Dimensional Analysis

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October 7, 2011

Introduction

Dimensional analysis is concerned with the relationships amongst the dimensions of the parameters and variables that occur in an equation. By dimension we mean the fundamental quantities of the system. For example the dimensions of most mechanical systems are length, mass, and time. The most basic idea of dimensional analysis is that, in order for an equation that is to represent a natural phenomena to be meaningful, the terms appearing on each side of the equality must have the same dimensions. What is more, dimensional analysis is a first step into the idea of invariance (or symmetry) of differential equations which is useful in constructing solutions (e.g. similarity solutions).

The fundamental theorem of dimensional analysis is the so called Buckingham Pi Theorem first proved by the American engineer E. Buckingham in 1914 [2]. The basic idea of the theorem is that relations between natural quantities can be expressed in an equivalent form that is comprised entirely of dimensionless quantities. On reason why this theorem is important is that natural laws (i.e. equations) should be independent of the units used in the expressions of the law. Consider for example Newton's second law expressed as

$$F = ma. \tag{1}$$

Since force has dimensions $\frac{\text{mass} \cdot \text{length}}{\text{time}^2}$ and acceleration has dimensions $\frac{\text{length}}{\text{time}^2}$ both sides of (1) have the same dimensions and do not depend on the specific system of units used to measure the quantities involved. We say that (1) is dimensionally homogeneous.

Dimensional analysis is also useful in deriving the relationships between quantities. For example, suppose that the speed s of a ball depends only on its radius r and the time t and no other physical quantities. Then we have a functional relationship

$$s = f(r, t). \tag{2}$$

Now, dimensional analysis tells us that the expression $f(r, t)$ should have the dimensions of speed, i.e. $\frac{\text{length}}{\text{time}}$. The only possible way to combine r and t to get the dimensions of speed is through their ratio $\frac{r}{t}$. For example if we assumed an expression such as

$$s = ar + bt$$

then a, b would have to have units which contradicts our assumption that the only physical quantities that s depends on are r and t. Thus we must have

$$s = c\frac{r}{t} \tag{3}$$

where c is a pure number.

The next section is concerned with the formal statement of the Buckingham Pi theorem. Following that we give some simple examples of its applications. Finally in the last section we give an application of dimensional analysis to the reduction of variables in a partial differential equation (PDE) and discuss the relationship between dimensional analysis and invariance under groups of scaling. For further information on dimensional analysis and realted topics see [1, 4].

Main Result

The discussion of the Buckingham Pi theorem follows [1, 3]. The main idea of the approach to the theorem taken here is to transform the problem of reducing equations into equivalent dimensionless equations into a problem of linear algebra. This leads to an algorithm for reducing a dimensionally homogeneous expression into equivalent expression with a smaller number of dimensionless variables [3].

Assumptions of Dimensional Analysis

We assume

1. A quantity u of interest is determined by n measurable quantities (the independent variables and parameters) $\{x_1, x_2, \ldots, x_n\}$ as

   $$u = f(x_1, x_2, \ldots, x_n). \tag{4}$$

   We assume that this expression is dimensionally homogeneous, i.e. independent of the choice of units.

2. The quantities $\{u, x_1, x_2, \ldots, x_n\}$ are measured in terms of m fundamental dimensions labeled $\{L_1, L_2, \ldots, L_m\}$.

3. Let W represent any of $u, x_1, x_2, \ldots, x_n$, then we denote by $[W]$ the dimension of W. This can be written as

   $$[W] = L_1^{p_1} L_2^{p_2} \cdots L_m^{p_m}, \tag{5}$$

for some real numbers p_i, $i = 1, \ldots, m$ which we can arrange into an m-dimensional column vector

$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix}, \tag{6}$$

the dimension vector of W. A quantity is said to be *dimensionless* if and only if $[W] = 1$. Let

$$\mathbf{P_j} = \begin{pmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{mj} \end{pmatrix}, \tag{7}$$

$j = 1, \ldots, n$ be the dimension vector of x_j, $j = 1, \ldots, n$. Then we have the $m \times n$ dimension matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{P_1}| & \mathbf{P_2}| & \cdots| & \mathbf{P_n}| \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{pmatrix}. \tag{8}$$

4. For any set of fundamental dimensions $\{L_1, L_2, \ldots, L_m\}$, one can choose a system of units for measuring the value of any quantity W. Note that a dimensionless quantity is invariant under a scaling of units.

