximum for all real x at $x = 0$, we mean that $f[0] \geq f[x]$ for all x . That is reasonable enough. The graph in Figure 11.2:4 of $f[x] = \frac{1}{x^2+1}$ has its max at $x = 0$. Everyday talk might say that the minimum of this function is $y = 0$, but this is *not* the custom in mathematics. It is true that $f[x]$ tends to zero as x tends to either $+\infty$ or $-\infty$, and it is true that $f[x] > 0$ for all x , but since $f[x]$ never actually takes the value zero, we say $f[x]$ does not have a minimum. Perhaps it would be better to say it does not attain its minimum.

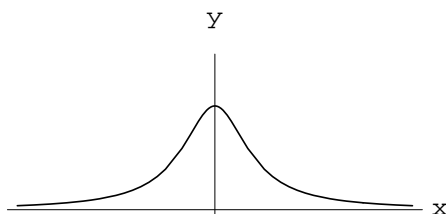


Figure 11.2:4: Max, but no min attained

Limiting values are not entirely the difficulty. The graph of $g[x] = \frac{x}{x^2+1}$ in Figure 11.2:5 has both a max and a min and also has limiting values. Precisely speaking, there are points x_{min} and x_{MAX} such that for all other x , $y_{min} = g[x_{min}] < g[x] < g[x_{MAX}] = y_{MAX}$. Find these points (x_{min}, y_{min}) and (x_{MAX}, y_{MAX}) on the graph.

Having pre-assigned endpoints simplifies max-min theory because endpoints eliminate limiting behavior. We discuss this in the next section. For now, we want to explore the “critical point condition.” The question for this section is, “What slope could the graph have at a max or min?” The answer is clear graphically, except for a proviso.

The next result rules out many possible places that might be maxima or minima. That’s right - rules out points. It does not tell you where they are but where they are not. It is usually stated in the following logically correct but confusing way, but it means that we need to examine only points x_0 such that $f'[x_0] = 0$ for the possibility that they are a max or min. These points are “critical” in our investigation of extrema, but they may or may not be extrema.

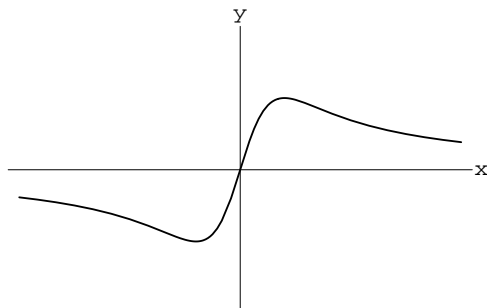


Figure 11.2:5: Max and min attained

Theorem 11.1 *Interior Critical Points*

Suppose $f[x]$ is a smooth function on some interval. If $f[x]$ has a maximum or a minimum at a point x_0 inside the interval, then $f'[x_0] = 0$.

PROOF:

We give a geometric proof. Suppose $f'[x_0] \neq 0$. We know that we will see a straight line of slope $f'[x_0] \neq 0$ if we look at the graph under a powerful microscope. Now, **if** we can move even a small amount to either side of x_0 , **then** we can make $f[x]$ both larger and smaller than $f[x_0]$ by moving both ways. If x_0 is inside an interval over which we maximize or minimize, then x_0 cannot be the max or the min.

(This proof is correct, but we can elaborate on it some more. If $f'[x_0] \neq 0$, then $f[x]$ is either increasing or decreasing in an interval containing x_0 . This is intuitively clear if you think of looking in a microscope and seeing a linear graph of slope $f'[x_0]$, but you can prove it algebraically in the project on Taylor's Formula.)

This completes the proof by use of the contrapositive. Yuck! The contrapositive? Why not find $f'[x]$, set it equal to zero, and solve. Then the solutions will be the max and min. Why not? Because that is wrong. Notice that the proof of the theorem is little more than a negative statement of where the extrema cannot be. Implications can be confusing and we want to make it clear that the theorem does **not** say that "if $f'[x_0] = 0$, then x_0 is a max or a min." Logically, A implies B does **not** mean that B implies A .

Correct reasoning is important in everyday life as well as in mathematics. The logic of implication can be confusing and that is what is behind the erroneous everyday conclusion that A implies B means that B also implies A . Let's see why this is so. "If it is raining, then there are clouds." is a correct statement.

$$A \Rightarrow B$$

where A stands for "it is raining" and B stands for "there are clouds." Certainly

$$B \Rightarrow A \text{ is false}$$

because $B \Rightarrow A$ says “if there are clouds, then it is raining.”

A way to explain implication is

$$A \Rightarrow B \quad \equiv \quad (\text{not}A) \text{ or } B$$

This is a funny way to think of it, but it is equivalent. When A is true, the implication means that B must also be true. When A is false, B can be either true or false. The other statement has the same properties, so it is equivalent to the implication. If A is true, $\text{not}A$ is false so the other part of the “or” must be true, B . If A is false, $\text{not}A$ is true; so B may be either true or false.

Now, we can do some logical calculus to see the contrapositive

$$A \Rightarrow B \quad \equiv \quad (\text{not}B) \Rightarrow (\text{not}A)$$

because $(\text{not}A) \text{ or } B \equiv (\text{not not}B) \text{ or } (\text{not}A)$.

Exercise Set 11.2

1. Find the maximum and minimum values of the function $f[x] = 3 - 2x$ for $-1 \leq x \leq 2$. What is the slope of the graph $y = f[x]$ at these points? Sketch the graph and mark the max and min.
2. Find the max and min of $f[x] = x^3$ for $-1 \leq x \leq 2$. Graph the function. Is $f'[x] = 0$ at the max or at the min? Where is $f'[x] = 0$?
3. In the Interior Critical Point Theorem 11.1, A stands for the statement “ $f[x]$ has an extremum at x_0 inside the interval” and B stands for “ $f'[x_0] = 0$.” The logical statement is of the form $A \Rightarrow B$. Give an example function to show that $B \Rightarrow A$ is false in this case.
4. Let B be the statement “ $f'[x_0] = 0$.” Let A be the statement “ $f[x_0]$ is an interior extremum for the interval I .” The interior critical point theorem above is the statement $A \Rightarrow B$. State the contrapositive theorem $(\text{not}B) \Rightarrow (\text{not}A)$ in English.
5. *Symbolic Minimum Box*
In Exercise 11.1.1, you showed that the area A of a square-base box with no top is

$$A = \frac{400}{s} + s^2$$

- (a) Show symbolically that the one and only (real) value of s where the slope of the A vs. s graph is zero is $s_0 = \sqrt[3]{200} \approx 5.84804$.
- (b) Make a slope table for the graph (as in Section 9.3).
- (c) Explain why your computation and the shape table PROVES that the minimum occurs when $s = s_0 = \sqrt[3]{200}$ and $A = 60\sqrt[3]{5} \approx 102.599$

11.3 Max - min with Endpoints

A smooth function $f[x]$ on a compact interval $[a, b]$ must have both a max and a min. This fact makes the job easier when we have endpoints.

Various intervals with and without endpoints arise in max-min problems. It is convenient to have some notation for the various cases. The basic notation is that round brackets, (or), cut off “just before” the endpoint, while square brackets, [or], include the endpoint. Here we will use the love knot symbol ∞ to mean intervals “keep going.” (The infinity symbol ∞ cannot stand for an extremely large hyperreal number because it **violates** rules of algebra, contrary to the Algebra Axiom for hyperreals given in the background material. For example, $\infty + \infty = \infty$ and $\infty \times \infty = \infty$ are false for both real and hyperreal numbers.) It is not a number, but ∞ is a convenient symbol.

Definition 11.1 *Notation:* If a and b are numbers,

$$\begin{array}{ll}
 a) \quad [a, b] = \{x : a \leq x \leq b\} & b) \quad (a, b) = \{x : a < x < b\} \\
 c) \quad [a, b) = \{x : a \leq x < b\} & d) \quad (a, b] = \{x : a < x \leq b\} \\
 e) \quad (-\infty, b] = \{x : x \leq b\} & f) \quad (-\infty, b) = \{x : x < b\} \\
 g) \quad [a, \infty) = \{x : a \leq x\} & h) \quad (a, \infty) = \{x : a < x\}
 \end{array}$$

This is simple. Try it out in Exercise 11.3.1.

The most important intervals are the ones of finite length that include their endpoints, $[a, b]$, for numbers a and b . These intervals are sometimes described as “closed and bounded,” because they have the endpoints and have bounded length. A shorter name is “compact” intervals.

