Chapter 10
Velocity, Acceleration, and Calculus

The first derivative of position is velocity, and the second derivative is acceleration. These derivatives can be viewed in four ways: physically, numerically, symbolically, and graphically.

The ideas of velocity and acceleration are familiar in everyday experience, but now we want you to connect them with calculus. We have discussed several cases of this idea already. For example, recall the following (restated) Exercise car from Chapter 9.

Example 10.1 *Over the River and Through the Woods*

We want you to sketch a graph of the distance traveled as a function of elapsed time on your next trip to visit Grandmother.

Make a qualitative rough sketch of a graph of the distance traveled, s, as a function of time, t, on the following hypothetical trip. You travel a total of 100 miles in 2 hours. Most of the trip is on rural interstate highway at the 65 mph speed limit. You start from your house at rest and gradually increase your speed to 25 mph, slow down and stop at a stop sign. You speed up again to 25 mph, travel for a while and enter the interstate. At the end of the trip you exit and slow to 25 mph, stop at a stop sign, and proceed to your final destination.

The correct "qualitative" shape of the graph means things like *not* crashing into Grandmother's garage at 50 mph. If the end of your graph looks like the one on the left in Figure 10.1:1, you have serious damage. Notice that Leftie's graph is a straight line, the rate of change is constant. He travels 100 miles in 2 hours, so that rate is 50 mph. Imagine Grandmother's surprise as he arrives!

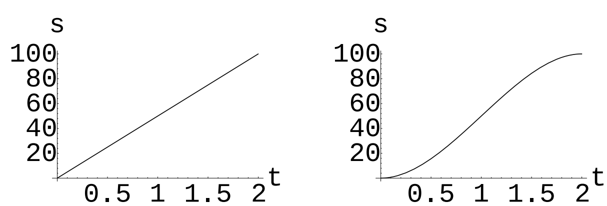

Figure 10.1:1: Leftie and Rightie go to Grandmother's

The graph on the right slows to a stop at Grandmother's, but Rightie went though all the stop signs. How could the police convict her using just the graph?

She passed the stop sign 3 minutes before the end of her trip, 2 hours less 3 minutes = 2 - 3/60 = 1.95 hrs. Graphs of her distance for short time intervals around $t = 1.95$ look like Figure 10.1:2

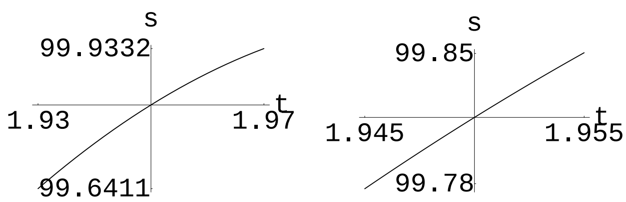

Figure 10.1:2: Two views of Righties moving violation

Wow, the smallest scale graph looks linear? Why is that? Oh, yeah, microscopes. What speed will the cop say Rightie was going when she passed through the intersection?

$$\frac{99.85 - 99.78}{1.955 - 1.945} = 7 \quad \text{mph}$$

He could even keep up with her on foot to give her the ticket at Grandmothers. At least Rightie will not have to go to jail like Leftie did.

You should understand the function version of this calculation:

$$\frac{99.85 - 99.78}{1.955 - 1.945} = \frac{s[1.955] - s[1.945]}{1.955 - 1.945} = \frac{s[1.945 + 0.01] - s[1.945]}{(1.945 + 0.01) - 1.945}$$
$$= \frac{s[t + \Delta t] - s[t]}{\Delta t} = \frac{0.07}{0.01}$$

Exercise Set 10.1

1. *Look up your solution to Exercise 9.2.1 or resolve it. Be sure to include the features of stopping at stop signs and at Grandmother's house in your graph. How do the speeds of 65 mph and 25 mph appear on your solution? Be especially careful with the slope and shape of your graph. We want to connect slope and speed and bend and acceleration later in the chapter and will ask you to refer to your solution.*

2. *A very small-scale plot of distance traveled vs. time will appear straight because this is a magnified graph of a smooth function. What feature of this straight line represents the speed? In particular, how fast is the person going at* $t = 0.5$ *for the graph in Figure 10.1:3? What feature of the large-scale graph does this represent?*

