Geometric generation of permutation sequences

Dennis Roseman

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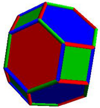
March 26, 2009



Overture

Music

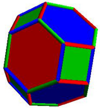
Original motivation: apply mathematics to the composition of music.

Mathematics Focus:

Some geometry of the n-dimensional permutahedron.

Visualization Focus

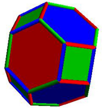
Higher dimensional visualization including braids used as a visualization tool.



From Abstract Mathematics to Musical Composition

Two very different musical examples:

- change ringing
- Nomos Alpha of Xenankis



From *Change Ringing* by Wilfrid G. Wilson

CHANGE RINGING

The Art and Science of Change Ringing
on Church and Hand Bells

by

WILFRID G. WILSON

Master of the London County Association of Change Ringers
1963–65
Vice President of the Oxford Diocesan Guild
of Church Bell Ringers
Member of the Central Council of Church Bell Ringers

OCTOBER HOUSE INC.
New York

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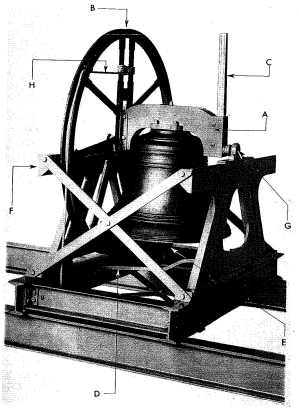

1. A BELL AT REST

A, Headstock; B, Wheel; C, Stay; D, Slider; E, Clapper; F, Frame; G, Gudgeons and Bearings; H, Rope

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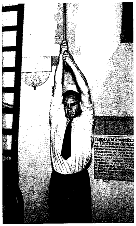
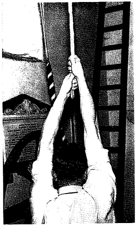


3a. Showing, from front, the correct hold on the tail end, with arms stretched and close to ears.



3b. Showing, from back, the correct hold on the tail end.

3c. Showing, from back, (a) left arm stretched (b) rope over back of hand (c) tail end between thumb and sally.

3d. Showing, from back, (a) Correct hold of tail end (b) both arms stretched (c) rope hanging straight (d) position of hands on sally.

From *Change Ringing* by Wilfrid G. Wilson

CHAPTER FIVE

Plain Bob—The Method

The simplest of the even bell methods and the one best suited for use as an introduction to the inexhaustible complexities and problems of change ringing is Plain Bob. The basis of the method is the plain hunt (see Chapter Four) and the variations from it are the simplest possible by which to produce additional changes. A thorough knowledge of Plain Bob is essential to a study of other aspects of change ringing.

On Four Bells

Very little change ringing is practised on four bells, but the whole principle of Plain Bob can be seen clearly and concisely on this number, thus making it easier to understand the method on the higher numbers of bells.

Starting from rounds (1 2 3 4) write out a plain hunt on four bells (as on p. 14) until the first lead of the treble (number 1) is reached. Then, while the treble leads twice, let the bell which is second strike twice in that position. This is called making second place. It is then impossible without clashing for the bells in the third and fourth positions to return to their original positions as they otherwise would, so they dodge with each other. Each of these two bells takes a step backward in its hunting course. This block of changes from the time the treble left the front until it got back to it again is called a lead, and the backstroke row when the treble is leading is called the lead end.* We now have a new lead end 1 3 4 2 – see row A p. 19.

From this new lead end write out a plain hunt again until the treble is once more leading. As these changes start from 1 3 4 2 instead of from 1 2 3 4 they will be different ones. Then at the second lead of the treble make similar variations – the bell that is second makes second place and the other two bells dodge with each other in 3–4 (i.e. the third and fourth positions). This completes another lead and produces another lead end, 1 4 2 3, marked B.

* For note on this see p. 27.

From *Change Ringing* by Wilfrid G. Wilson

1	2	3	4	
2	1	4	3	
2	4	1	3	
4	2	3	1	
4	3	2	1	
3	4	1	2	
3	1	4	2	
1	3	2	4	
1	3	4	2	A
3	1	2	4	
3	2	1	4	
2	3	4	1	
2	4	3	1	
4	2	1	3	
4	1	2	3	
1	4	3	2	
1	4	2	3	B
4	1	3	2	
4	3	1	2	
3	4	2	1	
3	2	4	1	
2	3	1	4	
2	1	3	4	
1	2	4	3	
1	2	3	4	C

From this new lead end write out another plain hunt until the treble reaches the front again. Make similar variations to those at the other lead ends and we are back at rounds (row C). We have produced all the possible 24 changes on four bells, we have no repetitions and we have missed none out. This block of changes (three complete leads) is called a plain course of Plain Bob Minimus, sometimes Plain Bob Singles. The major factor in this plain course is the continued plain hunting of the treble and you will find that there is a whole large class of methods in which the treble plain hunts continually among the other bells – however many there may be.

Now on the table of 24 changes draw a line through all the 2' s. It will be this shape, though less squashed up:

From *Formal Music* by I. Xenakis—Nomos Alpha

Formalized Music

THOUGHT AND MATHEMATICS IN COMPOSITION

Revised Edition

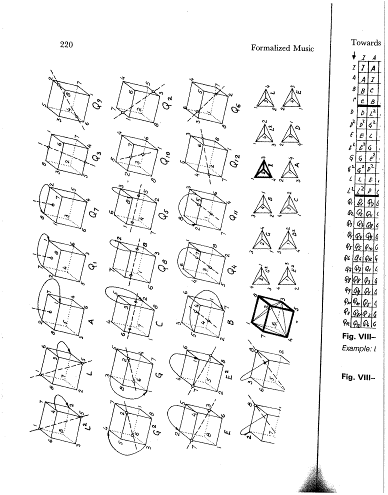
Iannis Xenakis

Additional material compiled and edited
by Sharon Kanach

HARMONOLOGIA SERIES No. 6

PENDRAGON PRESS
HILLSDALE, NY

From *Formal Music* by I. Xenakis—Nomos Alpha



From *Formal Music* by I. Xenakis—Nomos Alpha

ized Music

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Organization In-Time

II. The mapping of the eight forms onto the letters C_i change cyclically in the order $\alpha, \beta, \gamma, \alpha, \ldots$ after each three substitutions of the cube.