The Situation for a Mechanical System

Before stating the conclusions of the Buckingham Pi theorem let us consider the situation of a mechanical system so that the only fundamental dimensions are length L, mass M, and time T. Suppose that u depends on the physical parameters or variables $\{x_1, x_2, \ldots, x_n\}$. Then we have that for each $i = 1, \ldots, n$

$$[x_i] = L^{\ell_i} M^{m_i} T^{t_i}. \tag{9}$$

Thus the dimension vector for each x_i is

$$\mathbf{P_i} = \begin{pmatrix} \ell_i \\ m_i \\ t_i \end{pmatrix}, \tag{10}$$

and the dimension matrix is

$$\mathbf{A} = \begin{pmatrix} \ell_1 & \ell_2 & \cdots & \ell_n \\ m_1 & m_2 & \cdots & m_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix}. \tag{11}$$

For example from above (1) we have

$$[m] = M^1 \tag{12}$$

$$[a] = L^1 T^{-2} \tag{13}$$

and hence a dimension vectors

$$\mathbf{P}_m = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{14}$$

$$\mathbf{P}_a = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \tag{15}$$

and dimension matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -2 \end{pmatrix}. \tag{16}$$

Conclusion of Buckingham Pi

The assumptions above have the following conclusions:

1. The relation

$$u = f(x_1, x_2, \ldots, x_n), \tag{17}$$

can be expressed in terms of dimensionless quantities.

2. The number of dimensionless quantities is

$$k + 1 = n + 1 - \operatorname{rank}(\mathbf{A}). \tag{18}$$

3. Since $\mathbf{A}$ has $\operatorname{rank}(\mathbf{A}) = n - k$, there are k linearly independent solutions of

$$\mathbf{A}\mathbf{z} = 0$$

which we denote by $\mathbf{z^1}, \mathbf{z^2}, \ldots, \mathbf{z^k}$. Let $\mathbf{a}$, an m column vector, be the dimension vector of u, and let $\mathbf{y}$, an n column vector be a solution of

$$\mathbf{Ay} = -\mathbf{a}. \tag{19}$$

Then (17) simplifies to

$$\pi = g(\pi_1, \pi_2, \ldots, \pi_k) \tag{20}$$

where

$$\pi = u x_1^{y_1} x_2^{y_2} \cdots x_n^{y_n} \tag{21}$$

$$\pi_i = x_1^{\mathbf{z^i}_1} x_2^{\mathbf{z^i}_2} \cdots x_n^{\mathbf{z^i}_n}, \quad i = 1, \ldots, k. \tag{22}$$

That is,

$$u = x_1^{-y_1} x_2^{-y_2} \cdots x_n^{-y_n} g(\pi_1, \pi_2, \ldots, \pi_k). \tag{23}$$

Example with Newton's Law

According to the theorem, we can reduce 1 to an equivalent expression

$$\pi_F = g \equiv 1, \tag{24}$$

since the dimension of the null space of 16 is 0. That is, the only solution of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{z} = 0, \tag{25}$$

is $\mathbf{0}$ we have

$$\pi_F = 1, \tag{26}$$

where

$$\pi_F = F m^{-1} a^{-1}. \tag{27}$$

This is because the solution of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{y} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \tag{28}$$

is

$$\mathbf{y} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}. \tag{29}$$

Example with Explosions

Let r be the radius of a shock wave. For an atomic explosion it is assumed that

$$r = f(E, t, \rho_0, P_0), \tag{30}$$

where

1. E - energy of explosion,
2. t - elapsed time after explosion,
3. ρ_0 - initial air density,
4. P_0 - initial air pressure.

Thus in the notation of the theorem

1. $u = r$,
2. $x_1 = E$,
3. $x_2 = t$,
4. $x_3 = \rho_0$,
5. $x_4 = P_0$.

Also

1. $[u] = [r] = L^1 M^0 T^0$,
2. $[x_1] = [E] = L^2 M^1 T^{-2}$,
3. $[x_2] = [t] = L^0 M^0 T^1$,
4. $[x_3] = [\rho_0] = L^{-3} M^1 T^0$,
5. $[x_4] = [P_0] = L^{-1} M^1 T^{-2}$.

Then

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ -2 & 1 & 0 & -2 \end{pmatrix}, \tag{31}$$

and

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{32}$$

Now

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ -2 & 1 & 0 & -2 \end{pmatrix} \mathbf{z} = 0 \tag{33}$$

has general solution

$$z = s \begin{pmatrix} -\frac{2}{5} \\ \frac{6}{5} \\ -\frac{3}{5} \\ 1 \end{pmatrix} \tag{34}$$

thus the number k of linearly independent solutions is 1, thus we expect there to be 1 dimensionless quantity π_1. Now, the general solution of

$$\mathbf{Ay} = -\mathbf{a}, \tag{35}$$

is

$$\mathbf{y} = \begin{pmatrix} -\frac{1}{5} \\ -\frac{2}{5} \\ \frac{1}{5} \\ 0 \end{pmatrix} + s \begin{pmatrix} -\frac{2}{5} \\ \frac{6}{5} \\ -\frac{3}{5} \\ 1 \end{pmatrix}, \tag{36}$$

taking $s = 0$ gives

$$\pi = rE^{-\frac{1}{5}} t^{-\frac{2}{5}} \rho_0^{\frac{1}{5}} P_0^0 = r \left(\frac{\rho_0}{Et^2} \right)^{\frac{1}{5}}. \tag{37}$$