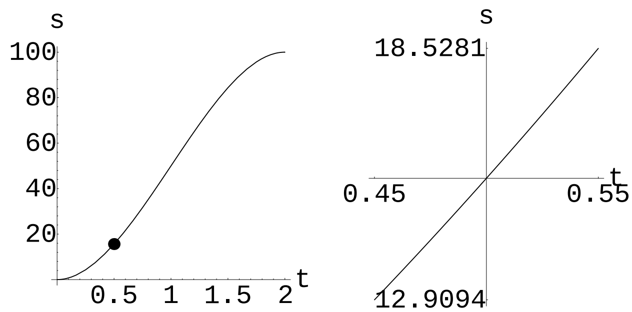

Figure 10.1:3: A microscopic view of distance

Velocity and the First Derivative

Physicists make an important distinction between speed and velocity. A speeding train whose speed is 75 mph is one thing, and a speeding train whose velocity is 75 mph on a vector aimed directly at you is the other. Velocity is speed plus direction, while speed is only the instantaneous time rate of change of distance traveled. When an object moves along a line, there are only two directions, so velocity can simply be represented by speed with a sign, + or −.

3. *An object moves along a straight line such as a straight level railroad track. Suppose the time is denoted t, with $t = 0$ when the train leaves the station. Let s represent the distance the train has traveled. The variable s is a function of t, $s = s[t]$. We need to set units and a direction. Why? Explain in your own words why the derivative $\frac{ds}{dt}$ represents the instantaneous velocity of the object. What does a negative value of $\frac{ds}{dt}$ mean? Could this happen? How does the train get back?*

4. *Krazy Kousin Keith drove to Grandmother's, and the reading on his odometer is graphed in Figure 10.1:4. What was he doing at time $t = 0.7$? (HINT: The only way to make my odometer read less is to back up. He must have forgotten something.)*

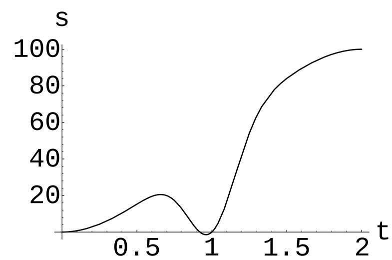

Figure 10.1:4: Keith's regression

5. *Portions of a trip to Grandmother's look like the next two graphs.*

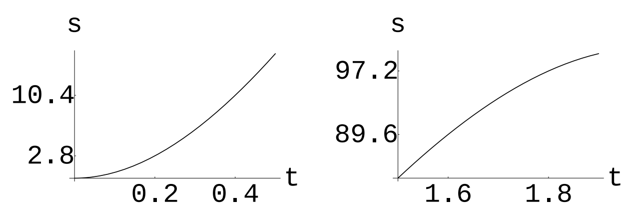

Figure 10.1:5: Positive and negative acceleration

Which one is "gas," and which one is "brakes"? Sketch two tangent lines on each of these graphs and estimate the speeds at these points of tangency. That is, which one shows slowing down and which speeding up?

10.2 Acceleration

Acceleration is the physical term for "speeding up your speed... " Your car accelerates when you increase your speed.

Since speed is the first derivative of position and the derivative of speed tells how it "speeds up." In other words, the second derivative of position measures how speed speeds up... We want to understand this more clearly. The first exercise at the end of this section asks you to compare the symbolic first and second derivatives with your graphical trip to Grandmother's. A numerical approach to acceleration is explained in the following examples. You should understand velocity and acceleration numerically, graphically, and symbolically.

Example 10.2 *The Fallen Tourist Revisited*

Recall the tourist of Problem 4.2. He threw his camera and glasses off the Leaning Tower of Pisa in order to confirm Galileo's Law of Gravity. The Italian police videotaped his crime and recorded the following information:

t = time (seconds)	s = distance fallen (meters)
0	0
1	4.90
2	19.6
3	44.1

We want to compute the average speed of the falling object during each second, from 0 to 1, from 1 to 2, and from 2 to 3? For example, at $t = 1$, the distance fallen is $s = 4.8$ and at $t = 2$, the distance is $s = 18.5$, so the change in distance is $18.5 - 4.8 = 13.7$ while the change in time is 1. Therefore, the average speed from 1 to 2 is 13.7 m/sec,

$$\text{Average speed} = \frac{\text{change in distance}}{\text{change in time}}$$