III. The same is true for the cube of the letters C_i.

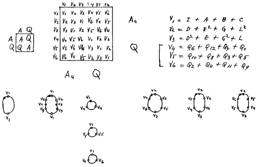

Fig. VIII–8

From *Formal Music* by I. Xenakis—Nomos Alpha

Geometric generation of permutation sequences

Dennis Roseman

Permutahedron

Change Ringing

Bouncing

Problem List

Cell Structure

coloring edges

coloring facets

Braids

Beam Calculation

Edges in layers

Tiling

Organization In-Time

IV. The products $K_l^r \times C_j$ and $K_l \times C_m$ are the result of the product of two graphs of closed transformations of the cube in itself. The mapping of the graphs is one-to-one and sounded successively; for example:

$$\downarrow \begin{matrix} C_i \\ K_j \end{matrix} \longrightarrow \downarrow \begin{matrix} \text{graph } (D \ \overrightarrow{\ \ } \ Q_{12}) \\ \text{graph } (D \ \overrightarrow{\ \ } \ Q_3) \end{matrix}$$

(See Figs. VIII–9, 10.)

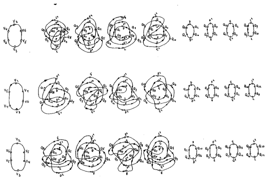

Fig. VIII–9

V. Each C_i is mapped onto one of the cells of $H \times X$ according to two principles: maximum expansion (minimum repetition), and maximum contrast or maximum resemblance. (See Fig. VIII–11.)

Music and Mathematics

Definition

A **musical composition** is a family of sequences of related musical events.

Time and voices

Progression in time is related to succession in a sequence; each sequence represents a "voice".

The mathematical objects we chose are permutations.

Construct families of sequences length k of permutations of order n , where k and n are independent.

Permutations Geometrically: the Permutahedron

The Permutahedron

1. Take the $n!$ permutations S_n to be all permutations of $(1, 2, \ldots, n)$

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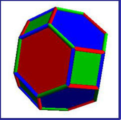
The Permutahedron

1. Take the $n!$ permutations S_n to be all permutations of $(1, 2, \ldots, n)$
2. They are n-tuples—plot them as points in R^n.



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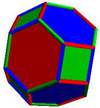
3. Take the convex hull.



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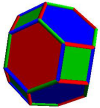
1. Take the $n!$ permutations S_n to be all permutations of $(1, 2, \ldots, n)$
2. They are n-tuples—plot them as points in R^n.
3. Take the convex hull.
4. The resulting polytope is the **permutahedron** $\mathcal{P}(n)$

Low Dimensional Cases

Some Examples:

1. $\mathcal{P}(2)$ is the line segment in R^2 with endpoints $(1, 2)$ and $(2, 1)$.



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Low Dimensional Cases

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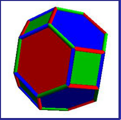
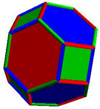


Low Dimensional Cases

Some Examples:

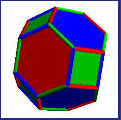
1. $\mathcal{P}(2)$ is the line segment in R^2 with endpoints $(1,2)$ and $(2,1)$.
2. $\mathcal{P}(3)$ is a hexagon in R^3 in the plane .



Low Dimensional Cases

Some Examples:

1. $\mathcal{P}(2)$ is the line segment in R^2 with endpoints $(1, 2)$ and $(2, 1)$.
2. $\mathcal{P}(3)$ is a hexagon in R^3 in the plane subset of the plane $x + y + z = 6$.



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3. $\mathcal{P}(4)$ is a truncated octahedron in R^3 .

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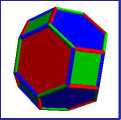
2. $\mathcal{P}(3)$ is a hexagon in R^3 in the plane .
3. $\mathcal{P}(4)$ is a truncated octahedron in R^3 subset of the hyperplane $x + y + z + w = 10$.



The Permutahedron of order 2

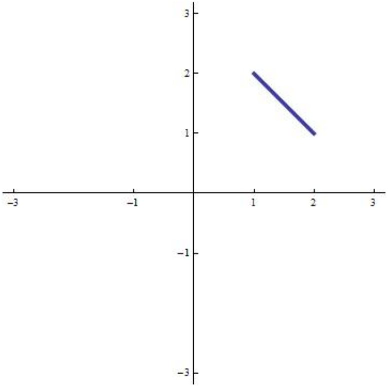


Figure: The two permutations $(1, 2)$ and $(2, 1)$: a line segment in R^2

The Permutahedron of order 3

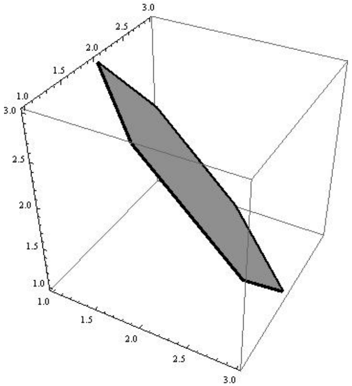


Figure: Hexagon in R^3 of the six permutations of order 3: $(1,2,3),(2,1,3),(3,1,2),(3,2,1),(2,3,1),(1,3,2),(1,2,3)$

The Permutahedron of order 4

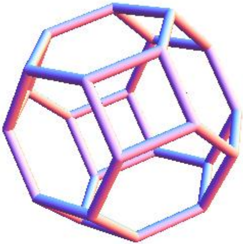


Figure: The 24 permutations of order 4 determine a truncated octahedron in R^4 which we show in R^3

Change Ringing n bells

Definition

A sequence Σ of permutations is a **change ringing** composition if

- Σ begins and ends with the identity permutation of S_n
- Otherwise each of the $n!$ order n permutations occurs exactly one time
- Two consecutive permutations of Σ differ by switching two consecutive integers.

Ringing Changes on Three Bells

1	2	3
2	1	3
3	1	2
3	2	1
2	3	1
1	3	2
1	2	3

Table: One way to ring changes on 3 bells; the second reverses the order.

Ringing Changes Geometrically

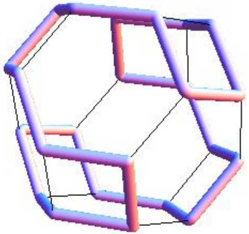


Figure: The change *Double Canterbury Pleasure Minimus* corresponds to a Hamiltonian path in the edge set of $\mathcal{P}(4)$

Change Ringing

Critique of Change Ringing

- Change ringing is very limited—hard to get non-trivial examples.

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- Change ringing is very limited—hard to get non-trivial examples.
- There is no relationship between one change and another
- Each permutation is treated equally. Musically one expects to make choices.
- There is a fixed length to a ring of changes
- The difficulty of calculation increases rapidly with n

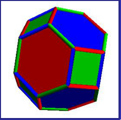
A new way to get a sequences of permutations

Bouncing a light in a mirrored $\mathcal{P}(4)$

A new way to get a sequences of permutations

Bouncing a light in a mirrored $\mathcal{P}(4)$

- Build room in the shape of $\mathcal{P}(4)$ with all walls made of mirror.