Theorem 11.2 *The Extreme Value Theorem*

If $f[x]$ is a continuous real function on the real compact interval $[a, b]$, then f attains its maximum and minimum; that is, there are real numbers x_m and x_M such that $a \leq x_m \leq b$, $a \leq x_M \leq b$, and for all x with $a \leq x \leq b$

$$f[x_m] \leq f[x] \leq f[x_M]$$

PROOF:

We will show how to locate the maximum. You can find the minimum. We begin with an approximate maximum. Partition the interval into steps of size Δx ,

$$a < a + \Delta x < a + 2\Delta x < \cdots < b$$

and define a function

$$M[\Delta x] = \text{the } x \text{ of the form } x_1 = a + k\Delta x$$

so that $f[x]$ is maximal *for the partition* at $x = M[\Delta x]$,

$$f[M[\Delta x]] = f[x_1] = \max\{f[x] : x = a + h\Delta x, h = 0, 1, \dots, n\}$$

This function is the discrete maximum from among a finite number of possibilities, so that $M[\Delta x]$ has two properties: (1) $M[\Delta x]$ is one of the partition points. (2) All other partition points $x = a + h\Delta x$ satisfy $f[x] \leq f[M[\Delta x]]$.

Next, we partition the interval into tiny steps,

$$a < a + \delta x < a + 2\delta x < \dots < b$$

with $\delta x \approx 0$, and consider the discrete maximizing function $M[\delta x]$ along the fine partition. We know that (1) $x_1 = M[\delta x]$ is one of the points in the small partition. (2) $f[x] \leq f[x_1]$ for all other partition points x . (3) Every other number x_2 in $[a, b]$ is within δx of some partition point, $x_2 \approx x$.

Continuity of f means that $f[x] \approx f[x_2]$, so we have

$$f[x_2] \approx f[x] \leq f[x_1]$$

which says f almost attains its maximum at x_1 . This completes the intuitive proof. In the Mathematical Background on CD, we show how to complete it. (Essentially, we only need to fix the location of x_1 .)

You can make the computer mimic the proof for partitions of step Δx . You might hope to take Δx smaller and smaller and see what $M[\Delta x]$ converges toward. There really are serious problems with using this as a general purpose algorithm. It is very inefficient. It can oscillate as you make Δx smaller if there are several places near the maximum so the process might not converge. Still, you might want to try the proof directly on some simple functions.

11.3.1 Summary of the Theory on Compact Intervals

I am not a candidate for governor. That means that you do not have to consider me in making your vote or in your betting pool. We have two theorems about differentiable functions $f[x]$. One says that interior points of $[a, b]$ cannot be extrema unless $f'[x] = 0$, and the other says that there must be a max and a min. So what is left?

CANDIDATES:

If $f[x]$ is a differentiable function on the real compact interval $[a, b]$, then there is a max and a min amongst the points:

- 1) Endpoints, $x = a$ and $x = b$
- 2) Critical points $x = c$ such that $f'[c] = 0$

To find a max and a min, isolate the candidates, examine their record and vote. That is,

Procedure 11.1 *Isolate the Candidates:*

If you have a differentiable function $f[x]$ to extremize over a compact interval $[a, b]$:

- (a) Compute $f'[x]$. (Be sure it is defined on all of $[a, b]$.)
 - (b) Find the critical points, that is, all solutions c of $f'[c] = 0$ with $a < c < b$.
 - (c) Make a table of the values $f[x]$ at $x =$ endpoints and critical points.
 - (d) Select the largest and smallest values of the function at the candidate points.
-

It is often helpful to compute the microscopic slope table after you have found all the critical points in step (2). This is not essential when you have endpoints, but shows clearly where the graph is increasing and decreasing. The Extreme Value Theorem guarantees that there will be both a max and a min, whereas the Interior Critical Point Theorem says that we only need to check critical points and endpoints.

Example 11.1 *Find the maximum and minimum of*

$$f[x] = x^3 - 6x^2 + 9x + 1$$

on the interval $[0, 5]$.

SOLUTION:

First, we isolate the possible candidates. The endpoints are

$$x = 0 \quad \text{and} \quad x = 5$$

The interior critical points are found by first computing $f'[x]$ and then finding all solutions of the equation $f'[x] = 0$.

$$\begin{aligned} \frac{df}{dx} = f'[x] &= 3x^2 - 12x + 9 \\ &= 3(x - 1)(x - 3) \end{aligned}$$

The derivative is always defined, so $f[x]$ is continuous and differentiable on $[0, 5]$.

The solutions of $f'[x] = 0$ are

$$3(x - 1)(x - 3) = 0 \quad \Leftrightarrow \quad x = 1 \quad \text{or} \quad x = 3$$

This isolates the candidates, so we compute their values:

| Candidate | Value |
|-----------|----------|
| $x =$ | $f[x] =$ |
| 0 | 1 |
| 1 | 5 |
| 3 | 1 |
| 5 | 21 |

We see from the table of values that the maximum of f is 21 and that occurs at $x = 5$, $f[5] = 21$. The minimum is 1 and it occurs at two places, $f[0] = f[3] = 1$.

It is instructive to go on and compute the microscopic slope table. The formula $f'[x] = 3x^2 - 12x + 9 = 3(x-1)(x-3)$ makes $f'[x] > 0$ on $(-\infty, 1)$, $f'[x] < 0$ on $(1, 3)$ and $f'[x] > 0$ on $(3, 5)$. This is shown on Figure 11.3:6.

| | | | | | | |
|----|---|---|---|---|---|----|
| x | 0 | 1 | | 3 | | 5 |
| y | 1 | 5 | | 1 | | 21 |
| y' | ↗ | — | ↘ | — | ↗ | ↗ |

Figure 11.3:6: Slope Table for $f = x^3 - 6x^2 + 9x + 1$

The slope table shows that the left endpoint is a possible min, the critical point $x = 1$ is a possible max, the critical point $x = 3$ is another possible min, and the endpoint $x = 5$ is a possible max. The table of values is needed to decide which possibilities occur. A the computer graph is shown in Figure 11.3:7.

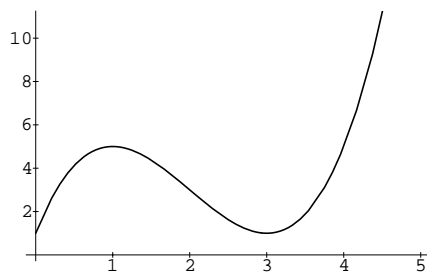


Figure 11.3:7: $y = x^3 - 6x^2 + 9x + 1$

It might be helpful to see how endpoint conditions arise in a simple application. We take that up in the next example, but first review a procedure for word problems:

Procedure 11.2 *To solve applied max-min exercises:*

- (a) Read the question and decide on a list of variables - main variables and auxiliary ones that help in translation. List the variables with units.
 - (b) Translate each phrase of the problem into a statement about your variables.
 - (c) State the question in terms of your variables. Try to give a compact interval over which you seek the max or min. Look hard for endpoints because they make the problem much easier. Unfortunately, it is not always possible to find a compact interval. In that case, you have less theory to help you solve your problem and it may not even have a solution.
 - (d) Apply the max-min theory that you can, computing limiting values and graphs as needed. When you have a compact interval, you only need to isolate the candidates as above. In the noncompact case, you will have to examine slopes and perhaps even limits.
 - (e) Interpret your solution.
-

To solve real-world max-min problems, you often need to formulate a clear statement of the question before you can begin to translate the information and state the problem mathematically. Often a drawing helps in the formulation.

Example 11.2 *We have a 12 inch square piece of thin material and want to make an open box by cutting small squares from the corners of our material and folding the sides up. The question is, "Which cut produces the box of maximum volume?"*

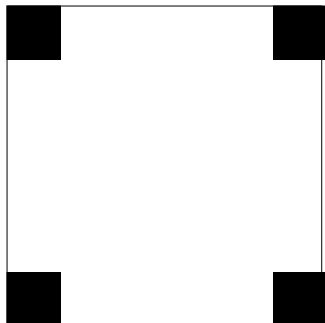


Figure 11.3:8: Max of volume

First, we draw the diagram Figure 11.3:8 and assign variables

x = the length of the cut on each side of the little squares
 V = the volume of the folded box

The length of the base after two cuts along each edge of size x is $12 - 2x$. The depth of the box after folding is x , so the volume is

$$V = x(12 - 2x)(12 - 2x) = 4x^3 - 48x^2 + 144x$$

Endpoints arise mathematically from the geometry. A cut must have positive length, $0 \leq x$, and even $x = 0$ means we have nothing to fold up. Two equal length cuts from 12 inches can each have a maximum length of 6 inches, and again $x = 6$ means we have no base left on our box.

MAXIMIZE:

$$V = 4x^3 - 48x^2 + 144x \quad x \in [0, 6]$$

The two endpoints $x = 0$ and $x = 6$ make $V[x] = 0$. Mathematically, these are minimum points, and so, are of no interest in the application. All we are left with for maximum candidates are interior critical points.

$$\frac{dV}{dx} = 12x^2 - 96x + 144 = 12(x - 2)(x - 6)$$

and

$$\frac{dV}{dx} = 0 \quad \Leftrightarrow \quad x = 2 \quad \text{or} \quad x = 6$$

There is only one interior critical point, $x = 2$, where $V = 128$. This must be the maximum, because we have already checked all other candidates.