Time interval	Average speed $= \frac{\Delta s}{\Delta t}$
$[0, 1]$	$v_1 = \frac{4.90 - 0}{1 - 0} = 4.90$
$[1, 2]$	$v_2 = \frac{19.6 - 4.90}{2 - 1} = 14.7$
$[2, 3]$	$v_3 = \frac{44.1 - 19.6}{3 - 2} = 24.5$

Example 10.3 *The Speed Speeds Up*

These average speeds increase with increasing time. How much does the speed speed up during these intervals? (This is not very clear language, is it? How should we say, "the speed speeds up"?)

Interval to interval	Rate of change in speed
$[0,1]$ to $[1,2]$	$a_1 = \frac{14.7 - 4.90}{?} = \frac{9.8}{?}$
$[1,2]$ to $[2,3]$	$a_2 = \frac{24.5 - 14.7}{?} = \frac{9.8}{?}$

The second speed speeds up 9.8 m/sec during the time difference between the measurement of the first and second average speeds, but how should we measure that time difference since the speeds are averages and not at a specific time?

The tourist's camera falls "continuously." The data above only represent a few specific points on a graph of distance vs. time. Figure 10.2:6 shows continuous graphs of time vs. height and time vs. $s =$ distance fallen.

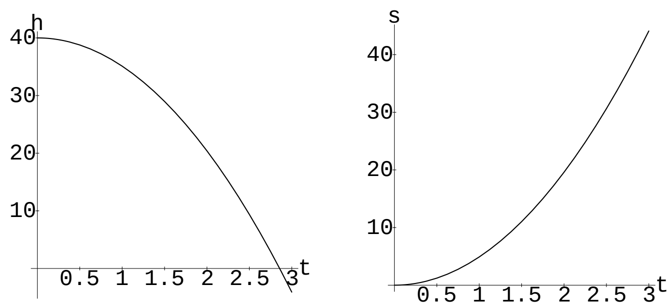

Figure 10.2:6: Continuous fall of the camera

The computation of the Example 10.2 finds $s[1] - s[0]$, $s[2] - s[1]$, and $s[3] - s[2]$. Which continuous velocities do these best approximate? The answer is $v[\frac{1}{2}]$, $v[\frac{3}{2}]$, and $v[\frac{5}{2}]$ - the times at the midpoint of the time intervals. Sketch the tangent at time 1.5 on the graph of s vs. t and compare that to the segment connecting the points on the curve at time 1 and time 2. In general, the symmetric difference

$$\frac{f[x + \frac{\delta x}{2}] - f[x - \frac{\delta x}{2}]}{\delta x} \approx f'(x)$$

gives the best numerical approximation to the derivative of

$$y = f[x]$$

when we only have data for f. The difference quotient is best as an approximation at the midpoint. The project on Taylor's Formula shows algebraically and graphically what is happening. Graphically, if the curve bends up, a secant to the right is too steep and a secant to the left is not steep enough. The average of one slope below and one above is a better approximation of the slope of

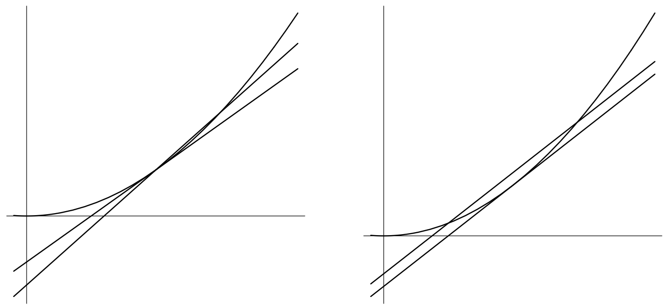

Figure 10.2:7: $[f(x+\delta x) - f(x)]/\delta x$ vs. $[f(x+\delta x) - f(x-\delta x)]/\delta x$

the tangent. The average slope given by the symmetric secant, even though that secant does not pass through $(x, f[x])$. The general figure looks like Figure 10.2:7.

