22m:033 Notes:
3.1 Introduction to Determinants

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1 When does a 2×2 matrix have an inverse?

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ we have found that A is invertible if and only if the determinant

$$a_{11}a_{22} - a_{12}a_{21}$$

is not zero. This was part of a homework problem and the method was to use our method of calculating inverse and pay attention to the algebra at the end of the calculation.

So the calculation begins with constructing

If

$$\begin{pmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{pmatrix}.$$

Assume that $a_{11} \neq 0$; then we can multiply the top row by $\frac{1}{a_{11}}$. (If it is 0 we will need an argument for this case, but we will omit this detail here.)

$$\begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{1}{a_{11}} & 0 \\ a_{21} & a_{22} & 0 & 1 \end{pmatrix}.$$

Next we multiply row 1 by $-a_{21}$ and add to row 2 and get

$$\begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{1}{a_{11}} & 0 \\ 0 & (a_{22} - a_{21}\frac{a_{12}}{a_{11}}) & \frac{-a_{21}}{a_{11}} & 1 \end{pmatrix}.$$

Rewriting the expression in the second row we get

$$\begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{1}{a_{11}} & 0 \\ 0 & \frac{a_{11}a_{22}-a_{21}a_{12}}{a_{11}} & \frac{-a_{21}}{a_{11}} & 1 \end{pmatrix}.$$

Our next step would be to multiply row 2 by $\frac{a_{11}}{a_{11}a_{22}-a_{21}a_{12}}$ which would give us a 1 in the second column of the second row. HOWEVER we can only do if and only if $a_{11}a_{22} - a_{21}a_{12} \neq 0$. This expression $a_{11}a_{22} - a_{21}a_{12}$ is our determinant. At this point we have determined our condition for deciding whether or not A has an inverse.

2 Determinant of a 3×3 matrix

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. We ask: For what values of the a_{ij} will be invertible?

We can do the same thing—it is just a longer calculation.

1. Assume that $a_{11} \neq 0$; then we can multiply the top

row by $\frac{1}{a_{11}}$

2. Next multiply row 1 by $-a_{21}$ and add to row 2
3. Then multiply row 1 by $-a_{31}$ and add to row 3
4. ...

If we did this we would find that we can invert A if and only if

$$-a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33} \neq 0$$

Definition 2.1 *If A is a 3×3 matrix, the determinant $\triangle$ of A is defined*

$$\triangle = -a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33}$$

Remark 2.2 If we examine the terms of this expression for $\triangle$ we can observe:

1. each the six terms is a product of three numbers say $a, b,$ and c
2. no two of these numbers are in the same column, no two are in the same row
3. half (three) the terms appear with a "+" and half with a "−"

We ask: what is a rule which could tell us which of the terms appear with a "+" and which with a "−"?

3 The Determinant in the $n \times n$ case

There are some interesting ways to define a sign convention to these terms. Also we can extend the definition of determinant for the $n \times n$ case. Our text chooses the following (inductive) definition:

Definition 3.1 *If A is an $n \times n$ matrix, let A_{ij} denote the $(n-1) \times (n-1)$ matrix formed by deleting the i-th row and the j-th column from A. The **determinant of** A, denoted* $\det A$ *is defined:*

$$\det A = (-1)^{i+j} \sum_{j=1}^{n} a_{1j} A_{1j}$$

Remark 3.2 To understand the sign $(-1)^{i+j}$ it is helpful to note that this is positive if $i+j$ is even and negative if $i+j$ is odd and that if we mark locations in a matrix with "+" and "-" accordingly we get a "checkerboard" pattern:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

In our definition the first row of A has been singled out for special attention. There is no need for this. We could alternatively used any row as the next theorem shows. In fact we could also have a rule based on columns. First we introduce a definition.

Definition 3.3 *Given a square matrix A, the* ***(i, j)-cofactor of*** *A is*

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Proposition 3.4 *If A is an $n \times n$ matrix, then we can calculate* $\det A$ *using a formula based on any row and any column as follows:*

The calculation based on the i-th row is:

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$

The calculation based on the j-th column is:

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \blacksquare$$

Notation: a second common notation for $\det A$ is $|A|$. Thus for a 2×2 matrix we write:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 3.5 We first calculate using the definition:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45-48)-2(36-42)+3(32-35) = 3-2(-6)+3(-3) = 0$$

We began discussion of determinant as a way of deciding if a matrix had an inverse. Since the determinant of our matrix is zero we conclude that the matrix is not invertible. In a way this is something that we have already seen.

