22m:033 Notes:
5.1 Eigenvectors and eigenvalues

Dennis Roseman
University of Iowa
Iowa City, IA

http://www.math.uiowa.edu/∼roseman

April 13, 2010

1 Understanding linear transformations

Example 1.1 Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$ and consider the associated linear transformation $T_A(\overrightarrow{x}) = A\overrightarrow{x}$. Then T_A is a map from 3-dimensional space to itself.

$$T_A\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x \\ -y \\ \frac{z}{3} \end{pmatrix}.$$

We can describe this geometrically as follows.

What T_A does is

- stretch by a factor of 2 in the x-coordinate
- reverse sign in the y-coordinate
- shrink by using factor of $\frac{1}{3}$ in the z-coordinate

Here is an even even simpler example

Example 1.2 Let $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and consider the associated linear transformation $T_B(\overrightarrow{x}) = B\overrightarrow{x}$. Then T_B is a map from 3-dimensional space to itself.

$$T_B\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ 2y \\ 3z \end{pmatrix}.$$

We can describe this geometrically as follows.

What T_B does is

- makes no change in the x-coordinate
- stretch by a factor of 2 in the y-coordinate
- stretch by a factor of 3 in the z-coordinate

Example 1.3 Let $C = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ and consider the associated linear transformation $T_C(\overrightarrow{x}) = A\overrightarrow{x}$.

There does not appear to be any simple way to describe what T_C does. It certainly seems very different from the transformation T_B of Example 1.2.

What we will see in this section is a way of discovering and articulating "transformations T_A and T_B are fundamentally very similar in some sense".

2 Definitions

Definition 2.1 *If A is a square matrix, a non-zero vector $\overrightarrow{x}$ such that $A\overrightarrow{x} = \lambda\overrightarrow{x}$ for some number λ is called an* ***eigenvector of*** *A. If such a λ exists it is called an* ***eigenvalue*** *and we say that $\overrightarrow{x}$ is an* ***eigenvector corresponding to eigenvalue*** λ***.***

Recall the standard basis of R^3 is:

$$\overrightarrow{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \overrightarrow{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \overrightarrow{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Remark 2.2 A linear transformation is determined by what it does on for a basis. The first few examples we look at can be easily understood using the standard basis.

When we get to more complicated looking examples we will see the need for using other bases. That is, by using a "complicated" basis, an apparently "complicated" linear transformation becomes "simpler".

it is easy to calculate and in very simple examples will tell us enough that we can understand the transformation.

Example 2.3 So in Example 1.1 we note that

- $T_A(\overrightarrow{e_1}) = 2\overrightarrow{e_1}$
- $T_A(\overrightarrow{e_2}) = -\overrightarrow{e_2}$
- $T_A(\overrightarrow{e_3}) = \frac{1}{3}\overrightarrow{e_3}$

Using our definitions we can say:

- e_1 is an eigenvector for A with eigenvalue 2
- e_2 is an eigenvector for A with eigenvalue -1
- e_3 is an eigenvector for A with eigenvalue $\frac{1}{3}$

Example 2.4 In Example 1.2 we note that

- $T_B(\overrightarrow{e_1}) = 1\overrightarrow{e_1}$
- $T_B(\overrightarrow{e_2}) = 2\overrightarrow{e_2}$
- $T_B(\overrightarrow{e_3}) = 3\overrightarrow{e_3}$

Using our definitions we can say:

- e_1 is an eigenvector for A with eigenvalue 1
- e_2 is an eigenvector for A with eigenvalue 2
- e_3 is an eigenvector for A with eigenvalue 3

It is important to note that a matrix might have no eigenvectors at all.

Example 2.5 Let $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This has no eigenvectors at all. We could show this algebraically, but it is simpler to note that this corresponds to rotation in the plane counterclockwise by $\frac{\pi}{2}$.

Example 2.6 Let $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$$T_D\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

- $T_D(\overrightarrow{e_1}) = \overrightarrow{e_1}$
- $T_D(\overrightarrow{e_2}) = \overrightarrow{e_2}$
- $T_D(\overrightarrow{e_3}) = \overrightarrow{0}$

From this we note that

- e_1 is an eigenvector for A with eigenvalue 1
- e_2 is an eigenvector for A with eigenvalue 1
- e_3 is an eigenvector for A with eigenvalue 0

One important thing we notice is that 0 is a possible eigenvalue (BUT, by definition, $\overrightarrow{0}$ is not a possible eigenvector).

We also note that any vector of the form $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ will be an eigenvector with eigenvalue 1—for example all of these:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1066 \\ -\pi^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 55 \\ -6666 \\ 0 \end{pmatrix}, \cdots$$

Example 2.7 Let $E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

$$T_E\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y \\ x \\ -z \end{pmatrix}.$$

- $T_E(\overrightarrow{e_1}) = \overrightarrow{e_2}$
- $T_E(\overrightarrow{e_2}) = \overrightarrow{e_1}$
- $T_E(\overrightarrow{e_3}) = -\overrightarrow{e_3}$

From this we note that

- e_1 is an not eigenvector for A
- e_2 is an not eigenvector for A
- e_3 is an eigenvector for A with eigenvalue -1

One important thing we notice is that an eigenvalue can be negative.