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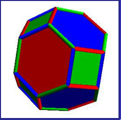
- Build room in the shape of $\mathcal{P}(4)$ with all walls made of mirror.
- From inside the room shine a "generic" laser beam from point x_0 in direction λ_0.



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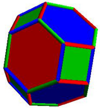
- The beam as it reflects will hit successive walls giving a sequence of points $x_1, x_2, \ldots$.



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- Since the beam is generic there will be a unique vertex (permutation) π_i of $\mathcal{P}(4)$ nearest to x_i.

A new way to get a sequences of permutations

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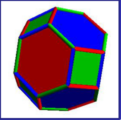
- The beam as it reflects will hit successive walls giving a sequence of points $x_1, x_2, \ldots$.
- Since the beam is generic there will be a unique vertex (permutation) π_i of $\mathcal{P}(4)$ nearest to x_i.
- Thus we generate our sequence of permutations $S(x_0, \lambda_0) = (\pi_1, \pi_2, \ldots)$



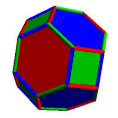
Bounce points

Definition

The points $x_1, x_2, \ldots$ are called either **intersection points** (they are calculated as an intersection of a ray and $\partial\mathcal{P}(n)$) or **bounce points** (since our beam bounces there).

Bouncing for $\mathcal{P}(n)$

Cleary we can define this process for permutations of order n.



Lets bounce

Geometric generation of permutation sequences

Dennis Roseman

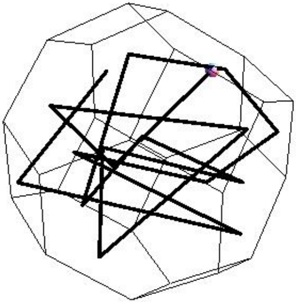
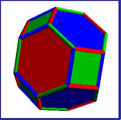


Figure: An example of 16 bounces



Corresponding Sequence of Permutations

2	1	4	3
2	1	4	3
1	3	2	4
2	4	3	1
3	1	2	4
3	2	1	4
2	3	1	4
2	4	1	3
2	3	4	1
4	2	1	3
3	4	1	2
3	4	1	2
1	3	4	2
1	3	4	2
4	1	3	2
1	2	3	4
1	4	2	3

1	4	2	3
3	4	2	1
4	2	1	3
1	2	4	3
1	3	4	2
3	2	4	1
4	1	3	2
3	1	2	4
3	1	2	4
2	4	3	1
1	4	2	3
1	4	2	3
2	1	3	4
2	1	3	4
4	1	3	2
2	4	3	1

Lets bounce

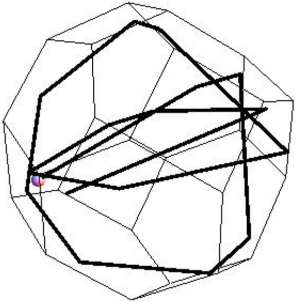


Figure: Another example of 16 bounces

The numbers we use are not necessarily pitches

Figure: Any knob or input/output on the Control Panel of this Moog corresponds to a number. Photo by Kevin Lightner

Bouncing, Ringing

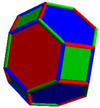
Comparing Bouncing to Change Ringing

- **Change ringing**: finite number of possibilities

Bouncing, Ringing

Comparing Bouncing to Change Ringing

- **Bouncing**:infinite number of possibilities



Bouncing, Ringing

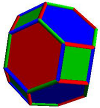
Comparing Bouncing to Change Ringing

- **Bouncing**:infinite number of possibilities
- **Change ringing** is hard is it to calculate.

Bouncing, Ringing

Comparing Bouncing to Change Ringing

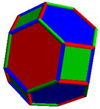
- **Bouncing**:infinite number of possibilities
- **Bouncing** is easy to calculate



Bouncing, Ringing

Comparing Bouncing to Change Ringing

- **Bouncing**:infinite number of possibilities
- **Bouncing** is easy to calculate as we will see



Bouncing, Ringing

Comparing Bouncing to Change Ringing

- **Bouncing**:infinite number of possibilities
- **Bouncing** is easy to calculate
- **Change Ringing**—no relationship between one change and another

Bouncing, Ringing

Comparing Bouncing to Change Ringing

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- **Bouncing** is easy to calculate
- **Bouncing**: if one varies x and λ one gets related permutation sequences $S(x, \lambda)$

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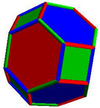
- **Change Ringing**: each permutation is treated equally



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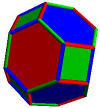

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- **Bouncing**:infinite number of possibilities
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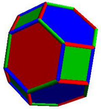
- **Bouncing**: if one varies x and λ one gets related permutation sequences $S(x, \lambda)$
- **Bouncing**: distinct sequences have distinct characteristics
- **Change Ringing**: difficulty of calculation increases rapidly with n



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- **Bouncing**:infinite number of possibilities
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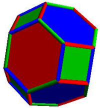
- **Bouncing**: if one varies x and λ one gets related permutation sequences $S(x, \lambda)$
- **Bouncing**: distinct sequences have distinct characteristics
- **Bouncing**: calculation is quadratic with respect to n

The rest of the talk: mathematics and visualization

Questions we now address

1. How fast can we calculate the bouncing path for fairly high orders—8, 12, 16, 32?
2. How can we visualize the calculational process and the results?
3. What is the geometry of a high dimensional permutahedron?
4. How does the geometry of the permutahedron change with n?



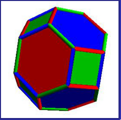
Basic approach

Change Ringing

1. Construct a special polygonal path *in* the wireframe of a permutahedron.
2. The sequence of vertices *on* that path is the desired permutation sequence.

Bouncing, an alternative

1. Take a generically generated generic path in R^n that *avoids* the wireframe
2. Obtain the sequence of permutations by "digitizing" to permutations *near* the path.



Vertices of the Permutahedron

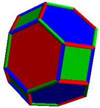
Theorem

There are $n!$ vertices in $\mathcal{P}(n)$.

All vertices of $\mathcal{P}(n)$ all lie on an $(n-2)$-sphere with center $\mathcal{C}_n$, the centroid of $\mathcal{P}(n)$, and radius ρ_n.

Definition

This sphere is the **permutahedral sphere of order** n and ρ_n the **permutahedrdal radius**. The distance between $\mathcal{C}_n$ and the centroid of Y_α is the **inner permutahedrdal radius**.