Consider the set of equations:

$$\begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 0 \\ 7x + 8y + 9z &= 0 \end{aligned}$$

and suppose we ask: does this set of equations have a unique solution? This has a matrix M:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

We have seen in previous class notes that the row reduced form of this is:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

and we conclude that M does not have a unique solution.

We could have concluded the fact that M does not have a unique solution as follows. We calculated that the determinant was zero. This implies that M is not invertible and this implies (using one of the conclusions of the Inverse Theorem) that our equation $M\overrightarrow{x} = \overrightarrow{0}$ does not have a unique solution.

Getting back the the calculation of the determinant, we could use the second row to calculate:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -4\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

$$= -4(18-24)+5(9-21)-6(8-14) = -4(-6)+5(-12)-6(-6) =$$

$$24 - 60 + 36 = 0$$

Here is a calculation based on the last column:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} + 9\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 3(32-35)-6(8-14)+9(5-8) = 3(-9)-6(-6)+9(-3) = 0$$

Example 3.6 Next we calculate the determinant of a 4×4 matrix. Note in the matrix the second column is mostly zeros. This can simplify the calculation a little if we base or calculation on this column:

$$\begin{vmatrix} 1 & 0 & -2 & 1 \\ 1 & 2 & -2 & 1 \\ -2 & 0 & -1 & -3 \\ 1 & 0 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & -2 & 1 \\ -2 & -1 & -3 \\ 1 & 1 & 2 \end{vmatrix} = 2(-2) = -4$$

But even if we base calculation on first column, we get same answer:

$$\begin{vmatrix} 1 & 0 & -2 & 1 \\ 1 & 2 & -2 & 1 \\ -2 & 0 & -1 & -3 \\ 1 & 0 & 1 & 2 \end{vmatrix} =$$

$$1 \begin{vmatrix} 2 & -2 & 1 \\ 0 & -1 & -3 \\ 0 & 1 & 2 \end{vmatrix} -1 \begin{vmatrix} 0 & -2 & 1 \\ 0 & -1 & -3 \\ 0 & 1 & 2 \end{vmatrix} +(-2) \begin{vmatrix} 0 & -2 & 1 \\ 2 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} -1 \begin{vmatrix} 0 & -2 & 1 \\ 2 & -2 & 1 \\ 0 & -1 & -3 \end{vmatrix} =$$

$$(1)(2)(-2+3)-1(0)+(-2)(-2)(-4-1)-1(-2)(6+1) =$$

$$2 - 0 - 20 + 14 = -4$$

4 Determinants of matrices with an organized pattern of zeros

Definition 4.1 *In a square matrix $A = (a_{ij})$ the **main diagonal of** A are the locations where $i = j$. Visually this is a diagonal that goes from the upper left to the lower right corners.*

*A **diagonal matrix** is one whose non diagonal elements are all zero. A **matrix is upper triangular** if all the entries below the main diagonal are zero. A **matrix is lower triangular** if all the entries above the main diagonal are zero. A **matrix is triangular** if it is either an upper triangular matrix or a lower triangular matrix.*

Because of all the zeros, It is easy calculate determinants of these matrices:

Proposition 4.2 *If M is a diagonal matrix then $det(M)$ is the product of all of the entries on the diagonal.* ∎

and similarly that

Proposition 4.3 *If M is a triangular matrix then*

$det(M)$ is the product of all of the entries on the diagonal. ∎

Example 4.4

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 8 & 10 \\ 0 & 2 & 6 & 9 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ -100 & 13 & 3 & 0 \\ 66 & 1028 & -999 & 4 \end{vmatrix} = 1{\cdot}2{\cdot}3{\cdot}4 = 24$$

Question 4.5 There are 6 alternate ways to calculate the determinant of the 4×4 matrix given in Example 3.6. Show these in detail as done in that example.