The best times to associate to our average speeds in comparison to the continuous real fall are the midpoint times:

Time	Speed $= \dfrac{\Delta s}{\Delta t}$
0.50	$v_1 = v[0.50] = 4.90$
1.50	$v_2 = v[1.50] = 14.7$
2.50	$v_3 = v[2.50] = 24.5$

This interpretation gives us a clear time difference to use in computing the rates of increase in the acceleration:

Time	Rate of change in speed
Ave[0.50&1.50] = 1	$a[1] = \dfrac{14.7 - 4.90}{1.50 - 0.50} = 9.8$
Ave[1.50&2.50] = 2	$a[2] = \dfrac{24.5 - 14.7}{2.50 - 1.50} = 9.8$

We summarize the whole calculation by writing the difference quotients in a table opposite the various midpoint times as follows:

First and Second Differences of Position Data			
Time	Position	Velocity	Acceleration
0.00	0.00		
0.50		4.90	
1.00	4.90		9.8
1.50		14.7	
2.00	19.6		9.8
2.50		24.5	
3.00	44.1		

Table 10.1: One-second position, velocity, and acceleration data

---

Exercise Set 10.2

The first exercise seeks your everyday interpretation of the positive and negative signs of $\frac{ds}{dt}$ *and* $\frac{d^2s}{dt^2}$ *on the hypothetical trip from Example 10.1. We need to understand the mechanical interpretation of these derivatives as well as their graphical interpretation.*

1. *Look up your old solution to Exercise 9.2.1 or Example 10.1 and add a graphing table like the ones from the Chapter 9 with slope and bending. Fill in the parts of the table corresponding to* $\frac{ds}{dt}$ *and* $\frac{d^2s}{dt^2}$ *using the microscopic slope and smile and frown icons including* $+$ *and* $-$ *signs. Remember that* $\frac{d^2s}{dt^2}$ *is the derivative of the function* $\frac{ds}{dt}$*; so, for example, when it is positive, the function* $v[t] = \frac{ds}{dt}$ *increases, and when it is negative, the velocity decreases. We also need to connect the sign of* $\frac{d^2s}{dt^2}$ *with physics and the graph of* $s[t]$*. Use your solution graph of time,* t*, vs. distance,* s*, to analyze the following questions.*

   (a) *Where is your speed increasing? Decreasing? Zero? If speed is increasing, what geometric shape must that portion of the graph of* $s[t]$ *have? (The graph of* $v[t]$ *has upward slope and positive derivative,* $\frac{dv}{dt} = \frac{d^2s}{dt^2} > 0$*, but we are asking how increase in* $v[t] = \frac{ds}{dt}$ *affects the graph of* $s[t]$*.)*

   (b) *Is* $\frac{ds}{dt}$ *ever negative in your example? Could it be negative on someone's solution? Why does this mean that you are backing up?*

   (c) *Summarize both the mechanics and geometrical meaning of the sign of the second derivative* $\frac{d^2s}{dt^2}$ *in a few words. When* $\frac{d^2s}{dt^2}$ *is positive .... When* $\frac{d^2s}{dt^2}$ *is negative ....*

   (d) *Why must* $\frac{d^2s}{dt^2}$ *be negative somewhere on everyone's solution?*

   There are more accurate data for the fall of the camera in half-second time steps:

Accurate Position Data			
Time	Position	Velocity	Acceleration
0.000	0.000		
0.500	1.233		
1.000	4.901		
1.500	11.03		
2.000	19.60		
2.500	30.63		
3.000	44.10		

Table 10.2: Half-second position data

2. *Numerical Acceleration*
Compute the average speeds corresponding to the positions in Table 10.2 above and write them next to the correct midpoint times so that they correspond to continuous velocities at those times. Then use your velocities to compute accelerations at the proper times. Simply fill in the places where the question marks appear in the velocity and acceleration Tables 10.3 and 10.4 following this exercise. The data are also contained in the Gravity program so you can complete this arithmetic with the computer in Exercise 10.3.3. HINTS: We begin the computation of the accelerations as follows. First, add midpoint times to the table and form the difference quotients of position change over time change:

Differences of the Half-Second Position Data			
Time	Position	Velocity	Acceleration
0.000	0.000		
0.250		$\frac{1.223-0}{0.5} = 2.446$	
0.500	1.233		
0.750		$\frac{4.901-1.223}{1.0-0.5} = 7.356$	
1.000	4.901		
1.250		$\frac{?-?}{?} =?$	
1.500	11.03		
1.750		?	
2.000	19.60		
2.250		?	
2.500	30.63		
2.750		?	
3.000	44.10		