Generators of the Symmetric Group

Definition

An **elementary transposition** is a permutation that interchanges consecutive integers,

Note:

This is interchange of *consecutive integers* (wherever they are) not interchange of integers in *consecutive positions* (whatever the integers are).

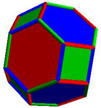
Edges of the Permutahedron

Definition

The union of edges of $\mathcal{P}(n)$ is called the **wireframe** of $\mathcal{P}(n)$

Example

The four edges from $(1, 2, 3, 4, 5)$ go to $(2, 1, 3, 4, 5)$, $(1, 3, 2, 4, 5)$, $(1, 2, 4, 3, 5)$, and $(1, 2, 3, 5, 4)$.



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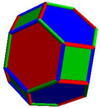
Basic general edge facts

- Two permutations are connected by an edge if and only if coordinates differ by a switch of two coordinates of consecutive value.



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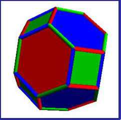
- Thus any edge corresponds to an elementary transposition.



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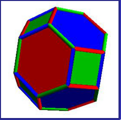

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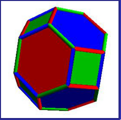
Basic general edge facts

- Two permutations are connected by an edge if and only if coordinates differ by a switch of two coordinates of consecutive value.
- Thus any edge corresponds to an elementary transposition.
- Thus all edges have length $\sqrt{2}$.
- The order of any vertex is $(n-1)$.



Visualizing the edges

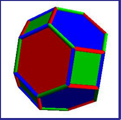
Rotate and project to low dimensions

We can generically rotate a wireframe of any order permutahedron then project homemorphically into R^3.

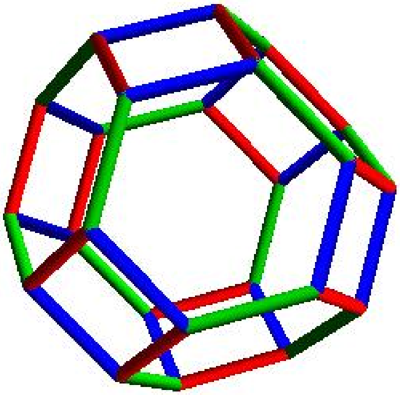
We can generically rotate a wireframe of any order permutahedron then project non-homemorphically into R^2 and still get a meaningful image.

Color the edges

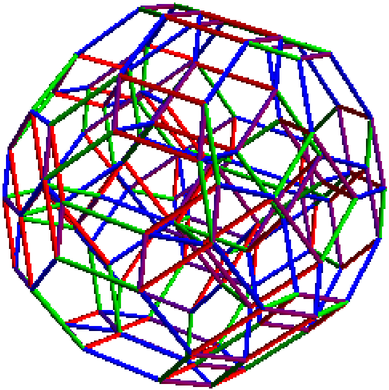
We can use $(n-1)$ colors on the edges to code the corresponding transpositions.



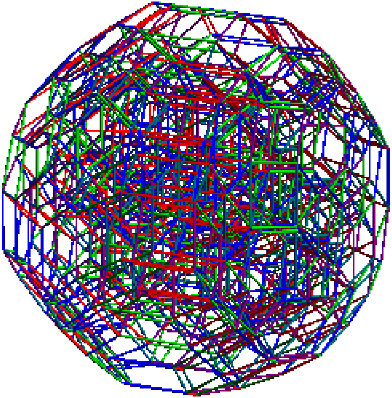
Coloring Edges: Order 4

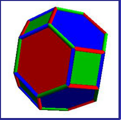


Coloring Edges: Order 5



Coloring Edges: Order 6





Cells of the Permutahedron

Proposition

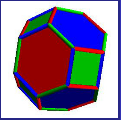
An k-cell of $\mathcal{P}(n)$ is either subgroup which is a product of k symmetric groups or a coset of one of such subgroup. Here $P(0) = \{1\}$.

Definition

Let $Y_\alpha = \{(x_1, \ldots, x_n) \in S_n : x_1 = 1\}$ and $Y_\omega = \{(x_1, \ldots, x_n) \in S_n : x_n = n\}$. We call Y_α the **first Young subgroup** of S_n, Y_ω the **last Young subgroup** of S_n.

Remark

Y_α and Y_ω are isomorphic to S_{n-1}



Re-examining $\mathcal{P}(4)$

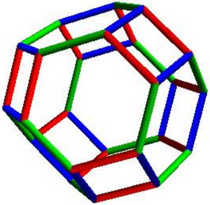
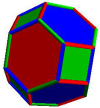


Figure: The edges of one color are all the cosets of a Young subgroup of order two. The hexagons are all cosets of the two Young subgroups isomorphic to $\mathcal{P}(3)$. The squares are cosets of $\mathcal{P}(2) \times \mathcal{P}(2)$.



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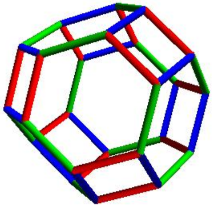

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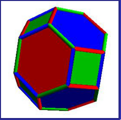

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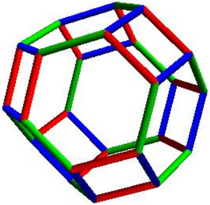

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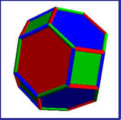


Facets of the Permutahedron

Definition

The **facets** are the the $(n-2)$-cells of $\mathcal{P}(n)$.

Proposition

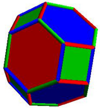
$\mathcal{P}(n)$ has $2^n - 2$ facets

Example

So $\mathcal{P}(8)$ has 254 facets and $\mathcal{P}(12)$ has 4094.

Implication

The number of facets is exponential in n. Our light beam calculation should not be based on examination of all facets.



A duality of facets and edges

Transposition colors for facets

At a vertex v of facet F you see $(n-1)$ edges all of distinct colors.

One of these colors is *not* an edge of F.

This color will identify our corresponding elementary transposition.

Coloring the facets

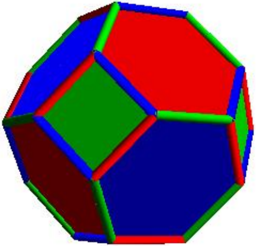


Figure: Here we color the three generators: σ_1 σ_2 σ_3

Coloring the facets

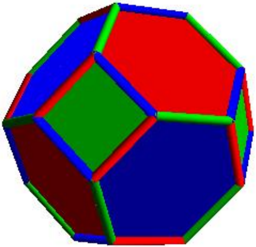
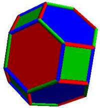


Figure: Here we color the three generators: **red green blue**



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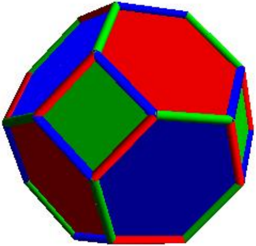

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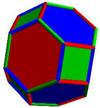

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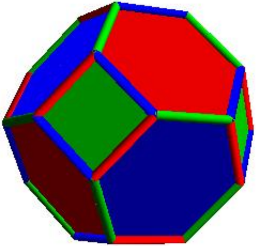
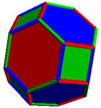


Figure: Here we color the three generators: σ_1 σ_2 σ_3 The facet color is the unique color *not* an edge color of the facet.



