22m:033 Notes:
1.2 Row Reduction and Echelon Forms

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1 Echelon form and reduced Echelon form

Definition 1.1 *The first non-zero entry in a row of a matrix is called a* ***leading entry****.*

A zero row has no leading entry.

Definition 1.2 *A rectangular matrix is in* ***echelon form*** *(or* ***row echelon form****) if the first three conditions are met*

1. *All rows of zeros constitute the bottom most rows of the matrix,*
2. *Each leading entry lies in a column to the right of the leading entry of the previous row.*

   If we, in addition, we have the next two conditions met, we say the matrix is in ***reduced echelon form*** *(or* ***reduced row echelon form****)*
3. *The leading entry in each non-zero row is 1*
4. *In a column containing a leading entry, all other entries are 0.*

Remark 1.3 The text has a third condition for echelon form " If a column has a leading entry then all entries in that column and below that entry are zero" and they point out that they do not need it. For clarity, we omit this in our definition.

The two conditions for echelon form might be paraphrased as:

1. "zero rows have been pushed to the bottom"
2. "rows arranged so that number of initial zeros always increases as you go down in rows"

The following result is basic to much of what we will do in this course:

Proposition 1.4 *(Uniqueness of Reduced Echelon Form) Each matrix is row equivalent to only and only one reduced echelon matrix.*

We will not prove this in class. A proof is found in Appendix A of our text, based on a result of a chapter we will not cover.

2 Pivot Postitions

The following definition is technical, but is often used in this text as well as others.

Definition 2.1 *A **pivot position of a matrix** A is the location of a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position. A pivot is a non-zero entry of A in a pivot position.*

Remark 2.2 Note that the terms pivot position and pivot column refer to the original matrix if we somehow knew (at least something about) what reduced form would be. So in general you cannot just look at a matrix and easily see what the pivot positions or columns are without going through (as least part of) a row reduction calculation.

3 The Row Reduction Algorithm

1. Begin with the leftmost zero column.

2. Interchange rows if needed to make sure that there is a non-zero entry at the top of this column.

3. Use row replacement operations of create zeros below this entry

4. Ignore that top row and any rows if any above it. Repeat the three steps above for the remaining matrix.

   Using the steps above will give a matrix in echelon form. to get a matrix in reduced echelon form we also do

5. Beginning with the last non-zero row, working upwards and do the left, create zeros above the pivot and make 1 by scaling

4 Solving Linear equations via row reduction

Most students find the process of row reducing a given matrix easy and it gives a good way of solving systems of equations.

The "hard part" is understanding what the solutions to the system are. For this we need to take a reduced echelon matrix and think about what the corresponding equations are.

4.1 Unique solution

Suppose our reduced echelon matrix (where the columns correspond the variables $a, b, c,$ and d) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

This first row corresponds to $1a + 0b + 0c + 0d = 0$, in other words $a = 0$. We see there is a unique solution: $a = 0, b = -5, c = 3,$ and $d = -1$.

4.2 Inconsistent equations

However if the reduced augmented matrix had a row with all zeros except in the last column, then the equations would be inconsistent. For example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Here that bottom row corresponds to the equation

$$0a + 0b + 0c + 0d = -1 \text{ or } 0 = -1.$$

This is impossible and so this means that a solution does not exist.

4.3 Case of infinitely many solutions

Suppose our reduced augmented matrix is:

$$\begin{pmatrix} 1 & 0 & -1 & 3 & 0 \\ 0 & 1 & 0 & 2 & -5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This corresponds the system of equations:

$$\begin{aligned} a - c + 3d &= 0 \\ b + 2d &= -5 \\ c &= 3 \\ 0 &= 0 \end{aligned}$$

We will adopt the routine of going from the bottom equations towards the top.

The fourth equation tells us nothing, the third that c must be 3. The second equation relates b and d.

There are two ways to proceed at this point, the difference is only in the notation.

4.3.1 The "free variable" style

The value of d is not determined and is termed "free". Then $b = -2d - 5$.

The top equation can be written: $a = c - 3d = 3 - 3d$

So we simply say the solutions are:

- $c = 3$
- d is free
- $b = -2d - 5$
- $a = 3 - 3d$

4.3.2 Using other parameters style

Another way to express this is to say that we let $d = t$. Where t is a parameter.

Then the solutions will be expressed.

- $c = 3$
- $d = t$
- $b = -2t - 5$
- $a = 3 - 3t$

This is used in the text in Section 1.5 and called the parametric vector form.

4.3.3 More than one parameter (free variable)

The example we just did needed only one parameter (free variable), however there could be many needed in some cases.

For example suppose the reduced echelon matrix of a system with variables x, y, z and w, in that order is:

$$\begin{pmatrix} 1 & 0 & -1 & 3 & 0 \\ 0 & 1 & 0 & 2 & -5 \end{pmatrix}$$

These equations are:

$$\begin{aligned} x - z + 3w &= 0 \\ y + 2w &= -5 \end{aligned}$$

In this case we let $w = t$ and $z = s$

and

$$\begin{aligned} x &= s - 3t \\ y &= -2t - 5 \end{aligned}$$