Moreover, the theorem implies that π has the form $\pi = g(\pi_1)$ therefore

$$r = \left(\frac{Et^2}{\rho_0} \right)^{\frac{1}{5}} g(\pi_1) \tag{38}$$

where π_1 is a dimensionless quantity.

Application to PDEs

Suppose that we have the quantities $u, x_1, x_2, \ldots, x_n$ where u is the solution of a boundary value problem and $x_1, x_2, \ldots, x_n$ are the independent variables and parameters of the equation. Suppose that there are ℓ independent variables and $n-\ell$ parameters. Then we can form $\mathbf{A}_1$, the $m \times \ell$ dimension matrix for the independent variables and $\mathbf{A}_2$, the $m \times (n-\ell)$ dimension matrix for the parameters. Then the dimension matrix for the boundary value problem is

$$\mathbf{A} = \left(\left. \mathbf{A_1} \right| \mathbf{A_2} \right). \tag{39}$$

Applying the Buckingham Pi theorem allows us to reduce the number of independent variables by $\text{rank}(\mathbf{A}_1)$ and the number of parameters by $\text{rank}(\mathbf{A}_2)$.

Consider the boundary value problem for heat conduction

$$\rho c \frac{\partial u}{\partial t} - K \frac{\partial^2 u}{\partial x^2} = 0, \qquad -\infty < x < \infty, \quad t > 0, \tag{40}$$

$$u(x,0) = \frac{Q}{\rho c}\delta(x), \tag{41}$$

$$\lim_{x \to \pm\infty} u(x,t) = 0. \tag{42}$$

The solution u to this boundary value problem will depend on the variables x, t and the parameters ρ, c, K, Q. We use so called "thermal units" for the fundamental dimensions, that is

1. $L_1 = \text{length} = L$,
2. $L_2 = \text{time} = T$,
3. $L_3 = \text{mass} = M$,
4. $L_4 = \text{temperature} = H$,
5. $L_5 = \text{calories} = C$.

We then have

1. $[u] = L^0 M^0 T^0 H^1 C^0$,
2. $[x_1] = [x] = L^1 M^0 T^0 H^0 C^0$,
3. $[x_2] = [t] = L^0 M^0 T^1 H^0 C^0$,
4. $[x_3] = [\rho] = L^{-3} M^1 T^0 H^0 C^0$,
5. $[x_4] = [c] = L^0 M^{-1} T^0 H^{-1} C^1$,

6. $[x_5] = [K] = L^{-1}M^0T^{-1}H^{-1}C^1$,
7. $[x_6] = [Q] = L^{-2}M^0T^0H^0C^1$.

Thus the dimension matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -3 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}. \tag{43}$$

The dimension of the null space of $\mathbf{A}$ is one so there is one dimensionless quantity π_1. We consider the solution

$$\mathbf{z} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} \tag{44}$$

of $\mathbf{Az} = 0$, and the solution

$$\mathbf{y} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix}, \tag{45}$$

of

$$\mathbf{Ay} = -\mathbf{a}. \tag{46}$$

Then by the theorem

$$\pi_1 = x^1 t^{-\frac{1}{2}} \rho^{\frac{1}{2}} c^{\frac{1}{2}} K^{-\frac{1}{2}} Q^0. \tag{47}$$

We define

$$\kappa = \frac{K}{\rho c} \tag{48}$$

and write

$$\pi_1 = \eta := \frac{x}{\sqrt{\kappa t}}. \tag{49}$$

Thus we have that

$$\pi = u x^0 t^{\frac{1}{2}} \rho^{\frac{1}{2}} c^{\frac{1}{2}} K^{\frac{1}{2}} Q^{-1} = u \frac{\sqrt{\rho c K t}}{Q}, \tag{50}$$

or

$$u = \frac{Q}{\sqrt{\rho c K t}} g(\eta). \tag{51}$$

Substitution of this into the original boundary value problem leads to the ordinary differential equation

$$2g''(\eta) + \eta g'(\eta) + g(\eta) = 0, \tag{52}$$

with appropriate conditions. Thus we have reduced the number of independent variables.

References

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