Exercise Set 11.3

Finding the interior critical points requires that we find all solutions of

$$f'[x] = 0$$

Our textbook exercises are contrived to make this problem fairly easy. Factor or use the quadratic formula, something basic usually works. In “real-world” problems, this step can be quite difficult, but remember that the computer has FindRoot[.] and Solve[.] to help you.

1. Describe of the following intervals and sketch them as segments on the real line:

$$\text{a) } (0, \infty) \quad \text{b) } (0, 1) \quad \text{c) } (-1, 3] \quad \text{d) } [-2, 5]$$

2. Find the maximum and minimum of each of the following functions over the specified interval.

$$\text{(a) } f[x] = x^3 - 3x^2 + 3x + 2, \quad x \in [-2, 2]$$

$$\text{(b) } f[x] = x^3 - 6x + 4, \quad x \in [-3, 3]$$

$$\text{(c) } f[x] = x^3 + 7x + 4, \quad x \in [-10, 10]$$

$$\text{(d) } f[x] = x^3 + 7x + 4, \quad x \in [0, 10]$$

$$\text{(e) } f[x] = \frac{x}{x^2+1}, \quad x \in [-5, 5]$$

3. A homeowner has a long strip of 10 inch wide metal and wants to make a rain gutter by folding the sides up to form a rectangular cross-section with an open top shown in Figure 11.3:9. Where should she fold in order to get the maximum cross sectional area?

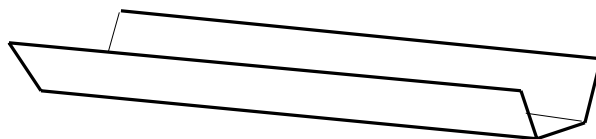


Figure 11.3:9: Folded gutter

4. Find the dimensions of the right circular cylinder with the largest volume that can be enclosed in a sphere of radius 24 inches. (What are the endpoints of the intervals for your variables? Is this question a continuous max-min on a compact interval?)

5. Geometric Endpoints?

In Exercise 11.2 you PROVED that the function $A = \frac{400}{s} + s^2$ with derivative $\frac{dA}{ds} = 2s - \frac{400}{s^2}$ has a minimum at $s = \sqrt[3]{200}$. There is no maximum for the area of this box and we want you to understand this both intuitively and mathematically.

- (a) Why are there no endpoints to the interval of possible s -values in this problem?
- (b) What happens to A as s approaches “limiting” values? (That is, what is the mathematical limit of A as s approaches each of the meaningful limits of the problem?)
- (c) Why does A have no maximum? For example, how could you build a box with a volume of 100 cubic inches and an area of 1 square mile (4014489600 square inches)?

6. *Fictitious Farming*

- (a) A farmer has 100 feet of fence and wants to make a rectangular holding pen. What dimensions should he make it in order to maximize the area?
- (b) Suppose the farmer makes the rectangular holding pen using an existing fence for one side. What dimensions should he use then to maximize the area within his 100 feet of new fence?

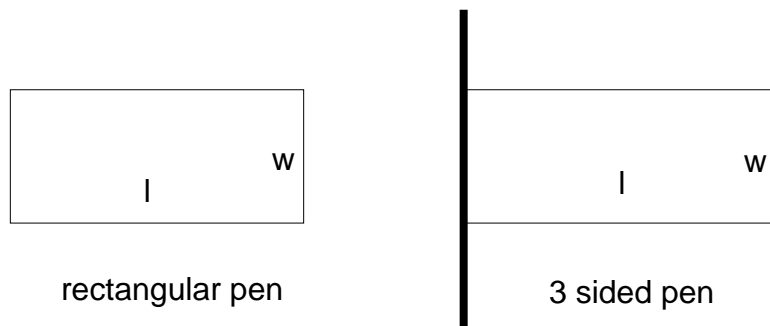


Figure 11.3:10: Holding Pens

7. *Mathematical Farming*

A math professor moves to the country and wants to fence some property so that he will enclose a rectangle of exactly 625 square feet of land. He likes fencing, so find the maximum amount of fence he can use in doing this. In particular, what are the dimensions of the math professor’s 625 square foot tract if he uses 1 mile of fence? (What’s wrong with this question intuitively and mathematically? Real farmers would have more sense, of course. What are the hypotheses of our max-min procedure?)

8. Find the max and min of

$$f[x] = \frac{\sqrt{x^2 - \frac{x^4}{2}}}{x} \quad \text{for } x \in [-1, 1]$$

Be careful and use the computer to plot the graph if you have trouble analyzing the function. What are the hypotheses of our max-min procedure?

9. Write a computer program to compute the maximizing function in the proof of the Extreme Value Theorem, $M[\Delta x]$. There is a start for you in the program *ExtremeValue*.

11.4 Max - min without Endpoints

The moral of this section is that you need to investigate more graphical properties when you have less theory.

The math professor who tries farming in Exercise 11.3 is an example of a max-min problem over a non-compact interval. The problem is silly in order to help get your attention and to help you reason intuitively about what is wrong with the optimization question. If the length of the field is l (feet) and the width is w , then the area of 625 makes $l = 625/w$ and the length of the fence $p = 2l + 2w = 2(w + 625/w)$. In other words, we are asking to MAXIMIZE $f[w] = w + 625/w$ for all positive values of w . (Of course, the function $f[w] = w + 625/w$ is not continuous at $w = 0$ and in fact, $f[w]$ tends to infinity as w tends to zero. We solve this in Example 11.3.)

There is no maximum. The silly math professor can use as much fence as he wants and still only enclose 625 square feet. (One student suggested that the solution was to have the math professor move back to town and rent an apartment.) The point is that lack of an endpoint, whether it is at infinity or zero, causes extra mathematical problems. The Extreme Value Theorem does not apply, and we need to use additional information from microscopic shape tables and the resulting graphs.

Without the hypotheses of the Candidates Procedure 11.4 - that we are working with a differentiable function $f[x]$ on a compact interval $[a, b]$ - there may not be either a max or a min. If the function is not differentiable, it might not be continuous and then there might not be a max or min. If the problem does not have the variable restricted to a compact interval, there may be problems as you approach an endpoint that is not there.

Example 11.3 Find the max and min of

$$f[x] = x + \frac{625}{x}$$

for all positive values, $x \in (0, \infty)$.

SOLUTION:

We know that we cannot have an interior max or min unless $f'[x] = 0$ or $f'[x]$ does not exist. In this problem,

$$f'[x] = 1 - \frac{625}{x^2}$$

which is defined on the open interval $(0, \infty)$. The positive critical value is

$$f'[x] = 0 \quad \Leftrightarrow \quad x = 25$$

When $0 < x < 25$, $f'[x] < 0$ and when $x > 25$, $f'[x] > 0$, so the microscopic slope table is down-over-up as shown in Figure 11.4:11.

| | | | | | | |
|----|---|---|----|---|---|------------|
| x | 0 | | 25 | | | + infinity |
| y | | | | | | |
| y' | * | ↘ | — | ↗ | ↗ | * |

Figure 11.4:11: Slope table for $y = x + \frac{625}{x}$

The slope table proves that $f[25] = 50$ is a minimum. What about a maximum? There is not any and now we have to reason directly. We need to know

$$\lim_{x \downarrow 0} f[x] \quad \text{and} \quad \lim_{x \rightarrow \infty} f[x]$$

because the function increases as we move from 25 in either of these directions. How can we compute these limits? The idea of $\lim_{x \rightarrow 0} f[x]$ is to see what happens as x gets smaller and smaller. We cannot just plug in $x = 0$, because the function is undefined. We can take a positive tiny $0 < \delta \approx 0$ and plug that in:

$$f[\delta] = \delta + \frac{625}{\delta} \approx \frac{625}{\delta} = 625 \frac{1}{\delta}$$

The reciprocal of a positive tiny number is an extremely large positive number, so $f[\delta]$ is positive and extremely large. In other words, $f[x]$ can be made as large as we please by taking x closer and closer to 0. We write this result

$$\lim_{x \downarrow 0} f[x] = \infty$$

What happens as x gets larger and larger? We write this $x \rightarrow \infty$, but recall that ∞ is not an ordinary (real or hyperreal) number. It is only a symbol to indicate “larger and larger.” To compute $\lim_{x \rightarrow \infty} f[x]$, take an extremely large positive number such as $H = \frac{1}{\delta}$ and compute

$$f[H] = H + \frac{625}{H} \approx H$$

which is extremely large again, so

$$\lim_{x \rightarrow \infty} f[x] = \infty$$

This function has no maximum - you can make it as large as you please by either taking x near enough to zero or taking x very large. A computer graph of $y = f[x]$ is shown on Figure 11.4:12.