Table 10.3: Half-second velocity differences

Next, form the difference quotients of velocity change over time change:

Second Differences of the Half-Second Position Data			
Time	Position	Velocity	Acceleration
0.000	0.000		
0.250		2.446	
0.500	1.233		$\frac{7.356-2.446}{0.75-0.25} = 9.820$
0.750		7.356	
1.000	4.901		$\frac{?-?}{?} = ?$
1.250			
1.500	11.03		?
1.750			
2.000	19.60		?
2.250			
2.500	30.63		?
2.750			
3.000	44.10		

Table 10.4: Half-second acceleration differences

10.3 Galileo's Law of Gravity

The acceleration due to gravity is a universal constant, $\frac{d^2s}{dt^2} = g$.

Data for a lead cannon ball dropped off a tall cliff are contained the the computer program **Gravity**. The program contains time-distance pairs for $t = 0$, $t = 0.5$, $t = 1.0$, $\cdots$, $t = 9.5$, $t = 10$. A graph of the data is included in Figure 10.3:8.

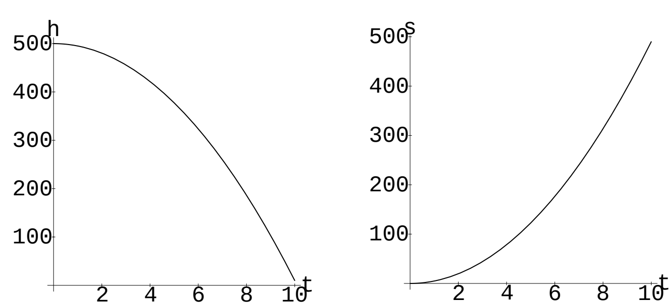

Figure 10.3:8: Free fall without airfriction

Galileo's famous observation turns out to be even simpler than the first conjecture of a linear speed law (which you will reject in an exercise below). He found that as long as air friction can be neglected, the rate of increase of speed is constant. Most striking, the constant is universal - it does not depend on the weight of the object.

Galileo's Law of Gravity
The acceleration due to gravity is a constant, g, independent of the object.

The value of g depends on the units of time and distance, $9.8m/sec^2$ or $32ft/sec^2$.

"Fluency" in calculus means that you can express Galileo's Law with the differential equation

$$\frac{d^2s}{dt^2} = g$$

The first exercise for this section asks you to clearly express the law with calculus.

Exercise Set 10.3

1. *Galileo's Law and* $\frac{d^2s}{dt^2}$
   Write Galileo's Law, "The rate of increase in the speed of a falling body is constant." in terms of the derivatives of the distance function $s[t]$. *What derivative gives the speed? What derivative gives the rate at which the speed increases?*

   We want you to verify Galileo's observation for the lead ball data in the Gravity program.

2. *Numerical Gravitation*
   (a) *Use the computer to make lists and graphs of the speeds from 0 to* $\frac{1}{2}$ *second, from* $\frac{1}{2}$ *to 1, and so forth, using the data of the Gravity program. Are the speeds constant? Should they be?*
   (b) *Also use the computer to compute the rates of change in speed. Are these constant? What does Galileo's Law say about them?*

3. *Galileo's Law and the Graph of* $\frac{ds}{dt}$
   Galileo's Law is easiest to confirm with the data of the Gravity program by looking at the graph of $\frac{ds}{dt}$ *(because error measurements are magnified each time we take differences of our data). What feature of the graph of velocity is equivalent to Galileo's Law?*

In the Problem 4.2 you formulated a model for the distance an object has fallen. You observed that the farther an object falls, the faster it goes. The simplest such relationship says, "The speed is proportional to the distance fallen." This is a reasonable first guess, but it is not correct. We want you to see why. (Compare the next problem with Problem 8.5.)