Two group presentations:

Presentation of Order n Braid Group

Generators: $\sigma_1, \ldots, \sigma_{n-1}$

Relations:

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $j \neq i \pm 1$
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

Presentation of Order n Symmetric Group

Generators: $\sigma_1, \ldots, \sigma_{n-1}$

Relations:

- $\sigma_i = \sigma_i^{-1}$
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $j \neq i \pm 1$
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

Inverses and elementary transpositions

From the symmetric group to the braid group

A finite sequence of elementary transpositions $\tau_1, \tau_2, \ldots, \tau_n$ corresponds to a word in the *symmetric group*:

$$\tau_1 \ \tau_2 \ \cdots \ \tau_n.$$

But if (somehow) we can distinguish elementary transpositions from their inverses we would obtain a word in the *braid group*:

$$\tau_1^{\epsilon_1} \ \tau_2^{\epsilon_2} \ \cdots \ \tau_n^{\epsilon_n}.$$

Signs for transpositions: one of many methods

Definition

A **braid sign convention** is a function that associates to any bounce point x_i of any bouncing path $\epsilon(x_i) = \pm 1$.

Example

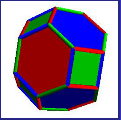
Let $\overrightarrow{N}$ be the vector from the identity permutation to the reverse of the identity. Define the *sign at* x_i to be the sign of the dot product $\overrightarrow{x_{i-1}x_i} \cdot \overrightarrow{N}$. Think of the identity as the "south pole". *Positive* means we were heading north before we "bounced"; *negative* means heading "south".

A bouncing path braid

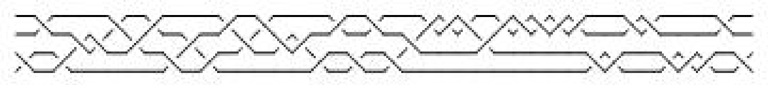
Definition

Given a bouncing path $x_0, x_1, x_2, x_3 \ldots$ and a braid sign convention we obtain a **bounce path braid**— the braid given by the word in the braid group:

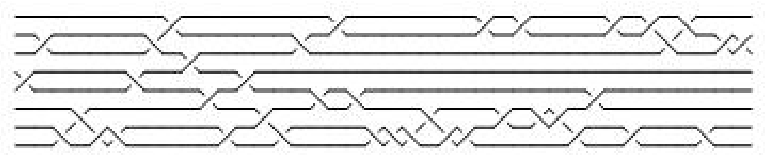
$$\sigma(x_1)^{\epsilon_1}\ \sigma(x_2)^{\epsilon_2}\ \sigma(x_3)^{\epsilon_3} \cdots$$



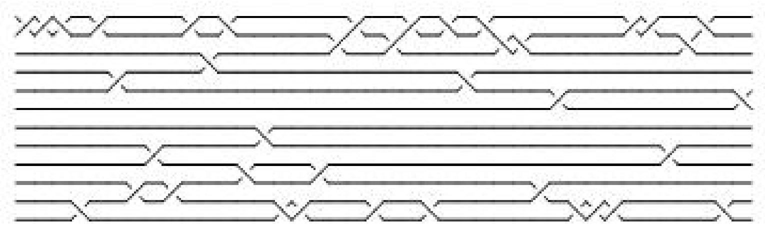
Example of Bounce Braid for Order 4

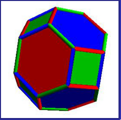


Example of Bounce Braid for Order 8

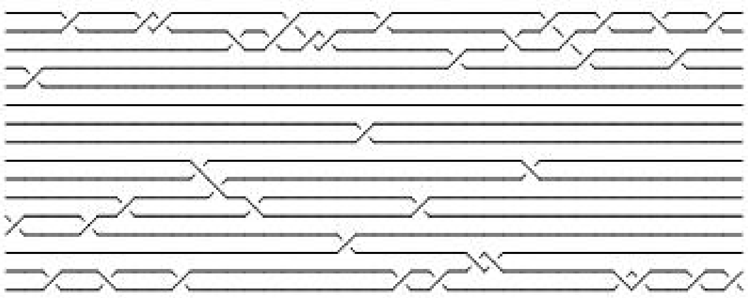


Example of Bounce Braid for Order 12





Example of Bounce Braid for Order 16



Bounce Braids

Braid as an Aid

In general we need to look at the bounce permutations *together* with the bounce braid.

The bounce path indicates *where* the bounce occurs, the braid tells us something about *how* the "type" of bounce.

The nearest permutation to a point

Definition

A **generic point** $z = (z_1, \ldots, z_n)$ of R^n will have n distinct coordinate values.

The **rank of** z_i, denoted $r(z_i)$, is one plus the number of coordinates of z smaller than z_i.

Definition

The **rank vector** $\rho(z) = (r(z_1), \ldots, r(z_n))$

In other words:

Simply Put: The rank of z is the closest permutation to z.

Or not: A generic point is mapped to a chamber of the real n-braid arrangement

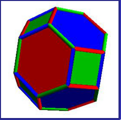
Finding the intersection of a ray and $\partial\mathcal{P}(n)$

The key

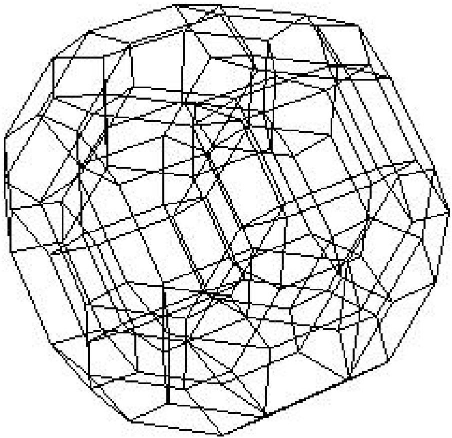
Focus on the plane P that contains the three points

1. the centroid C of $\mathcal{P}(n)$,
2. the initial point x_0
3. the tip of our vector $x_0 + \lambda$.

We then project the wireframe of $\mathcal{P}(n)$ onto this plane.