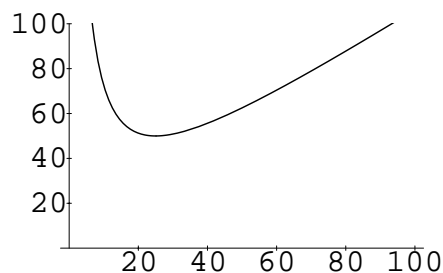


Figure 11.4:12: Min only attained

Our next example comes up as part of an economic model later in this chapter.

Example 11.4 Find the max and min of

$$T[p] = (p - 0.5) \left(\frac{1000}{1 + p^2} \right)$$

for $0.5 \leq p$ or $p \in [0.5, \infty)$

SOLUTION:

The first step is to find the derivative of $T[p] = 500(2p - 1)(1 + p^2)^{-1}$ using the Product Rule and Chain Rule (or quotient rule, if you prefer):

$$\begin{aligned} T'[p] &= 500 \frac{d(2p - 1)}{dp} (1 + p^2)^{-1} + 500(2p - 1) \frac{d([1 + p^2]^{-1})}{dp} \\ &= 1000(1 + p^2)^{-1} + 500(2p - 1)(-1)(1 + p^2)^{-2}(2p) \\ &= 1000 \frac{1 + p^2}{(1 + p^2)^2} - 500 \frac{(2p - 1)(2p)}{(1 + p^2)^2} \\ &= (-p^2 + p + 1) \frac{1000}{(1 + p^2)^2} \end{aligned}$$

The critical points are where $-p^2 + p + 1 = 0$, so by the quadratic formula,

$$p = \frac{-1 \pm \sqrt{1 + 4}}{-2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

or $p \approx 1.618$ and $p \approx -0.618$. We are only interested in values $p \in [0.5, \infty)$, so our only critical point is $p = \frac{1}{2} + \frac{\sqrt{5}}{2} \approx 1.618$.

The sign of $T'[p]$ for $0.5 \leq p < 1.618$ is $T'[p] > 0$. The sign for $p > 1.62$ is $T'[p] < 0$, so the microscopic slope table is up-over-down.

| | | | | | | |
|----|-----|---|------|---|---|------------|
| p | 0.5 | | 1.62 | | | + infinity |
| T | | | | | | |
| T' | / | / | — | \ | \ | * |

Figure 11.4:13: Slope table for $T = (p - 0.5)\left(\frac{1000}{1+p^2}\right)$

This proves that the maximum occurs at the critical value, but what about the minimum? At the endpoint $p = 0.5$, $T[0.5] = 0$. At all other values of $p \in [0.5, \infty)$, $T[p] > 0$, so zero is the minimum. What happens as p tends to infinity?

$$\lim_{p \rightarrow \infty} T[p] = ?$$

Take $p = P$ extremely large and estimate

$$\begin{aligned} T[p] &= (2P - 1) \frac{500}{1 + P^2} \\ &= 500 \frac{\left(\frac{2}{P} - \frac{1}{P^2}\right)}{\frac{1}{P^2} + 1} \end{aligned}$$

The quantities $\frac{2}{P}$ and $\frac{1}{P^2}$ are tiny when P is extremely large, so

$$T[p] = 500 \frac{\varepsilon}{\delta + 1}$$

for two tiny numbers, ε and δ . The product $500\varepsilon \approx 0$ and $\frac{1}{\delta+1} \approx 1$, so

$$T[P] \approx 0 \quad \text{and} \quad \lim_{p \rightarrow \infty} T[p] = 0$$

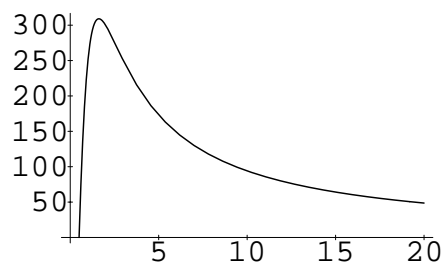


Figure 11.4:14: Min and max

Exercise Set 11.4

1. Find the maximum and the minimum (if they exist) of the following functions for all real x where they are defined. Are the extrema attained or only limiting values?

a) $f[x] = 3x^4 - 4x^3$ b) $f[x] = x^3 + 3x^2 + 4x + 5$

c) $f[x] = \frac{x}{x^2 + 4}$ d) $f[x] = \frac{x^2}{x^2 + 4}$

e) $f[x] = \frac{x^2}{x^3 + 4}$ f) $f[x] = \frac{x^3}{x^4 + 27}$

2. Find the maximum and the minimum of the following functions for all real x where they are defined. Are the extrema attained or only limiting values? Use the computer if you have technical difficulties. You should be able to find the shapes, but perhaps not the exact limiting values.

a) $f[x] = \frac{\text{Sin}[x]}{x}$ b) $f[x] = \frac{\text{Cos}[x]}{x}$

c) $f[x] = \frac{\text{Log}[x]}{x}$ d) $f[x] = x \text{Log}[x]$

e) $f[x] = xe^x$ f) $f[x] = xe^{-x^2}$

3. Complete the analysis of the max and min of

$$f[x] = \frac{\sqrt{x^2 - x^4/2}}{x}$$

Where is it defined? What are its max and min, if any? Are they attained or only approached?

11.5 Supply and Demand

In this section, we study simplified monopoly economies and show how a manufacturer maximizes profit.

“Monopoly” means a manufacturer can choose whatever price she wants without fear of competition. Our marketplace will be populated by a large number of individual consumers with no

organized cooperation. Their aggregate demand for the product will be determined by the price. If the price is low, then the consumers will buy a great deal of the product. But, as the price rises, the demand will fall. Thus, the demand function will be some sort of decreasing function of price.

Our manufacturer wants to set the price that will maximize her total profit. She cannot simply make the price as high as she wishes because demand will drop. A high price on a few units may well produce less profit than a smaller price on many. On the other hand, a very low price may sell many units but produce no profit because of the costs of manufacturing.

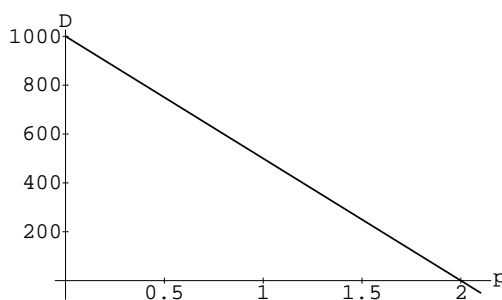


Figure 11.5:15: A linear demand function

Our main simplification is in giving an explicit formula for the number of units of the product that consumers want to purchase or “demand.” As an example, we begin by working with the linear demand function, $D[p] = 1000 - 500p$.

Example 11.5 *Maximum Profit for Demand $D[p] = 1000 - 500p$*

For this demand function, the maximum possible demand is 1000 units when the product is given away. The maximum price is 2 dollars after which no one wants the product. Thus, we are interested in a compact range of prices $0 \leq p \leq 2$.

Now, suppose that the manufacturing cost (i.e., raw materials and labor) for each unit of the product is 20 cents. The unit profit at price p will be $(p - 0.20)$ and the total profit $T[p]$ at the price p will be the number of units sold at the price p multiplied by the profit per unit. In general, for any demand function $D[p]$,

$$T[p] = D[p](p - \text{cost per item})$$

In this example

$$T[p] = (1000 - 500p)(p - 0.20)$$

This further restricts the range of prices to $0.20 \leq p \leq 2$ because the manufacturer will not sell at a loss. The next figure is a graph of this simple quadratic function.

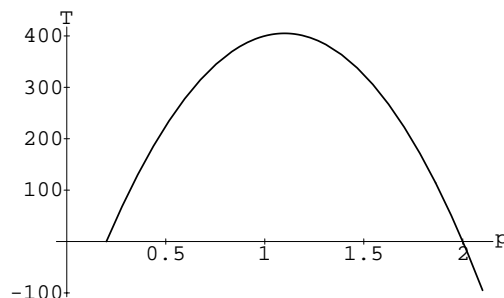


Figure 11.5:16: Total profit vs. price

Our manufacturer wants to find the price p that maximizes her profit.

$$\text{Max}[T[p] : 0.20 \leq p \leq 2]$$

Notice that she loses money to sell for less than 20 cents and that no one will buy for 2 dollars or more.