Problem 10.1

Try to find a constant to make the conjecture of Problem 4.2 match the data in the program ***Gravity****, that is, make the differential equation*

$$\frac{ds}{dt} = k\, s$$

predict the position of the falling object. This will not work, but trying will show why. There are several ways to approach this problem. You could work first from the data. Compute the speeds between 0 and $\frac{1}{2}$ *seconds, between* $\frac{1}{2}$ *and 1 second, and so forth, then divide these numbers by* s *and see if the list is approximately the same constant. The differential equation* $\frac{ds}{dt} = k\, s$ *says it should be, because*

$$v = \frac{ds}{dt} = k\, s \qquad \Leftrightarrow \qquad \frac{v}{s} = k \quad \textit{is constant.}$$

There is some help in the ***Gravity*** *program getting the computer to compute differences of the list. Remember that the time differences are* $\frac{1}{2}$ *seconds each. You need to add a computation to divide the speeds by* s*,*

$$\frac{s[t+\frac{1}{2}] - s[t]}{\frac{1}{2}} \approx \frac{ds}{dt} \quad \textit{at } t + \frac{1}{4}$$

each divided by $s[t]$*. Notice that there is some computation error caused by our approximation to* $\frac{ds}{dt}$ *actually being best at* $t + \frac{1}{4}$ *but only having data for* $s[t]$*. Be careful manipulating the lists with Mathematica because the velocity list has one more term than the acceleration list. Another approach to rejecting Galileo's first conjecture is to start with the differential equation. We can solve* $\frac{ds}{dt} = k\, s$ *with the initial* $s[0] = 0$ *by methods of Chapter 8, obtaining* $s[t] = S_0\, e^{k\, t}$*. What is the constant* S_0 *if* $s[0] = 0$*. How do you compute* $s[0.01]$ *from this? See Bugs Bunny's Law of Gravity, Problem 8.5 and Exercise Set 8.2. The zero point causes a difficulty as the preceding part of this problem shows. Let's ignore that for the moment. If the data actually are a solution to the differential equation,* $s = S_0\, e^{k\, t}$*, then*

$$\mathrm{Log}[s] = \mathrm{Log}[S_0\, e^{k\, t}] = \mathrm{Log}[S_0] + \mathrm{Log}[e^{k\, t}] = \sigma_0 + k\, t$$

so the logarithms of the positions (after zero) should be linear. Compute the logs of the list of (non-zero) positions with the computer and plot them. Are they linear?

10.4 Projects

Several Scientific Projects go beyond this basic chapter by using Newton's far-reaching extension of Galileo's Law. Newton's Law says $F = m\, a$, the total applied force equals mass times acceleration. This allows us to find the motion of objects that are subjected to several forces.

10.4.1 The Falling Ladder

Example 7.14 introduces a simple mathematical model for a ladder sliding down a wall. The rate at which the tip resting against the wall slides tends to infinity as the tip approaches the floor. Could a real ladder's tip break the sound barrier? The speed of light? Of course not. That model neglects the physical mechanism that makes the ladder fall - Galileo's Law of Gravity. The project on the ladder asks you to correct the physics of the falling ladder model.

10.4.2 Linear Air Resistance

A feather does not fall off a tall cliff as fast as a bowling ball does. The acceleration due to gravity is the same, but air resistance plays a significant role in counteracting gravity for a large, light object. A basic project on Air Resistance explores the path of a wooden ball thrown off the same cliff as the lead ball we just studied in this chapter.

10.4.3 Bungee Diving and Nonlinear Air Resistance

Human bodies falling long distances are subject to air resistance, in fact, sky jumpers do not keep accelerating but reach a "terminal velocity." Bungee jumpers leap off tall places with a big elastic band hooked to their legs. Gravity and air resistance act on the jumper in his initial "flight," but once he reaches the length of the cord, it pulls up by an amount depending on how far it is stretched. The Bungee Jumping Project has you combine all these forces to find out if a jumper hits the bottom of a canyon or not.

10.4.4 The Mean Value Math Police

The police find out that you drove from your house to Grandmother's, a distance of 100 miles in 1.5 hours. How do they know you exceeded the maximum speed limit of 65 mph? The Mean Value Theorem Project answers this question.

10.4.5 Symmetric Differences

The Taylor's Formula (from the project of that name) shows you why the best time for the velocity approximated by $(s[t_2]-s[t_1])/(t_2-t_1)$ is at the midpoint, $v[(t_2+t_1)/2]$. This is a general numerical result that you should use any time that you need to estimate a derivative from data. The project shows you why.