Moduli spaces of representations of tame finite-dimensional algebras

Ryan Kinser

University of Iowa

Banff International Research Station
September 29, 2022

Push the button/Appuyez sur le bouton!

Tame algebras 1

- $\mathbb{k}$: algebraically closed field of characteristic 0
- A: finite dimensional associative $\mathbb{k}$-algebra ...
  (Up to Morita equivalence, $A \simeq \mathbb{k}Q/I$ for quiver Q and admissible ideal I)
- **Definition.** A is *tame* if, for each dimension d:
  – there exists a finite number of A-$\mathbb{k}[X]$-bimodules M_i which are free of rank d as right $\mathbb{k}[X]$-modules,
  – such that every indecomposable A-module of dimension d is isomorphic to $M_i \otimes_{\mathbb{k}[X]} \mathbb{k}[X]/(X - \lambda)$ for some i and some $\lambda \in \mathbb{k}$.
- Informally: each bimodule M_i determines a "1-parameter family" of A-modules (*example coming*)

Tame algebras 2

Example: Kronecker quiver

$Q: \bullet \rightrightarrows \bullet$ Indecomposables:

$$\Bbbk^n \underset{\begin{bmatrix}0\\ I_n\end{bmatrix}}{\overset{\begin{bmatrix}I_n\\ 0\end{bmatrix}}{\rightrightarrows}} \Bbbk^{n+1}, \qquad \Bbbk^{n+1} \underset{\begin{bmatrix}0 & I_n\end{bmatrix}}{\overset{\begin{bmatrix}I_n & 0\end{bmatrix}}{\rightrightarrows}} \Bbbk^{n},$$

$$\Bbbk^n \underset{J_n(\lambda)}{\overset{I_n}{\rightrightarrows}} \Bbbk^n \quad (\lambda \in \Bbbk), \qquad \Bbbk^n \underset{I_n}{\overset{J_n(0)}{\rightrightarrows}} \Bbbk^n$$

$$\hookrightarrow \Bbbk[X]^n \underset{J_n(X)}{\overset{I_n}{\rightrightarrows}} \Bbbk[X]^n$$ evaluated at $X = \lambda$.

Tame algebras 3

- For algebras of global dimension 1, the tame algebras are exactly $A \simeq \Bbbk Q$ where Q is an affine Dynkin quiver.

- For higher global dimension, no classification of tame algebras. Special cases:
  – 1 vertex/simple module: Ringel, 1974
  – 2 vertices/simples: Hoshino and Miyachi, 1988

- There are a few robust general results about tame algebras, for example Crawley-Boevey 1988, 1991.

⋆ Most methods to prove an algebra is tame do **not** produce an explicit description of its indecomposable representations!

Moduli Spaces of Representations 1 (Overview)

If we want to geometrically organize representations of tame algebras, there is "too much freedom" in the choice of 1-parameter families above.

General idea: Apply Geometric Invariant Theory (GIT) to construct projective varieties that "naturally" parametrize isomorphsim classes of representations (A.D. King, 1994)

Today's Goal: Understand these for tame algebras, particularly the "1-parameter families"

Moduli Spaces of Representations 2 (Stability)

- A *weight* on Q is a tuple of integers indexed by the vertices of Q:

$$\theta \in \mathbb{Z}^{\{\text{vertices of } Q\}}.$$

  A weight induces a function $\theta\colon \operatorname{rep}(A) \to \mathbb{Z}$, where $\theta(V) = \theta \cdot \underline{dim} V$.

- V is *θ-semistable* if $\theta(W) \leq \theta(V) = 0$ for all $W \leq V$

- V is *θ-stable* if it is θ-semistable and $\theta(W) \neq 0$ for all $0 < W < V$

Lemma. The full subcategory $\operatorname{rep}_\theta^{ss}(A)$ of θ-semistable representations is abelian and closed under extensions.
Its simple objects are precisely the θ-stable representations.

Informal slogan: The weight θ picks out an abelian subcategory of $\operatorname{rep}(A)$ with a richer collection of simples

Moduli Spaces of Representations 3 (Technical)

(Some details for the AG folks, not necessary for rest of talk.)

Given $A = \Bbbk Q/I$ and dim vector $\underline{\mathrm{d}}$, there is an affine variety $\mathrm{rep}_A(\underline{\mathrm{d}})$ parametrizing reps of A of dim vector d in a fixed basis.
The natural base change group $GL(\underline{\mathrm{d}})$ acts on $\mathrm{rep}_A(\underline{\mathrm{d}})$ by "conjugation".

For $Z \subseteq \mathrm{rep}_A(\underline{\mathrm{d}})$ irreducible and $GL(\underline{\mathrm{d}})$-invariant closed subvariety, let

$$SI(Z)_\theta := \{f \in \Bbbk[Z] \mid g \cdot f = \theta(g)f,\ \forall g \in GL(\underline{\mathrm{d}})\}$$

$$\text{and then} \quad \mathcal{M}_\theta^{ss}(Z) := \mathrm{Proj}\,\Big(\bigoplus_{n \geq 0} SI(Z)_{n\theta}\Big),$$

called *moduli space of θ-semistable reps in Z*.