She wants to find the maximum of the function $T[p] = (1000 - 500p)(p - 0.20)$ on the interval $[0.20, 2]$. We already know that the endpoints represent points of zero profit or the mathematically necessary, but economically uninteresting minima. The next step is to differentiate the function $T[p]$. (We use the Product Rule rather than expand the expression.)

$$\begin{aligned} T'[p] &= (1000 - 500p)(1) + (p - 0.20)(-500) \\ &= 1000 - 500p - 500p + 100 \\ &= 1100 - 1000p \end{aligned}$$

Next, we solve the equation

$$\begin{aligned} T'[p] &= 0 \\ 1100 - 1000p &= 0 \\ 1000p &= 1100 \\ p &= 1100/1000 \\ p &= 1.10 \end{aligned}$$

Thus, $T'[p]$ is zero at $p = 1.10$ and only there. Notice, in addition, that $T'[p]$ is positive if $p < 1.10$ and $T'[p]$ is negative if $p > 1.10$. This means that the microscopic slope table is up-over-down, $T[p]$ is increasing from $p = 0.20$ to $p = 1.10$ and decreasing from $p = 1.10$ to $p = 2.00$. Of course, we spoiled this analysis by showing the graph above first. You can look at it to see the slope information. We could have done the slope analysis without the graph - and might need to with a complicated demand function.

We have already seen that $T[0.20] = 0$ (or $T[0] < 0$) and that $T[2.00] = 0$. There is only one critical point inside $(0.20, 2)$ so the Candidates procedure says the max must be at $p = 1.10$, even without the slope analysis or the graph.

The profit maximizing price is \$ 1.10, and, at this price, the manufacturer will make a total profit of $T[1.10] = 405$ dollars.

Example 11.6 *A Parametric Maximum*

You will notice an interesting phenomenon in the Exercise 11.5.3. Exactly half of the rise in the unit manufacturing cost is passed on to the consumer. Is it true in general that a manufacturer always passes half of a cost increase to her customers? Let us begin by analysing a general linear demand function. That is, use the demand function

$$D[p] = a - bp$$

where a and b are arbitrary positive constants. The graph of such a function, with $D = a$ if $p = 0$ and $D = 0$ at $p = a/b$, is shown in Figure 11.5:17.

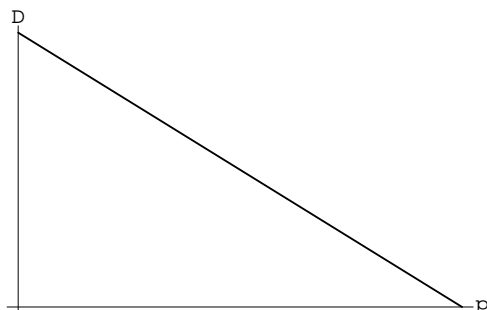


Figure 11.5:17: A general linear demand function

Let the constant c represent the unit manufacturing cost for the product. Thus, the profit per unit would be $(p - c)$ and the total profit would be

$$T[p] = (p - c)D[p] = (p - c)(a - bp)$$

where a , b , and c are all positive constants.

Solving the optimization problem with the parameters a and b will tell us more than we would have known by solving only numerical examples. The numerical examples suggested that half the cost increases were passed on in the linear demand case, but the parameter solution will show it. Optimization with parameters often yields important insights beyond those that you can see from special numerical cases. The parameters may be confusing so that working a numerical case helps to get started, but frequently they are indispensable for gaining scientific insight.

We want to find the profit-maximizing price as a function of the parameters. The Candidates Procedure says to check the endpoints and interior critical points. We are to maximize $T[p]$ over the compact interval $[c, a/b]$, but we know that $T = 0$ at the endpoints $p = c$ and $p = a/b$. Hence, we calculate

$$\begin{aligned} T'[p] &= (p - c)(-b) + (a - bp)(1) \\ &= -bp + bc + a - bp \\ &= -2bp + bc + a \end{aligned}$$

To find the maximum, we set $T'[p] = 0$ and solve

$$\begin{aligned} -2bp + bc + a &= 0 \\ 2bp &= bc + a \\ p &= \frac{bc + a}{2b} \\ &= \frac{a}{2b} + \frac{c}{2} \end{aligned}$$

This price maximizes profit because some point must be by the Extreme Value Theorem and there is only one candidate left. (You could make a slope table, up-over-down, turning at this price.)

This answers our question for any linear demand function. If our first cost is c_1 , our first maximizing price is

$$p_1 = \frac{a}{2b} + \frac{c_1}{2}$$

If our second cost is c_2 , our second price is

$$p_2 = \frac{a}{2b} + \frac{c_2}{2}$$

and

$$p_2 - p_1 = \frac{a}{2b} + \frac{c_2}{2} - \frac{a}{2b} + \frac{c_1}{2} = \frac{c_2 - c_1}{2}$$

What about nonlinear demand functions? See Exercise 11.5.4.

Exercise Set 11.5

1. Suppose that the unit manufacturing cost for the product with demand $D[p] = 1000 - 500p$ rises to 0.30 per item. What price should the manufacturer charge to maximize her profit now? (In the text example she charged 1.10 at a cost of 0.20.) How much of the rise in the manufacturing cost is passed on to the consumer?
2. Suppose the demand function for another product is $D[p] = 2000 - 500p$. What price should the manufacturer charge to maximize his profit in the following cases?
 - The cost of producing each item is 0.30.

- The cost of producing each item is 0.50. How much of the rise in the manufacturing cost is passed on to the consumer?
3. Suppose the demand function for a third product is $D[p] = 1500 - 100p$. What price should the manufacturer charge to maximize profit in the following cases?
- The cost of producing each item is 0.40.
 - The cost of producing each item is 0.50. How much of the rise in the manufacturing cost is passed on to the consumer?
4. Suppose that the demand for a particular commodity is given by the function

$$D[p] = \frac{1000}{1 + p^2}$$

(See Example 11.4 for the max-min analysis of this exercise.) Find the profit maximizing price in the following cases:

- The unit cost of production is $c = 0.50$
 - The unit cost of production is $c = 1.00$
 - The unit cost of production is $c = 1.50$
5. In view of the cases of the previous exercise what can you say about our earlier question on how much of a cost of a materials increase is passed to consumers in the nonlinear case?

In Chapter 20, we investigate a dynamic model of price adjustment in an economy where producers do not have a monopoly. Instead, they are willing to produce goods in quantities depending on the price. That introduces the supply side.

11.6 Constrained Max-Min

This section solves max-min problems with a constraint. For example, we might want to know the minimum distance between two curves. We are constrained to choose points on the curves.

Many max-min applications are easier to solve using implicit differentiation (first mentioned in Section 7.2). Review those simple examples to remind you how to use implicit differentiation and then we can apply the method to max-min with constraints.

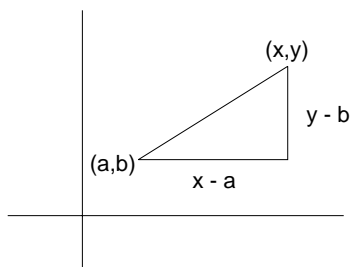


Figure 11.6:18: Coordinate Distance

11.6.1 The Distance Formula

Recall that the formula for the distance between the points with coordinates (x, y) and (a, b) is

$$\text{distance} = \sqrt{(x - a)^2 + (y - b)^2}$$

This is simply a coordinate expression of the Pythagorean theorem that the length of the hypotenuse is the square root of the sum of the squares of the lengths of the two legs of a right triangle. (English is an awkward way to express this.)

Example 11.7 Points on a Circle Nearest a Point

Find the points on the unit circle $x^2 + y^2 = 1$ nearest and farthest from $(1, 1)$. We will use implicit differentiation in this basic example and ask you to generalize it and check your answer with common sense geometric reasoning in Problem 11.3.

The distance from (x, y) to $(1, 1)$ is given by $\sqrt{(x - 1)^2 + (y - 1)^2}$ but we will maximize and minimize the square of this distance. The square will be largest and smallest where the distance is, and differentiating the square is much simpler. This makes our problem:

Find

$$\max \text{ and } \min [D = (x - 1)^2 + (y - 1)^2 : x^2 + y^2 = 1]$$

The differential of the square distance is

$$dD = 2(x - 1) dx + 2(y - 1) dy$$

The differential of the constraint (when $y \neq 0$) is

$$2x dx + 2y dy = 0$$

$$y dy = -x dx$$

$$dy = -\frac{x}{y} dx$$

Substituting this into the dy -expression in dD gives

$$\begin{aligned} dD &= 2(x-1) dx + 2(y-1) dy \\ &= 2(x-1) dx + 2(y-1) \left(-\frac{x}{y} dx\right) \\ &= 2 \left((x-1) - \frac{x(y-1)}{y} \right) dx \\ \frac{dD}{dx} &= 2 \left(\frac{y(x-1) - x(y-1)}{y} \right) \end{aligned}$$

The critical points where $\frac{dD}{dx} = 0$ (and $y \neq 0$) are:

$$\begin{aligned} \frac{dD}{dx} = 0 &= 2 \left(\frac{y(x-1) - x(y-1)}{y} \right) \\ y(x-1) &= x(y-1) \\ x &= y \end{aligned}$$

But the point must be on the circle, so we need to have both

$$x = y \quad \text{and} \quad x^2 + y^2 = 1$$

We can easily solve these simultaneously, obtaining $x = \pm \frac{1}{\sqrt{2}}$.