Projection wireframe $\mathcal{P}(5)$



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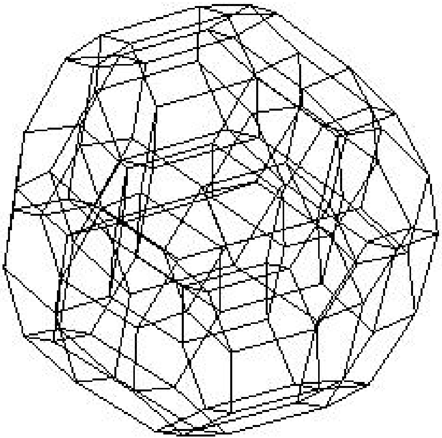

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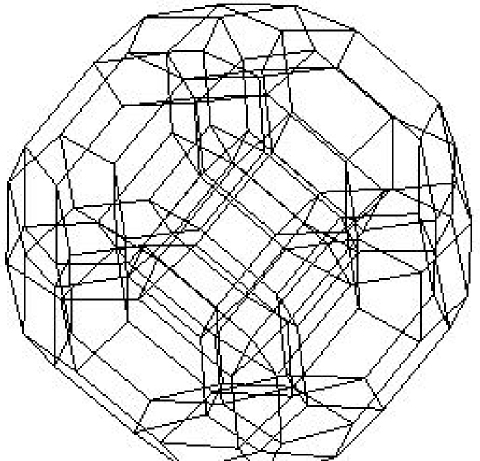

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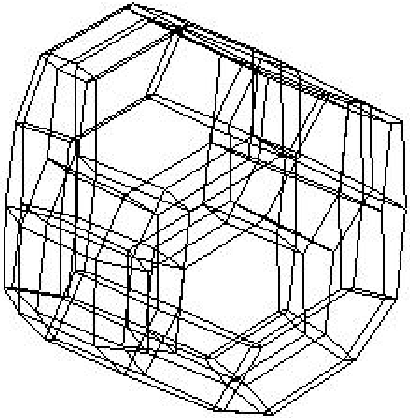

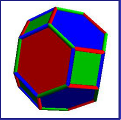

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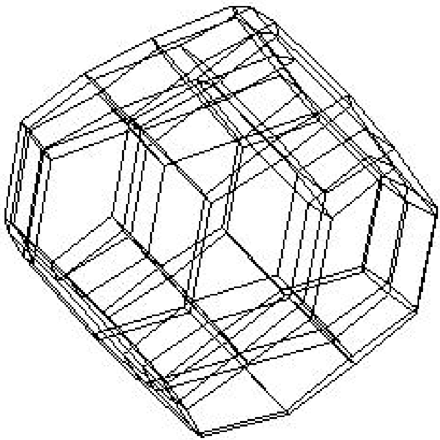

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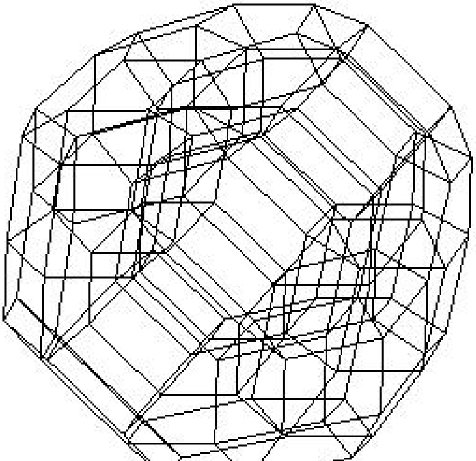
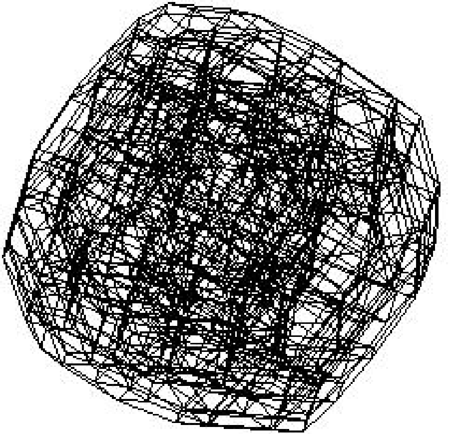
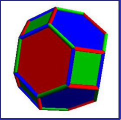


Projection wireframe $\mathcal{P}(6)$





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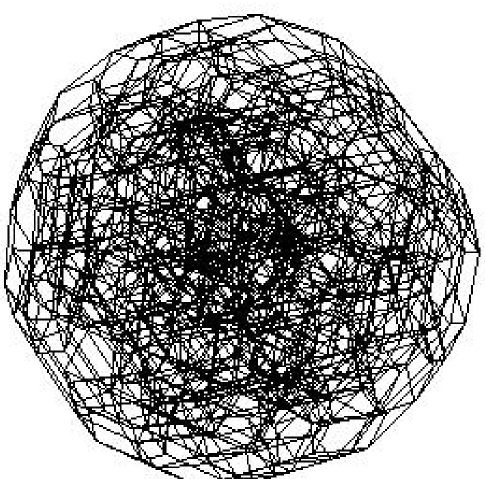
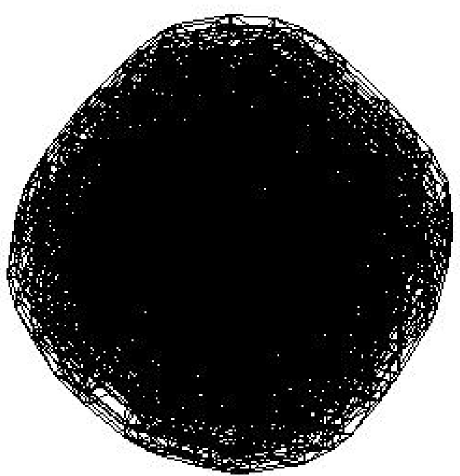


Projection wireframe $\mathcal{P}(7)$



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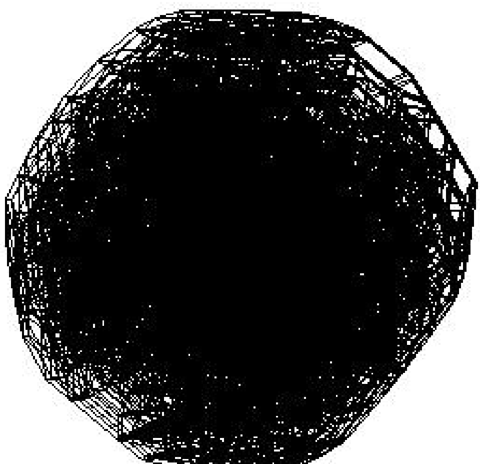