Also we can find eigenvectors, they just are not the standard basis vectors.

For example we can check that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue 1 and that

$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue - 1.

Here is another example:

Example 2.8 In Example 1.3 we claimed that Let $C = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ of Example 1.2 were similar in some fundamental way.

We note that C has the same eigenvalues as B, namely $1, 2, 3$. In fact the C has three eigenvectors:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

3 Strategy for finding eigenvectors and eigenvalues

It might seem that in order to find eigenvectors and eigenvalues, we first should find eigenvectors and when we do it will be clear what the eigenvalues are.

So if (somehow) someone told us that $\vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector for $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ we could easily figure out the what the corresponding eigenvalue must be:

$$\begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$$

So this eigenvalue is -2.

4 Finding eigenvectors if you know the eigenvalue—the eigenspace

However, it is generally easier to find eigenvalues first and then eigenvectors.

We will first show how to find eigenvectors IF we know eigenvalues. Then in the next section we will learn how to find eigenvalues.

Remark 4.1 If A has one eigenvector with eigenvalue λ how many does it have?

The answer is "infinitely many". In fact if A is $n \times n$ let E_λ denote all vectors in R^n which are eigenvectors for A with eigenvalue λ.

We can see that $\overrightarrow{u} \in E_\lambda$ and c is any non zero number then $c\overrightarrow{u} \in E_\lambda$ since

$$A(c\overrightarrow{u}) = c(A\overrightarrow{u}) = c(\lambda\overrightarrow{u}) = \lambda(c\overrightarrow{u})$$

We look at a numerical example. If $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$ and

$\overrightarrow{u} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$

Then

$$\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -24 \\ 20 \end{pmatrix} = -4 \begin{pmatrix} 6 \\ -5 \end{pmatrix}$$

So we see $\overrightarrow{u}$ is an eigenvector with eigenvalue -4. If we let $c = 2$ we calculate: Then

$$\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 12 \\ -10 \end{pmatrix} = \begin{pmatrix} -48 \\ 40 \end{pmatrix} = -4 \begin{pmatrix} 12 \\ -10 \end{pmatrix}$$

Similarly we note that if $\overrightarrow{u}, \overrightarrow{v} \in E_\lambda$ and $\overrightarrow{u} + \overrightarrow{v} \neq \overrightarrow{0}$ then $\overrightarrow{u} + \overrightarrow{v} \in E_\lambda$.

So E_λ is "almost" a subspace of R^n. The only "problem" is that $0 \notin E_\lambda$.

Since it is most convenient to frame subsets as subspaces, we use an indirect definition.

We first note that for an eigenvector $\overrightarrow{x}$ we have:

$$A\overrightarrow{x} = \lambda\overrightarrow{x}$$

or equivalently

$$A\overrightarrow{x} - \lambda\overrightarrow{x} = \overrightarrow{0}.$$

If we let $I = I_n$ Now

$$\overrightarrow{x} = I\overrightarrow{x}$$

and so

$$\lambda\overrightarrow{x} = \lambda I\overrightarrow{x}.$$

Putting this all together we get:

$$A\overrightarrow{x} - \lambda I\overrightarrow{x} = \overrightarrow{0}$$

which we finally can rewrite as:

$$(A - \lambda I)\overrightarrow{x} = \overrightarrow{0}$$

Definition 4.2 *If λ is an eigenvalue for square matrix A, the null space of $A-\lambda I$ is the* ***eigenspace of** A **corresponding to** λ.*

One of the best ways to describe a subspace is to give a basis for that subspace. So in many of our problems we will be asked to find a basis for the eigenspace. Sometimes we just want to know the dimension of an eigenspace.

Example 4.3 Recall Example 2.6 $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

What we found out was that the matrix has two eigenvalues and that

- The eigenspace corresponding to eigenvalue 1 has basis $\{\overrightarrow{e_1}, \overrightarrow{e_2}\}$
- The eigenspace corresponding to eigenvalue 0 has basis $\{\overrightarrow{e_3}\}$.

Example 4.4 Here is problem 15. Given $A = \begin{pmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{pmatrix}$ find an eigenspace for $\lambda = 3$

Then

$$A - 3I = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{pmatrix}$$

which clearly row reduces to

$$A - 3I = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we write $\overrightarrow{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and write free variables as parameters: $z = t, y = s$, then $x = -2s - 3t$. So the null space consists of all vectors of the form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s - 3t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

So a basis for this eigenspace is:

$$\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \}.$$