The points of $\mathcal{M}_\theta^{ss}(A, \underline{\mathrm{d}}) := \mathcal{M}_\theta^{ss}(\mathrm{rep}_A(\underline{\mathrm{d}}))$ are in bijection with isoclasses of direct sums of θ-stable reps of A (i.e. semi-simple objects of $\mathrm{rep}_\theta^{ss}(A)$ of dim. vect. $\underline{\mathrm{d}}$).

Moduli Spaces of Representations 4 (Kronecker example)

Example: Kronecker quiver

$$Q : \bullet \rightrightarrows \bullet$$

For $\underline{d} = (d_1, d_2)$ let $\theta = (d_2, -d_1)$

$$\Bbbk^n \overset{\begin{bmatrix} I_n \\ 0 \end{bmatrix}}{\underset{\begin{bmatrix} 0 \\ I_n \end{bmatrix}}{\rightrightarrows}} \Bbbk^{n+1}, \qquad \Bbbk^{n+1} \overset{\begin{bmatrix} I_n & 0 \end{bmatrix}}{\underset{\begin{bmatrix} 0 & I_n \end{bmatrix}}{\rightrightarrows}} \Bbbk^n,$$

Each $\mathcal{M}^{ss}_{\theta}(Q, \underline{d})$ is single point

$$\Bbbk^n \overset{I_n}{\underset{J_n(\lambda)}{\rightrightarrows}} \Bbbk^n \quad (\lambda \in \Bbbk), \qquad \Bbbk^n \overset{J_n(0)}{\underset{I_n}{\rightrightarrows}} \Bbbk^n$$

Can compute $\mathcal{M}^{ss}_{\theta}(Q, (1,1)) \simeq \mathbb{P}^1$

and in general $\mathcal{M}^{ss}_{\theta}(Q, (n,n)) \simeq \mathbb{P}^n$.

Some history of moduli of reps of tame algebras

The works cited below build evidence for the following conjecture:

Conjecture. Let A be a tame algebra. Then any irreducible component of $\mathcal{M}^{ss}_{\theta}(A, \underline{\mathrm{d}})$ is isomorphic to $\mathbb{P}^{e_1} \times \cdots \times \mathbb{P}^{e_r}$ for some $e_i \geq 0$.

The conjecture has been proven for:

- algebras of global dimension 1 (Domokos-Lenzing '02)
- tilted algebras (Chindris '13)
- quasi-tilted algebras (Bobiński '14)
- acyclic gentle algebras (Carroll-Chindris '15)

The techniques are often representation-theoretical, using homological algebra, Schofield semi-invariants, and Auslander-Reiten theory to understand how the spaces of semi-invariants $SI(Z)_{n\theta}$ grow with n.

Some recent advances 1 (A key tool)

The Moduli Decomposition Theorem (joint with Calin Chindris, 2018).
(This works for arbitrary finite dimensional algebras, not just tame.)
Very informally, the 3 parts of this theorem describe how $\mathcal{M}_\theta^{ss}(Z)$ is determined by moduli spaces associated to the θ-Jordan-Holder factors of representations in Z...*up to a finite, birational morphism*.

In the special case of tame algebras it implies:
$\star$ if we can show Z_θ^{ss} is a *normal variety* whenever $\mathcal{M}_\theta^{ss}(Z)$ is an irreducible component of $\mathcal{M}_\theta^{ss}(A, \underline{d})$, then the Conjecture holds for the algebra A.

Some recent advances 2 (Specific results)

Theorem. *(joint work with Carroll, Chindris, Weyman, 2020)*
The Conjecture is true for *special biserial algebras*.

Informally, all indecomposable representations of a special biserial algebra $A = \Bbbk Q/I$ can be obtained by "pushing forward" representations of Dynkin types $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ onto Q while "avoiding" I.

Theorem. *(Cody Gilbert, arXiv:2208.00336, August 2022)*
The Conjecture is true for *clannish algebras*.

Clannish algebras are a generalization of special biserial where $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ are replaced with $\mathbb{D}$ and $\widetilde{\mathbb{D}}$. The class includes skewed-gentle algebras.

In contrast with earlier results, these proofs are almost entirely geometric, relying on a lucky/magical connection with affine Schubert varieties.

Ideas for more progress on Conjecture

Next we need to get a foothold above specific classes of algebras, but not necessarily at the level of full generality. Some ideas of things to look at are:

- For arbitrary tame A, restrict to irreducible components $Z \subseteq \text{rep}_A(\underline{d})$ such that a general representation has projective dimension 1 and trivial endomorphism ring. For such $\underline{d}$, there is a canonical weight $\theta = \langle \underline{d}, \cdot \rangle_A$ where $\langle \, , \cdot \rangle_A$ is the Euler form of A.
  $\Rightarrow$ Prove $\mathcal{M}_\theta^{ss}(Z) \simeq \mathbb{P}^1$ in this case. *(This is an idea of Chindris.)*
- Approach general case via covering theory of algebras (P. Gabriel, E. Green, early 1980s).
  (a) Prove the conjecture for strongly simply connected tame algebras, where more tools are available, for example good behavior of the Tits quadratic form (Brüstle-de la Peña-Skowroński, 2010).
  (b) Separately, study general behavior of moduli spaces with respect to covering functors.