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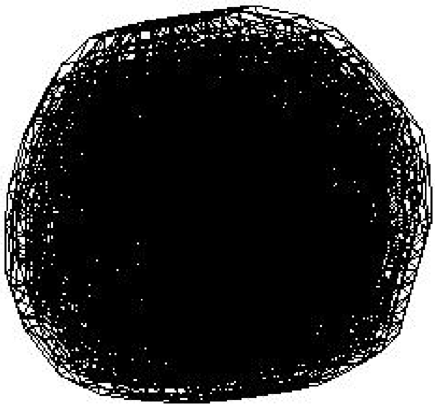

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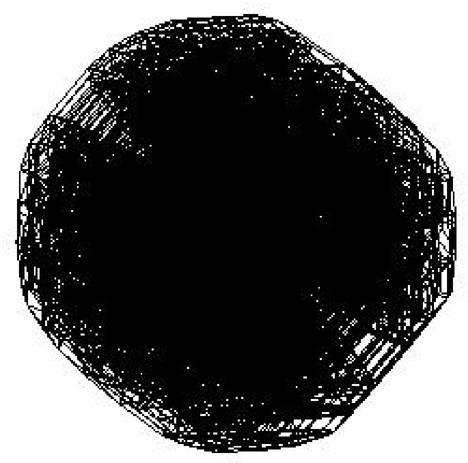

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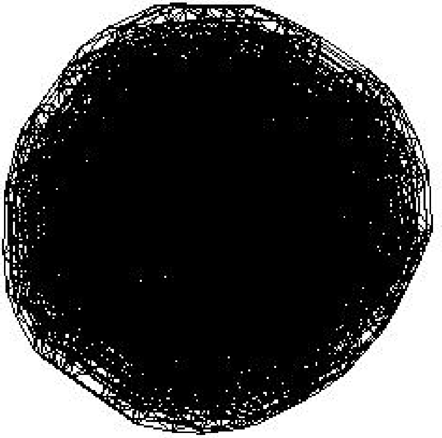

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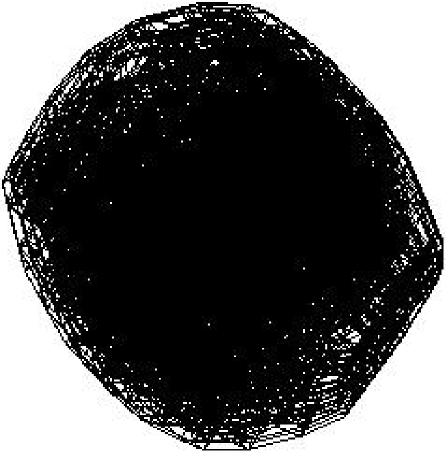

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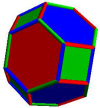

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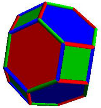

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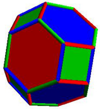

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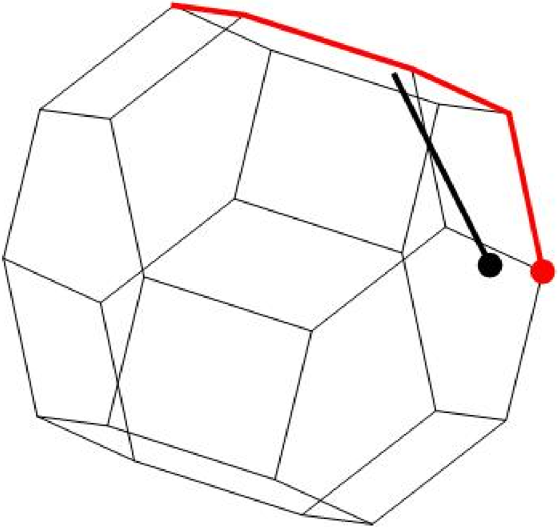
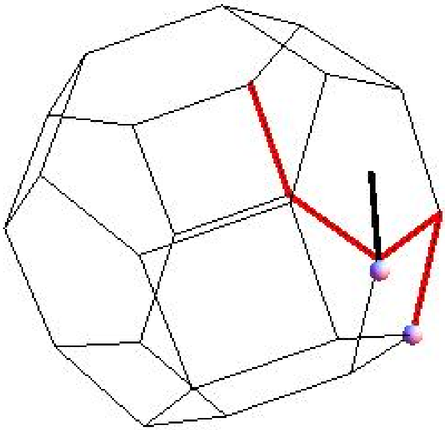
  (In fact the bounding polygonal path of the projection is the projection of a simple closed polygonal path of $\mathcal{P}(n)$ edges)

Creating great path: projection to the plane P



Same as previous figure after rotation in R^3



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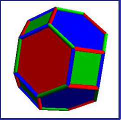

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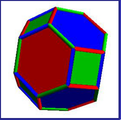
7. At each p_i find the intersection point of the ray with hyperplanes determined by facets at p_i.
8. By convexity of $\mathcal{P}(n)$ the closest such intersection point is our bounce point.

Yet more braids!

A braid from edges, not facets

1. Put together the great paths used in calculating a bounce sequence.
2. This sequence of edges of $\partial\mathcal{P}(n)$ gives a sequence of elementary transpositions.
3. There are ways to define signs to this sequence giving yet more braids.



Very briefly—one way to get signs

Example

1. Consider the polygonal path $\beta = P \cap \partial\mathcal{P}(n)$
2. orient β using λ
3. orient the $(n-3)$-cells of $\partial\mathcal{P}(n)$
4. use intersection numbers of the β with those cells to get our sign

Note:

There is a quick indirect way to calculate this from the construction of the great path.

Sorting networks

A connection to topic in computer science

There is a relationship between a colored great path which joins two antipodal permutations and the concept of a **sorting network**.

A different bouncing braid

A braid based on edges near bounce

1. Take bounce points $x_1, x_2, \ldots x_n$ where x_i lies in facet F_i
2. Let e_i be the edge of F_i closest to x_i with associated an elementary transposition $\Sigma(x_i)$
3. There are a number of ways to assign a braid sign convention ϵ_i
4. This gives a braid word

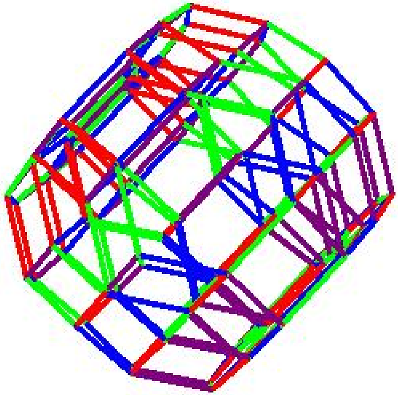
$$\Sigma(x_1)^{\epsilon_1}\Sigma(x_2)^{\epsilon_2}\ldots\Sigma(x_n)^{\epsilon_n}$$

A different bouncing braid

There is not time to go into detail . . .