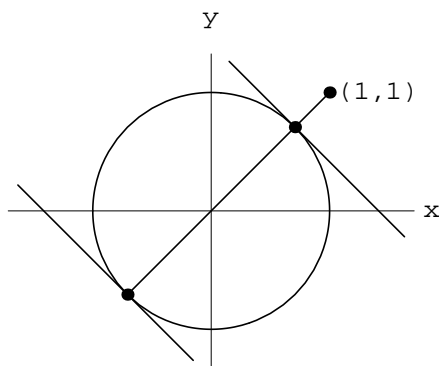


Figure 11.6:19: Points on $x^2 + y^2 = 1$ nearest and farthest from $(1, 1)$

Example 11.8 *Endpoints in the Implicit Distance Problem*

Once you see the solution to the previous problem on Figure 11.6:19, it is geometrically obvious. However, there are two gaps in the reasoning we used and it is helpful to fill these gaps in a problem that is obvious.

First, we neglected the case $y = 0$ when we computed the differential (since we divided by y .) In Chapter 19 we take a more geometrical approach (that does not require dividing), but, for now, we could simply solve the constraint differential for

$$dx = -\frac{y}{x}dy$$

and substitute into dD , obtaining

$$\frac{dD}{dy} = 2\frac{-y(x-1) + x(y-1)}{x}$$

Solving $\frac{dD}{dy} = 0$ and the original ellipse equation gives the same answer when $x \neq 0$ and we cannot have both $x = 0$ and $y = 0$.

The second gap in our reasoning is not just a technical problem. Max-min theory works most effectively in the case where the independent variable of a smooth function runs over a compact interval. In that case, we can guarantee that there is a max and a min and find them by checking endpoints and critical points. Since we have given the problem implicitly, it is a little harder to justify use of this theory. The variable x does range over the compact interval $[-1, 1]$, but we do not have the square distance given as a smooth function $D = f[x]$. We could apply the theory on the top and the bottom of the circle where $D = f[x]$ may be solved as a continuous function on a compact interval. Those details show that there has to be a max and a min and that the critical values we have found are the only candidates. (A more systematic approach is given in Chapter 19.)

Solving the systems of equations that arise in constrained max-min problems can become quite technical, but do not forget the computer.

Example 11.9 *Help from the Computer*

Find the points on the ellipse

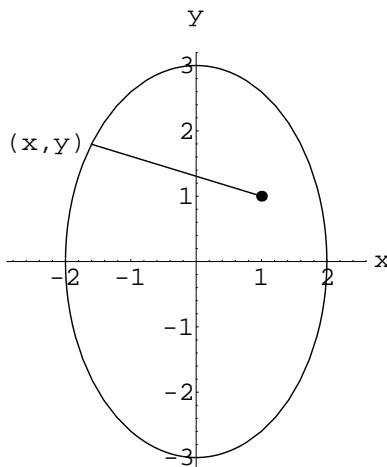
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

that are nearest and farthest from the point $(1, 1)$. The distance from a point (x, y) to $(1, 1)$ is illustrated on Figure 11.6:20.

SOLUTION:

The distance is given by $\sqrt{(x-1)^2 + (y-1)^2}$, but it is easier to extremize the square,

$$D = (x-1)^2 + (y-1)^2$$

Figure 11.6:20: $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

The differential of the squared distance is

$$dD = 2(x - 1)dx + 2(y - 1)dy$$

and the differential of the constraint equation

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

is the equation

$$\begin{aligned} \frac{1}{2}x \, dx + \frac{2}{9}y \, dy &= 0 \\ dy &= -\frac{9x}{4y}dx \end{aligned}$$

provided $y \neq 0$. We substitute this expression for dy into the differential dD , obtaining

$$dD = 2(x - 1)dx + 2(y - 1) \left(-\frac{9x}{4y}\right) dx$$

so

$$\frac{dD}{dx} = \frac{4y(x - 1) - 9x(y - 1)}{2y}$$

and we have $\frac{dD}{dx} = 0$ when the numerator is zero. The numerator also involves y because we used implicit differentiation, so we must also satisfy the original ellipse equation. The computer finds the simultaneous solutions of

$$4y(x - 1) = 9x(y - 1) \quad \text{and} \quad \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

to be

$$(x, y) \approx (1.82, 1.25) \quad \text{and} \quad (-.494, -2.91)$$

[and two complex roots].

The solution is shown on Figure 11.6:21. Notice the perpendicularity of the segments and tangents. See Problem 11.2. This example is also solved using a parametric equation in Problem 16.5.

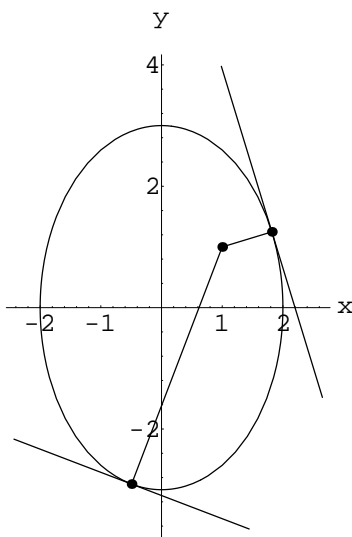


Figure 11.6:21: Points on $(\frac{x}{2})^2 + (\frac{y}{3})^2 = 1$ nearest and farthest from $(1, 1)$

In this problem, we know that there is a max and a min because the variables x and y are confined to compact intervals, $[-2, 2]$ and $[-3, 3]$. This is a little vague because the distance is not given by an explicit formula, but complete reasoning can be done along the lines of the circle example above. Sometimes it is not so easy to see where the endpoint conditions enter a problem. Here is another constrained maximization problem to illustrate this.

Example 11.10 *Hidden Endpoints*

A farmer wants to build a silo with cylindrical sides and a hemispherical top. He has a fixed budget of 10,000 dollars. The sides cost 3 dollars per square foot, but the top costs 9 dollars per square foot because of the labor and materials used in making it spherical. What proportions should the farmer make the silo?

SOLUTION:

Draw a diagram of the silo and let the radius be r and the height h (measured in feet)

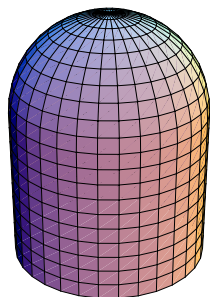


Figure 11.6:22: silo

The area of the cylindrical side is $A_{side} = 2\pi r h$, so the cost of the side is $C_{side} = 6\pi r h$. The area of the hemispherical top is $A_{top} = 2\pi r^2$ and its cost is $C_{top} = 18\pi r^2$. The sum is the total cost, which we set equal to \$ 10,000: $6\pi r h + 18\pi r^2 = 10000$, and divide both sides by 6π to obtain

$$r h + 3r^2 = \frac{10000}{6\pi}$$

whose differential yields the equation

$$h dr + r dh + 6r dr = (6r + h)dr + r dh = 0$$

and

$$r dh = -(6r + h)dr$$

The volume of the cylindrical part of the silo is $V_{cyl} = \pi r^2 h$, and the volume of the hemispherical top is $V_{top} = \frac{2}{3}\pi r^3$, so the total volume is

$$V_{total} = \pi r^2 h + \frac{2}{3}\pi r^3$$

with differential

$$\begin{aligned} dV_{total} &= \pi \left(2r h dr + r^2 dh + \frac{2}{3} 3r^2 dr \right) \\ &= \pi \left((2rh + 2r^2) dr + r(r dh) \right) \end{aligned}$$

substituting $r dh = -(6r + h)dr$, we obtain

$$dV_{total} = \pi (rh - 4r^2) dr$$

or

$$\frac{dV_{total}}{dr} = \pi r (h - 4r)$$

The critical values, $\frac{dV_{total}}{dr} = 0$ occur at $r = 0$ and $h = 4r$. The answer to the question is that the height should be four times the radius.

This method obscures the role of endpoints somewhat because it does not make the dependence between h and r explicit. We can reason that there are “endpoints” for r if we choose it as the independent variable, however. First, if $r = 0$, the silo has no volume. The limit of V_{total} as r tends to zero should be zero, but this is not completely obvious because h must tend to infinity in order to hold the cost at 10,000 dollars. However, the microscope comes to our rescue again. The derivative $\frac{dV_{total}}{dr} = \pi r (h - 4r)$ is positive when r is small and positive and h is large and the cost is fixed. This means V_{total} increases for small values of r and cannot tend to a larger value than the one we found. (Can you write V_{total} explicitly in terms of r and find the limit of V_{total} as r tends to zero?)