To give a sense that the color of an edge tells something about the nature of the bounce consider the following graphics.

A projection of labeled wireframe of $\mathcal{P}(5)$



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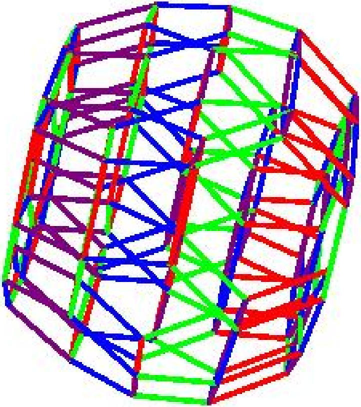

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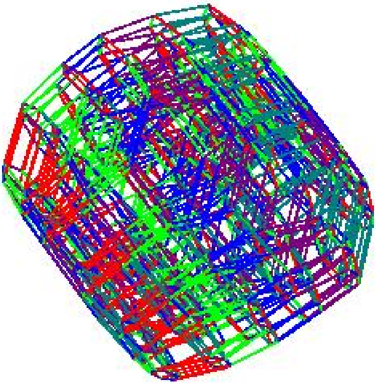
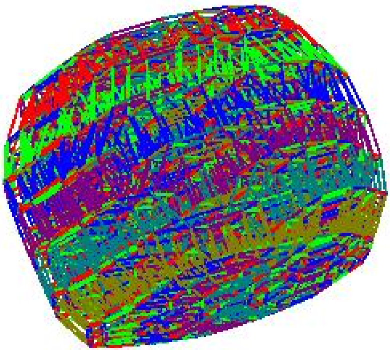
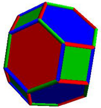


A projection of labeled wireframe of $\mathcal{P}(7)$



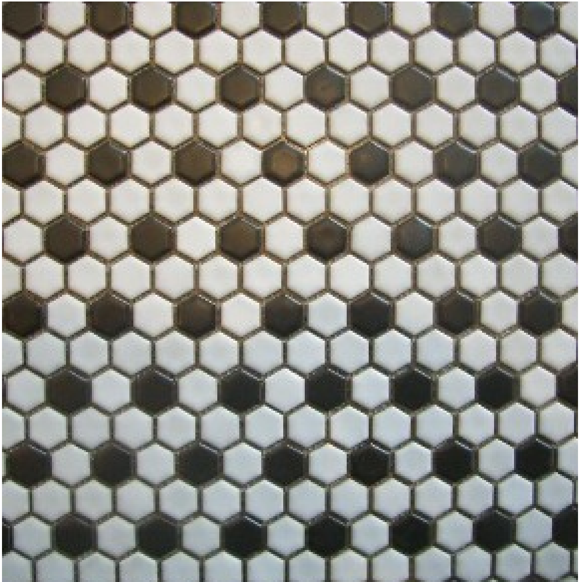


An alternative to bouncing: permutahedral tiles

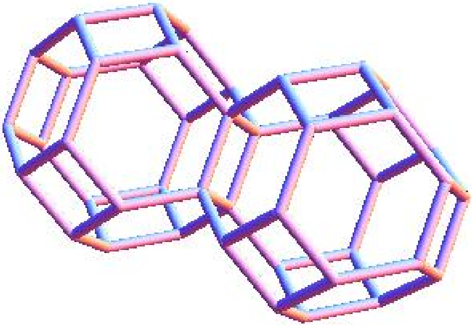
Theorem

R^n can be tiled with translated copies $\mathcal{P}(n)$

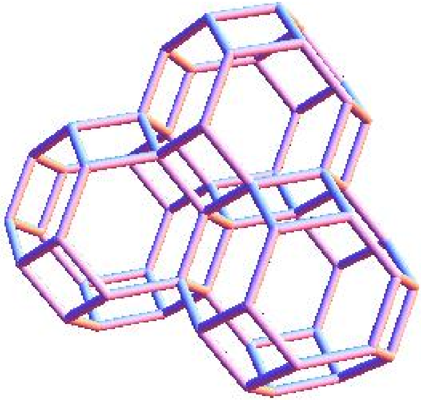
A black and white tiling of the plane by hexagons



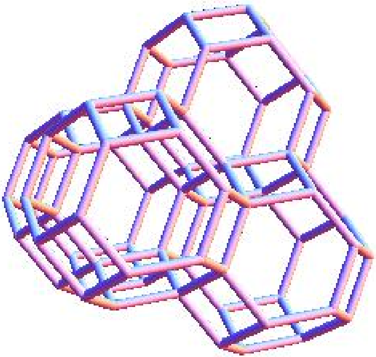
Fitting two adjacent $\mathcal{P}(4)$s



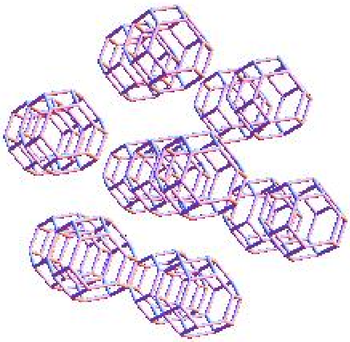
Fitting three adjacent $\mathcal{P}(4)$s



Fitting four adjacent $\mathcal{P}(4)$s



A “black and white tiling of R^3 by $\mathcal{P}(4)$s



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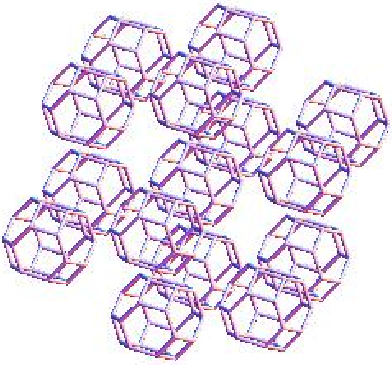

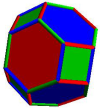

Review

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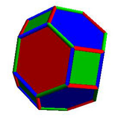

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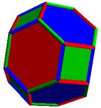

Review

Geometric generation of permutation sequences

Dennis Roseman

Permutahedron

Change Ringing

Bouncing

Problem List

Cell Structure

coloring edges

coloring facets

Braids

Beam Calculation

Edges in layers

Tiling

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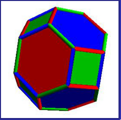

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4. How does the geometry of the permutahedron change with n? It does not become rounder. In fact in some directions it "flattens out".

Thank you