At the other extreme, if $h = 0$ (so the silo is a hemisphere), you can still spend all the money. Without solving, we can see that there is a maximal value of the radius.

Exercise Set 11.6

1. Find the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius r .

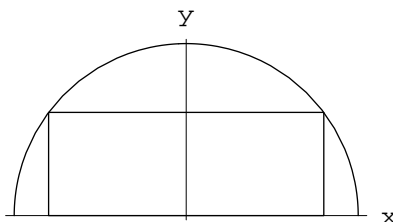


Figure 11.6:23: Rectangle inscribed in semicircle

2. A manufacturer wants to design a cylindrical can that will hold 1 liter of liquid (1 liter is 1000.0028 cubic centimeters). The total area of the outside of the can will be the sum of two disks at the top and bottom plus the cylindrical side. Express this in terms of radius and height. What should the dimensions of the can be in order to minimize the total area of the can and, hence, the cost of materials?

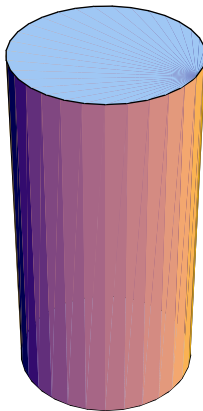


Figure 11.6:24: Cylindrical can

3. *Electronic Minima*

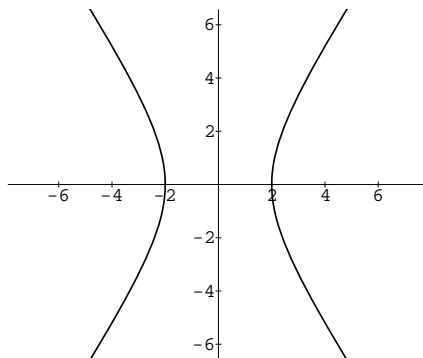
Find the point closest to $(x, y) = (1, 1)$ on each branch of the hyperbola

$$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{3}\right)^2 = 1$$

The computer says that the simultaneous solutions of

$$4y(x - 1) + 9x(y - 1) = 0 \quad \text{and} \quad (x/2)^2 - (y/3)^2 = 1$$

are $(x, y) = (2.07215, 0.813035)$ & $(x, y) = (-2.03981, 0.601566)$ and four complex roots.

Figure 11.6:25: $(\frac{x}{2})^2 - (\frac{y}{3})^2 = 1$

Sometimes we can find extrema without calculus, and it is a good idea for you to compare a simple example with several methods.

Problem 11.1 *A Geometric Minimum*

1. We have a line L and a point p not on the line. Draw an example. The point q on the line L that is closest to p is the one such that the segment pq is perpendicular to L . Add the segment to your figure. Any other point r on L must be farther from p than q . Use basic geometry to show why. (HINT: the Pythagorean Theorem.)
2. Suppose the line is given by the equation $y = -\frac{1}{2}x$ and the point has coordinates $(1, 1)$. Find the coordinates of the point q . HINT: The equation of the line through p and q has slope 2 (the negative reciprocal of the slope of L) and passes through $(1, 1)$. Find the equation. Make the algebraic statement that $q = (x, y)$ lies on both lines and solve.
3. Use calculus to minimize the square of the distance from $(1, 1)$ to a point $r = (x, y)$ on the line $y = -\frac{1}{2}x$.

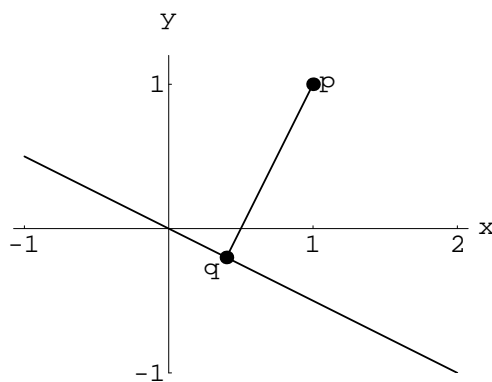


Figure 11.6:26: The distance from a point to a line

Problem 11.2 *Microscopic Geometric Minimum*

Suppose the point (x, y) minimizes the distance from a point (a, b) to a curve. Why does the segment from (a, b) to (x, y) intersect the curve at right angles? (HINT: Imagine viewing the point (x, y) under a powerful microscope.)

The next problem follows up on Example 11.7.

Problem 11.3 *Another Geometric Minimum*

Let a , b , and r denote constants. Use calculus to show that the points on the circle

$$x^2 + y^2 = r^2$$

of radius r that are nearest and farthest from the point (a, b) lie on the line through the center of the circle and the point (a, b) . Notice that the slope of the radial line from $(0, 0)$ to a point (x, y) on the circle is y/x and the slope of the line from (a, b) to (x, y) is $(y - b)/(x - a)$. What is the slope of the tangent to the circle at (x, y) ? Why does the equation

$$\frac{y}{x} = \frac{y - b}{x - a} \quad \text{or} \quad y(x - a) = x(y - b)$$

say that the segment from (a, b) to (x, y) meets the circle at right angles?

11.7 Max-min with Parameters

This section looks at max-min problems with a parameter. Often scientific max-min problems ask how the maximum depends on one or more parameters. This is how Wein's Temperature Law is derived from Planck's Radiation Law in the projects. Sharing half the cost of a price increase for linear demand was proved with parameters in the economics section of this chapter.

We begin with a simple contrived example to illustrate how to work with a strictly symbolic max-min question.

A Parametric Max-Min Problem

The function $f[x] = x^3 + cx$ has a fixed but unknown parameter c . Find the maximum and minimum of this function for $-2 \leq x \leq 2$.

First, the function is continuous everywhere because we can compute its derivative, $f'[x] = 3x^2 + c$, by rules and get formula valid for all real x . This means that the Extreme Value Theorem applies to $f[x]$ on $[-2, 2]$. The extrema must exist and, by the Critical Points Theorem, must occur either at endpoints or critical points.

Next, we want to know how the shape of the graph depends on the parameter. Compute $f'[x] = 3x^2 + c$ and ask when the graph is level, $f'[x] = 0$:

$$\begin{aligned} 3x^2 + c &= 0 \\ x^2 &= -c/3 \\ x &= \pm\sqrt{-c/3} \end{aligned}$$

Now, consider the cases: $c > 0$ and $c < 0$.

CASE 1: $c > 0$

The number $\sqrt{-c/3} = i\sqrt{|c/3|}$ is complex. There are no real solutions to the equation: “ $y = f[x]$ is level” $\Leftrightarrow f'[x] = 0 \Leftrightarrow x = \pm i\sqrt{|c/3|}$.

At $x = 0$, $f'[x] = c > 0$, so the graph $y = f[x]$ is always increasing. This is the easiest case for max-min. Since the function is increasing, $f[-2]$ is minimal and $f[2]$ is maximal for the x -interval $[-2, 2]$.

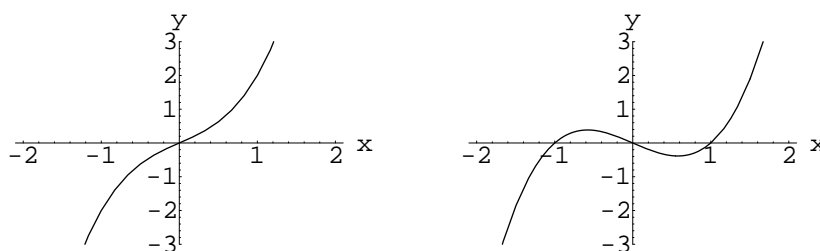


Figure 11.7:27: Examples of $y = x^3 + cx$ with $c > 0$ and $c < 0$

CASE 2: $c < 0$

The numbers $\pm\sqrt{-c/3} = \pm\sqrt{|c/3|}$ are real - one positive, one negative. The graph $y = f[x]$ is level at these two points and nowhere else.

At $x = 0$, $f'[x] = c < 0$, so the graph is decreasing near zero. When $|x|$ is huge, $3x^2$ is bigger than c , so the graph is increasing and the shape table is “up” before $-\sqrt{|c/3|}$, “over” at $-\sqrt{|c/3|}$, “down” until $\sqrt{|c/3|}$, “over” at $\sqrt{|c/3|}$, and up after that.

The candidates for max and min are $x = -2$, $x = +2$ and the critical values, *if* they lie in the interval $[-2, 2]$. Note that $\sqrt{-c/3} \leq 2 \Leftrightarrow -12 \leq c \leq 0$.

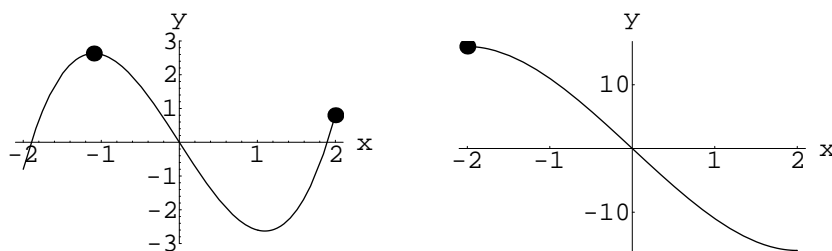


Figure 11.7:28: Examples of $y = x^3 + cx$ with $c < 0$ and $c \ll 0$

The more interesting question in this specific case, however, is: When is the highest critical value greater than the highest endpoint value? By analyzing the shape table, “up” - “down” - “up”, we want to solve $f[-\sqrt{|c/3|}] \geq f[2]$ and find that this happens when $c \leq -3$. The maximum

of $f[x] = x^3 + cx$ for $-2 \leq x \leq 2$ is at $x = 2$; for $c \geq -3$, at $x = -\sqrt{|c|/3}$ for $-12 \leq c \leq -3$; and at $x = -2$ for $c \leq -12$.

The minimum occurs at the negative value since the function is odd, $f[-x] = -f[x]$, and the interval is symmetric $|x| \leq 2$.

Here are some simple exercises on max-min with parameters. Note that some are general cases of previous exercises. We will see other important examples of the use of parameters in the following problems and in the projects.

Exercise Set 11.7

1. A manufacturer wants to design a cylindrical can that will hold the fixed (but unknown) amount V cubic centimeters. The total area of the outside of the can will be $A = 2\pi r^2 + 2\pi rh$ where r is the radius of the cylinder and h is its height. What should the dimensions of the can be in order to minimize the total area of the can and, hence, the cost of materials? (See Exercise 11.6.2)

Show that the cylindrical can (cylinder of height h and radius r with top and bottom) of fixed surface area A of maximal volume V has $h = 2r$, the height equal to the diameter.

2. Find the dimensions of the right circular cylinder with the largest surface area that can be enclosed in a sphere of area S . (See Exercise 11.3.4.)
3. A farmer wants to enclose a rectangular field with fixed (but unknown) area A square feet with the least possible amount of fencing.
 - (a) What should the dimensions of the field be?
 - (b) What is the minimal perimeter in terms of the parameter A ?
 - (c) If p denotes the perimeter of a rectangular field of any dimensions with an area of A , prove the inequality

$$4\sqrt{A} \leq p \quad \text{with equality only for a square}$$

- (d) What is the largest area A of a field with a perimeter that is fixed at a constant p ?
4. (See Exercise 11.3.6.)

A Family of Curves

The shape of the graph $y = f[x] = 2cx^2 - x^4$ for $x > 0$ depends on the value of c . Which values of c give $f[x]$ a positive maximum for $x \geq 0$? (HINT: Find the critical values in terms of c . When are they real numbers? Prove your case by making a shape table as in Section 9.3.)

5. Find the dimensions of the right circular cylinder with the largest volume that can be enclosed in a sphere of volume V .

6. Find the dimensions of the right circular cylinder with the largest surface area that can be enclosed in a sphere of surface area A .

The arithmetic mean is the usual average

$$\frac{a+b}{2}$$

The geometric mean

$$(a \times b)^{\frac{1}{2}}$$

is a sort of “multiplicative average” of positive numbers. Also, notice that if $\alpha = \text{Log}[a]$ and $\beta = \text{Log}[b]$, then $\frac{\alpha+\beta}{2} = \frac{1}{2}(\text{Log}[a] + \text{Log}[b]) = \text{Log}[(a \times b)^{\frac{1}{2}}]$.

Problem 11.4 *The Arithmetic and Geometric Means*

1. Use high school algebra to prove

$$\sqrt{ab} \leq \frac{a+b}{2}$$

by squaring both sides and doing some algebra to put the inequality in the form

$$0 \leq a^2 - 2ab + b^2 = (a-b)^2$$

2. For which positive values of a and b is

$$\sqrt{ab} = \frac{a+b}{2}$$

3. The inequality $\sqrt{ab} \leq \frac{a+b}{2}$ is equivalent to

$$a \leq \frac{1}{b} \left(\frac{a+b}{2} \right)^2$$

Treat a as a constant and use calculus to minimize the function

$$f(b) = \frac{1}{b} \left(\frac{a+b}{2} \right)^2$$

and show that the minimum is a when $b = a$.

4. Use calculus to prove

$$(abc)^{\frac{1}{3}} \leq \frac{a+b+c}{3}$$

for positive numbers, with equality only if $a = b = c$. First, this inequality is equivalent to

$$ab \leq \frac{1}{c} \left(\frac{a+b+c}{3} \right)^3$$

so treat a and b as parameters and minimize the function

$$f[c] = \frac{1}{c} \left(\frac{a+b+c}{3} \right)^3$$

showing that the minimum is greater than ab , unless $a = b = c$ when you get equality.

11.7.1 Resonance of a Linear Oscillator

Resonance of a vibrating system is a maximal response to forcing. Many old cars hum loudly at a speed like 46 mph, but are quieter at both slower and faster speeds. The vibration at 46 mph is maximal, at least for an interval of frequencies (a local maximum). You can hear the resonance.

Cars with worn-out shock absorbers also oscillate. In Chapter 23 we show that an idealized car's front end when forced with the pure sinusoid of frequency ω ,

$$\text{Sin}[\omega t]$$

has a response of amplitude

$$A[\omega] = \sqrt{\frac{1}{(s - m\omega^2)^2 + (c\omega)^2}}$$

when it consists of a spring of constant s , a shock absorber with damping constant c , and mass m .

Problem 11.5 Resonant Frequency of a Linear Oscillator

Find the maximum of $A(\omega)$ for positive forcing frequencies ω . To simplify the problem, minimize the square of the denominator

$$B[\omega] = (s - m\omega^2)^2 + (c\omega)^2$$

by showing that it has critical values at zero and the roots of a simple quadratic equation. The idealized front end will oscillate on its own at a certain "natural frequency" when $4ms > c^2$. You do not get an interior maximum unless you have the condition $2ms > c^2$. Why is that? The natural frequency is what you observe when a car with worn shocks goes over a bump and then back onto smooth (unforced) pavement. The car keeps bouncing up and down for a long time. This natural frequency is

$$\omega_N = \sqrt{\frac{s}{m} - \frac{c^2}{4m^2}}$$

How does this compare to the maximal amplitude frequency (where B is minimized, $\omega_M = \sqrt{\frac{s}{m} - \frac{c^2}{2m^2}}$)? In particular, what happens in the particular cases

| m | c | s |
|-----|-----|---------------|
| 5 | 4 | $\frac{5}{4}$ |
| 5 | 4 | 5 |

How do the frequencies compare when c is very small, that is, in the limit as your shocks become completely useless? What happens to the amplitude $A[\omega]$ in this limit?

11.8 Projects

11.8.1 Optics and Least Time

Many interesting basic results in optics can be proved using minimization. Fermat's Principle says that light travels along the path that requires the least time. From this and max-min theory we can show that light reflects off mirrors at equal angles and is refracted according to Snell's Law which can be expressed:

$$\frac{\sin[\alpha]}{\sin[\beta]} = \frac{u}{v}$$

Using the computer, we can add many beautiful pictures showing light focusing at a point when reflecting off a parabolic mirror or creating a spherical aberration, or light caustic, when reflecting off a spherical mirror.

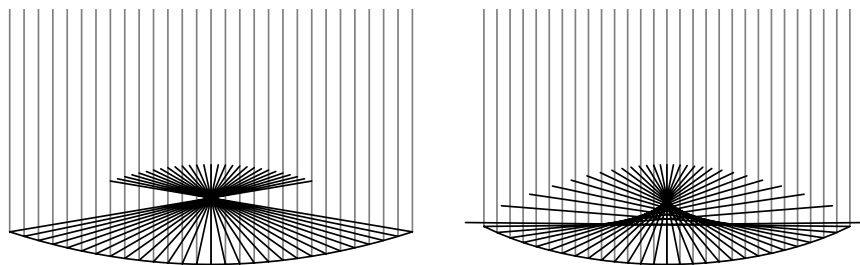


Figure 11.8:29: Parabolic and Spherical Mirrors

11.8.2 Monopoly Pricing

Optimal pricing is studied in a model of a monopoly which can charge different kinds of customers different prices. This is based on cable TV prices for separate homes and apartments.

11.8.3 Max-min in S-I-R Epidemics

We cannot find explicit formulas for the functions $s[t]$ and $i[t]$ of the epidemic model of Chapter 2, and this project shows how you can find max-min information anyway.